Constraint augmentation in pseudo-singularly perturbed linear programs

Article *in* Mathematical Programming · April 2010

DOI: 10.1007/s10107-010-0388-0

4 authors:

K. E. Avrachenkov
National Institute for Research in Computer Science and Control

Regina Burachik
University of South Australia

Jerzy A. Filar
The University of Queensland

Vladimir Gaitsgory
Macquarie University

Constraint Augmentation in Pseudo-Singularly Perturbed Linear Programs

K. Avrachenkov*, R.S. Burachik , J.A. Filar and V. Gaitsgory†

ABSTRACT

In this paper we study a linear programming problem with a linear perturbation introduced through a parameter $\varepsilon > 0$. We identify and analyze an unusual asymptotic phenomenon in such a linear program. Namely, discontinuous limiting behavior of the optimal objective function value of such a linear program may occur even when the rank of the coefficient matrix of the constraints is unchanged by the perturbation. We show that, under mild conditions, this phenomenon is a result of the classical Slater constraint qualification being violated at the limit and propose an iterative, constraint augmentation approach for resolving this problem.

1 Introduction

Most of Optimization and Operations Research text books contain a discussion of "sensitivity analysis" of linear programs in situations where the coefficients of the objective function, or of the right hand side of constraints, or of non-basic columns are perturbed by some parameter. Some books also contain a cautionary note about situations where basic columns of the coefficient matrix are also affected by such a perturbation. Indeed, the interest in the latter situation gave rise to a line of research devoted to analyzing the asymptotic behavior of mathematical programs under perturbations (e.g., see [8], [4],[6]). In the case of perturbed linear programs the main distinction identified was between the so-called "singular perturbation" where the rank of the coefficient matrix of the linear constraints is different at $\varepsilon = 0$ and at $\varepsilon > 0$, and the "regular perturbation" when these ranks coincide. We note also that some of the mathematical techniques used in the preceding references were in a similar spirit as some of the techniques developed in [1] for the analysis of an infinite dimensional linear program and subsequently in [2] in the context of computing the Drazin inverse. Of course, the need for the latter arises naturally when considering the problem of inverting matrix functions that are singular at the origin (see also, [3]).

In this paper we identify and analyze an unusual asymptotic phenomenon that is undetected by the preceding dichotomous distinction. Namely, discontinuous limiting behavior of the optimal objective function value of such a linear program may occur even when the rank of the coefficient matrix of the constraints is unchanged by the disturbance. In such a case, we shall say that the perturbation is *pseudo-singular* or *weakly singular* [6].

We show that, under mild conditions, this pseudo-singularity is a result of the classical Slater constraint qualification being violated in the limit and propose an iterative, multi-layered

*K.E. Avrachenkov (k.avrachenkov@sophia.inria.fr) is with INRIA, 2004 route des lucioles - BP 93 FR-06902 Sophia Antipolis Cedex, France

†R.S. Burachik (Regina.Burachik@unisa.edu.au), J.A. Filar (Jerzy.Filar@unisa.edu.au) and V.G. Gaitsgory (Vladimir.Gaitsgory@unisa.edu.au) are with the Centre for Industrial and Applied Mathematics, University of South Australia, Mawson Lakes, SA 5095, Australia. The work on this project was supported by the ARC Linkage International Grants LX0560049 and LX0881972 and partially by ARC Discovery Grants DP0664330 and DP0666632.

technique for resolving this problem. The latter is inspired by a dimension completing procedure described in [3].

More precisely, we consider the family of linear programming problems parameterized by $\varepsilon \geq 0$:

$$\max\{\langle c^{(0)} + \varepsilon c^{(1)}, x\rangle \ : \ x \in \theta(\varepsilon)\} \stackrel{\text{def}}{=} F^*(\varepsilon), \tag{1.1}$$

where

$$\theta(\varepsilon) \stackrel{def}{=} \{x \in \mathbb{R}^n \ : \ (A^{(0)} + \varepsilon A^{(1)})x = b^{(0)} + \varepsilon b^{(1)}, \quad x \geq 0\} \tag{1.2}$$

with $c^{(0)}, c^{(1)} \in \mathbb{R}^n$, $b^{(0)}, b^{(1)} \in \mathbb{R}^m$ and $A^{(0)}, A^{(1)} \in \mathbb{R}^{m\times n}$ $(n \geq m)$. Together, (1.1)-(1.2) define a *perturbed linear program* (*LP*) if $\varepsilon > 0$; it is called *unperturbed LP* if $\varepsilon = 0$. The set $\theta(\varepsilon)$ is called the *feasible set* of the perturbed problem and, similarly, $\theta(0)$ is the feasible set of the unperturbed LP.

The goal of the present work is to construct, if possible, an independent of ε linear programming problem such that its optimal solutions are *limiting optimal* for (1.1)-(1.2) in the sense prescribed by Definition 1.1.

Definition 1.1 *A vector $x \in \mathbb{R}^n$ is called* limiting optimal *for the perturbed linear program (1.1)-(1.2) if $x \in \lim_{\varepsilon\downarrow 0}\theta(\varepsilon)$ and $\lim_{\varepsilon\downarrow 0} F^*(\varepsilon) = \langle c^{(0)}, x\rangle$.*

Definition 1.2 *A linear program will be called a limiting LP if all its optimal solutions are limiting optimal for (1.1)-(1.2).*

In Definition 1.1 and in what follows, $\lim_{\varepsilon\downarrow 0}\theta(\varepsilon)$ is understood in Kuratowski-Painleve's sense (e.g., see [5, Definition 2.2.4] or [10, Chapter 5]). For readers' convenience we recall the definition of $\lim_{\varepsilon\downarrow 0}\theta(\varepsilon)$ at the end of this section. Note that from the subsequent consideration it follows that, under mild conditions, $\lim_{\varepsilon\downarrow 0}\theta(\varepsilon)$ exists but it may not be equal to $\theta(0)$:

$$\lim_{\varepsilon\downarrow 0}\theta(\varepsilon) \neq \theta(0), \tag{1.3}$$

and we aim at giving an explicit characterization of this limit. Note also that a possible discontinuity of $\theta(\varepsilon)$ at $\varepsilon = 0$ may lead to a discontinuity of the optimal value:

$$\lim_{\varepsilon\downarrow 0} F^*(\varepsilon) \neq F^*(0), \tag{1.4}$$

that takes place when none of the optimal solutions of the unperturbed problem is limiting optimal for (1.1).

In [8] and [6] the appearance of such discontinuities had been attributed to the fact that the matrix $A^{(0)}$ may not have a full ($rank[A^{(0)}] < m$). By contrast to these earlier developments, in the present paper we show that the latter may occur even in the case when $A^{(0)}$ has a full rank ($rank[A^{(0)}] = m$), and we construct the limiting LP for this pseudo-singular case.

The paper consists of five sections, and the material is organized as follows. Section 1 is this introduction. In Section 2, we first use an elementary example to illustrate that the discontinuity (1.4) may occur if the unperturbed problem does not satisfy a Slater condition (see Example 2.1). We then introduce a new linear program by adding an extra layer of constraints to the unperturbed LP, and we state our first main result (Theorem 2.1) establishing that, under certain assumptions, this new LP is the limiting LP for (1.1). A main assumption of Theorem 2.1 is an *extended Slater condition of order* 1. The proof of the theorem is given in Section 3.

In Section 4, we demonstrate with an example that the extended Slater condition of order 1 may not be satisfied (see Example 4.3), and we state and prove our second main result (Theorem 4.2) establishing that the LP obtained by adding k ($k \geq 1$) extra layers of constraints to the unperturbed LP is the limiting LP for (1.1) if the extended Slater condition of order k is satisfied. In Section 5, we introduce a more general *lexicographic Slater condition* that is equivalent to the fulfillment of the Slater condition in the perturbed problem, and we show that the augmentation of the unperturbed LP with extra layers of constraints leads the limiting LP for (1.1) in this case as well (see Theorem 5.3 and Proposition 5.2).

To finish this section, we recall the definition of $\lim_{\varepsilon\downarrow 0}\theta(\varepsilon)$.

Definition 1.3 *It is said that* $\lim_{\varepsilon\downarrow 0}\theta(\varepsilon)$ *exists and is equal to* θ *if*

$$\liminf_{\varepsilon\downarrow 0}\theta(\varepsilon) = \limsup_{\varepsilon\downarrow 0}\theta(\varepsilon) = \theta,$$

where

$$\liminf_{\varepsilon\downarrow 0}\theta(\varepsilon) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \ : \ \forall\, r > 0, \ \exists\, \varepsilon_0 > 0 \ \textit{s.t.} \ \ B(x,r)\cap\theta(\varepsilon) \neq \emptyset \ \ \forall\, \varepsilon < \varepsilon_0\},$$

and

$$\limsup_{\varepsilon\downarrow 0}\theta(\varepsilon) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \ : \ \forall\, r, \varepsilon_0 > 0 \ \ \exists\, \varepsilon < \varepsilon_0 \ \ \textit{s.t.} \ \ B(x,r)\cap\theta(\varepsilon) \neq \emptyset\};$$

$B(x,r)$ *is an open ball centered at* x *with the radius* r.

2 Extended Slater condition of order 1

Let us introduce the following three assumptions.

Assumption (H_0): *There exists a positive* γ_0 *and a bounded set* $B \subset \mathbb{R}^n$ *such that* $\emptyset \neq \theta(\varepsilon) \subset B$ *for every* $\varepsilon \in (0, \gamma_0]$.

Assumption (H_1): *The matrix* $A^{(0)}$ *has rank* m.

Assumption (H_2) *For all* ε *sufficiently small* $\varepsilon > 0$, *the rank of* $A^{(0)} + \varepsilon A^{(1)}$ *is equal to* m.

Note that Assumption (H_1) implies Assumption (H_2).

The unperturbed problem is said to satisfy Slater condition if

$$\theta(0) \cap \mathbb{R}^n_{++} \neq \emptyset \ , \ \text{where} \ \ \mathbb{R}^n_{++} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \ : \ x > 0\}.$$

In [8], it has been shown that if Assumptions (H_0) and (H_1) are valid and if the Slater condition (2) is satisfied, then the unperturbed LP is the limiting problem for the perturbed program (1.1). That is, every optimal solution of the former is limiting optimal for the latter.

The following example is to demonstrate that the discontinuity of $\theta(\varepsilon)$ at $\varepsilon = 0$ may occur (that is, (1.3) may take place) if the Slater condition does not hold, with Assumptions (H_0) and (H_1) being satisfied.

Example 2.1 Let $A^{(0)} = [1\,0\,0]$ and $A^{(1)} = [0\,1\,1]$, $b^{(0)} = 0$ and $b^{(1)} = 1$. That is,

$$\theta(\varepsilon) = \{x = (x_1, x_2, x_3) \ : \ x_1 + \varepsilon x_2 + \varepsilon x_3 = \varepsilon, \quad x_1, x_2, x_3 \geq 0 \ \}.$$

It is easy to see that the feasible set $\theta(\varepsilon)$ has three basic solutions: $x^I = (\varepsilon, 0, 0)^T$, $x^{II} = (0, 1, 0)^T$ and $x^{III} = (0, 0, 1)^T$ and that it can be parameterized as follows

$$\theta(\varepsilon) = \{x = (x_1, x_2, x_3) \ : \ x_1 = t\varepsilon \ , \quad x_2 + x_3 = 1 - t \ , \quad t \in [0,1] \ , \quad x_2, x_3 \geq 0 \ \}.$$

As can be easily seen, in this case,

$$\lim_{\varepsilon\downarrow 0}\theta(\varepsilon) = \{x \ : \ x_1 = 0, \quad x_2 + x_3 \leq 1, \quad x_2, x_3 \geq 0 \ \}, \tag{2.1}$$

and this limit is not equal to

$$\theta(0) = \{x \ : \ x_1 = 0, \, x_2, x_3 \geq 0\}.$$

Note that Assumptions (H_0) and (H_1) hold true, but the Slater condition for the unperturbed problem is not valid in this example (since no $x > 0$ satisfies $A^{(0)}x = b^{(0)}$).

Assume that (H_1) and (H_2) are satisfied and define the set

$$J_0 := \{i \in \{1, \ldots, n\} \ : \ \exists x \in \theta(0) \ \text{ such that } \ x_i > 0\}. \tag{2.2}$$

According to this definition, if $j \notin J_0$, then $x_j = 0$ for every $x \in \theta(0)$. Moreover, if $J_0 \neq \emptyset$, convexity of $\theta(0)$ implies that there exists $\hat{x} \in \theta(0)$ such that $\hat{x}_j > 0$ for every $j \in J_0$. Note that J_0 can be determined by solving n independent linear programming problems $\max_{x \in \theta(0)} x_j$, where $j = 1, \ldots, n$.

Consider the following linear program

$$\max\{\langle c^{(0)}, x^0 \rangle \ : \ x^0 \in \theta_1\} \stackrel{\text{def}}{=} F_1^*, \tag{2.3}$$

where

$$\theta_1 \stackrel{\text{def}}{=} \{x^0 \ : \ \exists \ (x^0, x^1) \in \Theta_1\}, \tag{2.4}$$

and

$$\Theta_1 = \{(x^0, x^1) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ x^0 \in \theta(0), \ \ A^{(0)}x^1 + A^{(1)}x^0 = b^{(1)}, \ \ x_j^1 \geq 0 \ \forall j \notin J_0\}. \tag{2.5}$$

Note that,

$$\theta_1 \subset \theta(0) \quad \text{ and therefore } \quad F_1^* \leq F^*(0).$$

Slater condition (2) is equivalent to having $J_0 = \{1, 2, \ldots, n\}$. If this is the case, then $\theta_1 = \theta(0)$ (provided that Assumption (H_1) is satisfied), and the problem (2.3) is equivalent to the unperturbed problem. If the Slater condition is not satisfied, these two problems are not equivalent. To deal with this case, let us introduce the following extended version of the Slater condition.

Definition 2.4 *We say that the* extended Slater condition of order 1 *(or, for brevity, ES-*1*) is satisfied if there exists $(\hat{x}^0, \hat{x}^1) \in \Theta_1$ such that $\hat{x}_j^1 > 0$ for every $j \notin J_0$ and $\hat{x}_j^0 > 0$ for every $j \in J_0$.*

Example 2.2 In Example 2.1, $J_0 = \{2, 3\}$, and from (2.5) the set Θ_1 for this example is

$$\Theta_1 = \{(x^0, x^1) \in \mathbb{R}^3 \times \mathbb{R}^3 \ : \ x^0 \in \theta(0), \ \ x_1^1 + x_2^0 + x_3^0 = 1, \ \ x_1^1 \geq 0\}.$$

The ES-1 condition is satisfied in this case by taking as $(\hat{x}^0, \hat{x}^1) \in \Theta_1$ as follows: $\hat{x}_1^0 = 0$, $\hat{x}_2^0 = \hat{x}_3^0 = \frac{1}{3}$ and $\hat{x}_1^1 = \frac{1}{3}$, $\hat{x}_2^1 = \hat{x}_3^1 = 0$. It is easy to verify that in this case

$$\theta_1 = \{x^0 \ : \ x_1^0 = 0, \ \ x_2^0 + x_3^0 \leq 1, \ \ x_2^0, x_3^0 \geq 0 \ \}.$$

That is, θ_1 coincides with $\lim_{\varepsilon \downarrow 0} \theta(\varepsilon)$ (see (2.1)).

The following theorem, which is our first main result, will be proved in Section 3.

Theorem 2.1 *Let Assumptions (H_0) and (H_2) be satisfied. Then*

$$\limsup_{\varepsilon \downarrow 0} \theta(\varepsilon) \subset \theta_1 \tag{2.6}$$

and

$$\limsup_{\varepsilon \downarrow 0} F^*(\varepsilon) \leq F_1^* \ . \tag{2.7}$$

*If, in addition, Assumption (H_1) and the ES-*1 *condition are satisfied, then*

$$\theta_1 \subset \liminf_{\varepsilon \downarrow 0} \theta(\varepsilon) \tag{2.8}$$

and the following equalities are valid:

$$\lim_{\varepsilon \downarrow 0} \theta(\varepsilon) = \theta_1, \tag{2.9}$$

$$\lim_{\varepsilon \downarrow 0} F^*(\varepsilon) = F_1^* \ . \tag{2.10}$$

Also, if x^0 *is an optimal solution of the problem* (2.3)*, then it is limiting optimal for the perturbed problem* (1.1)*.*

Remark 2.1 Note that (2.7) is implied by (2.6); (2.9) is implied by (2.6) and (2.8); and (2.10) is implied by (2.9). Also, by (2.9) and (2.10), if (x^0, x^1) is an optimal solution of the problem (2.3), then $\langle c^{(0)}, x^0 \rangle = F_1^*$, and $x^0 \in \lim_{\varepsilon \downarrow 0} \theta(\varepsilon)$, which proves that x^0 is an limiting optimal solution of the perturbed problem (1.1) (see Definition 1.1). Thus, for proving Theorem 2.1, it is enough to establish inclusions (2.6) and (2.8).

Remark 2.2 From Theorem 2.1 it follows that (2.3) is a limiting LP in the sense of Definition 1.2 if (H_0), (H_1) and ES-1 hold.

3 Proof of Theorem 2.1

Let us introduce some notations and establish certain auxiliary results. Given a finite set S, denote by $|S|$ the number of elements of S. Let $S_m := \{J \subset \{1, 2, \ldots, n\} \ : \ |J| = m\}$, so $|S_m| = \binom{n}{m}$. Given a matrix $D \in \mathbb{R}^{m \times n}$ and an index set $J \in S_m$, the matrix $D_J \in \mathbb{R}^{m \times m}$ is constructed by extracting from D the set of m columns indexed by the elements of J. In a similar way, given a vector $x \in \mathbb{R}^n$ and $J \in S_m$, we denote by x_J the vector of $\mathbb{R}^m$ constructed by extracting from x the coordinates x_j, $j \in J$ (that is, $x_J \stackrel{\text{def}}{=} \{x_j\}, \ j \in J$).

Given $J \in S_m$, define $P_J : \mathbb{R} \to \mathbb{R}$ as

$$P_J(\varepsilon) := \det{(A^{(0)} + \varepsilon A^{(1)})_J},$$

note that $P_J(\cdot)$ is a polynomial of degree m (possibly identically equal to zero).

Lemma 3.1 *Let* $J \in S_m$*. Only one of the following possibilities takes place:*

(I) There exists $\gamma(J) > 0$ *such that* $(A^{(0)} + \varepsilon A^{(1)})_J$ *is nonsingular for all* $\varepsilon \in (0, \gamma(J))$*.*

(II) $(A^{(0)} + \varepsilon A^{(1)})_J$ *is singular for all* $\varepsilon \geq 0$*.*

Also, if Assumption (H_2) *is satisfied, then there exists* $\gamma_1 > 0$ *such that* S_m *can be partitioned as* $S_m = \Omega_1 \cup \Omega_2$ *with* $\Omega_1 \cap \Omega_2 = \emptyset$ *and*

$$\Omega_1 := \{J \in S_m \ : \ (A^{(0)} + \varepsilon A^{(1)})_J \ \textit{is nonsingular for} \ \varepsilon \in (0, \gamma_1)\} \neq \emptyset,$$

$$\Omega_2 := \{J \in S_m \ : \ (A^{(0)} + \varepsilon A^{(1)})_J \ \textit{is singular for all} \ \varepsilon \geq 0\}.$$

Proof. Assume (I) does not hold. Hence there exists a sequence $\varepsilon_k \downarrow 0$ such that $P_J(\varepsilon_k) = 0$. This implies that the polynomial P_J has countable roots, and therefore it must be identically zero. This implies that (II) holds. On the other hand, when (II) does not hold, this means that $P_J(\cdot)$ is a nontrivial polynomial of degree m, so it has no more than m distinct roots. This readily implies that (I) must hold. Regarding the last statement of the lemma, note that there is a finite number of $J \in S_m$, so we can take

$$\gamma_1 := \min_{J \in S_m} \{\gamma(J) \ : \ det(A^{(0)} + \varepsilon A^{(1)})_J \neq 0 \ \ \forall \ \varepsilon \in (0, \gamma(J))\}.$$

The fact that $\Omega_1 \neq \emptyset$ follows from (H_2). $\square$

Lemma 3.2 *Let Ω_1, γ_1 be as defined in Lemma 3.1 and let*

$$x_J(\varepsilon) := [(A^{(0)} + \varepsilon A^{(1)})_J]^{-1}(b_0 + \varepsilon b_1) \tag{3.1}$$

for $J \in \Omega_1$ and $\varepsilon \in (0, \gamma_1)$. If there exists a sequence $\varepsilon_i \downarrow 0$ such that

$$\lim_{\varepsilon_i \downarrow 0} \|x_J(\varepsilon_i)\| < \infty, \tag{3.2}$$

then:

(i) There exist $\gamma_2 > 0$ ($\gamma_2 \leq \gamma_1$) and $M > 0$ such that the following power series expansion is valid

$$x_J(\varepsilon) = \sum_{l=0}^{\infty} \varepsilon^l u_J^l, \tag{3.3}$$

where $u_J^l \in \mathbb{R}^m$ are such that $\|u_J^l\| \leq \frac{M}{\gamma_2^l}$ and $\varepsilon \in (0, \gamma_2)$, with ε^l and γ_2^l standing for the power l of ε and, respectively, γ_2.

(ii) There exists $\gamma_3 > 0$ ($\gamma_3 \leq \gamma_2$) such that only one of the following two options may take place:

(I) $x_J(\varepsilon) \geq 0$ for all $\varepsilon \in (0, \gamma_3)$, or

(II) $x_J(\varepsilon) \not\geq 0$ for all $\varepsilon \in (0, \gamma_3)$.

Proof. The proof of the statement (i) follows from the fact that $x_J(\varepsilon)$ defined in (3.1) allows a Laurent series expansion and from the fact that, due to (3.2), the coefficients near negative powers of ε are equal to zero (details can be found in [6]). Let us prove the statement (ii). From (3.3) it follows that

$$\lim_{\varepsilon \downarrow 0} x_J(\varepsilon) = u_J^0 \ . \tag{3.4}$$

Assume now that (ii) is not true. Then one can find a sequence $\{\varepsilon_k\}$ and a sequence $\{\delta_k\}$ such that $\varepsilon_k \downarrow 0$, $\delta_k \downarrow 0$, and such that $x_J(\varepsilon_k) \geq 0$ for all k and $x_J(\delta_k) \not\geq 0$ for all k. That is, $x_j(\varepsilon_k) \geq 0$ and $x_j(\delta_k) < 0$ for all k and at least one $j \in J$. Note that from (3.4) it follows that $u_j^0 = 0$ for this j (here and in what follows u_j^l stands for the j^{th} component of u_J^l). Let an integer $p \geq 1$ be such that $u_j^0 = u_j^1 = \ldots = u_j^{p-1} = 0$ and $u_j^p \neq 0$. By (3.3),

$$x_j(\varepsilon) = \sum_{l=p}^{\infty} \varepsilon^l u_j^l = \varepsilon^p (u_j^p + \varepsilon \sum_{l=p+1}^{\infty} \varepsilon^{l-(p+1)} u_j^l). \tag{3.5}$$

Considering (3.5) with $\varepsilon = \delta_k$ and taking into account the fact that $x_j(\delta_k) < 0$, we conclude that $u_j^p < 0$. Note also that the second term inside the parenthesis in (3.5) tends to zero thanks to the uniform boundedness of $\{u^l\}$ established in part (i). The fact that $u_j^p < 0$, and (3.3) yield $x_j(\varepsilon) < 0$ for all $\varepsilon > 0$ small enough. This contradicts the fact that $x_j(\varepsilon_k) \geq 0$ for all k. □

Proof of Theorem 2.1. Let $\Omega_1^b(\varepsilon) \subset \Omega_1$ be such that $J \in \Omega_1^b(\varepsilon)$ if and only if the column numbers contained in J define a basic feasible solution of (1.2). Since the set $\theta(\varepsilon)$ of feasible solutions of (1.2) is non-empty and bounded (Assumption (H_0)), then $\Omega_1^b(\varepsilon) \neq \emptyset$, and from the fact that there exists a sequence $\varepsilon_i \downarrow 0$ such that $J \in \Omega_1^b(\varepsilon_i)$ it follows that (3.2) is satisfied. This implies that the statements (i) and (ii) of Lemma 3.2 are valid and, hence, $J \in \Omega_1^b(\varepsilon)$ for all $\varepsilon > 0$ small enough. Based on this argument, one can come to the conclusion that $\Omega_1^b(\varepsilon)$ is independent of ε for $\varepsilon > 0$ small enough. That is, there exists a *nonempty* subset Ω_1^b of Ω_1 such that

$$\Omega_1^b(\varepsilon) = \Omega_1^b$$

and

$$x_J(\varepsilon) \geq 0 \quad \forall J \in \Omega_1^b \ , \qquad x_J(\varepsilon) \not\geq 0 \ \ \forall J \in \Omega_1 \setminus \Omega_1^b \tag{3.6}$$

for all $\varepsilon > 0$ small enough, where $x_J(\varepsilon)$ is defined by (3.1) (the second relationship being valid due to the fact that the existence of a sequence $\varepsilon_i \downarrow 0$ such that $x_J(\varepsilon_i) \geq 0$ implies that $J \in \Omega_1^b(\varepsilon_i) = \Omega_1^b$).

Let $J \in \Omega_1^b$ and let $x_J(\varepsilon)$ be the vector of basic components of the corresponding feasible basic solution $x(\varepsilon)$ of (1.2). That is (see (3.1)),

$$[A_J^{(0)} + \varepsilon A_J^{(1)}]x_J(\varepsilon) = (b^{(0)} + \varepsilon b^{(1)}), \tag{3.7}$$

and

$$x_j(\varepsilon) = 0 \ \ \forall j \notin J. \tag{3.8}$$

By substituting the expansion (3.3) into (3.7) and by equating the first two coefficients of the resulting expansions, we obtain the following two equations:

$$A_J^{(0)}\ u_J^0 = b_0\ , \qquad A_J^{(0)}\ u_J^1 + A_J^{(1)}\ u_J^0 = b^{(1)}\ . \tag{3.9}$$

Also, similarly to the proof of Lemma 3.2(ii) (see (3.4) and (3.5)), it can be shown that from the fact that $x_J(\varepsilon) \geq 0$ it follows that

$$u_J^0 \geq 0 \tag{3.10}$$

and that

$$u_j^1 \geq 0 \ \ \forall j \notin J_0^b \stackrel{\text{def}}{=} \{j \in J\ :\ u_j^0 > 0\}. \tag{3.11}$$

Define $(x^0, x^1) \in \mathbb{R}^n \times \mathbb{R}^n$ by the equations

$$x_j^0 \stackrel{\text{def}}{=} u_j^0\ ,\ \ x_j^1 \stackrel{\text{def}}{=} u_j^1 \ \ \forall j \in J, \qquad x_j^0 \stackrel{\text{def}}{=} x_j^1 \stackrel{\text{def}}{=} 0 \ \ \forall j \notin J\ .$$

Note that, by (3.4) and (3.8),

$$\lim_{\varepsilon \downarrow 0} x(\varepsilon) = x^0 \tag{3.12}$$

and, by (3.9)-(3.11)

$$A^{(0)}x^0 = b_0\ , \qquad A^{(0)}x^1 + A^{(1)}x^0 = b^{(1)}\ , \qquad x^0 \geq 0\ , \qquad x_j^1 \geq 0 \ \ \forall j \notin J_0^b\ . \tag{3.13}$$

Since $J_0^b \subset J_0$ (see (2.2) and (3.11)), the relationships (3.13) imply that

$$(x^0, x^1) \in \Theta_1, \ \ \text{which in turn yields} \ \ x^0 \in \theta_1\ . \tag{3.14}$$

That is, the limit (3.12) of the basic feasible solution $x(\varepsilon)$ of (1.2) defined by J belongs to θ_1. Because $x(\varepsilon) \in \theta(\varepsilon)$ for ε small enough, we have that (2.6) holds.

Take now an arbitrary $x^0 \in \theta_1$, so there exists $(x^0, x^1) \in \Theta_1$. To prove (2.8), we will find $\hat{x}(\varepsilon) \in \theta(\varepsilon)$ such that

$$\lim_{\varepsilon \downarrow 0} \hat{x}(\varepsilon) = x^0\ .$$

Consider $(x^0(\delta), x^1(\delta))$ defined as follows.

$$x^0(\delta) \stackrel{\text{def}}{=} (1-\delta)x^0 + \delta\hat{x}^0, \qquad x^1(\delta) \stackrel{\text{def}}{=} (1-\delta)x^1 + \delta\hat{x}^1, \qquad \delta \in (0,1), \tag{3.15}$$

where $(\hat{x}^0, \hat{x}^1)$ are as in the ES-1 condition). Note that $(x^0(\delta), x^1(\delta)) \in \Theta_1$ (due to convexity of Θ_1) and also that

$$x_j^0(\delta) \geq \delta\hat{x}_j^0 \geq \delta a \ \ \forall j \in J_0\ , \qquad x_j^1(\delta) \geq \delta\hat{x}_j^1 \geq \delta a \ \ \forall j \notin J_0, \tag{3.16}$$

where

$$a \stackrel{\text{def}}{=} \min\{\min_{j' \in J_0} \hat{x}_{j'}^0, \min_{j' \notin J_0} \hat{x}_{j'}^1\} > 0. \tag{3.17}$$

By Assumption (H_1), there exists $J \in S_m$ such that the matrix $A_J^{(0)}$ is invertible. Due to the invertibility of $A_J^{(0)}$, one has $A_J^{(0)} + \varepsilon A_J^{(1)} = A_J^{(0)}(\mathbb{I} + \varepsilon D_J)$, where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^m$ and $D_J \stackrel{\text{def}}{=} [A_J^{(0)}]^{-1} A_J^{(1)}$. Note that the matrix $\mathbb{I} + \varepsilon D_J$ is invertible and its inverse has a uniformly bounded norm for all $\varepsilon \in [0, \gamma_4]$, where $\gamma_4 > 0$ is small enough. Define $x(\delta, \varepsilon)$ by the equation

$$x(\delta,\varepsilon) \stackrel{\text{def}}{=} x^0(\delta) + \varepsilon x^1(\delta) + \varepsilon^2 x^2(\delta,\varepsilon), \tag{3.18}$$

where

$$x_J^2(\delta,\varepsilon) \stackrel{\text{def}}{=} -(\mathbb{I} + \varepsilon D_J)^{-1}[A_J^{(0)}]^{-1} A^{(1)}[x^1(\delta)] , \qquad x_j^2(\delta,\varepsilon) \stackrel{\text{def}}{=} 0 \ \forall j \notin J. \tag{3.19}$$

Because the norm of $(\mathbb{I} + \varepsilon D_J)^{-1}$ is uniformly bounded for $\varepsilon \in [0, \gamma_4]$, the vector $x(\delta, \varepsilon)$ is uniformly bounded for $\varepsilon \in [0, \gamma_4]$. Let us verify now that $x(\delta, \varepsilon) \in \theta(\varepsilon)$ for $\varepsilon > 0$ and $\delta > 0$ small enough. Indeed, it is straightforward to obtain

$$\begin{aligned}(A^{(0)} + \varepsilon A^{(1)})x(\delta,\varepsilon) \;&=\; A^{(0)}x^0(\delta) + \varepsilon[A^{(0)}x^1(\delta) + A^{(1)}x^0(\delta)] \\ &\quad + \varepsilon^2[A^{(0)}x^2(\delta,\varepsilon) + A^{(1)}x^1(\delta)] + \varepsilon^3 A^{(1)}x^2(\delta,\varepsilon).\end{aligned}$$

Since $(x^0(\delta), x^1(\delta)) \in \Theta_1$, we have that $A^{(0)}x^0(\delta) = b^0$ and $A^{(0)}x^1(\delta) + A^{(1)}x^0(\delta) = b^1$. So the above expression simplifies to

$$(A^{(0)} + \varepsilon A^{(1)})x(\delta,\varepsilon) \;=\; b^0 + \varepsilon b^1 + \varepsilon^2[(A^{(0)} + \varepsilon A^{(1)})x^2(\delta,\varepsilon) + A^{(1)}x^1(\delta)].$$

We claim that the expression between the square parentheses above is zero. Indeed, due to (3.19),

$$\begin{aligned}(A^{(0)} + \varepsilon A^{(1)})x^2(\delta,\varepsilon) \;&=\; (A_J^{(0)} + \varepsilon A_J^{(1)})x_J^2(\delta,\varepsilon) = A_J^{(0)}(\mathbb{I} + \varepsilon D_J)x_J^2(\delta,\varepsilon) \\ &=\; -A_J^{(0)}(\mathbb{I} + \varepsilon D_J)(\mathbb{I} + \varepsilon D_J)^{-1}[A_J^{(0)}]^{-1}A^{(1)}[x^1(\delta)] \\ &=\; -A^{(1)}\, x^1(\delta)\end{aligned}$$

$$\Rightarrow \qquad (A^{(0)} + \varepsilon A^{(1)})x(\delta,\varepsilon) = b^0 + \varepsilon b^1 \ . \tag{3.20}$$

Thus, to prove that $x(\delta, \varepsilon) \in \theta(\varepsilon)$, we need to show that $x(\delta, \varepsilon) \geq 0$ for $\varepsilon > 0$ and $\delta > 0$ small enough. As a consequence of the uniform boundedness of the expansions in (3.18)-(3.19) it is possible to find a positive scalar c, bounded above, such that

$$c \;\geq \max_{j=1,\ldots,n} \{|x_j^1(\delta) + \varepsilon x_j^2(\delta,\varepsilon)| \ , |x_j^2(\delta,\varepsilon)|\}$$

for all $\varepsilon \in [0, \gamma_4]$ and all $\delta \in (0, 1)$. Now, using (3.16)-(3.18) we can write

$$x_j(\delta,\varepsilon) = x_j^0(\delta) + \varepsilon[x_j^1(\delta) + \varepsilon x_j^2(\delta,\varepsilon)] \geq \delta a - \varepsilon c \qquad \forall j \in J_0 \ ; \tag{3.21}$$

$$x_j(\delta,\varepsilon) = x_j^0(\delta) + \varepsilon[x_j^1(\delta) + \varepsilon x_j^2(\delta,\varepsilon)] \geq \varepsilon[x_j^1(\delta) + \varepsilon x_j^2(\delta,\varepsilon)] \geq \varepsilon[\delta a - \varepsilon c] \qquad \forall j \notin J_0 \ . \tag{3.22}$$

Take now $0 < \gamma_5 < \gamma_4$ such that for all $\varepsilon < \gamma_5$ we have $2\varepsilon(c/a) < 1$, and set $\delta \stackrel{\text{def}}{=} 2\varepsilon(c/a)$. Now (3.21) and (3.22) yield

$$x(\delta,\varepsilon) = x(2\varepsilon(c/a),\varepsilon) \stackrel{\text{def}}{=} \hat{x}(\varepsilon) > 0 \ \ \forall \varepsilon \in (0,\gamma_5) \qquad \Rightarrow \qquad x(\delta,\varepsilon) = \hat{x}(\varepsilon) \in \theta(\varepsilon) \ \ \forall \varepsilon \in (0,\gamma_5) \ .$$

Also, by (3.15) and (3.18),

$$\lim_{\varepsilon \downarrow 0} \hat{x}(\varepsilon) = x^0 \ .$$

Since $x^0 \in \theta_1$ is arbitrary, (2.8) holds. This completes the proof of Theorem 2.1. $\square$

4 Extended Slater conditions of higher orders

In this section we extend our analysis to the case when the ES-1 condition is not satisfied. To this end, let us introduce some additional notations and definitions. Firstly, let

$$J_1 \stackrel{\text{def}}{=} \{j \notin J_0 \ : \ \exists\ (x^0, x^1) \in \Theta_1 \ \text{ such that } \ x_j^1 > 0\} \ .$$

If

$$J_0 \cup J_1 = \{1, \ldots, n\}, \tag{4.1}$$

then the ES-1 condition is equivalent to the condition that there exists $(\hat{x}^0, \hat{x}^1) \in \Theta_1$ such that

$$\hat{x}_j^0 > 0 \ \ \forall j \in J_0, \qquad \hat{x}_j^1 > 0 \ \ \forall j \in J_1 = (J_0)^c.$$

By Theorem 2.1, the ES-1 condition implies that every optimal solution of problem (2.3) is limiting optimal for the perturbed problem (1.1). In this situation, adding one extra layer of constraints to the unperturbed problem through the construction of Θ_1 is all we need for constructing a limiting LP for (1.1)-(1.2) in the sense of Definition 1.2.

However, if (4.1) does not hold, then one layer of additional constraints is not enough for constructing a limiting LP, and an extension of the ES-1 condition is needed. This motivates the introduction of an ES condition of an order higher than 1. To this end, let us define the set $\Theta_2 \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ by the equation

$$\Theta_2 \stackrel{\text{def}}{=} \{(x^0, x^1, x^2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ : (x^0, x^1) \in \Theta_1, \ A^{(0)} x^2 + A^{(1)} x^1 = 0, \ \ x_j^2 \geq 0 \ \forall j \notin J_0 \cup J_1\}$$

and let

$$J_2 \stackrel{\text{def}}{=} \{j \notin (J_0 \cup J_1) \ : \ \exists\ (x^0, x^1, x^2) \in \Theta_2 \ \text{ such that } \ x_j^2 > 0\} \ .$$

Altogether, given the pair (Θ_1, J_1), we can define (Θ_2, J_2) as above. If $J_0 \cup J_1 \cup J_2 = \{1, \ldots, n\}$ we stop, otherwise we continue inductively as follows. Assume that (Θ_l, J_l) has been defined for $l = 1, \ldots, k-1$ and that $\cup_{l=0}^{k-1} J_l \neq \{1, \ldots, n\}$, define (Θ_k, J_k) as follows:

$$\begin{aligned} \Theta_k \ &\stackrel{\text{def}}{=} \ \{(x^0, \ldots, x^{k-1}, x^k) \in (\mathbb{R}^n)^{k+1} : \ (x^0, \ldots, x^{k-1}) \in \Theta_{k-1}, \\ &\qquad A^{(0)} x^k + A^{(1)} x^{k-1} = 0, \ x_j^k \geq 0 \ \ \forall j \notin \cup_{l=0}^{k-1} J_l\} \end{aligned} \tag{4.2}$$

$$J_k \stackrel{\text{def}}{=} \{j \notin \cup_{l=0}^{k-1} J_l \ : \ \exists\ (x^0, \ldots, x^k) \in \Theta_k \ \text{ such that } \ x_j^k > 0\} \ .$$

Note that, by construction, the fact that $(x^0, x^1, \ldots, x^l) \in \Theta_l$ implies that

$$x_j^0 = x_j^1 = \ldots = x_j^l = 0 \ \ \forall j \notin \cup_{r=0}^{l} J_r \ , \qquad l = 0, 1, \ldots, k \ , \tag{4.3}$$

which, in particular, implies that

$$x_j^0 = x_j^1 = \ldots = x_j^{l-1} = 0 \ \ \forall j \in J_l \ , \qquad l = 1, \ldots, k \ . \tag{4.4}$$

Consider now the following linear programs

$$\max\{\langle c^{(0)}, x^0 \rangle \ : \ x^0 \in \theta_l\} = F_l^* \ , \quad l = 0, 1, \ldots, k \ , \tag{4.5}$$

where

$$\theta_l \stackrel{\text{def}}{=} \{x^0 \ : \ (x^0, x^1, \ldots, x^l) \in \Theta_l\} \ , \qquad l = 1, \ldots, k \ ; \qquad \theta_0 \stackrel{\text{def}}{=} \theta(0) \ . \tag{4.6}$$

Note that

$$\theta_k \subset \cdots \subset \theta_1 \subset \theta_0 \qquad \text{which yields} \qquad F_k^* \leq \ldots \leq F_1^* \leq F_0^* \stackrel{\text{def}}{=} F^*(0) \ .$$

Definition 4.5 *We say that the* extended Slater condition of order k (*or, the ES-k condition*) *is satisfied if there exists* $(\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^k) \in \Theta_k$ *such that*

$$\hat{x}^l_j > 0 \quad \forall j \in J_l \ , \qquad l = 0, 1, \ldots k \ . \tag{4.7}$$

and if

$$J_0 \cup \ldots \cup J_k = \{1, \ldots, n\}. \tag{4.8}$$

Note that some of J_l , $l = 0, 1, \ldots, k-1$, in (4.8) can be empty but it is assumed that $J_k \neq \emptyset$.

Remark 4.3 Assuming that (4.8) is satisfied, the validity of (4.7) (for a given $k \geq 1$) can be verified by solving n linear programs:

$$max\{x^l_j \ : \ (x^0, x^1, \ldots, x^k) \in \Theta_k\}, \qquad \forall j \in J_l \ , \qquad l = 0, 1, \ldots k \ .$$

Due to convexity of Θ_k, (4.7) is satisfied if the optimal values of the above programs are positive.

Example 4.3 Let

$$A^{(0)} = \begin{bmatrix} 1 & -2 & -1 \\ 1 & -2 & 0 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad b^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad b^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is straightforward to verify that in this case the feasible set $\theta(\varepsilon)$ has two basic solutions $x^I = (2, 1, 0)^T$ and $x^{II} = (\varepsilon, 0, \varepsilon^2)^T$, and it is representable as their convex hull:

$$\begin{aligned} \theta(\varepsilon) &= \{tx^I + (1-t)x^{II} \ : \ t \in [0,1]\} \\ &= \{x \in \mathbb{R}^3 \ : \ x_1 = \varepsilon + t(2-\varepsilon), \quad x_2 = t, \quad x_3 = (1-t)\varepsilon^2, \quad t \in [0,1] \ \} \end{aligned} \tag{4.9}$$

The expressions for $\theta(0)$ and Θ_1 can be verified to be as follows:

$$\theta(0) = \{x \in \mathbb{R}^3 \ : \ x_1 - 2x_2 = 0, \ x_1 \geq 0, \ x_2 \geq 0, \quad x_3 = 0\} \qquad \Rightarrow \qquad J_0 = \{1, 2\} \ ; \tag{4.10}$$

$$\begin{aligned} \Theta_1 &= \{(x^0, x^1) \in \mathbb{R}^3 \times \mathbb{R}^3 \ : \ x^0 \in \theta(0), : \ x^1_1 - 2x^1_2 - x^1_3 + x^0_1 - x^0_2 = 1, \\ &\qquad x^1_1 - 2x^1_2 + x^0_2 = 1, \ x^1_3 \geq 0\} \\ &= \{(x^0, x^1) \in \mathbb{R}^3 \times \mathbb{R}^3 \ : \ x^0 \in \theta(0) \\ &\qquad x^0_1 - x^0_2 = 1 - (x^1_1 - 2x^1_2), \quad x^0_2 = 1 - (x^1_1 - 2x^1_2), \quad x^1_3 = 0\}. \end{aligned} \tag{4.11}$$

The last expression above shows that $x^1_3 = 0$ and hence $J_1 = \emptyset$, so the ES-1 condition is not satisfied in the present example. Using (4.10) and (4.11) and introducing the notation $s \stackrel{\text{def}}{=} x^1_1 - 2x^1_2$, one can obtain that

$$\theta_1 = \{x^0 \in \theta(0) \ : \ x^0_1 - x^0_2 = 1 - s, \quad x^0_2 = 1 - s, \quad x^0_3 = 0, \quad s \in (-\infty, 1] \ \} \ .$$

By definition, we have that

$$\begin{aligned} \Theta_2 \quad &= \quad \{(x^0, x^1, x^2) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \ : (x^0, x^1) \in \Theta_1, \ x^2_1 - 2x^2_2 - x^2_3 + x^1_1 - x^1_2 = 0, \\ &\qquad x^2_1 - 2x^2_2 + x^1_2 = 0, \quad x^2_3 \geq 0\}. \end{aligned}$$

Subtracting one of the equations from another in the above expression, we obtain

$$\begin{aligned} \Theta_2 \quad &= \quad \{(x^0, x^1, x^2) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \ : (x^0, x^1) \in \Theta_1, \ x^1_1 - x^1_2 = -(x^2_1 - 2x^2_2 - x^2_3) \\ &\qquad x^1_1 - 2x^1_2 = x^2_3, \quad x^2_3 \geq 0\}. \end{aligned}$$

By direct substitution, we see that $(\hat{x}^0, \hat{x}^1, \hat{x}^2) \in \Theta_2$, with $\hat{x}^0 = (1, \frac{1}{2}, 0)^T$, $\hat{x}^1 = (\frac{1}{2}, 0, 0)^T$, $\hat{x}^2 = (0, 0, \frac{1}{2})^T$. Hence, $J_2 = \{3\}$ and $J_0 \cup J_1 \cup J_2 = \{1, 2, 3\}$. That is, the ES-2 condition is satisfied in this example. By further analyzing the expressions for Θ_1, θ_1 and Θ_2, one can obtain that

$$\theta_2 = \{x^0 \in \theta(0) \,:\, x_1^0 - x_2^0 = 1 - s, \quad x_2^0 = 1 - s, \quad x_3^0 = 0, \quad s \in [0, 1]\ \} \ .$$

Finding the explicit expressions for x_1^0 and x_2^0 in terms of $1 - s$ and re-denoting $1 - s$ as t, one obtains that

$$\theta_2 = \{x^0 \in \theta(0) \,:\, x_1^0 = 2t, \quad x_2^0 = t, \quad x_3^0 = 0, \quad t \in [0, 1]\ \} \ .$$

This expression, together with (4.9), imply that $\lim_{\varepsilon \downarrow 0} \theta(\varepsilon) = \theta_2$.

Theorem 4.2 *Let Assumptions (H_0), (H_2) be satisfied. Then*

$$\limsup_{\varepsilon \downarrow 0} \theta(\varepsilon) \subset \theta_k \tag{4.12}$$

and

$$\limsup_{\varepsilon \downarrow 0} F^*(\varepsilon) \leq F_k^* \ . \tag{4.13}$$

If, in addition, Assumption (H_1) and the ES-k condition are satisfied, then

$$\theta_k \subset \liminf_{\varepsilon \downarrow 0} \theta(\varepsilon) \tag{4.14}$$

and the following equalities are valid

$$\lim_{\varepsilon \downarrow 0} \theta(\varepsilon) = \theta_k \ , \tag{4.15}$$

$$\lim_{\varepsilon \downarrow 0} F^*(\varepsilon) = F_k^* \ . \tag{4.16}$$

Also, in this case, if x^0 is an optimal solution of the problem (4.5) with $l = k$, then x^0 is limiting optimal for the perturbed problem (1.1). That is, (4.5) with $l = k$ is limiting LP in the sense of Definition 1.2.

Proof. The fact that (4.13) is implied by (4.12) and that (4.15) is implied by (4.12) and (4.14) as well as the fact that (4.16) is implied by (4.15) follow from the definitions of the respective limits. Also, the fact that x^0 is limiting optimal for (1.1) if it is an optimal solution of (4.5) with $l = k$ readily follows from (4.15) and (4.16) (as indicated in Remark 2.1). Hence we proceed to prove (4.12) and (4.14). We use the notations and results from Section 3. We start with proving (4.12).

Let $J \in \Omega_1^b$ (see (3.1) and (3.6)) and let $x_J(\varepsilon)$ be the vector of basic components of the corresponding feasible basic solution $x(\varepsilon)$. That is, (3.7) and (3.8) are valid. By substituting the expansion (3.3) into (3.7) and equating the coefficients of the powers ε^l, $l = 0, 1, \ldots, k$, we obtain the following equations:

$$A_J^{(0)}\ u_J^0 = b^{(0)} \ , \qquad A_J^{(0)}\ u_J^1 + A_J^{(1)}\ u_J^0 = b^{(1)} \ , \qquad A_J^{(0)}\ u_J^l + A_J^{(1)}\ u_J^{l-1} = 0, \quad l = 2, \ldots, k,$$

which yields

$$A^{(0)} x^0 = b^{(0)} \ , \qquad A^{(0)} x^1 + A^{(1)} x^0 = b^{(1)} \ , \qquad A^{(0)} x^l + A^{(1)} x^{l-1} = 0, \quad l = 2, \ldots, k, \tag{4.17}$$

where $x^0, x^1, \ldots, x^k$, are defined by the equations

$$x_j^l \stackrel{\text{def}}{=} u_j^l \ , \quad \forall j \in J, \qquad x_j^l \stackrel{\text{def}}{=} 0 \quad \forall j \notin J \ , \qquad l = 0, 1, \ldots, k.$$

Using the above notations and (3.8), one can rewrite the expansion (3.3) in the form (that includes also the non-basic components)

$$x(\varepsilon) = \sum_{l=1}^{k} \varepsilon^l x^l + O(\varepsilon^{k+1}). \tag{4.18}$$

In the proof of Theorem 2.1 it has been established that the inclusion $(x^0, x^1) \in \Theta_1$ is valid (see (3.14)). Also, from (3.6) and (4.18) it follows, in particular, that

$$x_j^2 \geq 0 \quad \forall j \in \omega_1 \stackrel{\text{def}}{=} \{j' \in \{1, \ldots, n\} \ : \ x_{j'}^0 = x_{j'}^1 = 0\}. \tag{4.19}$$

By (4.3), $\{1, \ldots, n\} \setminus (J_0 \cup J_1) \ \subset \omega_1$, and, hence, (4.19) and (4.17) imply

$$\left.\begin{array}{l} x_j^2 \geq 0 \quad \forall j \notin J_0 \cup J_1 \\ \\ A^{(0)}x^2 + A^{(1)}x^1 = 0 \end{array}\right\} \ \Rightarrow \ (x^0, x^1, x^2) \in \Theta_2.$$

We now prove by induction that

$$(x^0, x^1, \ldots, x^k) \in \Theta_k, \tag{4.20}$$

assuming that the inclusion

$$(x^0, x^1, \ldots, x^{k-1}) \in \Theta_{k-1},$$

holds. From (3.6) and (4.18) it follows that

$$x_j^k \geq 0 \quad \forall j \in \omega_{k-1} \stackrel{\text{def}}{=} \{j' \in \{1, \ldots, n\} \ : \ x_{j'}^0 = \ldots = x_{j'}^{k-1} = 0\},$$

which yields

$$x_j^k \geq 0 \quad \forall j \notin \cup_{r=0}^{k-1} J_r \ , \tag{4.21}$$

the latter being implied by the fact that, due to (4.3),

$$\{1, \ldots, n\} \setminus (\cup_{r=0}^{k-1} J_r) \subset \omega_{k-1} \ .$$

Altogether, (4.21), along with (4.17), prove (4.20), and this completes the induction argument. Hence, we proved that $(x^0, x^1, \ldots, x^k) \in \Theta_k$. In particular, this and (4.6) yield $x^0 \in \theta_k$. Taking limits in (4.18) for $\varepsilon \downarrow 0$ we see that

$$\lim_{\varepsilon \downarrow 0} x(\varepsilon) = x^0 \in \theta_k.$$

That is, the limit x^0 of the basic feasible solution of (1.2) defined by J (see (3.12)) belongs to θ_k. Inclusion (4.12) follows now from the fact that any basic feasible solution of (1.2) corresponds to some $J \in \Omega_1^b$ and that any feasible solution of (1.2) (i.e., any element of $\theta(\varepsilon)$) is a convex combination of the basic solutions.

To prove (4.14), take an arbitrary $x^0 \in \theta_k$, so there exists $(x^0, x^1 \ldots, x^k) \in \Theta_k$. Define $(x^0(\delta), x^1(\delta), \ldots, x^k(\delta))$ by the equation

$$(x^0(\delta), x^1(\delta), \ldots, x^k(\delta)) \stackrel{\text{def}}{=} (1-\delta)(x^0, x^1, \ldots, x^k) + \delta(\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^k), \tag{4.22}$$

where $\delta \in (0,1)$ and $\hat{x}^l, \ l = 0, 1, \ldots, k$, satisfy (4.7). Note that

$$(x^0(\delta), x^1(\delta), \ldots, x^l(\delta)) \in \Theta_l \ , \qquad l = 0, 1, \ldots, k \ ,$$

which, by (4.4), implies that

$$x_j^0(\delta) = x_j^1(\delta) = x_j^{l-1}(\delta) = 0 \ , \quad \forall j \in J_l \ , \qquad l = 1, \ldots, k \ . \tag{4.23}$$

Also, by (4.7),

$$x_j^l(\delta) \geq \delta \hat{x}_j^l \geq \delta a \ \ \forall j \in J_l \ , \quad l = 0, 1, \ldots, k$$

where

$$a \stackrel{\text{def}}{=} \min_{l=0,1,\ldots,k} \{\min_{j \in J_l} \hat{x}_j^l\} > 0. \tag{4.24}$$

Let $J \in S_m$ be such that the matrix $A_J^{(0)}$ is invertible and let $D_J \stackrel{\text{def}}{=} [A_J^{(0)}]^{-1} A_J^{(1)}$. Then $A_J^{(0)} + \varepsilon A_J^{(1)} = A_J^{(0)}(\mathbb{I}+\varepsilon D_J)$, with the matrix $\mathbb{I}+\varepsilon D_J$ being invertible and its inverse uniformly bounded for $\varepsilon \geq 0$ small enough. Define $x(\delta, \varepsilon)$ by the equation

$$x(\delta, \varepsilon) \stackrel{\text{def}}{=} \sum_{l=0}^{k} \varepsilon^l x^l(\delta) + \varepsilon^{k+1} x^{k+1}(\delta, \varepsilon), \tag{4.25}$$

where

$$x_J^{k+1}(\delta, \varepsilon) \stackrel{\text{def}}{=} -(\mathbb{I} + \varepsilon D_J)^{-1} [A_J^{(0)}]^{-1} A^{(1)} [x^k(\delta)] \ , \qquad x_j^{k+1}(\delta, \varepsilon) \stackrel{\text{def}}{=} 0 \ \forall j \notin J. \tag{4.26}$$

Let us show that $x(\delta, \varepsilon) \in \theta(\varepsilon)$ for $\varepsilon > 0$ and $\delta > 0$ small enough. Via a direct substitution (similarly to the way (3.20) was established) one can obtain that

$$(A^{(0)} + \varepsilon A^{(1)}) x(\delta, \varepsilon) = b^0 + \varepsilon b^1 \ .$$

Having in mind (4.23) and (4), we obtain from (4.25):

$$\begin{aligned} x_j(\delta, \varepsilon) &= \textstyle\sum_{r=l}^{k} \varepsilon^r x_j^r(\delta) + \varepsilon^{k+1} x_j^{k+1}(\delta, \varepsilon) \\ &\geq \varepsilon^l [x_j^l(\delta) + \varepsilon(\textstyle\sum_{r=l+1}^{k} \varepsilon^{r-(l+1)} x_j^r(\delta) + \varepsilon^{k-l} x_j^{k+1}(\delta, \varepsilon)\ )] \\ &\geq \varepsilon^l [\delta a - \varepsilon c] \qquad \forall j \in J_l \ , \qquad l = 0, 1, \ldots, k \ , \end{aligned} \tag{4.27}$$

Where $c > 0$ is a constant such that

$$c \ \geq \max_{l=0,1,\ldots,k} \max_{j=1,\ldots,n} \{ | \sum_{r=l+1}^{k} \varepsilon^{r-(l+1)} x_j^r(\delta) + \varepsilon^{k-l} x_j^{k+1}(\delta, \varepsilon) | \}$$

for all $\varepsilon > 0$ small enough. Such a $c > 0$ exists thanks to (4), and c is bounded above as a consequence of the uniform boundedness of the coefficients in the expressions (4.25)-(4.26). Take now (as in the last part of the proof of Theorem 2.1) ε small enough such that $2\varepsilon(c/a) < 1$ and fix $\delta \stackrel{\text{def}}{=} 2\varepsilon(c/a)$. Now (4.27) and (4.8) yield

$$x(\delta, \varepsilon) = x(2\varepsilon(c/a), \varepsilon) \stackrel{\text{def}}{=} \hat{x}(\varepsilon) > 0 \quad \text{ and hence } \quad \hat{x}(\varepsilon) \in \theta(\varepsilon),$$

for all ε small enough. Also, by (4.22) and (4.25),

$$\lim_{\varepsilon \downarrow 0} \hat{x}(\varepsilon) = x_0 \ .$$

Since $x^0 \in \theta_k$ is arbitrary, the above implies (4.14). This completes the proof. □

Remark 4.4 In the course of the proof of Theorem 4.2 it has been established, in particular, that

$$\exists \ \hat{x}(\varepsilon) \in \theta(\varepsilon) \ , \quad \hat{x}(\varepsilon) > 0 \ , \tag{4.28}$$

with the last inequality being valid for all $\varepsilon > 0$ small enough (see (4)). Note that (4.28) is a Slater condition for the perturbed problem. The following result establishes that, if (4.28) is valid, then the equality (4.8) (which is a part of the condition ES-k) is satisfied.

Proposition 4.1 *Let Assumptions* (H_0) *and* (H_2) *be satisfied. If the Slater condition (4.28) holds , then equality (4.8) is valid for some* $k \geq 0$.

Proof. For each $J \in \Omega_1^b$, we denote by $X_J(\varepsilon)$ the basic feasible solution of (1.2) that corresponds to $J \in \Omega_1^b$ (unlike the $x_J(\varepsilon)$ used above, $X_J(\varepsilon)$ includes both basic and non-basic components of the solution). Define also a vector $\{\tilde{\lambda}_J\}_{J\in\Omega_1^b}$ such that

$$\sum_{J\in\Omega_1^b} \tilde{\lambda}_J = 1, \qquad \tilde{\lambda}_J > 0 \ \ \forall J \in \Omega_1^b.$$

Then (4.28) implies that

$$\tilde{X}(\varepsilon) \stackrel{\text{def}}{=} \sum_{J\in\Omega_1^b} \tilde{\lambda}_J X_J(\varepsilon) \ > \ 0 \ . \tag{4.29}$$

Since each of $X_J(\varepsilon)$ allows a power series expansion (see Lemma 3.2 (i)), so does $\tilde{X}(\varepsilon)$:

$$\tilde{X}(\varepsilon) \stackrel{\text{def}}{=} \sum_{l=0}^{\infty} \varepsilon^l \tilde{X}^l \tag{4.30}$$

or, componentwise,

$$\tilde{x}_j(\varepsilon) \stackrel{\text{def}}{=} \sum_{l=0}^{\infty} \varepsilon^l \tilde{x}_j^l \ , j = 1, \ldots, n \ ,$$

where $\tilde{x}_j(\varepsilon)$, $\tilde{x}_j^l$ stand for the components of $\tilde{X}(\varepsilon)$ and $\tilde{X}^l$, respectively. For any $j \in \{1, \ldots, n\}$, let l_j be such that

$$\tilde{x}_j^0 = \ldots = \tilde{x}_j^{l_j-1} = 0, \qquad \tilde{x}_j^{l_j} > 0 \ , \tag{4.31}$$

with $l_j = 0$ if $\tilde{x}_j^0 > 0$. Also, let

$$\tilde{J}_0 \stackrel{\text{def}}{=} \{j \ : \ l_j = 0\} \ , \quad \tilde{J}_1 \stackrel{\text{def}}{=} \{j \ : \ l_j = 1\} \ , \ \ldots \ , \quad \tilde{J}_k \stackrel{\text{def}}{=} \{j \ : \ l_j = k\} \ , \tag{4.32}$$

where k is such that

$$\tilde{J}_k \neq \emptyset \ , \qquad \cup_{l=0}^k \tilde{J}_l = \{1, \ldots, n\} \ . \tag{4.33}$$

Note that, due to (4.29), such a k exists. Using (4.30) and the fact that $X(\varepsilon) \in \theta(\varepsilon)$, one can repeat the argument of the corresponding part of the proof of Theorem 4.2 (e.g., see (4.20)) to establish that $(\tilde{X}^0, \tilde{X}^1, \ldots, \tilde{X}^k) \in \Theta_k$. Hence,

$$\tilde{J}_0 \subset J_0 \ , \quad (\tilde{J}_1 \setminus J_0) \subset J_1 \ , \quad \ldots \quad , \quad (\tilde{J}_k \setminus \cup_{l=0}^{k-1} J_l) \subset J_k \ . \tag{4.34}$$

From the first two of the above inclusions it follows that

$$\tilde{J}_0 \cup \tilde{J}_1 \subset J_0 \cup \tilde{J}_1 = J_0 \cup (\tilde{J}_1 \setminus J_0) \subset J_0 \cup J_1 \ .$$

Let us use induction to prove that

$$\cup_{l=0}^k \tilde{J}_l \subset \cup_{l=0}^k J_l \ . \tag{4.35}$$

Assume that

$$\cup_{l=0}^{k-1} \tilde{J}_l \subset \cup_{l=0}^{k-1} J_l \ . \tag{4.36}$$

Then, using the last inclusion in (4.34), one can write down

$$\cup_{l=0}^k \tilde{J}_l \subset (\cup_{l=0}^{k-1} J_l) \cup \tilde{J}_k = (\cup_{l=0}^{k-1} J_l) \cup (\tilde{J}_k \setminus (\cup_{l=0}^{k-1} J_l)) \subset (\cup_{l=0}^{k-1} J_l) \cup J_k \ . \tag{4.37}$$

This proves (4.35), which along with (4.33) establish the validity of (4.8). □

Note that the fulfillment of (4.8) does not necessarily mean that there exists $(\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^k) \in \Theta_k$ such that (4.7) is satisfied. In the next section we introduce yet another type of Slater condition that is implied by (and, in fact, is equivalent to) the Slater condition for the perturbed problem (4.28).

5 Lexicographic Slater condition

Given $(x^0, x^1 \ldots, x^k) \in (\mathbb{R}^n)^{k+1}$, let $I_l(x^l) \subset \{1, ..., n\},\ l = 0, ..., k$, be defined as follows

$$I_l(x^l) \stackrel{\text{def}}{=} \{i \ :\ x_i^l > 0\}, \quad l = 0, ..., k. \tag{5.1}$$

Definition 5.6 *We shall say that a concatenated vector $(x^0, x^1 \ldots, x^k) \in (\mathbb{R}^n)^{k+1}$ is lexicographically nonnegative (L-nonnegative) and write*

$$(x^0, x^1 \ldots, x^k) \succeq 0$$

if $x^0 \geq 0$ and if

$$x_i^l \geq 0\ \forall i \notin I_0(x^0) \cup I_1(x^1) \cup \ldots \cup I_{l-1}(x^{l-1}), \quad l = 1, ..., k.$$

We shall say that a concatenated vector $(x^0, x^1 \ldots, x^k) \in (\mathbb{R}^n)^{k+1}$ is lexicographically positive (L-positive) and write

$$(x^0, x^1 \ldots, x^k) \succ 0$$

if it is L-nonnegative and

$$I_0(x^0) \cup \ldots \cup I_k(x^k) = \{1, \ldots, n\}.$$

Remark 5.5 As can be easily verified, $(x^0, x^1 \ldots, x^k)$ is L-nonnegative if and only if every row of the $n \times (k+1)$ matrix $Z = (x^0, x^1, ...x^k)$ has the property that it is either the zero row, or its first nonzero entry is positive; $(x^0, x^1 \ldots, x^k)$ is L-positive if, in addition to the above, the matrix Z does not contain the zero row. Also, $(x^0, x^1 \ldots, x^k)$ is L-nonnegative if and only if

$$x(\varepsilon) \stackrel{\text{def}}{=} x^0 + \varepsilon x^1 + \ldots \varepsilon^k x^k \geq 0$$

for all $\varepsilon > 0$ small enough; $(x^0, x^1 \ldots, x^k)$ is L-positive if and only if

$$x(\varepsilon) \stackrel{\text{def}}{=} x^0 + \varepsilon x^1 + \ldots \varepsilon^k x^k > 0$$

for all $\varepsilon > 0$ small enough.

Lemma 5.3 *If*

$$(\bar{x}^0, \bar{x}^1 \ldots, \bar{x}^k) \succeq 0, \qquad (\bar{\bar{x}}^0, \bar{\bar{x}}^1 \ldots, \bar{\bar{x}}^k) \succeq 0,$$

then, for any $\delta \in (0, 1)$,

$$(x^0(\delta), x^1(\delta) \ldots, x^k(\delta)) \stackrel{\text{def}}{=} (1-\delta)(\bar{x}^0, \bar{x}^1 \ldots, \bar{x}^k) + \delta(\bar{\bar{x}}^0, \bar{\bar{x}}^1 \ldots, \bar{\bar{x}}^k) \succeq 0.$$

Proof. Due to Remark 5.5, the proof is obvious $\square$

Define the sets Ξ_k , Γ_k by the equations

$$\begin{aligned} \Xi_k \ \stackrel{\text{def}}{=}\ & \{(x^0, x^1 \ldots, x^k) \in (\mathbb{R}^n)^{k+1} :\ A^{(0)}x^0 = b^0, \ \ A^{(0)}x^1 + A^{(1)}x^0 = b^1 \ \ A^{(0)}x^l + A^{(1)}x^{l-1} \\ & = \ 0, \ \ l = 2 \ldots, k; \quad (x^0, x^1 \ldots, x^k) \succeq 0 \ \ \}; \end{aligned} \tag{5.2}$$

$$\Gamma_k \stackrel{\text{def}}{=} \{x^0 \ :\ (x^0, x^1 \ldots, x^k) \in \Xi_k\}. \tag{5.3}$$

Note that, by Lemma 5.3, the set Ξ_k and, hence, the set Γ_k are convex.

Lemma 5.4 *Under Assumptions (H_0) and (H_2),*

$$\limsup_{\varepsilon \downarrow 0} \theta(\varepsilon) \subset \Gamma_k \ . \tag{5.4}$$

Proof. As in Proposition 4.1, let $X_J(\varepsilon)$ stand for the basic feasible solution of (1.2) that corresponds to $J \in \Omega_1^b$. By Lemma 3.2, $X_J(\varepsilon)$ allows a power series expansion

$$X_J(\varepsilon) = \sum_{l=0}^{k} X_J^l \varepsilon^l + O(\varepsilon^{k+1}),$$

where $X_J^l \in R^n$, $l = 0, \ldots, k$, are some constant vectors, and $|O(\varepsilon^{k+1})| \leq c\varepsilon^{k+1}$ ($c > 0$ is a constant). Since $X_J(\varepsilon) \in \theta(\varepsilon)$, it follows that $(X_J^0, \ldots, X_J^k) \in \Xi_k$ which, in turn, implies that

$$\lim_{\varepsilon \downarrow 0} X_J(\varepsilon) = X_J^0 \in \Gamma_k. \tag{5.5}$$

Let

$$\bar{x} \in \limsup_{\varepsilon \downarrow 0} \theta(\varepsilon) \tag{5.6}$$

That is, there exist sequences ε_s , $s = 1, 2, \ldots$ ($\varepsilon_s \downarrow 0$) and $x_s \in \theta(\varepsilon_s)$ such that

$$\lim_{s \to \infty} x_s = \bar{x}. \tag{5.7}$$

From the fact that $x_s \in \theta(\varepsilon_s)$ it follows that there exist $\lambda_{J,s} \in [0, 1]$, $J \in \Omega_1^b$ satisfying

$$\sum_{J \in \Omega_1^b} \lambda_{J,s} = 1$$

such that

$$x_s = \sum_{J \in \Omega_1^b} \lambda_{J,s} X_J(\varepsilon_s). \tag{5.8}$$

Without loss of generality, one may assume that there exist limits

$$\lim_{s \to \infty} \lambda_{J,s} \stackrel{\text{def}}{=} \lambda_J \in [0, 1], \qquad \sum_{J \in \Omega_1^b} \lambda_J = 1 \tag{5.9}$$

By (5.5), (5.7), (5.8) and (5.9),

$$\bar{x} = \sum_{J \in \Omega_1^b} \lambda_J X_J^0.$$

Hence, due to the fact that Γ_k is convex, (and due to (5.5)), one may conclude that

$$\bar{x} \in \Gamma_k \ . \tag{5.10}$$

Thus, it has been established that (5.6) implies (5.10). This proves (5.4). $\square$

Definition 5.7 *We shall say that the lexicographic Slater condition of order* k (*or, the LS-*k *condition*) *is satisfied if* Ξ_k *contains an L-positive element. That is,*

$$\exists \ (x^0, x^1, \ldots, x^k) \in \Xi_k \quad \textit{such that} \quad (x^0, x^1, \ldots, x^k) \succ 0. \tag{5.11}$$

Let us now introduce some notations that are needed for further consideration (and that, in particular, allow us to give another characterization of the LS-k condition; see Corollary 5.5.1 below).

Define the index sets J_0^* , $\ldots$, J_k^* as follows

$$J_0^* \stackrel{\text{def}}{=} \cup_{(x^0, x^1 \ldots, x^k) \in \Xi_k} \{I_0(x^0)\}, \qquad J_1^* \stackrel{\text{def}}{=} \cup_{(x^0, x^1 \ldots, x^k) \in \Xi_k} \{I_1(x^1)\} \setminus J_0^* \tag{5.12}$$

$$J_l^* \stackrel{\text{def}}{=} \cup_{(x^0, x^1 \ldots, x^k) \in \Xi_k} \{I_l(x^l)\} \setminus (J_1^* \cup \ldots \cup J_{l-1}^*) \ , \quad l = 2, \ldots, k, \tag{5.13}$$

where $I_l(x^l)$ are as in (5.1). Note that

$$\cup_{s=0}^{l} J_s^* = (\cup_{(x^0,x^1\ldots,x^k)\in\Xi_k}\{I_0(x^0)\}) \cup \ldots \cup (\cup_{(x^0,x^1\ldots,x^k)\in\Xi_k}\{I_l(x^l)\}). \tag{5.14}$$

and, in particular (with $l = k$),

$$\cup_{s=0}^{k} J_s^* = (\cup_{(x^0,x^1\ldots,x^k)\in\Xi_k}\{I_0(x^0)\}) \cup \ldots \cup (\cup_{(x^0,x^1\ldots,x^k)\in\Xi_k}\{I_k(x^k)\}). \tag{5.15}$$

Lemma 5.5 *There exists $(\hat{x}^0, \ldots, \hat{x}^k) \in \Xi_k$ such that*

$$\hat{x}_j^l > 0 \quad \forall j \in J_l^* \quad (if \quad J_l^* \neq \emptyset), \quad \forall\ l = 0, \ldots, k. \tag{5.16}$$

Proof. Due to (5.14),

$$x_j^l = 0 \quad \forall j \notin (J_0^* \cup \ldots \cup J_l^*)\ ,\ \ \forall (x^0, x^1 \ldots, x^k) \in \Xi_k\ , \quad l = 0, \ldots, k-1.$$

Hence, by the definition of L-nonnegativity (see Definition 5.6),

$$x_j^{l+1} \geq 0 \quad \forall j \notin (J_0^* \cup \ldots \cup J_l^*)\ ,\ \ \forall (x^0, x^1 \ldots, x^k) \in \Xi_k\ , \quad l = 0, \ldots, k-1. \tag{5.17}$$

Also, by the definition of Ξ_k (see (5.2)),

$$x^0 \geq 0 \quad \forall (x^0, x^1 \ldots, x^k) \in \Xi_k. \tag{5.18}$$

Further on, for any $l \in \{0, \ldots, k\}$ such that $J_l^* \neq \emptyset$ and for any $j \in J_l^*$, there exists $(x^0, \ldots, x^k)^{l,j} \in \Xi_k$ such that $x_j^l > 0$ (see (5.12) and (5.13)). Define $(\hat{x}^0, \ldots, \hat{x}^k)$ by the equation

$$(\hat{x}^0, \ldots, \hat{x}^k) \stackrel{\text{def}}{=} \sum_{l=0}^{k} \sum_{j\in J_l^*} \lambda_{l,j} (x^0, \ldots, x^k)^{l,j}, \tag{5.19}$$

where

$$\lambda_{l,j} > 0, \quad \forall j \in J_l^*\ , l = 1, \ldots, k\ , \qquad \sum_{l=0}^{k} \sum_{j\in J_l^*} \lambda_{l,j} = 1. \tag{5.20}$$

As can be readily seen, (5.19) and (5.20) imply (5.16). Also, due to convexity of Ξ_k,

$$(\hat{x}^0, \ldots, \hat{x}^k) \in \Xi_k\ .$$

$\square$

The following result is an immediate consequence of Lemma 5.5.

Corollary 5.5.1 *The LS-k condition is equivalent to the equality*

$$\cup_{l=0}^{k} J_l^* = \{1, \ldots, n\}. \tag{5.21}$$

Proof. Note that, by (5.16) ,

$$I_0(\hat{x}^0) = J_0^*\ , \qquad I_l(\hat{x}^l) \supset J_l^*, \quad l = 1, \ldots, k.$$

Hence, if (5.21) is satisfied, then $\cup_{l=0}^{k} I_l(\hat{x}^l) = \{1, \ldots, n\}$, and $(\hat{x}^0, \ldots, \hat{x}^k)$ is L-positive. That is, the LS-k condition is satisfied. Conversely, by (5.15),

$$I_0(x^0) \cup \ldots \cup I_k(x^k) \subset J_0^* \cup \ldots \cup J_k^* \qquad \forall (x^0, x^1, \ldots, x^k) \in \Xi_k\ .$$

Hence, by definition of L-positivity, the validity of (5.11) implies the validity of (5.21). $\square$

Define the polyhedral set Ξ_k^* by the equation

$$\Xi_k^* \stackrel{\text{def}}{=} \{(x^0, x^1 \ldots, x^k) \in (\mathbb{R}^n)^{k+1} : \ A^{(0)}x^0 = b^0, \quad A^{(0)}x^1 + A^{(1)}x^0 = b^1, \quad A^{(0)}x^l + A^{(1)}x^{l-1}$$
$$= 0, \quad l = 2 \ldots, k; \quad x^0 \geq 0; \quad x_j^l \geq 0 \ \ \forall j \notin (J_0^* \cup \ldots \cup J_{l-1}^*), \quad l = 1, \ldots, k\}. \tag{5.22}$$

Note that from (5.17) and (5.18) it follows that

$$\Xi_k^* \supset \Xi_k \ . \tag{5.23}$$

The following results establishes that Ξ_k^* is equal to the closure of Ξ_k. The fact that Ξ_k may not be closed follows from the observation that the limit of a sequence of L-nonnegative (or even L-positive) elements is not necessarily L-nonnegative.

Lemma 5.6 *The following relationship is valid*

$$\Xi_k^* = cl\Xi_k \ . \tag{5.24}$$

Proof. Take an arbitrary $(x^0, \ldots, x^k) \in \Xi_k^*$, and define $(x^0(\alpha), \ldots, x^k(\alpha))$ by the equation

$$(x^0(\alpha), \ldots, x^k(\alpha)) \stackrel{\text{def}}{=} (1-\alpha)(x^0, \ldots, x^k) + \alpha(\hat{x}^0, \ldots, \hat{x}^k), \qquad \alpha \in (0,1),$$

where $(\hat{x}^0, \ldots, \hat{x}^k)$ satisfies the inequalities (5.16). Due to these inequalities,

$$x_j^l(\alpha) > 0 \ \ \forall j \in J_l^* \quad (if \quad J_l^* \neq \emptyset), \quad \forall \ l = 0, \ldots, k \ , \qquad \forall \ \alpha \in (0,1). \tag{5.25}$$

Also, due to (5.17) and (5.18), and due to the definition of Ξ_k^* (see (5.22)),

$$x^0(\alpha) \geq 0; \qquad x_j^l(\alpha) \geq 0 \ \ \forall j \notin (J_0^* \cup \ldots \cup J_{l-1}^*), \quad l = 1, \ldots, k \ , \qquad \forall \ \alpha \in (0,1). \tag{5.26}$$

From (5.25) and (5.26) it follows that

$$(x^0(\alpha), \ldots, x^k(\alpha)) \succeq 0 \quad \forall \ \alpha \in (0,1),$$

which implies that

$$(x^0(\alpha), \ldots, x^k(\alpha)) \in \Xi_k \quad \forall \ \alpha \in (0,1).$$

Since $lim_{\alpha \downarrow 0}(x^0(\alpha), \ldots, x^k(\alpha)) = (x^0, \ldots, x^k)$ and since $(x^0, \ldots, x^k)$ is an arbitrary element of Ξ_k^*, (5.24) is proved. □

Theorem 5.3 *Let Assumptions (H_0), (H_1) and the LS-k Slater condition be satisfied. Then*

$$\lim_{\varepsilon \downarrow 0} \theta(\varepsilon) = \Gamma_k^* \stackrel{\text{def}}{=} \{x^0 \ : \ (x^0, x^1, \ldots, x^l) \in \Xi_k^*\} \tag{5.27}$$

and

$$\lim_{\varepsilon \downarrow 0} F^*(\varepsilon) = \max\{\langle c^{(0)}, x^0 \rangle \ : \ x^0 \in \Gamma_k^*\}. \tag{5.28}$$

Proof. Note that (5.28) is implied by (5.27). Note also that from (5.3), (5.4) and (5.23) it follows that

$$\limsup_{\varepsilon \downarrow 0} \theta(\varepsilon) \subset \Gamma_k^* \ .$$

Thus, to prove the theorem, one needs to show that

$$\Gamma_k^* \subset \liminf_{\varepsilon \downarrow 0} \theta(\varepsilon). \tag{5.29}$$

By Lemma 5.5, by the definition of the LS-k condition and by (5.23), there exists

$$(\hat{x}^0, \ldots, \hat{x}^k) \in \Xi_k^*$$

such that (5.16) and (5.21) are valid. Hence, to prove (5.29) one can follow exactly the same steps as those used in the proof of (4.14) (see the second part of the proof of Theorem 4.2) with the replacement of J_l by J_l^*. □

Proposition 5.2 *Under Assumptions* (H_0) *and* (H_1)*, the LS-*k *condition is satisfied for some* $k \geq 0$ *if and only if the Slater condition for the perturbed problem (4.28) is satisfied.*

Proof. The fact that the validity of the Slater condition for the perturbed problem (4.28) is implied by the validity of the LS-k condition can be established in the same way as the fact that the validity of the former is implied by the ES-k condition (see Remark 4.4). Let us prove the converse statement that the validity of the Slater condition for the perturbed problem (4.28) implies the validity of the LS-k condition.

Since Assumption (H_1) implies Assumption (H_2), all conditions of Proposition 4.1 are satisfied, and we may use some elements of the proof of the latter. Let $\tilde{X}(\varepsilon)$ be as in (4.29) and let $\tilde{X}^l,\ l = 0, \ldots, k$, be the first $k+1$ vectors of coefficients in the expansion (4.30). Also, let $\tilde{J}_l,\ l = 0, \ldots, k$, be as in (4.32) (with k being defined by (4.33)).

From (4.29) and (4.30) it follows that

$$\sum_{l=0}^{k} \varepsilon^l \tilde{X}^l \geq 0$$

for all ε small enough, which implies (see Remark 5.5) that

$$(\tilde{X}^0, \ldots, \tilde{X}^k) \in \Xi_k \ .$$

Hence, by (5.15) and the definition of $\tilde{J}_l$,

$$\cup_{l=1}^{k} \tilde{J}_l \subset \cup_{l=1}^{k} J_l^* \ .$$

This and (4.33) imply (5.21). □

Remark 5.6 The Slater condition for the perturbed problem (4.28) can always be assumed to be satisfied (without loss of generality). In fact, similarly to Lemma 3.2(ii), one can show that the set $\{1, \ldots, n\}$ is decomposed into two subsets:

$$\{1, \ldots, n\} = D_1 \cup D_2 \ ,$$

where D_1 is such that $\ x_j = 0$ for any $x \in \theta(\varepsilon)\ $ and D_2 is such that there exists $\hat{x}(\varepsilon) \in \theta(\varepsilon)$ with $\hat{x}_j(\varepsilon) > 0\ \ \forall j \in D_2$. In both cases it is assumed that $\varepsilon \in (0, \gamma)$, where γ is positive and sufficiently small. By removing the components x_j with $j \in D_1$ from (1.1)-(1.2), one obtains an equivalent perturbed LP, in which the Slater condition (4.28) is satisfied. Of course, there remains the question of efficient determination of the membership of D_1. In principle, in a similar spirit to Remark 4.3 the asymptotic simplex method of [6] could be used repeatedly to identify indices of variables that qualify. However, this is likely to be computationally expensive.

The last result of this section establishes relationships between J_l^*, Ξ_k^* and J_l, Θ_k introduced in Section 4.

Proposition 5.3 *The following relationships are valid:*

$$J_0^* \cup \ldots \cup J_l^* \subset J_0 \cup \ldots \cup J_l \quad \forall\ l = 0, \ldots, k, \tag{5.30}$$

and

$$\Xi_k^* \subset \Theta_k \ . \tag{5.31}$$

If the ES-k *condition is satisfied for some* $k > 0$*, then the LS-*k *is satisfied for this* k *and*

$$J_l = J_l^* \ , \quad l = 0, \ldots, k; \tag{5.32}$$

$$\Xi_k^* = \Theta_k. \tag{5.33}$$

Proof. Let $(\hat{x}^0, \ldots, \hat{x}^k) \in \Xi_k$ be such that (5.16) is satisfied. Since $J_0^* \subset J_0$, then, by (5.17), $\hat{x}_j^1 \geq 0 \;\forall j \notin J_0$ and $(\hat{x}^0, \hat{x}^1) \in \Theta_1$. Hence,

$$J_1^* \setminus J_0 \subset J_1 \ ,$$

and

$$J_0^* \cup J_1^* \subset J_0 \cup J_1^* = J_0 \cup (J_1^* \setminus J_0) \subset J_0 \cup J_1 \ .$$

Using induction argument, assume that $(\hat{x}^0, \ldots, \hat{x}^{l-1}) \in \Theta_{l-1}$ and

$$J_0^* \cup \ldots \cup J_{l-1}^* \subset J_0 \cup \ldots \cup J_{l-1} \ . \tag{5.34}$$

Then, by (5.17), $\hat{x}_j^l \geq 0 \;\; \forall j \notin (J_0 \cup \ldots \cup J_{l-1})$ and $(\hat{x}^0, \ldots, \hat{x}^l) \in \Theta_l$. Consequently,

$$J_l^* \setminus (J_0 \cup \ldots \cup J_{l-1}) \subset J_l \ ,$$

which, along with (5.34), imply that

$$\begin{aligned} J_0^* \cup \ldots \cup J_{l-1}^* \cup J_l^* \subset J_0 \cup \ldots \cup J_{l-1} \cup J_l^* &= J_0 \cup \ldots \cup J_{l-1} \cup (J_l^* \setminus (J_0 \cup \ldots \cup J_{l-1})) \\ &\subset J_0 \cup \ldots \cup J_{l-1} \cup J_l \ . \end{aligned}$$

Thus, (5.30) is established.

Since the set Θ_k defined recursively by (4.2) allows a representation in the form

$$\begin{aligned} \Theta_k \;\; = \;\; & \{(x^0, x^1 \ldots, x^k) \in (\mathbb{R}^n)^{k+1} : \; A^{(0)}x^0 = b^0, \;\; A^{(0)}x^1 + A^{(1)}x^0 = b^1 \;\; A^{(0)}x^l + A^{(1)}x^{l-1} \\ & = \; 0, \;\; l = 2 \ldots, k; \quad x^0 \geq 0; \quad x_j^l \geq 0 \;\; \forall j \notin (J_0 \cup \ldots \cup J_{l-1}), \;\; l = 1, \ldots, k\}, \end{aligned}$$

from (5.30) and from the definition of Ξ_k^* in (5.22) it follows that (5.31) is valid.

Assume now that the ES-k condition is satisfied, that is, there exists $(\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^k) \in \Theta_k$ such that the relationships (4.7) and (4.8) are valid. It is easy to see that the validity of these relationships imply that

$$(\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^k) \succ 0.$$

Since $(\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^k)$ satisfies the equations defining Θ_k, it follows that

$$(\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^k) \in \Xi_k \ . \tag{5.35}$$

That is, the LS-k condition is satisfied.

To prove (5.32), note that (see (5.1))

$$J_l \subset I_l(\hat{x}^l) \; \forall \quad l = 0, \ldots, k.$$

Consequently, by (5.35) (and by (5.14)),

$$J_0 \cup \ldots \cup J_l \subset J_0^* \cup \ldots \cup J_l^* \quad \forall \, l = 0, \ldots, k,$$

which, being compared with (5.30), leads to

$$J_0 \cup \ldots \cup J_l = J_0^* \cup \ldots \cup J_l^* \quad \forall \, l = 0, \ldots, k.$$

Due to the fact that both $J_l \ , \;\; l = 0, \ldots, k$, and $J_l^* \ , \;\; l = 0, \ldots, k$, are disjoint, the latter imply (5.32), which in turn, implies (5.33).

□

Remark 5.7 It follows from Proposition 5.3 that the ES-k condition implies the LS-k condition. We conjecture that the converse does not hold. Note, however, that in the general case, the verification of the LS-k condition may be more involved than that of the ES-k condition. The former amounts to the verification of the Slater condition for the "full" perturbed problem while the later is verifiable via the consideration of LP problems independent of ε; see Remark 4.3.

Remark 5.8 The second parts of Theorems 2.1, 4.2 and Theorem 5.3 postulate the validity of Assumption (H_1) (rank$[A^{(0)}] = m$). If this assumption is not satisfied, it is possible (under certain conditions) to equivalently reformulate the perturbed problem so that it will be satisfied. In fact, let

$$\text{rank}[A^{(0)}] = m - k, \qquad k > 0$$

(that is, the problem (1.1)- (1.2) is singularly perturbed in the sense of [6] and [8]) and let V be a $m \times k$ matrix, the columns of which constitute a basis of the linear space $\mathcal{V}$,

$$\mathcal{V} \stackrel{\text{def}}{=} \{v \in \mathrm{I\!R}^{\mathrm{m}} \; : \; v^T A^{(0)} = 0\}.$$

It is, perhaps, worth pointing out that the "shuffle algorithm" as presented in [2] might be used calculate V. Note also that from Assumption (H_0) it follows that

$$V^T b^{(0)} = 0. \tag{5.36}$$

Assume that the following condition is satisfied

$$\text{rank} \begin{bmatrix} A^{(0)} \\ V^T A^{(1)} \end{bmatrix} = m \; . \tag{5.37}$$

Let $\bar{A}^{(0)}$ be a submatrix of $A^{(0)}$ formed by $m - k$ linearly independent rows of $A^{(0)}$. Then from (5.37) it follows that

$$\text{rank} \begin{bmatrix} \bar{A}^{(0)} \\ V^T A^{(1)} \end{bmatrix} = m \; . \tag{5.38}$$

Denote by $\bar{A}^{(1)}$, $\bar{b}^{(0)}$, and $\bar{b}^1$ the submatrices of $A^{(1)}$, $b^{(0)}$, and b^1 whose rows correspond to the rows of the submatrix $\bar{A}^0$. Using (5.36), one can show that the feasible set of the perturbed problem $\theta(\varepsilon)$ (see (1.2)) can be presented in the form

$$\theta(\varepsilon) = \{x \in \mathrm{I\!R}^n \; : \; \begin{bmatrix} \bar{A}^{(0)} + \varepsilon \bar{A}^{(1)} \\ V^T A^{(1)} \end{bmatrix} x = \begin{bmatrix} \bar{b}^{(0)} + \varepsilon \bar{b}^{(1)} \\ V^T b^{(1)} \end{bmatrix}, \quad x \geq 0\},$$

which, due to (5.38), implies that an appropriately modified Assumption (H_1) is satisfied. A similar process can be continued further in case (5.37) is not satisfied. The latter, however, is beyond the scope of this paper.

Acknowledgement. We would like to thank Phil Howlett for his help in constructing Example 4.3. We would also like to thank an anonymous reviewer, whose perceptive remark led to new results reported in Section 5 that made the presentation more complete.

References

[1] Anstreicher,K.M., 1983, "Generation of feasible descent directions in continuous time linear programming", SOL Technical Report 83-18, Dept. of Operations Research, Stanford University, Palo Alto.

[2] Anstreicher,K.M., Rothblum, U.G., 1987, "Using Gauss-Jordan elimination to compute the undex, generalized nullspaces, and Drazin Inverse," *Linear Algebra and Its Applications*, v.85, pp. 221-239.

[3] Avrachenkov, K.E. , Haviv, M., Howlett, P.G., 2001, "Inversion of analytic matrix functions that are singular at the origin", *SIAM Journal on Matrix Analysis and Applications*, v.22, no.4, pp.1175-1189.

[4] Bonnans, J.F., Shapiro, A., 2000, *Perturbation analysis of optimization problems*, Springer Series in Operations Research, New York.

[5] Burachik, R.S., Iusem, A.N., 2008, *Set-Valued Mappings and enlargements of monotone operators*, Springer, Berlin.

[6] Filar, J.A., Altman, E., Avrachenkov, K.E., 2002, "An asymptotic simplex method for singularly perturbed linear programs", *Operations Research Letters*, v.30, no.5, pp.295-307.

[7] Mangasarian, O., 1994, *Nonlinear Programming*. Classics in Applied Mathematics, Society for Industrial and Applied Mathematics-SIAM Publishers, Philadelphia.

[8] Pervozvanskii, A.A., Gaitsgori, V.G., 1988, *Theory of suboptimal decisions: Decomposition and aggregation*, Kluwer, Dordrecht.

[9] Puterman, M. L., 1994, *Markov Decision Processes: Discrete Stochastic Dynamic Programming*, John Wiley and Sons, New York.

[10] Rockafellar, R. T., Wets, R.J.-B. 1998, *Variational Analysis*. Springer, Berlin.