Flow Control as Stochastic Optimal Control Problem with Incomplete Information

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Abstract— **The nonlinear stochastic control problem related with flow control is considered. The state of the link is described by controlled hidden Markov process while the loss flow is described by the counting process with the intensity depending on the current transmission rate and unobserved link state. The control is the transmission rate and have to be chosen as non-anticipating process depending on the observation of the loss process. The aim of the control is to achieve the maximum of some utility function taking into account the losses of transmitted information. Originally the problem belongs to a class of stochastic control with incomplete information, however, the optimal filtering equations giving the estimation of the current link state based on the observation of the loss process give the opportunity to reduce the problem to the standard stochastic control problem with full observations. Then, a necessary optimality condition is derived in the form of stochastic maximum principle which allows us to obtain explicit analytic expressions for the optimal control in some particular cases. The optimal and suboptimal controls are investigated and compared with the flow control schemes which is used in TCP/IP networks. In particular, the optimal control demonstrates a much smoother behavior than the currently used TCP/IP congestion control.**

Index Terms— **TCP/IP, Stochastic Processes, Stochastic Optimal Control.**

I. Introduction

THE transmission of most Internet data flows is governed by TCP (Transmission Control Protocol) [1], [14]. TCP has two principal aims: to ensure reliable data delivery and to control congestion level in the network. The TCP New Reno version, which is currently the most widely deployed TCP version, uses either packet losses or ECN (Explicit Congestion Notification, [27]) marks as an indication of the network congestion. Congestion Avoidance, the main TCP phase, corresponds to AIMD (Additive Increase Multiple Decrease) congestion control scheme with a binary feedback [26]. The evolution of the TCP sending rate can be viewed as a piece-wise deterministic process [2], [3], [6]. Namely, TCP increases the data sending rate linearly in time in the absence of congestion notifications and when a congestion notification arrives, TCP reduces the sending rate by half. Up to the present, several fundamental questions about TCP properties have not been answered. These questions are: What are the reasons behind the piece-wise deterministic TCP sending rate evolution? What should be an optimal increase rate in the absence of congestion notifications? Is it optimal do divide the sending rate by two if a congestion notification arrives? In this paper we provide answers to these questions from the point of view of stochastic optimal control with incomplete information [8]–[10]. Specifically, we describe a congested link as a Hidden Markov Model [9]. We note that a commonly used Gilbert link model [13] is a particular instance of Hidden Markov Models. As an optimization goal we choose to maximize the average utility function of the data sending rate minus a cost of transfer (or a cost of lost packets). Each congestion notification contributes some fixed amount to the cost of transfer. The present optimization criterion resembles the optimization criteria of [5], [16]–[18], [20], [22]. However, the models in these references are fluid models which are derived to calculate an average TCP sending rate and hence cannot be used to analyze the behavior of the instantaneous TCP sending rate. In fact, comparing [5], [20] and [17] one can see that different transmission rate control schemes can be proposed which solve the same optimization problem. The dependence of the cost counting process on the transmission rate makes our problem especially difficult as the intensity of jumps of the sending rate becomes dependent on the sending rate itself. Note that in [2]–[4] the intensity of jumps of the corresponding piece-wise deterministic process is independent of the sending rate. Only in [28] some asymptotics have been obtained for the piece-wise de-

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terministic Markov process with rate dependent jumps for small intensity of congestion notifications. However, [28] provides the performance evaluation of the current TCP New Reno and does not address the question of the optimal design of the congestion control mechanism. The present work explains the reasons for the piece-wise deterministic form of the sending rate evolution and provides means to set an optimal time-varying rate of the sending rate increase and an optimal size of the downward jumps.

The paper is organized as follows: In the next Section 2, we present the Hidden Markov Model of the congested path and the performance criterion. In Section 3, using the sufficient statistics set, we transform the originally incomplete information problem to the problem of the optimal control based on complete observations. In particular, we give the characterization of the optimal control problem solution in terms of piece-wise deterministic Markov control settings. In Section 4 we consider a further transformation of the problem with the aid of Girsanov's change of measure. The latter allows us to provide the characterization of the optimal control in terms of necessary optimality conditions. In particular, these conditions give the possibility to calculate the optimal control if the generator of the underlying Hidden Markov Model does not depend on the control. Then, in Section 5 by means of a simple numerical example we compare the optimal control with the "suboptimal" solution corresponding to the AIMD congestion avoidance scheme. We recall that AIMD is the principle phase of the current TCP New Reno version. Remarkably, the optimal control demonstrates a much smoother behavior than the AIMD congestion avoidance scheme. The paper is concluded in Section 6, where we discuss future possible developments of this research direction.

All technical proofs of the present work can be found in the accompanying INRIA Research Report [24].

II. Model description and the problem statement

A. Hidden Markov Model of the congested path

Suppose that all processes are defined on a probability space $\{\Omega, \mathcal{F}, P\}$. We assume that the state of a congested path in the Internet can be described by a Markov process X_t, whose state space is the set of unit vectors $\{e_1, ..., e_n\}$ of R^n. Write $p_t^i = P(X_t = e_i)$ and suppose there is a time-dependent family of generators $A^*(t,u)$[1] which also depends on the scalar control parameter $u \in U = [0, U_{max}], \quad U_{max} < \infty$. The control parameter u corresponds to current value of the transmission rate, such that the probability vector $p_t = \{p_t^1, ..., p_t^n\}^*$ satisfies the Kolmogorov forward equations

$$\frac{dp_t}{dt} = A(t, u_t) p_t.$$

We assume that matrix valued function $A(t,u)$ is continuous with respect to both variables, so the above equation has a unique solution for any measurable admissible control, such that $u_t \in U$, a.s..

Suppose that one can observe the binary congestion notification feedback which is correlated with the current state of the link X_t and the current value of the data sending rate u_t. We recall that the data is sent in packets in the Internet. To indicate the level of congestion the network sends the flow of congestion notifications back to a source. The congestion notifications in the Internet are either packet losses or ECN marks. To represent the flow of congestion notifications we use the generalized counting Poisson process N_t with the intensity

$$c(X_t)u_t =< c, X_t > u_t,$$

where $< \cdot, \cdot >$ is a scalar product, and $c = \{c(e_1), ..., c(e_n)\}^*$ is such that $c(e_i) > 0$, for all $e_i = 1, ..., n$.

We note that $\{X_t, N_t\}$ is a controlled jump Markov process with countable state space. The control u_t belongs to $D_{[0,T]}(U)$, the set of functions continuous from the left and with a finite limit from the right. In particular, we have

$$P\{X_{t+\Delta t} = e_i, N_{t+\Delta t} = j | X_t = e_k, N_t = l, u_t^{t+\Delta t}(\cdot)\}$$

$$= \begin{cases} o(\Delta t), \quad \text{if} \quad i \neq k, j \neq l; \\ a_{ik}(t, u_t)\Delta t + o(\Delta t), \quad \text{if} \quad i \neq k, j = l; \\ c(e_i)u_t \Delta t + o(\Delta t), \quad \text{if} \quad i = k, j = l+1; \\ 1 - \left[\sum_{k \neq i} a_{ik}(t, u_t) + c(e_i)u_t \right] \Delta t + o(\Delta t), \\ \text{if} \quad i = k, j = l, \end{cases} \tag{1}$$

with

$$o(\Delta t)/\Delta t \to 0, \quad \text{as} \quad \Delta t \to 0,$$

uniformly with respect to (t, u, i, j, k, l) and where $a_{ik}(t, u_t)$ is the ik-element of matrix $A(t,u)$. We would like to emphasize that the intensity of N_t depends linearly on the control u_t. In particular, this assumption is justified if for some fixed state of the underlying Hidden Markov Model either packets are lost independently

[1] Here and below A^* denotes the transposed matrix A.

with some probability [28] or ECN marks are sent by Random Early Detection routers with appropriately chosen averaging parameter [11].

Next, let us introduce a semimartingale representation of the above Markov jump process [7], [19]. Towards this goal, let us first introduce the following right-continuous sets of complete σ–algebras generated by X_t and N_t, respectively:

$$\mathcal{F}_t^X = \sigma\{X_s : s \in [0,t]\} \quad \mathcal{F}_t^N = \sigma\{N_s : s \in [0,t]\}$$

and

$$\mathcal{F}_t^{X,N} = \mathcal{F}_t^X \vee \mathcal{F}_t^N.$$

We suppose that the set of admissible control functions $\{u\}$ is the set of $\mathcal{F}_t^N$–predictable processes with values in U. It means that N_t is the number of jumps occurred until the current time t and if $\{\tau_1, ..., \tau_{N_t}\}$ are the jump times of N, then for $\tau_{N_t} < t \le \tau_{N_t+1}$ control $u_t = u(t, N_0^t)$ is a function of $\{\tau_1, ..., \tau_{N_t}\}$ and current time t [9]. Then, the process $\{X_t, N_t\}$ admits the following semimartingale representation

$$X_t = X_0 + \int_0^t A(s,u_s)X_s ds + M_t, \tag{2}$$

$$N_t = \int_0^t < c, X_s > u_s ds + \nu_t, \tag{3}$$

where X_0 is the initial state of the path, and $M_t := \{M_t^1, \cdots, M_t^n\}$, ν_t are square integrable $(\mathcal{F}_t^{X,N}, P)$ martingales with the following quadratic variations

$$\langle M \rangle_t = - \int_0^t [A(s,u_s)(diag\, X_s) + (diag\, X_s)A^*(s,u_s)]\, ds + \int_0^t (diag\, A(s,u_s)X_s)\, ds, \tag{4}$$

$$\langle \nu \rangle_t = \int_0^t < c, X_s > u_s ds, \tag{5}$$

$$\langle \nu, M \rangle_t = 0, \tag{6}$$

where $diag\, X$ denotes the matrix with diagonal entries $X^1, ..., X^n$. The equation (6) takes place, since the processes X_t and N_t are pure jump processes and due to (1) with probability one they do not jump at the same time moment.

We would like to note that even though the above Hidden Markov Model has some similarity to the congested path model proposed in [3], [4], an important difference between two approaches is that here the rate of congestion notifications depends on the data sending rate.

B. *Performance criterion*

The optimization goal is to maximize the time average utility function of the data sending rate minus a cost of transfer. Each congestion notification contributes some fixed amount to the cost of transfer. Let T be the duration of data transfer. Then, the performance criterion can be written as follows:

$$J[U(\cdot)] = E\left\{ \int_0^T f_0(u_s, X_s)ds - kN_T \right\}, \tag{7}$$

with

$$f_0(u,x) =< f_0(u), x >$$

where $f_0^*(u) = (f_0(u,e_1), ..., f_0(u,e_n))$, and each $f_0(\cdot, e_i)$ is a utility function of sending rate when the congested path is in state e_i. It is increasing and concave and satisfies the following assumption

$$\lim_{u\to\infty} [f_0(u,e_i) - cu] = -\infty, \tag{8}$$

for any $c > 0, e_i, i = 1, ..., n$.

The parameter k can be interpreted either as a cost of information loss or $kc(e_i)$ can be interpreted as a price of bandwidth when the congested path is in state e_i [20].

Remark 1: We would like to note that condition (8) allows to consider control from some unbounded set U. Indeed, since for any $i = 1, ..., n$ and given k the function $f_0(u,e_i) - kc(e_i)u$ is uniformly bounded, there exists u_{max}^i such that

$$f_0(u,e_i) - kc(e_i)u \le f_0(u_{max}^i, e_i) - kc(e_i)u_{max}^i$$

for $u \ge u_{max}^i$. Let $U_{max} = \max_i u_{max}^i$. Thus, if we take the modified uniform bounded control

$$\tilde{u}_t = \begin{cases} u_t, & \text{if} \quad u_t \le U_{max}, \\ U_{max}, & \text{if} \quad u_t > U_{max}, \end{cases}$$

instead of unbounded control u_t, the value of the performance criterion cannot be decreased.

We would like to emphasize that we obtain the indirect information concerning the link state through the loss process only. We cannot measure the state dependent loss process intensity directly, so we are coming to an instance of the joint process-observation problems [25].

III. Filtering equations and the problem reduction

Let us transform the stochastic optimal control problem (2)-(7) with incomplete information to a standard stochastic optimal control problem within the class of completely observable Markov processes. This transformation can be done by using the set of sufficient statistics

$$\pi_t^i = E\{X_t^i|\mathcal{F}_t^N\}, \quad i = 1, ..., n. \tag{9}$$

Theorem 1: Suppose that u_s is $\mathcal{F}_t^N-$ predictable control, then the components of vector $\pi_t = \{\pi_t^1, ..., \pi_t^n\}^* \in R^n$ satisfy the following system of stochastic differential equations

$$d\pi_t = A(t, u_t)\pi_t dt + \Gamma(\pi_{t-})d\tilde{\nu}_t, \tag{10}$$

where

$$\tilde{\nu}_t = N_t - \int_0^t \sum_{i=1}^n \pi_s^i c(e_i)u_s ds = N_t - \int_0^t < \pi_s, c > u_s ds,$$

is a square integrable martingale with quadratic variation

$$\langle \tilde{\nu} \rangle_t = \int_0^t < \pi_s, c > u_s ds,$$

and

$$\Gamma(\pi) = \frac{1}{\sum_{i=1}^n \pi^i c(e_i)} \begin{pmatrix} \pi^1 c(e_1) \\ \ldots\ldots\ldots \\ \pi^n c(e_n) \end{pmatrix} - \begin{pmatrix} \pi^1 \\ \ldots \\ \pi^n \end{pmatrix}. \tag{11}$$

At the same time the performance criterion can be expressed as follows

$$J[u(\cdot)] = E\left\{\int_0^T < \pi_s, f_0(u_s) - kcu_s > ds\right\}. \tag{12}$$

□

First observation of the problem (10)-(12) shows that it belongs to a class of piece-wise deterministic control settings [8] since the evolution of the state process π_t is deterministic between jumps, and the jumps occur with intensity

$$\lambda_t^u = \sum_{i=1}^n \pi_t^i c(e_i)u_t =< \pi_t, c > u_t.$$

Moreover, the jumps in itself are deterministic and described by relation

$$\Delta\pi_\tau = \Gamma(\pi_{\tau-}).$$

The dynamical programming approach gives the following characterization of the value function. Define a function

$$V(x, t) = \sup_{u(\cdot)} J[u(\cdot)|\pi_t = x],$$

where supremum should be taken over the set of $\mathcal{F}_t^N-$ predictable controls, and

$$J[u(\cdot)|\pi_t = x] = E\left\{\int_t^T \left[\sum_{i=1}^n \pi_s^i (f_0(u_s, e_i) - kc(e_i)u_s)\right] ds\right\}$$

conditioned by $\pi_t = x$. Define the equation

$$\begin{array}{c} 0 = V_t^{'}(x, t) + \sup_{u \in U}\{V_x^{'}(x, t)[A(t, u)x \\ -\Gamma(x) < c, x > u] \\ +[V(x + \Gamma(x), t) - V(x, t)] < c, x > u \\ + < f_0(u) - kcu, x >\} \end{array} \tag{13}$$

with terminal condition $V(x, T) = 0$.

Theorem 2: Assume that:

1) U is a compact set;
2) V is a piece-wise C^1 solution of (13) on
$$E = \{x : x^i \geq 0, \sum_{i=1}^n x^i = 1\} \times [0, T].$$
3) There exist $\hat{u}(x, t) \in U$, such that at each $(x, t) \in E$ the value in r.h.s. of (13) achieves the maximum at $u_0 = \hat{u}(x, t)$.
4) There exist a unique solution of equation
$$dx_t = A(t, \hat{u}(x_t, t))x_t dt + \Gamma(x_{t-})d\tilde{\nu}_t^{\hat{u}}.$$
Then $\hat{u}(x, t)$ is the optimal control and
$$V(x, t) = J[\hat{u}].$$

□

IV. Reduction of the problem with the aid of Girsanov's transformation and derivation of necessary optimality conditions

As was noticed above, the dependence of the driving counting process on the control makes the present problem very difficult for the analysis. In particular, it is not evident to compare the efficiency of two different controls since they generate different flows of losses and therefore driving counting processes are different. It appears useful to search for the equivalent problem setting where the driving process is control independent. The Girsanov's measure transformation gives such opportunity. We are using here the reduction of the filtering problem in terms of unnormalized conditional distribution of X_t^u [9]. Suppose that on a probability space $(\Omega, \mathcal{F}^N, \bar{P})$, $N_t, 0 < t \leq T$ is a standard Poisson process and the observations are given by Markov process X_t^u which satisfies equation (2) with some $\mathcal{F}^N$–predictable control $u_s > 0$, P-a.s. almost everywhere on $[0,T]$. Then $\bar{Q}_t = N_t - t$ is a $\mathcal{F}^N$–martingale under $\bar{P}$.

For any $\mathcal{F}^N-$ predictable control u_s, consider the family of random variables

$$\bar{\Lambda}_t^u = \left(\prod_{0<\tau\leq t} c(X_{\tau-}) u_{\tau-} \Delta N_\tau \right) \times$$
$$\times \exp\left(\int_0^t (1 - c(X_\tau)) u_\tau d\tau \right).$$

Then, it follows from the Ito's formula that $\bar{\Lambda}_t^u$ satisfies the next equation

$$\bar{\Lambda}_t^u = 1 + \int_0^t \bar{\Lambda}_{\tau-}^u (c(X_{\tau-}) u_{\tau-} - 1) d\bar{Q}_\tau. \qquad (14)$$

Since the control is positive, we have that $\Delta\bar{\Lambda}_t > -1$, P-a.s.. Furthermore, since the process $(c(X_{\tau-})u_{\tau-} - 1)$ is uniformly bounded, the solution of equation (14) is a $(\mathcal{F}_t^N, \bar{P})$ martingale. Next we define a new probability measure P by

$$\bar{E}\left[\left. \frac{dP}{d\bar{P}} \right| \mathcal{F}_t^N \right] = \bar{\Lambda}_t^u.$$

Using the Abstract Girsanov's Theorem [9, Ch.13], [30, Ch.6], one can transform the semimartingale characteristics as specified in the next proposition.

Proposition 1: On the probability space $(\Omega, \mathcal{F}^{X,N}, P)$, the process defined by

$$\nu_t = \bar{Q}_t + t - \int_0^t c(X_\tau) u_{\tau-} d\tau$$

is a square integrable martingale with quadratic variation

$$<\nu>_t = \int_0^t c(X_\tau) u_\tau d\tau, \quad \text{P-a.s..}$$

Moreover, for any $\{t_1, t_2, ..., t_k \leq t\}$ and any Borel function ϕ the following relation holds

$$E\left[\phi(X_{t_1}^u, ..., X_{t_k}^u; \tilde{N}_{t_1}^u, ..., \tilde{N}_{t_k}^u \right] =$$
$$\bar{E}\left[\bar{E}\{ \bar{\Lambda}_t^u \phi(X_{t_1}^u, ..., X_{t_k}^u; N_{t_1}, ..., N_{t_k}) | \mathcal{F}_t^N \} \right],$$

where the expectation E is calculated with respect to the measure P, and the expectation $\bar{E}$ is calculated with respect to the measure $\bar{P}$.

□

Thus, Girsanov's transformation of measure allows us to construct a weak solution $(X_t^u, \tilde{N}_t^u)$ for the controlled system (2),(3) on the probability space $(\Omega, \mathcal{F}^{X,N}, P)$. This weak solution coincides per trajectory with a strong solution (X_t^u, N_t) on the probability space $(\Omega, \mathcal{F}^{X,N}, \bar{P})$. In particular, by Proposition 1, since measure P is absolutely continuous with respect to measure $\bar{P}$, one can calculate the moments of the controlled process along the trajectories of the strong solution when the system is driven by a standard Poisson process. Therefore, we are able to formulate the following equivalent optimization problem.

Theorem 3: The unnormalized conditional distribution of X_t given N_t is then provided by

$$\begin{aligned} q_t^u &= \bar{E}[\bar{\Lambda}_t^u X_t | \mathcal{F}_t^N] \qquad (15) \\ &= q_0 + \int_0^t A(s, u_s) q_s ds + \int_0^t B(u_{s-}) q_{s-} d\bar{Q}_s, \end{aligned}$$

where

$$B(u) = diag(c)u - I,$$

with I as $n \times n$ identity matrix. The cost function is equal to

$$J[u(\cdot)] = \bar{E} \int_0^T < f_0(u_s) - kcu_s, q_s^u > ds. \qquad (16)$$

□

Remark 2: The correspondence between π_t and q_t^u is specified by relations

$$\pi_t = \frac{q_t^u}{\bar{\Lambda}_t^u} = \frac{q_t^u}{\sum_{i=1}^n (q_t^u)^i},$$

where $(q_t^u)^i$ is the i-th component of vector q_t^u.

Remark 3: Finally we obtain the new optimal control problem for system (15) with the cost function (16) which is equivalent to the original problem, but much more easy for implementation since the driving process does not depend on control and the equation (15) is linear in q^u.

Let us suppose that there exists a $\mathcal{F}^N$–predictable optimal control u_t^* and derive the necessary optimality condition. There is a well developed theory of necessary optimality conditions (see e.g., [15] and [29]) for control processes driven by Poisson type noises, however, in our case of linear dynamic one can obtain the optimality condition more easily by direct calculation of Gâteaux derivative of the performance criterion.

Define the function

$$H(\psi, q, u, t) =< f_0(u) - kcu, q > + < \psi, A(t, u)q > . \tag{17}$$

Theorem 4: Assume that $(u_t^*, q_t^{u^*})$ is an optimal process in the problem (15), (16). Then there exists $\mathcal{F}_t^N$ measurable, continuous and square integrable process ψ_t satisfying the backward stochastic differential equation

$$\begin{aligned} d\psi_t \;\; = \;\; & -[diag(c)u_{t-}\Delta\bar{Q}_t + I]^{-1} \times \\ & \times\{A(t, u_t^*)dt + B(u_t^*)d\bar{Q}_t\}^* \psi_{t-} + \\ & (\Phi^{u^*}(t-, 0))^{-1}\phi_t d\bar{Q}_t - [f_0(u_t^*) - kcu_t^*]dt, \end{aligned} \tag{18}$$

with some square integrable $\mathcal{F}^N$ predictable process ϕ_t and with terminal condition $\psi_T = 0$, such that optimal control satisfies the maximum condition

$$H(\psi_t, q_t^{u^*}, u_t^*, t) = \max_{u \in U} H(\psi_t, q_t^{u^*}, u, t). \tag{19}$$

□

Remark 4: If matrix-valued function A does not depend on u and function f_0 satisfy (8), then there exists a unique control

$$u_t^* = \underset{U}{argmax} < f_0(u) - kcu, q_t^u >,$$

where q_t^u is the corresponding solution of (15). Moreover one can immediately obtain the corresponding solution for the original problem statement, since

$$\underset{U}{argmax} < f_0(u) - kcu, q_t^u >=$$

$$\underset{U}{argmax} < f_0(u) - kcu, \pi_t > .$$

If the function f_0 is strictly convex, then this control will be piece-wise deterministic optimal, since the number of jumps in each interval is finite almost surely and between the jumps the control is a continuous function of the current state π_t.

Remark 5: Even though the control u_t^* satisfies the stochastic maximum principle, we cannot conclude for sure that it is an optimal control. The existence theorem is still missing. Therefore, we can say that this control is very likely to be optimal. In the next section, via numerical experiments, we shall provide yet another indirect confirmation of optimality of the constructed control.

V. Numerical example and comparison with AIMD scheme

Let us consider a Gilbert type path model with two states. Transitions between two states are governed by a generator

$$A^* = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$

The losses or ECN marks arrive according to the counting process N_t with intensities c_1 (c_2) in State 1(2). We suppose that $c_1 < c_2$. Thus, State 1 corresponds to a "good" state of the path and State 2 corresponds to a "bad" state of the path, respectively. We can consider only one equation for the random variable

$$\pi_t^1 = E(I\{X_t = 1\}|\mathcal{F}_t^N),$$

which has the following form

$$d\pi_t^1 = [-\lambda\pi_t^1 + (1 - \pi_t^1)\mu + u_t\pi_t^1(1 - \pi_t^1)(c_2 - c_1)]dt$$

$$-\frac{\pi_t^1(1 - \pi_t^1)(c_2 - c_1)}{\pi_t^1 c_1 + \pi_t^2 c_2} dN_t. \tag{20}$$

Therefore, between the jumps an application of control $u_t > 0$ makes conditional probability π^1 to increase steadily, and at the jumps this probability abruptly decreases.

We take the following standard utility function

$$f_0(u, x) = -\frac{a^2}{RTT^2(x)u}.$$

This utility function corresponds to the AIMD congestion avoidance mechanism in the case of small losses [21]. We would like to note that in the above utility function the Round Trip Time (RTT) parameter depends on the unobservable state of the path.

The optimal control, which maximizes

$$< f_0(u) - kcu, \pi_t >,$$

has the following form

$$u_{opt}(\pi_t) = \frac{a}{\sqrt{k}}\sqrt{\frac{\dfrac{\pi_t^1}{RTT^2(1)} + \dfrac{1 - \pi_t^1}{RTT^2(2)}}{c_1\pi_t^1 + c_2(1 - \pi_t^1)}}. \tag{21}$$

If $RTT(1) = RTT(2) = RTT$, $k = 1$, $c^1 = 0$, $c^2 = c$, we obtain

$$u_{opt}(\pi_t) = \frac{a}{RTT\sqrt{c(1 - \pi_t^1)}} = \frac{a}{RTT\sqrt{\hat{p}_t}},$$

where $\hat{p}_t$ is a packet loss probability conditioned on the observed history. It is very interesting that this formula coincides with the one giving the average transmission rate of TCP New Reno over Markovian path where the probability of loss replaced by its conditioned version. Thus, we have got a version of the famous "square root" formula [23].

To perform the numerical experiments, one needs to substitute (21) into (20) and solve this differential equation numerically. It is important to note that one must solve the differential equation and generate the counting process N_t^u simultaneously, as N_t^u depends on the sending rate u and the state of the path. Some examples of numerical solutions are presented in Figures 1-4. In particular, one can clearly see from Figure 4 that the performance of the optimal control is more "conservative" than that of AIMD. The following set of parameters was used for the numerical analysis:

$$\begin{gathered} \lambda = \mu = 2, \quad RTT(1) = RTT(2) = 0.5, \\ c_1 = 0.2, \quad c_2 = 1.0, \\ T = 10, \quad a = 1, \quad k = 1. \end{gathered} \tag{22}$$

The evolution of AIMD is modeled by the standard stochastic differential equation

$$du_t = \frac{dt}{RTT(X_t^u)} - \frac{u_{t-}}{2} dN_t^u, \tag{23}$$

with the initial condition $u_0 = u_{opt}(\pi_0)$ in order to remove the effect of transient behavior. In (23) the counting process N_t^u is generated the same way as for the optimal control.

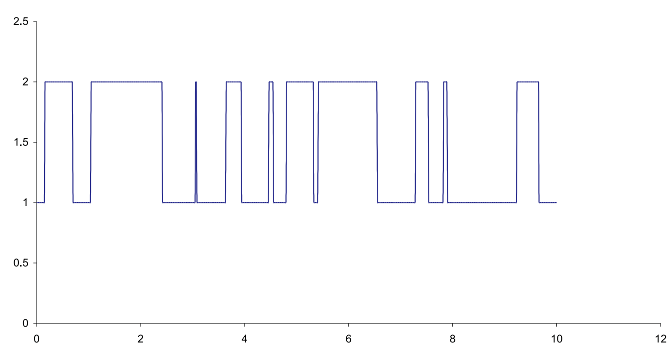

Fig. 1. The state of Markov process X_t.

We would like to note that by changing the parameter k, it is easy to tune the degree of "conservativeness" of the control. For instance, if we increase the cost of losses (marks) from 1 to 4, the average value of the transmission rate decreases and the control becomes nearly constant (see Figure 5).

And on contrary, the knowledge about the aprior information about the path characteristics, allows the optimal control increase the sending rate much faster than AIMD and at the same time to be enough prudent to avoid overshots. For instance, if it is known that the path

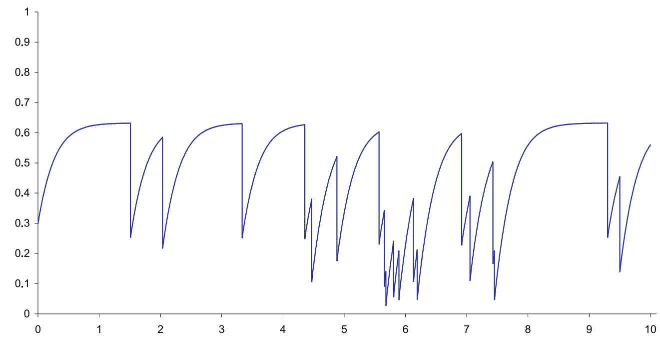

Fig. 2. Conditional probability π_t^1.

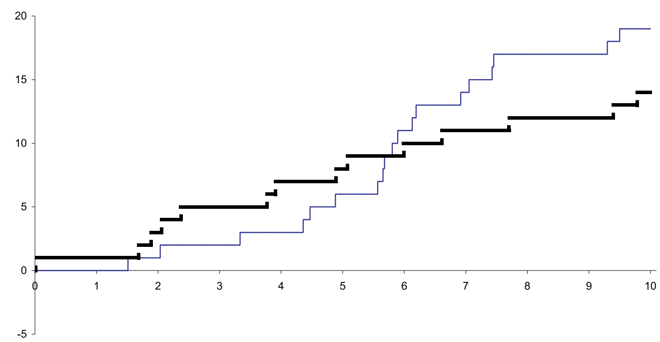

Fig. 3. Counting process N_t^u for the optimal control and the sub-optimal AIMD control.

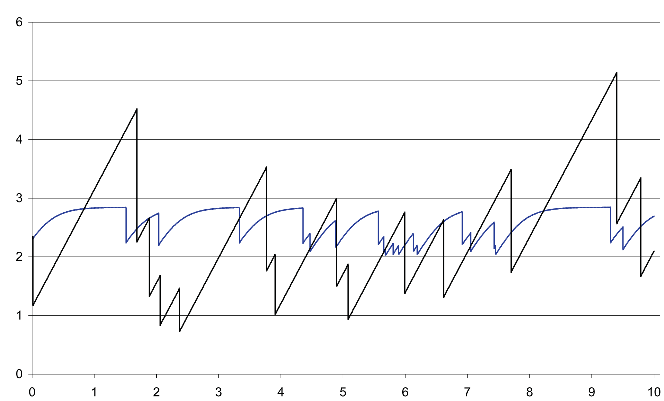

Fig. 4. The evolution of the optimal and AIMD controls.

is most of the time in a good state, e.g., $\lambda = 2, \mu = 6$ and $c_1 = 0.1$, the optimal control really takes advantage of this aprior information (see Figure 6). The superiority of the optimal control becomes even more pronounced when RTT changes significantly with the change of the underlying Markov chain (see Figure 7).

One can also use another approach for numerical experiments which is based on Girsanov's transformation.

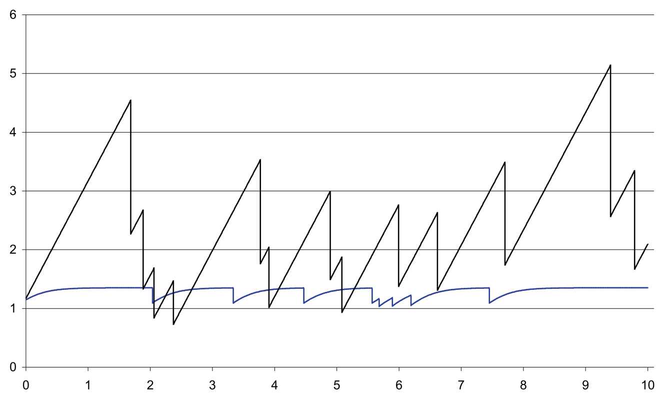

Fig. 5. The evolution of the optimal and AIMD controls ($k = 4$).

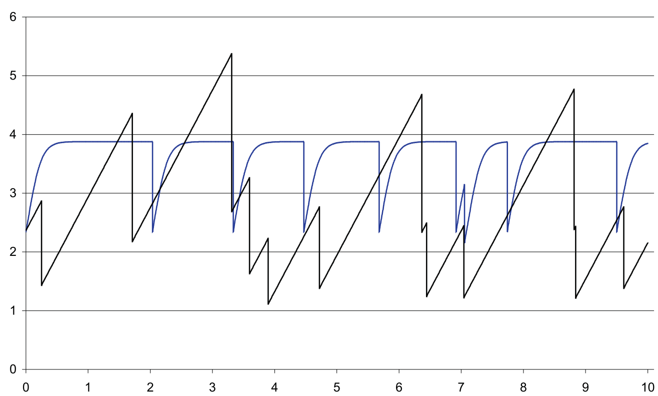

Fig. 6. The effect of aprior information on the control evolution.

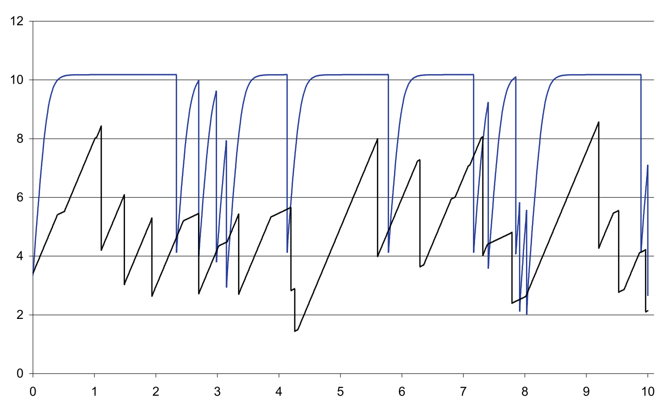

Fig. 7. The evolution of the optimal and AIMD controls ($RTT(1) = 0.2,\ RTT(2) = 1.0$).

In particular, this method is more computationally efficient, since one needs to generate only a homogeneous Poisson process. Furthermore, using this method it is more easy to compare the average values of the control and its variance for the optimal scheme and the sub-optimal AIMD. To calculate the average characteristics, we generate 100 trajectories. This results in the following values for the average performance criterion and sending rates:

$$\bar{J}[u_{opt}] = -2.86, \quad \bar{J}[u_{AIMD}] = -3.06,$$

$$\bar{u}_{opt} = 2.43, \quad \bar{u}_{AIMD} = 2.32.$$

Even though the differences between the respective average characteristics is less than 10%, the difference between the standard deviations is quite impressive (486%):

$$\sigma(u_{opt}) = 0.76, \quad \sigma(u_{AIMD}) = 4.45.$$

In other words, the optimal control produces much smaller variations of the sending rate and at the same time achieves the sending rate greater than that of AIMD. This nice property can be utilized in the context of the TCP-friendly protocols [12]. In Figures 8 and 9 we plot the average sending rates and their standard deviations along the time interval $[0, T]$ for the optimal and AIMD controls, respectively.

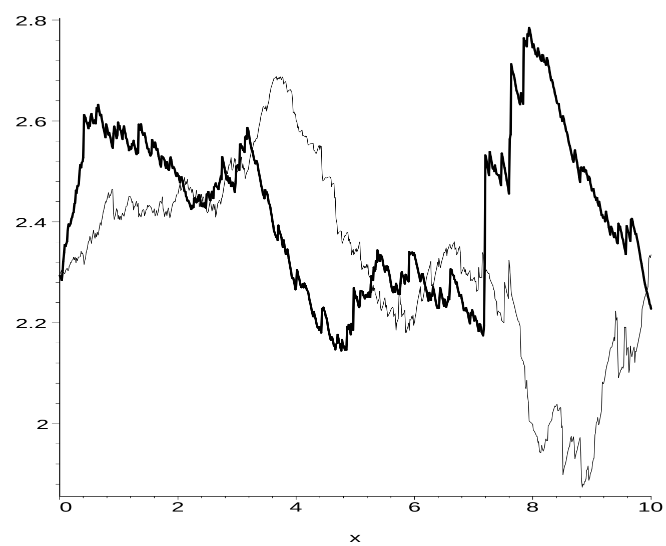

Fig. 8. The averages along $[0, T]$ of the optimal (thick line) and AIMD (thin line) controls.

VI. Conclusions and Future Research

The presented model allows us to answer several fundamental questions about the evolution of TCP sending rate. In particular, we show that for a Markovian path the optimal sending rate evolution is indeed piece-wise deterministic as it is the case in the current TCP implementation. However, we also show that the optimal increase is not linear and the optimal multiplicative decrease is not proportional to the instantaneous sending

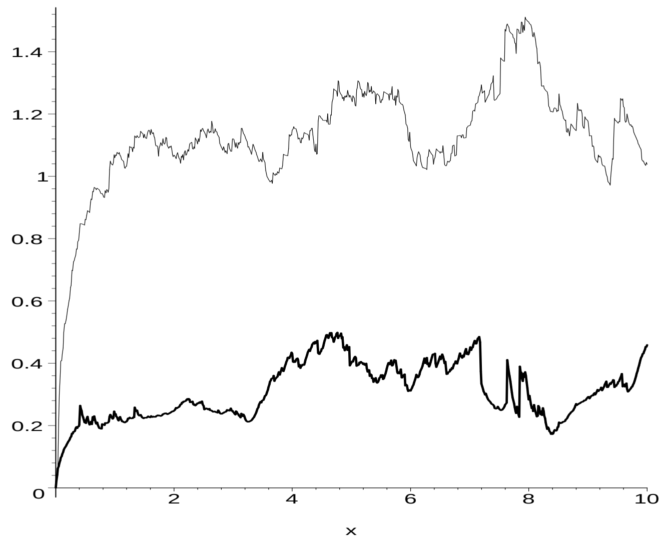

Fig. 9. The standard deviations along $[0, T]$ of the optimal (thick line) and AIMD (thin line) controls.

rate. The optimal control takes advantage from aprior information about the path characteristics. In particular, this allows congestion control to achieve much smaller variance of the sending rate evolution and at the same time to gain in the average throughput. The latter property of the optimal control can be exploited by the TCP-friendly rate control protocols. It seems that our results could be very helpful in the ensuing research directions:

1) the case of control dependent generators A, here a typical problem is the estimation of the buffer state and the flow control in the presence of unknown parasitic flows;
2) identification of the loss types: congestion versus transmission losses; in particular, in the context of wireless networks;
3) inclusion into the observation framework another parameters correlated with the state of the path, such as the RTT parameter.

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