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Continuous Polling with Rerouting: Performance and Modeling of Ferry assisted Wireless LANs

Veeraruna Kavitha

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Abstract In this paper we focus on a class of polling systems encountered while modeling a class of ferry based wireless local area networks (FWLAN). A ferry facilitates local communication between the nodes (or users) as well as the communication between the nodes and a base station (BS) that serves as the gateway with the external world. The ferry, while walking in a predetermined cyclic path, communicates with the static nodes of the network via a wireless link. The ferry is assumed to stop and communicate with a node that has a packet to send or to receive, when it is closest to that node. The location distribution of the node to which (or from which) a packet arrives is assumed to have a support of positive Lebesgue measure. These features imply that polling models with finite number of queues cannot be used to model the system. Further, in almost all studied continuous polling systems, the user leaves the system after his service is completed. But for every local data transfer, the ferry has to collect the data from the source and then deliver the same to the sink. It either delivers the data to the sink on its own or has to be guided by the BS for the same. Thus each transfer may require services at two or three independent locations. Such an application can be modeled by *polling systems with rerouting.* We study the continuous polling models with rerouting, via discretization approach and by using the known Pseudo conservation laws of discrete systems. We obtain its stationary expected workload as the limit of the same for a discrete system. Our results rely heavily on fixed point analysis of infinite dimensional operators.

Keywords Continuous Polling systems · Rerouting · Virtual Workload · Pseudo Conservation Laws · Local Area Network

Maestro, INRIA, Sophia Antipolis, France
LIA, University of Avignon, France
kvoletiv@sophia.inria.fr

1 Introduction

Polling systems are a special class of queueing systems wherein, while walking in a fixed cyclic path, a single server attends to a number of queues as and when it meets one. Continuous polling systems are the ones in which arrivals can occur and the service can be derived, anywhere in a continuum. Most of these systems are studied under standard *gated service*. The server attends to the users immediately, as and when it encounters one. We then have a *globally gated service* ([23]), wherein the server closes a global gate when it arrives say at 0 and tags all the users that arrived before the gate closure. Before retuning to 0, it only serves the tagged users. In all these systems, the user leaves the system after its service. But there can be applications, wherein the users require a second service or more.

In a Ferry assisted Wireless LAN (FWLAN), a message Ferry is a mobile relay station that serves as a "postman" to deliver (collect) messages to (from) the static or dynamic wireless nodes in an otherwise disconnected network. The ferry moves periodically in a cyclic route halts on its way only when it encounters a user with request. Every point on the ferry path is a potential stop and each stop is assigned an area that contains all points closer to that stop than to other stops. In this paper, we study one such FWLAN, where the ferry facilitates communication not only between the base station (a gateway to the external world) and the network users, but also facilitates the inter user or local communication. Every local data transfer demands the ferry to provide its service at two independent locations in the network: 1) ferry collects data from the source; and 2) ferry deposits the collected data at the sink. The mobile units like ferries, are complicated from a designer's perspective and it is often preferred to design such structures with minimal intelligence. We also consider a hybrid architecture, in which the ferry does facilitate local communication, but with the help of the base station. This architecture requires 3 independent services, an extra intermediate communication with the base station, in comparison with the previous architecture. To model such FWLANs we need systems that support more than one service.

In [22], we consider a polling system in which the user after its first service may be rerouted to a different point to obtain a second service. In this paper we further extend it to multiple but finite number of services. We now can model the hybrid architecture. In addition, we consider polling systems in which the arrival position distribution can be a mixed (mixture of continuous and discrete probability measures). This extension required significant changes in the convergence proofs in comparison with our previous paper ([22]). This extension facilitates modeling of many more variants (more practical systems) of Ferry aided Wireless LANs. We consider more practical examples of ferries that operate in a rectangular area and when the ferries can travel in a rectangular path, like for example the route of a bus. One can handle many more practically implementable paths and systems, for example zig zag paths.

Using the discretization approach of our earlier papers [23, 22], we obtain the stationary expected workload as the limit of the stationary expected workload of a discrete system. We discretize the continuous polling system in such a way that the Pseudo conservation laws of of Sidi et.al ([6]) for discrete polling systems with rerouting, are applicable. To the best of our knowledge this is the first paper which considers a continuous polling system with rerouting.

An important performance measure for FWLAN design is the expected waiting time, i.e., the average time an arrival has to wait before it's service starts. The waiting time depends on: 1) the over-all service times (the time required to complete an uplink or downlink request using wireless medium); 2) the walking time, i.e., the time taken by the ferry to traverse the cyclic path once. The over-all service times reduce if one uses longer ferry routes as, then, the users can be served by the ferry standing relatively at a shorter distance. However the walking time increases with longer routes. The objective of this paper is to study these trade-offs using stationary expected workload performance. By this, we minimize the expected waiting times at all users in the Pareto sense.

Related Work: Continuous polling systems were first introduced by Fuhrmann and Cooper [16], further explored by Coffman and Gilbert [12,13] and Kroese and Schmidt [20,17–19] in a series of works. Stability results are available in [24,21,11]. The continuous systems are usually analyzed under simplified conditions, which we refer as 'symmetric conditions': Poisson arrivals and every arrival picks up a uniformly-distributed landing site on the circle while the server is moving at a constant speed in a fixed direction. Further, the service times are identically distributed.

Snowplowing systems generalize many of the above assumptions and study a more general continuous system. For example, the incoming work-flow to the system is taken to be a general Lévy random measure and the walking times are assumed to be random (see for example [14,15]).

In literature, continuous polling systems have been usually analyzed with simple gated/exhaustive service. In [23] we study continuous polling systems which offer either globally gated service or gated service or a mixture of the two services or elevator service, without requiring symmetry. Even in this paper, we do not assume the symmetric assumptions and additionally consider rerouting.

Polling systems are widely used for modeling and analysis in various scenarios, like in communication systems, computer hardware and software, road traffic control. For example, in [29,28,30], authors analyze FWLANs. Apart from FWLANs, there are many other applications, in which customers may require more than one service. Various such applications are mentioned in [6], for example: 1) selective-repeat ARQ protocol used to recover from transmission errors; 2) computer system where a single processor often has the responsibility of performing many distinct tasks, such as computation, sending information to memory, retrieve information. The results of this paper can be applied in the aforementioned applications, whenever the arrivals are on a continuum. This paper (in contrast to [6]) supports many but finite number of reroutings and hence can model almost all the (continuous versions of the) applications mentioned in [6].

The system, model and the notations of the paper are introduced in section 2. In the same section we also brief upon the main result of the paper, Theorem 1. We discuss the discretization in section 3 and the proof of the Theorem 1 in sections 4 and 5. Ferry based wireless LANs are analyzed in section 6.

2 Continuous Polling Systems

We consider a continuous polling system, where the server moves continuously on a circle $\mathcal{C}$ with speed α and stops at a point only when it finds a user with request. The external arrival process is modeled by a Poisson process with intensity λ and every external/new arrival is associated with two marks: the position $R_0 \in \mathcal{C}$ distributed as P_{R_0} and the service times B_{r_0}. The service times in general can depend upon the position R_0 of the arrival. Let $b_{r_0}(q)$ $b_{r_0}^{(2)}(q)$ represent the conditional first two moments of the service time B_{r_0} conditioned on the event that the position of arrival is at point q. The service times of different users are independent of each other.

Every user is serviced the first time the server encounters him on the circle. After his service is completed, the user is either rerouted, independent of all the previous events, to a new point q' on the circle with probability $\epsilon_1 P_{R_1}(dq')$ or exits the system with probability $1-\epsilon_1$. *So the rerouting probabilities are independent of the position at which the previous service is obtained.* The rerouted users are serviced the next time the server meets them and then again either they exit with probability $1-\epsilon_2$ or are rerouted to another point with probability $\epsilon_2 P_{R2}(dq)$ independent of all the earlier events. This kind of rerouting can take place at maximum for N times with $N < \infty$ and using the rerouting probabilities $\{\epsilon_j P_{R_j}(dq)\}_{0 \le j < N}$. These arrival probabilities can be mixed, i.e.,

$$P_{R_j}(dq) = \sum_{i=1}^{M_j} p_{j,i} 1_{\{q_{j,i}\}}(q) + f_{R_j}(q)dq \text{ for all } 0 \le j < N. \tag{1}$$

With these, there can be more than one user waiting at the same point. In this case, the server first serves the user with maximum services completed and then the second maximum and so on. Further, the users belonging to the same type are serviced in FIFO (first in first out) order.

The users are rerouted independent of the previous happenings, however their service requirements for the $(j+1)$ service can depend upon the position of the j-rerouted (after completing j number of services) point R_j, which is distributed as P_{R_j}. Let $b_{r_j}(q)$, $b_{r_j}^{(2)}(q)$ represent the conditional first and second moments of the service time, B_{r_j}, demanded by the j-rerouted users given that the rerouted arrival is at point q. Let $\bar{b}_{r_j} := E_{R_j}[b_{r_j}(R)]$ (the expectations are with respect to P_{R_j}), for every $0 \le j < N$, represent the unconditional moments. Similarly define the second moments: $\bar{b}_{r_j}^{(2)} := E_{R_j}[b_{r_j}^{(2)}(R_j)]$.

<u>**Notations:**</u> The circular path is mapped to an interval $[0, |\mathcal{C}|]$ ($|\mathcal{C}|$ is length of $\mathcal{C}$) for the purposes of analysis. The variables like b_{r_j}, f_{R_j}, τ etc. represent nonnegative functions on interval $[0, |\mathcal{C}|]$ while terms like $b_{r_j}(q)$ or $\tau(q)$ represent their value at a point $q \in [0, |\mathcal{C}|]$. The bar of the same variable, for

example $\bar{b}_{r_j}$, represents the average w.r.t. to the corresponding position distribution. Similarly variables like $\tau^{(2)}$, $\tilde{\tau}^{(2)}$ etc. are functions on $[0, |\mathcal{C}|] \times [0, |\mathcal{C}|]$ while $\tau^{(2)}(q, q')$ represents the function value at (q, q'). The expectations are represented by symbol E and these expectations are either with respect to R_j for some j or the stationary measure of the process under consideration. E^0 represents the expectation with respect to Palm stationary measure. In situations of ambiguity we suffix E with variables like, R_j. Define the following (with $\epsilon_0 := 1$ $\hat{\epsilon}_j := \Pi_{i=0}^{j} \epsilon_i$):

$$\rho_{r_j}([a, c]) := \lambda \int_a^c b_{r_j}(q) P_{R_j}(dq), \quad \rho_{r_j} := \rho_{r_j}([0, |\mathcal{C}|]), \text{ for all } 0 \le j < N$$

$$\rho([a, c]) := \rho_{r_0}([a, c]) + \sum_{j=1}^{N-1} \hat{\epsilon}_j \rho_{r_j}([a, c]) \text{ and } \rho = \rho_{r_0} + \sum_{j=1}^{N-1} \hat{\epsilon}_j \rho_{r_j}.$$

2.1 Main Result

Virtual workload of a polling system is defined as the total workload corresponding to all the waiting users, i.e., the sum of the service times of all the waiting users. Not much theory is available for calculating the expected virtual workload of polling systems with arrivals in a continuum. In this section we derive new (stationary expected workload) results related to continuous polling systems with rerouting. Throughout we consider *stationary and ergodic systems.* We obtain the stationary expected virtual workload:

Theorem 1 *Assume that $\{b_{r_j}; j\}$ and $\{f_{R_j}; j\}$ are continuous functions. Then there exists a threshold λ_0 (given by equation (24) in Appendix A) and for all Poisson input arrivals at rates $\lambda < \lambda_0$, the expected stationary virtual workload for the continuous polling system with rerouting, V_{rrt}, equals (with $\hat{\rho}(q) := \rho([0, q])$, $\check{\epsilon}_j^k := \Pi_{i=j}^{k} \epsilon_i$):*

$$\begin{aligned} V_{rrt} = & \frac{\rho\lambda}{2(1-\rho)} \sum_{k=0}^{N-1} \hat{\epsilon}_k \bar{b}_{r_k}^{(2)} + \frac{\rho\lambda}{1-\rho} \sum_{l=0}^{N-1} \sum_{k=l+1}^{N-1} \hat{\epsilon}_l \bar{b}_{r_k} \bar{b}_{r_l} + \frac{\rho|\mathcal{C}|\alpha^{-1}}{2} \\ & + \frac{\lambda\alpha^{-1}}{1-\rho} \int_0^{|\mathcal{C}|} \int_0^{|\mathcal{C}|} \left(b_{r_0}(q) + \sum_{j=1}^{N-1} \hat{\epsilon}_j \bar{b}_{r_j} \right) \left(\hat{\rho}(y) - \hat{\rho}(q) + 1_{\{y<q\}} \rho \right) P_{R_0}(dq) dy \\ & + \sum_{j=0}^{N-2} \frac{\hat{\epsilon}_{j+1} \lambda \alpha^{-1}}{1-\rho} \int_0^{|\mathcal{C}|} \int_0^{|\mathcal{C}|} \left(b_{r_{j+1}}(q') + \sum_{k=j+2}^{N-1} \check{\epsilon}_{j+2}^k \bar{b}_{r_k} \right) (q' - q + |\mathcal{C}| 1_{\{q>q'\}}) \\ & \qquad P_{R_{j+1}}(dq') P_{R_j}(dq). \end{aligned} \quad (2)$$

Special Cases: It is easy to verify that, when $N = 2$ and when $\{P_{R_j}\}$ have densities, (2) matches with the stationary expected workload derived in [22, Theorem 1]. We now simplify this formula for some special cases.

1) Under symmetric conditions: Uniform arrivals, i.e., $\{P_{R_j}\}$ are all uniform. For every j, the service time moments $b_{r_j}(q)$, $b_{r_j}^{(2)}(q)$ are equal at all

points q and equal $\bar{b}_{r_j}$, $\bar{b}^{(2)}_{r_j}$. Then $\hat{\rho}(q) = \rho q/|\mathcal{C}|$ and (2) simplifies to:

$$V^{sym}_{rrt} = \frac{\rho\lambda}{2(1-\rho)} \sum_{k=0}^{N-1} \hat{\epsilon}_k \bar{b}^{(2)}_{r_k} + \frac{\rho\lambda}{1-\rho} \sum_{l=0}^{N-1} \sum_{k=l+1}^{N-1} \hat{\epsilon}_l \bar{b}_{r_k} \bar{b}_{r_l} + \frac{\rho|\mathcal{C}|\alpha^{-1}}{2(1-\rho)} + \sum_{j=0}^{N-2} \frac{\hat{\epsilon}_{j+1}\lambda\alpha^{-1}}{1-\rho} \left(\bar{b}_{r_{j+1}} + \sum_{k=j+2}^{N-1} \check{\epsilon}^k_{j+2} \bar{b}_{r_k} \right) \frac{|\mathcal{C}|}{2}. \tag{3}$$

2) Gated polling under symmetric conditions: (similar to case 1) is analyzed in [20]. By [20, Theorem 5.1], the stationary expected number of waiting users:

$$E[N]_g = \lambda \bar{b}_{r_0} + \frac{\lambda \left(\alpha^{-1} + \lambda \bar{b}^{(2)}_{r_0} \right)}{2\left(1 - \lambda \bar{b}_{r_0}\right)} \text{ with } |\mathcal{C}| = 1.$$

They also considered the user under service. Excluding the user under service and applying Wald's lemma the expected virtual workload equals :

$$V^{sym}_g = \bar{b}_{r_0} E[N]_{r_0} = \frac{\lambda \bar{b}_{r_0} \left(\alpha^{-1} + \lambda \bar{b}^{(2)}_{r_0} \right)}{2\left(1 - \lambda \bar{b}_{r_0}\right)}.$$

This matches with (3) when $|\mathcal{C}| = 1$, $N = 1$ (no rerouting).

3) Gated polling system under general conditions: In [23] we studied a mixed polling system which supports gated as well as globally gated service users. From Theorem 1, [23] the expected stationary virtual workload for pure gated service is obtained by substituting $p_{gg} = 0 = 1 - p_g$ and it equals:

$$V_g = \lambda \bar{b} \frac{\lambda \bar{b}^{(2)}}{2(1-\lambda\bar{b})} + \frac{\lambda \bar{b} |\mathcal{C}| \alpha^{-1}}{2} + \frac{|\mathcal{C}|\alpha^{-1}}{2(1-\lambda\bar{b})} \left(\lambda^2 \bar{b}^2 \right). \tag{4}$$

The first integral in the formula (2) with no rerouting ($N = 1$) equals[1]:

$$\frac{\lambda\alpha^{-1}}{1-\rho} \lambda \int_0^{|\mathcal{C}|} \int_0^{|\mathcal{C}|} b_{r_0}(q') \left(\hat{b}_{r_0}(q) - \hat{b}_{r_0}(q') + 1_{\{q<q'\}} \bar{b}_{r_0} \right) P_{R_0}(dq') dq$$
$$= \frac{\lambda\alpha^{-1}}{1-\rho} \lambda \left(\bar{b}_{r_0} \int_0^{|\mathcal{C}|} \hat{b}_{r_0}(q) dq - |\mathcal{C}| E_{R_0}[b_{r_0}(R_0) \hat{b}_{r_0}(R_0)] + \bar{b}_{r_0} \int_0^{|\mathcal{C}|} \left(\bar{b}_{r_0} - \hat{b}_{r_0}(q) \right) dq \right)$$
$$= \frac{\lambda\alpha^{-1}}{1-\rho} \lambda \bar{b}^2_{r_0} \frac{|\mathcal{C}|}{2}, \quad \text{where } \hat{b}_{r_0}(q) := \int_0^q b_{r_0}(y) P_{R_0}(dy).$$

[1] By interchanging the order of the two integrals,

$$E[b_{r_0}(R_0)\hat{b}_{r_0}(R_0)] = \int_0^{|\mathcal{C}|} \left(\int_0^{q'} b_{r_0}(q) P_{R_0}(dq) \right) b_{r_0}(q') P_{R_0}(dq')$$
$$= \int_0^{|\mathcal{C}|} \left(\int_q^{|\mathcal{C}|} b_{r_0}(q') P_{R_0}(dq') \right) b_{r_0}(q) P_{R_0}(dq)$$
$$= \int_0^{|\mathcal{C}|} \left(\bar{b}_{r_0} - \int_0^q b_{r_0}(q') P_{R_0}(dq') \right) b_{r_0}(q) P_{R_0}(dq) \quad \text{and so, } E[b_{r_0}(R_0)\hat{b}_{r_0}(R_0)] = \bar{b}^2_{r_0}/2.$$

Upon further simplification, (2) (with $N = 1$) matches with (4).

4) Globally gated systems under general conditions: A globally gated system (all the arrivals wait till the server reaches the global gate point 0 to get tagged and the server attends only the tagged users) can be obtained from our continuous polling system with rerouting by substituting $N = 2$, $P_{R_0}(dq) = 1_{\{q=0\}}$, $b_{R_0}(0) = 0$, $\epsilon_1 = 1$. With these (2) simplifies to[2]:

$$V_{gg} = \frac{\rho\lambda\bar{b}_{r_1}^{(2)}}{2(1-\rho)} + \frac{\rho|\mathcal{C}|\alpha^{-1}}{2}\frac{1+\rho}{1-\rho} + \lambda\alpha^{-1}E_{R_1}[Qb_{r_1}(Q)],$$

which matches with the formula derived for globally gated system in [23, eqn. (5)].

The main aim of this paper is to obtain the proof of Theorem 1. We later apply the result, formula (2), to analyze Ferry aided Wireless LANs in section 6. The proof is obtained in the following 3 major steps (as done in [23,22]). There are major changes in the proof, especially the third step, because of mixed probability measures modeling the distribution of the arrival positions as in (1).

1)Discretization: Continuous polling system with rerouting is converted to an appropriate discrete polling system with rerouting in section 3, for which the Pseudo conservation laws and hence the stationary expected virtual workload is known (see [6]).

Let $\delta^\sigma(q)$ for every point q on $\mathcal{C}$ represent the point, in the discrete system with σ discretization levels, standing at which the server attends the possible users of point q. Let δ^∞ represent the same for continuous system. Note that $\delta^\infty(q) = q$ for all q, i.e., that δ^∞ is the identity map.

2)Fixed point equations: We express the stationary moments of the time to reach $\delta^\sigma(q)$ and start the service of the external users, for every q on the circumference $\mathcal{C}$, starting from 0 (any reference point) as a fixed point (in the space of left continuous and right limit functions) of an affine linear operator in section 4. We obtain a common operator $((\mathcal{F}, \Theta)$ defined in section 4), which is further parametrized by σ. The fixed point of the common operator at $\sigma < \infty$ gives the required stationary moments for the discrete system while that at $\sigma = \infty$ (the identity map) corresponds to the continuous system. We show the continuity of these fixed points as $\sigma \to \infty$ via contraction mapping theorem and hence show that the stationary moments of the discrete system converge to that of the continuous system.

3)Alternate expression for Virtual Workload: We express the stationary expected virtual workload in terms of the stationary moments of Step 2. Note, that this common expression cannot be computed easily and is used only for proof. We show the continuity of this common expression as $\sigma \to \infty$ and hence

[2] note for example, as P_{R_0} is concentrated only at 0, the first integral of (2) simplifies to

$$\frac{\lambda\alpha^{-1}}{1-\rho}\int_0^{|\mathcal{C}|}(\bar{b}_{r_1})(\hat{\rho}(y))dy = \frac{\lambda\alpha^{-1}}{1-\rho}\lambda\bar{b}_{r_1}\left(|\mathcal{C}|\bar{b}_{r_1} - E[Qb_{r_1}(Q)]\right).$$

the convergence of the stationary expected virtual workload of the discrete system to that of the continuous system in section 5.

3 Discretization

For each integer σ, we construct a discrete polling system with $N\sigma$ queues such that the limit (as $\sigma \to \infty$) of the performance measure of this system converges to that of the continuous system of Theorem 1. The external arrivals to the system are served at σ number of queues while the rerouted users are served in the remaining $(N-1)\sigma$ queues: each σ of them are dedicated entirely for the j-rerouted users (i.e., the users that already received j services and are awaiting the $(j+1)$-th service), for $1 \leq j < N$. The circumference $|\mathcal{C}|$ is divided to σ equal segments $\{I_i\}_{i=1}^{\sigma}$ with $I_1 = [0, \mathcal{C}/\sigma]$. External Arrivals are as in the continuous system. Users arriving in area I_i are treated as though arriving in queue numbered $N(i-1)+N-j$ (for "j-rerouted" users) or Ni (for external users). For every i, the server stops upon reaching the starting point, $i^\sigma := (i-1)|\mathcal{C}|/\sigma$, of I_i and serves the users of I_i before moving further. Hence, $\delta^\sigma(q)$, the point standing at which the server attends the users at q, equals

$$\delta^\sigma(q) = \begin{cases} \sum_{i=1}^{\sigma} i^\sigma 1_{\{q \in I_i\}} \text{ with } I_i := [i^\sigma, (i+1)^\sigma) & \text{when } \sigma < \infty \\ q & \text{when } \sigma = \infty. \end{cases} \tag{5}$$

The server first attends the "$(N-1)$-rerouted" queue ($N(i-1)+1$ queue), using gated service: the server attends all those users that were rerouted for the $(N-1)$-th time, before the server reached i^σ. Then it serves $(N-2)$-rerouted queue and so on till 1-rerouted queue, all using gated service. After all the rerouted queues, the server attends the "external" queue once again using gated service, i.e., it serves all the external arrivals of I_i that arrived before the server completed with all the rerouted queues of I_i.

Within a queue, the server attends the users using a special order which we call as arrival position order. In this special order, the users within a queue are served in the order of their distance from the stop i^σ of the server, i.e., *the user at minimum distance is served first.* Further, the users if waiting at the same point, are served in FIFO order. So, the users are almost served in the same order as done in a continuous system. The main difference b/w the continuous system and the discretized system is that some of the users are postponed to the next cycle in the discretized system. This is mainly because of the combined effect of discretization and the gated service. But we will see that the effect of these difference users reduces to zero as σ tends to ∞.

Define $\{i,j\} := N(i-1)+N-j$. The external arrivals after their first service (at one of the $\{i,0\}$ numbered queues) are either rerouted to one of $\{k,1\}$ queues or exit. Similarly users of queues $\{i,j-1\}$ are either rerouted to queues $\{k,j\}$ or exit and these transitions happen according to the following probabilities (by independence):

$$P_{\{i,j-1\},\{k,j\}} = Prob\left(\text{User in } [i^\sigma, (i+1)^\sigma] \ \ j\text{-rerouted to } [k^\sigma, (k+1)^\sigma]\right)$$

$$= \epsilon_j P_{R_j}(I_k) \text{ for all } 1 \le j \le N-1, \ i,k \text{ and}$$
$$P_{\{i,j\},\{k,j'\}} = P_{\{i,j\},\{k,0\}} = 0 \text{ for all } i,k,j \text{ and } j' \text{ with } j' \ne j+1.$$

Poisson arrivals into the system occur with intensity λ and land in one of the $\{i,0\}$ queues: the arrivals in I_i form the external arrivals to queue $\{i,0\}$. Thus, the rate of external arrivals at different queues are:

$$\lambda_{\{i,0\}} = \lambda P_{R_0}(I_i) \text{ and } \lambda_{\{i,j\}} = 0 \text{ for all } i \le \sigma \text{ and } 0 < j < N.$$

An user at external (j-rerouted) queue $\{i,0\}$ ($\{i,j\}$), demands service B_{r_0} (B_{r_j}) and this service is conditioned on the event that the arrival is in I_i. Thus, the service time moments at different queues are:

$$b_{\{i,j\}} = \frac{E[B_{r_j} 1_{\{R\in I_i\}}]}{P_{R_j}(I_i)} \qquad b^{(2)}_{\{i,j\}} = \frac{E[B^2_{r_j} 1_{\{R\in I_i\}}]}{P_{R_j}(I_i)} \text{ for all } 0 \le j < N.$$

The overall arrival rates γ_i (resulting after rerouting) can be calculated solving [6, eqn. (2.1)] inductively as (recall $\hat{\epsilon}_j = \Pi_{k=0}^{j} \epsilon_k$):

$$\gamma_{\{i,0\}} = \lambda_{\{i,0\}} + \sum_{k}^{N\sigma} \gamma_k P_{k,\{i,0\}} \Rightarrow \gamma_{\{i,0\}} = \lambda_{\{i,0\}} \text{ for all } i \text{ and so}$$
$$\gamma_{\{i,j\}} = \lambda_{\{i,j\}} + \sum_{k=1}^{N\sigma} \gamma_k P_{k,\{i,j\}} = \epsilon_j P_{R_j}(I_i) \sum_{l=1}^{\sigma} \gamma_{\{l,j-1\}} \Rightarrow \gamma_{\{i,j\}} = \hat{\epsilon}_j \lambda P_{R_j}(I_i).$$

The overall service time requirements resulting from the first and the possible second service, $\tilde{b}$, can be calculated as below (solving [6, equations 2.3 and 2.4] by induction starting with $j = N-1$ and with $\check{\epsilon}_j^k := \Pi_{i=j}^{k} \epsilon_i$):

$$\tilde{b}_{\{i,j\}} = b_{\{i,j\}} + \sum_{k=j+1}^{N-1} \check{\epsilon}_{j+1}^k \bar{b}_{r_k}$$
$$\tilde{b}^{(2)}_{\{i,j\}} = b^{(2)}_{\{i,j\}} + \sum_{k=j+1}^{N-1} \check{\epsilon}_{j+1}^k \bar{b}^{(2)}_{r_k} + 2b_{\{i,j\}} \sum_{k=j+1}^{N-1} \check{\epsilon}_{j+1}^k \bar{b}_{r_k} + 2\sum_{l=j+1}^{N-2} \sum_{k=l+1}^{N-1} \check{\epsilon}_{j+1}^l \bar{b}_{r_k} \bar{b}_{rl}.$$

Define, $\rho_{\{i,j\}} := \gamma_{\{i,j\}} b_{\{i,j\}}$ and $\rho = \sum_{i,j} \rho_{\{i,j\}} = \lambda \bar{b}_{r_0} + \sum_{j=1}^{N-1} \hat{\epsilon}_j \lambda \bar{b}_{r_j}$.

Note that ρ is same for all σ. It represents the total work load in the system and the discrete system is stable only when $\rho < 1$ ([6]). This condition is guaranteed because: continuous polling system is assumed to be stable, so the stationary moments of Theorem 2 (given in section 4) exist and by the same theorem it is possible if and only if $\rho < 1$.

Expected Stationary Workload for discrete system: We thus have a $N\sigma$ stable polling system with σ queues experiencing the gated service with external arrivals and the remaining $(N-1)\sigma$ queues also experiencing the gated service but with only rerouted users. Further we have fixed walking times between queues, that between the queues of the same stop is zero, $r_{\{i,j\}} = 0$ when $j > 0$ and $r_{\{i,0\}} = |\mathcal{C}|\alpha^{-1}/\sigma$. This type of a discrete polling system with rerouting is considered in [6]. By Pseudo Conservation Laws of [6] the expected stationary workload of the σ-polling model with rerouting is (from [6, eqn. (6.4)] after removing the zero terms):

$$\begin{aligned} V^{\sigma}_{rrt} = & \frac{\sum_{i=1}^{\sigma} \lambda_{\{i,0\}} \tilde{b}^{(2)}_{\{i,0\}}}{2(1-\rho)} - \sum_{j=0}^{N-1} \sum_{i=1}^{\sigma} \gamma_{\{i,j\}} \left[\frac{b^{(2)}_{\{i,j\}}}{2} + \left(\tilde{b}_{\{i,j\}} - b_{\{i,j\}} \right) b_{\{i,j\}} \right] \\ & + \frac{\rho|\mathcal{C}|\alpha^{-1}}{2} + \frac{|\mathcal{C}|}{1-\rho} \sum_{i=1}^{\sigma} \frac{\alpha^{-1}}{\sigma} \sum_{l=1}^{\sigma} \lambda_{\{l,0\}} \tilde{b}_{\{l,0\}} \sum_{k=Nl}^{Ni} \rho_k \\ & + \frac{1}{1-\rho} \sum_{j=0}^{N-2} \sum_{i=1}^{\sigma} \sum_{l=1}^{\sigma} \gamma_{\{i,j\}} \epsilon_{j+1} P_{R_{j+1}}(I_l) \tilde{b}_{\{l,j+1\}} \sum_{k=Ni}^{Nl+j} r_k \end{aligned} \tag{6}$$

The results of [7,5,9,6] are valid for any work conserving order at each queue and hence the results are also valid for our *arrival position order.*

We will prove that the limit of the 'discrete' expected stationary virtual workload, V^{σ}_{rrt}, indeed equals that of the continuous system. This basically forms the proof of the Theorem 1 and is given in the next two sections. We conclude this section by computing the limit of (6) (proof in Appendix A):

Lemma 1 *The limit of V^{σ}_{rrt} (6) equals V_{rrt} given by (2) of Theorem 1.* □

4 Fixed point equations

Let 0 be any arbitrary point of the circumference, $\mathcal{C}$. We call the time period between two successful visits of the server at point 0 as cycle. Let $\phi^{\sigma}_n(q)$ represent the time at which the server starts the service of the external queue ($\{i,0\}$ numbered queue with only external arrivals) in the n^{th} cycle, to which the point q belongs. In case of the continuous system, this corresponds to the instance when the server reaches the point q, completes the service of all the rerouted users (if any) of point q and is ready to start with the external users of q, in the n^{th} cycle. Let $T^{\sigma}_n(q) := \phi^{\sigma}_n(q) - \phi^{\sigma}_n(0)$, represent the time taken by the server to travel starting from 0 while serving all the users (with requests) on the way till the time the service of the external queue, to which point q belongs, begins in the n^{th} cycle. Let $\mathcal{T}_{r_0}([a,c],T)$ represent the total workload of the Poisson (external) arrivals that arrived in strip $[a,c] \subset [0,|\mathcal{C}|]$, such that a user at point $q \in [a,c]$ can arrive during a period of time $T(q)$. Let $\mathcal{T}_{r_j}([a,c],T)$ represent the workload (in the j-th service) due to that fraction of the external arrivals, $\mathcal{T}_{r_0}([0,|\mathcal{C}|],T)$, which were j-rerouted to the strip $[a,c]$. Let

$$C^{\sigma}_n(q) \stackrel{\triangle}{=} \phi^{\sigma}_{n+1}(q) - \phi^{\sigma}_n(q) = T^{\sigma}_{n+1}(q) + T^{\sigma}_n(|\mathcal{C}|) - T^{\sigma}_n(q) \tag{7}$$

represent the cycle time w.r.t. q. With this, (note at $\sigma = \infty$, $|\mathcal{C}|/\sigma = 0$):

$$T_n^\sigma(q) = T_n^\sigma(\delta^\sigma(q)) = \delta^\sigma(q)\alpha^{-1} + \mathcal{T}_{r_0}([0, \delta^\sigma(q)), C_{n-1}^\sigma) \\ + \sum_{j=1}^{N-1} \mathcal{T}_{r_j}\left(\left[\frac{|\mathcal{C}|}{\sigma}, \delta^\sigma(q) + \frac{|\mathcal{C}|}{\sigma}\right), C_{n-1-j}^\sigma\right). \quad (8)$$

This is the most important equation and is derived as follows. In (8), the first term represents the time taken to travel the distance. The second term represents the time taken to complete service of the external arrivals at positions before $\delta^\sigma(q)$ while the third term represents the time taken to complete service of the rerouted users, that arrived at positions in between $\mathcal{C}|/\sigma$ and $\delta^\sigma(q) + |\mathcal{C}|/\sigma$. Note that $T_n^\sigma(q) = \phi_n^\sigma(0) - \phi_n^\sigma(q)$, hence is the time period between the time starts of gated service at external queues at stop 0 and the stop $\delta^\sigma(q)$ and so the time taken to service the rerouted users of the first stop, $\{\mathcal{T}_{r_j}([0, |\mathcal{C}|/\sigma])\}$, is not included in it. It instead includes the time taken to serve the rerouted users of the stop $\delta^\sigma(q)$, $\{\mathcal{T}_{r_j}([\delta^\sigma(q), \delta^\sigma(q) + |\mathcal{C}|/\sigma])\}$. *The + in the third term is in modulo $|\mathcal{C}|/\sigma$ sense.*

4.1 First Moments

We obtain integral representation of the first moments of $\mathcal{T}_{r_0}([0, q], T_n^\sigma)$, the workload, using [23, Lemma 1] whose statement is reproduced[3] here:

Lemma 2 *Let $T : [a, c] \mapsto \mathcal{R}^+$ be either monotone (increasing or decreasing) or constant nonnegative random function on interval $[a, c]$. Then,*

$$E[\mathcal{T}_{r_0}([a, c], T)] = \lambda \int_0^{|\mathcal{C}|} 1_{\{[a,c]\}} b_{r_0}(q)\tau(q)P_{R_0}(dq) \text{ where } \tau(q) := E[T(q)] \forall q,$$

$$= \lambda \left(\sum_{i=1}^{M_0} 1_{\{q_{j,i} \in [a,c])\}} \tau(q_{j,i}) b_{r_0}(q_{j,i}) p_{j,i} + \int_a^c b_{r_0}(q)\tau(q) f_{R_0}(q) dq \right) \quad \square$$

Lemma 2 is also true for open $((a, c))$, semi-open $((a, c]$, $[a, c))$ intervals and singletons $(\{a\})$. Similarly for rerouted users (proof in Appendix A):

Lemma 3 *With T as in Lemma 2, the expected workload due to j-rerouted users equals (with $\tau(q) := E[T(q)]$) for $j > 1$:*

$$E\left[\mathcal{T}_{r_j}([a, c], T)\right] = \hat{\epsilon}_j \rho_{r_j}([a, c]) \int_0^{|\mathcal{C}|} \tau(q) P_{R_0}(dq) \text{ with } \hat{\epsilon}_j := \Pi_{i=1}^j \epsilon_i.$$

[3] In [23] the proof is provided when P_{R_0} is a continuous, i.e., when $P_{R_0}(dq) = f_{R_0}(q)dq$. But the proof goes through in a similar way by replacing the Riemann sum with the sums defining the Riemann-Stieltjes integral, defined using the cumulative density function $F_{R_0}(q) := P_{R_0}([0, q])$, as in the proof of Lemma 1. Further there will be a corresponding change in the statement of Lemma to incorporate arrival distributions as in (1).

Let $\tau_n^\sigma(q) := E[T_n^\sigma(q)]$ represent the first moment of $T_n^\sigma(q)$ when the number of stops equal σ and let $\tau_n^\infty(q)$ represent the same for continuous system. Similarly, define $c_n^\sigma(q) := E[C_n^\sigma(q)]$. Note that cycle time $C_n^\sigma(q)$ is the sum of monotone increasing ($q \mapsto T_n^\sigma(q)$) and decreasing ($q \mapsto T_{n-1}^\sigma(|\mathcal{C}|) - T_{n-1}^\sigma(q)$) random function of $q \in [0, |\mathcal{C}|]$. Further the workload that arrived in two non overlapping intervals is the sum of the workloads that arrived in individual intervals. Thus by Lemmas 2 and 3:

$$\tau_n^\sigma(q) = \delta^\sigma(q)\alpha^{-1} + \lambda \int_0^{|\mathcal{C}|} 1_{\{[0,\delta^\sigma(q))\}} b_{r_0}(y) c_{n-1}^\sigma(y) P_{R_0}(dy) \tag{9}$$
$$+ \sum_{j=1}^{N-1} \hat{\epsilon}_j \rho_{r_j} \left(\left[\frac{|\mathcal{C}|}{\sigma}, \delta^\sigma(q) + \frac{|\mathcal{C}|}{\sigma} \right) \right) \int_0^{|\mathcal{C}|} c_{n-1-j}^\sigma(y) P_{R_0}(dy) \quad \text{and}$$
$$c_n^\sigma(q) = \tau_n^\sigma(q) + \tau_{n-1}^\sigma |\mathcal{C}| - \tau_{n-1}^\sigma(q).$$

Let $\mathcal{D}$ represent the Banach space of left continuous functions with right limits on $[0, |\mathcal{C}|]$ equipped with sup norm, $||\tau||_\infty := \sup_{q\in[0,|\mathcal{C}|]} |\tau(q)|$. Let $\mathbf{N} := \{1, 2, \cdots, \infty\}$ be an Euclidean metric space. Define function $\mathcal{F}$, parametrized by $\sigma \in \mathbf{N}$, where the image $\mathcal{F}(\tau; \sigma)$ is defined for any $\tau \in \mathcal{D}$, point-wise by:

$$\mathcal{F}(\tau;\sigma)(q) := \delta^\sigma(q)\alpha^{-1} + \tau(|\mathcal{C}|)\rho_{r_0}([0, \delta^\sigma(q)))$$
$$+\tau(|\mathcal{C}|) \sum_{j=1}^{N-1} \hat{\epsilon}_j \rho_{r_j} \left(\left[\frac{|\mathcal{C}|}{\sigma}, \delta^\sigma(q) + \frac{|\mathcal{C}|}{\sigma} \right) \right) \text{ for all } q \in [0, |\mathcal{C}|] \ .$$

Let $\tau_*^\sigma(q), c_*^\sigma(q)$, for $\sigma \in \mathbf{N}$, represent stationary moments corresponding to $\tau_n^\sigma(q)$, $c_n^\sigma(q)$ respectively. By Palm stationarity[4] and from equation (9), the stationary first moments of the discrete system is the fixed point of the parametrized function $\mathcal{F}$, at $\sigma < \infty$ while that of the continuous system is fixed point of the same function at $\sigma = \infty$.

The only limit point of the set $\mathbf{N}$ (in Euclidean metric) is ∞. Thus the function $\mathcal{F}$ is continuous in (τ, σ) (with $\tau \in \mathcal{D}$, $\sigma \in \mathbf{N}$) because: 1) $\mathcal{F}$ is

[4] For any stationary point process, for example in our case $\{\phi_n^\sigma(q)\}$ for any fixed $q \in \mathcal{C}$ (for further explanations we consider example case of $q = 0$ and discuss the cycle times $\{C_n^\sigma(0)\}_n$), there will be two associated probabilities: Stationary and Palm Stationary ([1]). In general, $\{\phi_n^\sigma(0)\}$ are defined such that $\phi_0^\sigma(0) \leq 0 < \phi_1^\sigma(0)$. Palm probabilities are the stationary probabilities obtained after conditioning on the event that $\{\phi_0^\sigma(0) = 0\}$ (see [1]). Throughout the paper, the expectation under Palm stationary measure is represented by E^0 and the corresponding moments are usually denoted with a $*$ as under-script. In [1], the stationary moment of the residual cycle $C_1^\sigma(0)$ (which we refer as $C_R^\sigma(0)$) as well as the past cycle $C_0^\sigma(0)$ (which we refer as $C_P^\sigma(0)$) is obtained in terms of Palm probabilities as

$$E[C_R^\sigma(0)] = E[C_P^\sigma(0)] = \frac{E^0\left[\left(C_1^\sigma\right)^2(0)\right]}{2E^0[C_1^\sigma(0)]}.$$

This result is also explicitly derived specifically for cycle times in a special polling system in Section 3.1 of [8]. This result will be used in the next subsection.
In Palm stationary regime, $\tau_n^\sigma, c_n^\sigma$ are same for all n, the common values are represented by τ_*^σ, c_*^σ and hence we get a fixed point operator, $\mathcal{F}$, to represent equation (9).

bounded linear in τ; 2) as $\sigma \to \infty$, $\delta^\sigma \to \delta^\infty$ in sup norm and $\mathcal{F}$ is continuous in $\delta \in \mathcal{D}$ (when $\mathcal{F}$ is viewed as a function of τ, δ, σ after replacing δ^σ with δ). Hence (see proof of [23, Theorem 2] for some more details) by contraction mapping theorem:

Theorem 2 *For any σ, the map $\mathcal{F}$ has an unique fixed point, τ_*^σ, if and only if $\rho = \rho_{r_0} + \sum_{j=1}^{N-1} \hat{\epsilon}_j \rho_{r_j} < 1$. Further, τ_*^σ, the stationary moments of discrete system with σ stops, converges to that of the continuous system:*

$$\sup_{q \in \mathcal{C}} |\tau_*^\sigma(q) - \tau_*^\infty(q)| \to 0 \text{ as } \sigma \to \infty.$$

Indeed because of modulo $|\mathcal{C}|/\sigma$ addition for all q

$$\tau_*^\sigma(|\mathcal{C}|) = \frac{\delta^\sigma(|\mathcal{C}|)}{\alpha(1-\rho)} \text{ and}$$

$$\tau_*^\sigma(q) = \frac{\delta^\sigma(q)}{\alpha} + \tau_*^\sigma(|\mathcal{C}|) \left(\rho_{r_0}\left([0, \delta^\sigma(q))\right) + \sum_{j=1}^{N-1} \hat{\epsilon}_j \rho_{r_j} \left(\left[\frac{|\mathcal{C}|}{\sigma}, \delta^\sigma(q) + \frac{|\mathcal{C}|}{\sigma} \right) \right) \right).$$

4.2 Second Moments

The equivalent of Lemma 2 for the second moments is obtained in [23, Lemma 2] and we reproduce it's statement (after modifying, as in Lemma 2, with a possible non continuous P_{R_0} and is true for any general interval):

Lemma 4 *Assume the hypothesis of Lemma 2. Further, let T_1 be any positive random variable with $E[T_1] < \infty$. Then for any interval I*

$$E[\mathcal{T}(I, T)T_1] = \lambda \int_0^{|\mathcal{C}|} 1_{\{q \in I\}} E[T(q)T_1] b_{r_0}(q) P_{R_0}(dq). \ \square$$

Using Lemma 4 and using the logic as in proof of Lemma 3 we can obtain the following lemma for the rerouted users:

Lemma 5 *Assume the hypothesis of Lemma 4. Then for any $1 \le j < N$,*

$$E\left[\mathcal{T}_{r_j}(I, T)T_1\right] = \hat{\epsilon}_j \rho_{r_j}(I) \int_0^{|\mathcal{C}|} E[T(q)T_1] P_{R_0}(dq). \quad \square$$

We will be working with the following $N+1$ second moments,

$$\tau_{j,n}^{(\sigma 2)}(q, q') := E[T_n^\sigma(q) T_{n-j}^\sigma(q')] \text{ with } 0 \le j \le N \text{ and let}$$
$$\top_n^{(\sigma 2)}(q, q') := \left[\tau_{0,n}^{(\sigma 2)}(q, q'), \tau_{1,n}^{(\sigma 2)}(q, q'), \cdots, \tau_{N,n}^{(\sigma 2)}(q, q')\right].$$

We obtain fixed point equations for these $N+1$ second moments as in the case of first moments. Taking expectation after multiplying equation (8) with $T^\sigma_{n-j}(q')$, for any $0 \le j \le N$, we get:

$$\tau^{(\sigma 2)}_{j,n}(q,q') = \delta^\sigma(q)\alpha^{-1}\tau^\sigma_{n-j}(q') + E\left[\mathcal{T}_{r_0}([0,\delta^\sigma(q)), C^\sigma_{n-1})T^\sigma_{n-j}(q')\right]$$
$$+ \sum_{j'=1}^{N-1} E\left[\mathcal{T}_{r_{j'}}\left(\left(\left[\frac{|\mathcal{C}|}{\sigma}, \delta^\sigma(q) + \frac{|\mathcal{C}|}{\sigma}\right), C^\sigma_{n-1-j'}\right) T^\sigma_{n-j}(q')\right].$$

Let $\tau^{(\sigma 2)}_{j,*}$ (for any $0 \le j \le N$), and $\top^{(\sigma 2)}_*$ respectively represent the stationary moments corresponding to $\tau^{(\sigma 2)}_{j,n}$ and $\top^{(\sigma 2)}_n$. By stationarity, the moments are fixed points of the following (using Lemmas 4 and 5):

$$\varTheta^j_{q,q'}(\top^{(2)};\sigma) = \varUpsilon_{q,q'}(\tau^\sigma_*) + \varOmega^j_{q,q'}(\top^{(2)};\sigma) \text{ for all } 0 \le j \le N, \qquad (10)$$

where τ^σ_* are given by Theorem 2 (note these are continuous as $\sigma \to \infty$) and the remaining operators are defined point wise (for every (q,q')) as

$$\varUpsilon_{q,q'}(\tau;\sigma) := \delta^\sigma(q)\alpha^{-1}\tau(q') \qquad (11)$$

$$\varOmega^j_{q,q'}(\top^{(2)};\sigma) \;=\; \lambda\int_0^{\delta^\sigma(q)}\left(\tau^{(2)}_j(q',y) + \psi^{q',|\mathcal{C}|}_{j,1} - \psi^{q',y}_{j,1}\right) b_{r_0}(y)P_{R_0}(dy) \qquad (12)$$
$$+\sum_{j'=1}^{N-1}\hat{\epsilon}_{j'}\rho_{r_{j'}}\left(\left[\frac{|\mathcal{C}|}{\sigma}, \delta^\sigma(q) + \frac{|\mathcal{C}|}{\sigma}\right]\right)\int_0^{|\mathcal{C}|}\left(\psi^{q',y}_{j,j'} + \psi^{q',|\mathcal{C}|}_{j,j'+1} - \psi^{q',y}_{j,j'+1}\right)P_{R_0}(dy)$$
$$\text{with } \psi^{y,y'}_{j,j'} \;=\; 1_{\{j>j'\}}\tau^{(2)}_{j-j'}(y,y') + 1_{\{j'\ge j\}}\tau^{(2)}_{j'-j}(y',y).$$

As explained in footnote 4, the stationary second moments of the discrete system are the fixed points of the map, $\varTheta := (\varTheta^0, \varTheta^1, \cdots, \varTheta^N)$, at $\sigma < \infty$, while that of the continuous system are fixed points of the same map at $\sigma = \infty$. We have the following convergence result (proof in Appendix A):

Theorem 3 *There exists a threshold λ_0 (given by equation (24) in the proof) and for all the Poisson arrival rates less than $\lambda \le \lambda_0$ as $\sigma \to \infty$ the stationary second moments (uniform convergence in q,q'),*

$$\sup_{q,q'\in[0,|C|]}\left|\top^{(\sigma 2)}_*(q,q') - \top^{(\infty 2)}_*(q,q')\right| \to 0. \qquad \square$$

5 Proof of Theorem 1

By Theorems 2 and 3, the first and second stationary moments of T^σ_n, of the discretized polling system, converge towards the corresponding ones of the continuous polling system. We obtain a common expression for the expected virtual workload using these moments and complete the proof.

From (7), the first two (Palm) stationary moments of cycle time w.r.t. the point q, $E^0[C_n^\sigma(q)]$ and $E^0\left(C_n^\sigma(q)\right)^2$ are:

$$c_*^\sigma(q) = \tau_*^\sigma(|\mathcal{C}|) \text{ and } c_*^{(\sigma 2)}(q) = 2\tau_{0,*}^{(\sigma 2)}(q,q) + \tau_{0,*}^{(\sigma 2)}(|\mathcal{C}|,|\mathcal{C}|) + 2\tau_{1,*}^{(\sigma 2)}(q,|\mathcal{C}|) \\ -2\tau_{1,*}^{(\sigma 2)}(q,q) - 2\tau_{0,*}^{(\sigma 2)}(q,|\mathcal{C}|).$$

Thus the stationary first moment of the residual of the cycle $C_n^\sigma(q)$ as seen by a random user is given by (see footnote 4),

$$E[C_R^\sigma(q)] = \frac{c_*^{(\sigma 2)}(q,q)}{2\left(c_*^\sigma(q)\right)^2}. \tag{13}$$

In the following we calculate the stationary expected workload due to external users and rerouted users separately, in terms of the stationary moments of the previous section, using Little's law and then the Wald's Lemma.

Workload due to External users: A randomly arriving external user, arriving at q, has to wait on average for: 1) residual of his own cycle $c_*^\sigma(q)$; 2) the time taken to service the external users waiting at q that arrived before him (FIFO); 3) in a discrete system (arrival position order service), till the external users waiting in $[\delta^\sigma(q), q))$ are served. The total time due to 2 and 3 points is given by $E\left[\mathcal{T}_{r_0}\left([\delta^\sigma(q), q]\,, \ddot{C}\right)\right]$, where $\ddot{C}(q') := C(q')$ (stationary cycle corresponding to $C_n(q')$) for all $q' \neq q$ and $\ddot{C}(q) := C_P^\sigma(q)$ (past cycle as in footnote 4 and note $E[C_R^\sigma(q)] = E[C_P^\sigma(q)]$). By Lemma 2 (see (1)),

$$E\left[\mathcal{T}_{r_0}\left([\delta^\sigma(q), q]\,, \ddot{C}\right)\right] = \tau_*^\sigma(|\mathcal{C}|)\rho_{r_0}\left([\delta^\sigma(q), q)\right) + \sum_{i=1}^{M_0} p_{0,i} 1_{\{q_{0,i}\}}(q) E[C_R^\sigma(q)]. \tag{14}$$

Thus the expected waiting time of an external user is:

$$E[W_{r_0}^\sigma](q) = E[C_R^\sigma(q)] + E\left[\mathcal{T}_{r_0}\left([\delta^\sigma(q), q]\,, \ddot{C}\right)\right]. \tag{15}$$

While that for a continuous system equals:

$$E[W_{r_0}^\infty](q) = E[C_R^\infty(q)] + \sum_{i=1}^{M_0} p_{0,i} 1_{\{q_{0,i}\}}(q) E[C_R^\infty(q)]. \tag{16}$$

By Little's law ([3]), the stationary expected number of waiting users (awaiting first service) that belong to infinitesimal segment $[q - dq, q + dq]$ equals, $\lambda E[W_{r_0}^\sigma](q) P_{R_0}(dq)$ and thus by Wald's Lemma, the stationary expected virtual workload (by independence) due to external users that belong to infinitesimal segment $[q - dq, q + dq]$ equals $\lambda E[W_{r_0}^\sigma](q) b_{r_0}(q) P_{R_0}(dq)$. Thus the expected stationary workload due to all the external users is:

$$V_{r_0}^\sigma = \int_0^{|\mathcal{C}|} \lambda E[W_{r_0}^\sigma](q) b_{r_0}(q) P_{R_0}(dq).$$

By lemma 7 (Appendix B) $\rho_{r_0}\left([\delta^\sigma(q), q)\right) \to 0$ as $\sigma \to \infty$. By Bounded Convergence Theorem (BCT) and Theorems 2, 3 terms like $E[C_R^\sigma(q)] \to E[C_R^\infty(q)]$

and hence the expected waiting time $E[W^\sigma_{r_0}](q) \to E[W^\infty_{r_0}](q)$ as $\sigma \to \infty$ for every q (see equations (13)-(16)). By BCT again, $V^\sigma_{r_0} \to V^\infty_{r_0}$.

Workload due to Rerouted users ($j > 0$): A j-rerouted user arrives just after completion of his j-th service and because of immediate rerouting to another point in the circle. Consider one such user who arrived at a point q in $\mathcal{C}$. His waiting time depends upon the point at which his j-th service was completed. Conditioned that the position of the j-th service was at q' (whose distribution is given by $P_{R(j-1)}(dq')$ because of independence), he will have to wait on average for: 1) if $q' < q$ then $\tau^\sigma_*(q) - \tau^\sigma_*(q')$ period of time; 2) if $q' > q$ then $\tau^\sigma_*(|\mathcal{C}|) - \tau^\sigma_*(q') + \tau^\sigma_*(q)$ period of time. The above gives the waiting time till the point at which the external arrivals belong to his strip $[\delta^\sigma(q), \delta^\sigma(q) + |\mathcal{C}|/\sigma)$ are served. But the rerouted user will be served before this time point and let $\nu^\sigma_j(q)$ represent the average of this time difference. This difference is calculated and its convergence is studied for all the polling systems in Lemma 6 of Appendix B. Then the stationary average waiting time of a j-rerouted user arrived at point q equals:

$$
\begin{aligned}
E[W^\sigma_{r_j}](q) &= \int_q^{|\mathcal{C}|} \tau^\sigma_*(|\mathcal{C}|) P_{R_{(j-1)}}(dq') + \int_0^{|\mathcal{C}|} \left(\tau^\sigma_*(q) - \tau^\sigma_*(q')\right) P_{R_{(j-1)}}(dq') - \nu^\sigma_j(q) \\
&= \tau^\sigma_*(q) + \tau^\sigma_*(|\mathcal{C}|) P_{R_{(j-1)}}([q, |\mathcal{C}|]) - \int_0^{|\mathcal{C}|} \tau^\sigma_*(q') P_{R_{(j-1)}}(dq') - \nu^\sigma_j(q)
\end{aligned}
$$

By Little's law ([3]) and Wald's lemma, as before, the stationary expected workload due to j-rerouted users is (note the effective arrival rate equals $\hat{\epsilon}_j \lambda P_{R_j}(q) dq$)

$$
V^\sigma_{r_j} = \lambda \hat{\epsilon}_j \int_0^{|\mathcal{C}|} E[W^\sigma_{r_j}](q) b_{r_j}(q) P_{R_j}(dq).
$$

By theorems 2, 3 and BCT (as done before for the external users) and further using Lemma 6 of Appendix B, we obtain: $V^\sigma_{r_j} \to V^\infty_{r_j}$.

Total workload: The total expected stationary virtual workload is the workload due to all types of users and hence $V^\sigma = V^\sigma_{r_0} + \sum_{j=1}^{N-1} V^\sigma_{r_j}$. From the above arguments, $V^\sigma \to V^\infty$, i.e., total stationary expected virtual workload of the discrete polling system converges towards that of the continuous polling system as $\sigma \to \infty$. The workload, V^σ, is obtained as an easily computable expression for each σ in section 3 using results of [6] as equation (6) and it's limit (2) thus represents the stationary expected workload of the continuous system. This completes the proof of Theorem 1. □

In the following, we use the formula (2) to obtain performance of some examples of ferry based wireless LANs (FWLAN).

6 Ferry based Wireless LAN

Static users are scattered in a geographical area Δ. The network is sparse and there is no direct global connectivity. The actual communication is facilitated by a ferry which moves in a closed cyclic path $\mathcal{C}$, placed inside Δ, repeatedly with constant speed α and serves as a postman. The ferry collects the data from the source users as and when it encounters one. We call this as uplink service. The uplink data also comes with the address of the destination user to which it is intended. The ferry downloads the data to the destined user the first time it meets the later, after collecting the uplink data. We refer this service as downlink service.

The base station (BS) forms the global gateway to the external world. The communication between the users of the network and the BS is also established via ferry as is done between two users of the node. That is, every download starts with uplink from BS to the ferry followed by the downlink from the ferry to the destined user and every upload starts with uplink from the source user to the ferry followed by the downlink from ferry to the BS.

Each point q in the cyclic path is assigned with a set of points $I(q) \subset \Delta$ and ferry stops at q if there is an user in $I(q)$ with either downlink or uplink request. For example, if we consider Δ to be an annular ring, the ferry is moving along a co-centric circle and if the sets $\{I(q)\}$ are decided based on the nearest point criterion then the Ferry will stop at a point on the circle if there is an active user located at the same angle (see Figure 1).

Ferry basically facilitates data transfer between two nodes of the area. The data transfer request arrivals are modeled as Poisson arrivals and each of these arrivals are associated with two marks: X the position of source distributed as P_X and Y the position of the destination/sink distributed as P_Y. Every such request requires fixed size, η, of data to be transfered from source X to destination Y. The position of the source and destination are independent of each other.

We use the following notations in this section. The points in the two dimensional area Δ are represented by x, y (if it is a sample point) or X, Y (if it is a random position). The points on the cyclic path $\mathcal{C}$ are represented by q or Q. We shall use the superscript u or d to denote uplink or downlink.

Ferry uses a wireless link to serve the users. It can receive/transmit the messages from/to the users at a distance of d from it at a rate $\kappa(d)$ for some decreasing function κ. Thus the total time required for transmitting a message of size η, when the user is located at $x \in \Delta$ and is associated with $q(x) \in \mathcal{C}$ is equal to its size divided by the service rate:

$$B(x) = \frac{\eta}{\kappa(||q(x) - x||)}. \tag{17}$$

In this paper *the objective function to be minimized will always be the expected virtual workload (which is obtained in Theorem 1 of section 2) and a ferry path which minimized the expected virtual workload is a Pareto optimal path for the multi-objective problem where the expected waiting times at different locations are to be minimized* (for details see the discussions in Appendix C of [23]).

We discuss design of optimal ferry paths $\mathcal{C}$ and optimal partitioning of the area into line segments $\{I(q); q \in \mathcal{C}\}$. The aim of this section is to obtain these objects in an optimal way that minimizes the virtual workload. Solving this problem in complete generality will be a very difficult task. Hence we instead obtain optimal ferry path among a special class of ferry paths.

Prior to discussing the optimality issues one first needs to map the FWLAN to a continuous polling, so that Theorem 1 can be used. This task is taken up immediately.

6.1 Mapping to a Continuous Polling system

We analyze this FWLAN using Theorem 1. We begin with identifying the components of the continuous polling system.

Server and path of the polling system : The ferry represents the server of the polling system. The ferry stops at a point q in its path only when there is a user with (downlink/uplink) request anywhere on the strip $I(q)$. Thus the entire strip $I(q)$ is modeled as a point on the server's path, in an equivalent continuous polling system.

Service times: An arrival (X, Y) is associated with the points $q(X)$, $q(Y)$ of the ferry route if X, Y lie in corresponding strips, i.e., if $X \in I(q(X))$ and $Y \in I(q(Y))$. Thus time required for uplink and downlink services respectively are:

$$B(X) = \frac{\eta}{\kappa(|X - q(X)|)} \text{ and } B(Y) = \frac{\eta}{\kappa(|Y - q(Y)|)}.$$

Note these two service times are independent of each other as required by Theorem 1. The moments of the uplink or downlink service times, in general depend upon the point $q \in \mathcal{C}$. For uplink (which is the first service required),

$$b_{r_0}(q) = b_u(q) = E[B(X)|q(X) = q] = E_X\left[\frac{\eta}{\kappa(|q - X|)} \middle| X \in I(q) \right],$$
$$b^{(2)}_{r_0}(q) = b^{(2)}_u(q) = E_X\left[\frac{\eta^2}{\kappa(|q - X|)^2} \middle| X \in I(q) \right].$$

Downlink (second service) moments $b_{r_1} = b_d$, $b^{(2)}_{r_1} = b^{(q)}_d$ can be defined in a similar way.

External and Rerouted arrivals: Every data transfer requires two services: starts with uplink service and ends with downlink service. A Poisson process models the arrival of a data transfer request and the same marks the arrival of an uplink service requirement. Hence, uplinks represent the external arrivals to the polling system that models the FWLAN. The completion of uplink service marks the arrival of a downlink service requirement and hence the down-links represent the rerouted arrivals. Here every uplink is converted to a downlink and hence $\epsilon_1 = 1$ and $N = 2$.

Position of arrival in the 'polling system' : The position of uplink arrival in FWLAN is given by P_X, a distribution over Δ. Every arrival in the strip $I(q)$ marks the arrival at point q of $\mathcal{C}$ in the equivalent polling system. Thus,

$$P_{R_0}(A) = P_X(\cup_{q \in A} I(q)) \text{ for any Borel set } A \subset \mathcal{C},$$

represents the external arrival distribution. For example, in an annular ring with circular path, i.e., for some $h_1 \le l \le h_2$

$$\Delta := \{x \in \mathcal{R}^2 : h_1^2 \le |x| \le h_2^2\} \text{ and } \mathcal{C} := \{q : ||q|| = l\},$$

and if $P_X \sim \mathcal{U}(\Delta)$ (uniform distribution) and $I(q) = \{x : \angle x = \angle q\}$ (with $\angle x$ representing the angle made by the line joining 0, x with the x axis), then P_{R_0} will be uniform over $\mathcal{C}$. Similarly $P_{R_1}(A) := P_Y(\cup_{q \in A} I(q))$, for all Borel $A \subset \mathcal{C}$.

Thus the FWLAN can be modeled by a continuous polling system with rerouting, Theorem 1 can be applied and the stationary expected virtual workload of the FWLAN can be calculated using (2) for any given cyclic path $\mathcal{C}$ and the corresponding line segments $\{I_q\}_{q \in \mathcal{C}}$. The Theorem can be applied only for those cases which satisfy the hypothesis of the Theorem, like for example the arrival rate should be less than λ_0 of Theorem 1.

In the following we consider some examples and compute the workload performance of FWLAN.

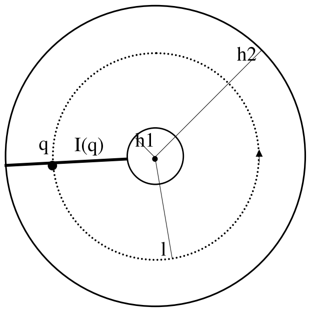


Fig. 1 Ferry in an annular ring

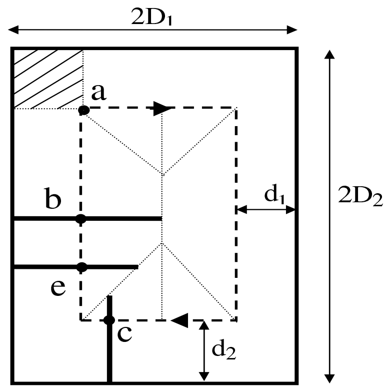


Fig. 2 Ferry in a rectangular area

6.2 Ferry in a Rectangular area

We consider a rectangular area Δ of length $2D_1$ and breadth $2D_2$ with a ferry which moves in a rectangular path as in Figure 2. This example could not be modeled using our previous results (see [22]) as it requires the arrival position measures that are mixed. But with Theorem 1 of this paper it is possible to model this example. We consider the following configuration of FWLAN: 1) uniform arrivals i.e., $P_X \sim \mathcal{U}(\Delta)$ and so is P_Y; 2) rate function resulting from the losses in wireless medium considers only the direct path for attenuation and is calculated assuming a receive transmit antennae difference of 1 and pathloss factor β (where d is the distance between the user and the ferry):

$$\kappa(d) = (1 + d^2)^{-\beta/2} \text{ for every } d \ge 0.$$

The ferry moves on rectangular path, $\mathcal{C}_{d_1,d_2}$, which is completely defined in terms of d_1 and d_2, the distances of the ferry path from the sides of the rectangular area as in Figure 2. The best possible thing with κ as above, is to associate every arrival with the nearest point on the ferry route, i.e., X is associated with $q(X) := \arg\min_{q \in \mathcal{C}_{d_1,d_2}} |q - X|$. Hence, $\{I(q)\}$ will either be rectangular region of area $d_1 \times d_2$ for the corner points (for example point a in the figure) or line segments (for example the line segments passing through points b, c and e of the ferry path in the figure). Let point a represent 0 of the line segment representing the entire cycle path. In the following we consider a suboptimal problem with $d_1 = d_2$. This will simplify the explanations, will result in a one dimensional optimization problem and if required one can easily extend all the below computations to the case with $d_1 \neq d_2$. Let $d = d_1 = d_2$. With the ferry moving in the direction as shown in the figure, the arrival probabilities $P_{R_0} = P_{R_1} =: \Psi$ are common and are given by

$$
\begin{aligned}
\Psi(dq) &= \sum_{i=0}^{3} p_0 1_{q_i} + p_1 f_\psi(q) dq \text{ where } p_0 := d^2 p_1, \quad p_1 := \frac{1}{4D_1 D_2}, \\
q_0 &= 0, \quad q_1 = 2(D_1 - d), \quad q_2 = q_1 + 2(D_2 - d), \\
q_3 &= q_2 + 2(D1 - d), \quad q_4 = q_3 + 2(D_2 - d) \text{ and} \\
f_\psi(q) &= \begin{cases} d + (q - q_i) \text{ if } \; q_i < q < q_i + D_1 - d & \text{for some } 0 \le i \le 3 \\ d + (q_i - q) \text{ if } \; q_i - (D_1 - d) < q < q_i & \text{for some } 1 \le i \le 4 \\ D_1 & \text{else.} \end{cases}
\end{aligned}
$$

The service time moments are same for both the services and can be calculated as (with $b := b_{r_1} = b_{r_0}$ and $b^{(2)} := b^{(2)}_{r_0} = b^{(2)}_{r_1}$ representing the common moments):

$$
\begin{aligned}
b(q_i) &= \frac{\eta}{d^2} \int_0^d \int_0^d (1 + x_1^2 + x_2^2)^{\frac{\beta}{2}} dx_1 dx_2 \text{ for every } 0 \le i \le 3 \\
b(q) &= \frac{\eta}{f_\psi(q)} \int_{-d}^{f_\psi(q) - d} (1 + l^2)^{\frac{\beta}{2}} dl \quad \text{for every } q \notin \{q_i\} \text{ and} \\
b^{(2)}(q_i) &= \frac{\eta^2}{d^2} \int_0^d \int_0^d (1 + x_1^2 + x_2^2)^{\beta} dx_1 dx_2 \text{ for every } 0 \le i \le 3 \\
b^{(2)}(q) &= \frac{\eta^2}{f_\psi(q)} \int_{-d}^{f_\psi(q) - d} (1 + l^2)^{\beta} dl \quad \text{for every } q \notin \{q_i\}.
\end{aligned}
$$

The average moments can be computed using Ψ and $\{b(q)\}$ and $\{b^{(2)}(q)\}$. For example, because of symmetry by interchanging the integrals and using change of variable formula, the first moment simplifies to (note that $|\mathcal{C}| = 4(D_2 - d) + 4(D_1 - d)$)

$$
\begin{aligned}
\bar{b} &= 4 p_0 b(q_0) + 8 \eta p_1 \int_0^{D_1 - d} \int_{-d}^{q} (1 + l^2)^{\frac{\beta}{2}} dl dq + 2 \eta p_1 2 (D_2 - D_1) \int_{-d}^{D_1 - d} (1 + l^2)^{\frac{\beta}{2}} dl \\
&= \frac{\eta}{D_1 D_2} \left(|\mathcal{C}| h(\beta, d) + 8 \frac{(1 + (D_1 - d)^2)^{\frac{\beta}{2}+1} - (1 + d^2)^{\frac{\beta}{2}+1}}{\beta/2 + 1} + 4 g(\beta, d) \right) \text{ with}
\end{aligned}
$$

$$h(\beta,d) := \int_0^{D_1} (1+(l-d)^2)^{\frac{\beta}{2}} dl \quad \text{and} \quad g(\beta,d) \quad := \quad \int_0^d \int_0^d (1+x_1^2+x_2^2)^{\frac{\beta}{2}} dx_1 dx_2.$$

Aim is to find the optimal ferry rectangular path:

$$d^* = \arg \min_{0<d\leq D_1} V_{fwlan}(d)$$

where $V_{fwlan}(d)$ represents the expected stationary workload in FWLAN when the ferry moves in rectangular path defined by $d_1 = d_2 = d$. This is calculated by substituting the common moments $\{b(q)\}$, $\{b^{(2)}(q)\}$ and common $P_{R_0} = P_{R_1} = \Psi$ into the stationary expected workload given by (2), which simplifies to ($\rho = 2\lambda \bar{b}$):

$$\begin{aligned} V_{fwlan}(d) = & \frac{\rho\lambda}{1-\rho}\left(\bar{b}^{(2)} + \bar{b}^2\right) + \rho|\mathcal{C}|\alpha^{-1}\frac{2+\rho}{4(1-\rho)} \\ & + \lambda\alpha^{-1}\left(E_\Psi[Qb(Q)] - \bar{b}E_\Psi[Q] + |\mathcal{C}|E_\Psi[\hat{b}(Q)]\right). \end{aligned} \tag{18}$$

Here E_Ψ represents the expectation w.r.t. Ψ, for example for any integrable function g,

$$E_\Psi[g(Q)] = \int_0^{|\mathcal{C}|} g(q)\Psi(dq) = \sum_{i=0}^{3} g(q_i)p_0 + p_1 \int_0^{|\mathcal{C}|} g(q) f_\psi(q) dq.$$

We thus obtained the stationary expected workload performance of FWLAN for every value of d. This expression however can't be simplified and one has to compute the optimal path only via numerical examples.

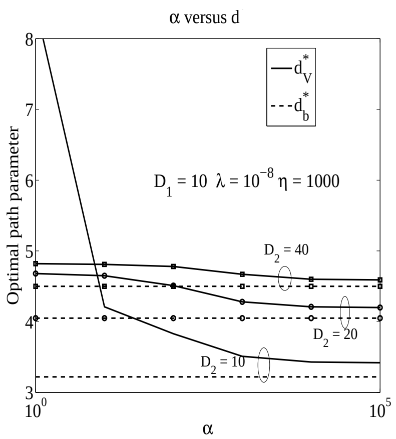
Numerical Examples

We find the optimal path parameter d^* (which we refer as d_V^* to reinforce that it is the minimizer of expected workload V_{fwlan}) for few numerical examples of FWLAN with ferry operating in a rectangular area. We also compute d_b^* the optimizer of the first moment of the service time, $\bar{b}$ (see Figure 3). In Figure 3 we plot the optimal parameter as a function of the ferry speed α for different values of the longer dimension D_2. We set $\beta = 2$, $\eta = 1000$, $D_1 = 10$ and $\lambda = 10^{-8}$. We notice that the two optimizers d_V^* and d_b^* move closer to each other either as the speed α increases or as the longer dimension D_2 increases. This could partially be explained observing the equation (18) for V_{fwlan}. The first term of (18) depends upon d mainly via the service moments and hence will be optimized by a d^* which will be close to d_b^*. The rest of the terms are weighted by a common factor α^{-1} and hence the influence of all these terms on the optimizer reduces for large ferry speeds. More important observation is that the optimal d^* approaches $D_1/2$ as D_2 increases.

The same inferences are reinforced again in Figure 4. Here we plot the optimal path parameter as a function of D_2 for various ferry speeds α. In this example we set $D_1 = 100$. We notice that the optimal path parameter converges to one fourth the lower dimension $D_1/2$ as the larger dimension increases. Further the optimizers are close to d_b^* as either the speed increases or D_2 increases. Thus we make the following two interesting observations:

O.1) The optimizer d_V^*, for most of the cases, is close to the optimizer of the first moment of the overall service times $\bar{b}$;

O.2) For thin or long strips of area the optimal ferry path parameter is one fourth of the lower dimension.



Minimizer for workload:

$$d_V^* := \arg \min_{0 \le d \le D_1} V_{fwlan}(d).$$

Minimizer for service moment:

$$d_b^* := \arg \min_{0 \le d \le D_1} \bar{b}.$$

Fig. 3 Optimal path for a Ferry moving in a rectangular area

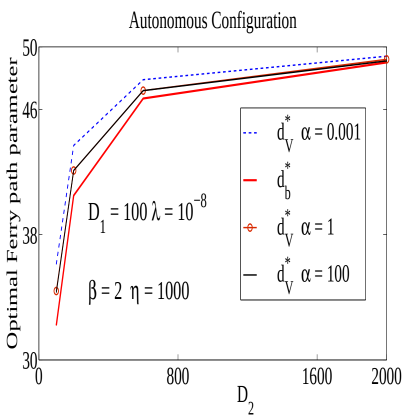


Fig. 4 Optimal path parameter for Autonomous architecture

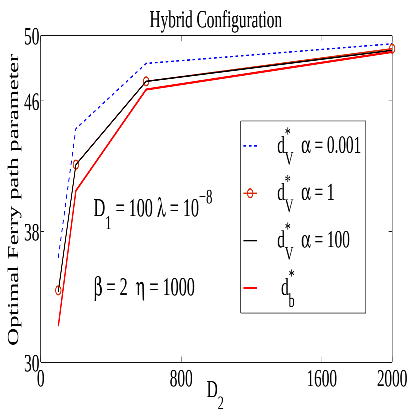


Fig. 5 Optimal path parameter for Hybrid architecture

6.3 Ferry in an Annular ring

Here we consider still a simpler configuration of FWLAN, that in a annular ring $\Delta := \{x \in \mathcal{R}^2 : h_1^2 \le |x| \le h_2^2\}$. Remaining setting are as in the previous example. The ferry moves on one concentric circle of radius l in the annular ring (Figure 1), i.e., $\mathcal{C}_l = \{q \in \mathcal{R}^2 : ||q|| = l\}$. Again, with κ as above, every arrival is associated with the nearest point on the ferry route, i.e., X is associated with $q(X) := \arg\min_{q \in \mathcal{C}_l} |q - X|$. Hence, $I(q) = \{x \in \Delta : \angle(x) = \angle(q)\}$, the angular segments for all $q \in \mathcal{C}_l$, (see figure 1). Aim is to find the optimal radius:

$$l^* = \arg \min_{l \in [h_1, h_2]} V_{fwlan}(l; h_1, h_2)$$

where $V_{fwlan}(l; h_1, h_2)$ represents the expected stationary workload in FWLAN when the ferry moves in $\mathcal{C}_l$.

Since $\{I(q)\}$ are angular segments, for calculating the service time moments one will require the radius $\Gamma = ||X||$ of the arrival. Under the assumptions of this section, $\Gamma \sim 2rdr/(h_2^2 - h_1^2)$ and further because of symmetry the service time moments will be independent of the position $q \in \mathcal{C}_l$, are same for uplink and downlink, but depend upon the ferry path radius l. The common service moments are:

$$\bar{b} = \eta \int_{h_1}^{h_2} \left(1 + (r - l)^2\right)^{\beta/2} \frac{2rdr}{h_2^2 - h_1^2},$$
$$\bar{b}^{(2)} = \eta^2 \int_{h_1}^{h_2} \left(1 + (r - l)^2\right)^{\beta} \frac{2rdr}{h_2^2 - h_1^2}.$$

The stationary expected workload (2) for this setting is simplified and is given by (3). Using this, $\rho = 2\lambda\bar{b}$ and

$$V_{fwlan}^{sym}(l; h_1, h_2) = \frac{\rho\lambda \left(\bar{b}^{(2)} + \bar{b}^2\right)}{1 - \rho} + \frac{6\pi l \rho \alpha^{-1}}{4(1 - \rho)}. \quad (19)$$

Once again, one has to perform numerical computations using the above formula to obtain the optimal radius. However one can get the following asymptotic characteristic of l^* from the formula itself: 1) as the propagation coefficient β tends to zero l^* tends to h_1, i.e., the optimal path for the ferry is the inner circle. (2) as the speed of the ferry, α, increases to infinity, the second term in the formula becomes negligible and hence optimal radius will be determined only by the service time moments and so the optimal radius will be above the middle of the ring, i.e., larger than $(h_1 + h_2)/2$. *Interestingly in contrast to the previous (rectangle) example, this optimal path (valid for the cases with large ferry speeds) is not in the center of the annular ring. This is because the uniform arrivals in the annular ring do not translate into uniform arrivals in the angular direction. The density of arrival distribution, as one moves along the same angle, increases linearly with the distance from the inner circle.*

6.4 Hybrid Architecture

The previous architecture of FWLAN is called Autonomous architecture as the ferry itself facilitates the local communications. But in certain situations, the ferries are built with minimal intelligence, the rerouting tasks are taken up only by the base station (BS). This architecture with finite number of ferry stops is discussed in [28]. In this case every local data transfer happens in three phases, uplink from the source node to the ferry, transfer to BS and BS to ferry back with the sink address and downlink to the sink node. All the other details remain as in the previous section.

The polling systems considered in this paper, can model this architecture also. Note here that our previous results ([22]) could not model this architecture. Most of the mapping aspects remain the same as in the previous section, we point out only the differences. Every arrival demands 3 services, uplink, downlink and BS-transfer service. Thus there are 3 reroutings, i.e., $N = 3$, with $\epsilon_1 = \epsilon_2 = 1$. As in the previous section, P_{R_0} and P_{R_2} are defined using P_X and P_Y respectively. However P_{R_1} is concentrated only at the BS (which we assume is associated with 0 of the ferry path), i.e., $P_{R_1}(\{0\}) = 1$. The service moments $b_{r_0} = b_{r_2}$ are same as the common moments b defined in the two previous sections and so are the second moments. If the BS is at d_b distance from the ferry path, then $b_b := \bar{b}_{r_1} = b_{r_1}(0) = \eta_1/\kappa(d_b)$ and $\bar{b}^{(2)}_{r_1} = b^{(2)}_{r_1}(0) = b_{r_1}(0)^2$ where η_1 can be different from η.

Rectangular Area of Figure 2: In this case, when P_X is same as P_Y

$$\rho = 2\lambda\bar{b} + b_b\lambda, \text{ and } \hat{\rho}(q) = b_b\lambda + 2\lambda\hat{b}(q) \text{ for all } q.$$

The stationary workload for the FWLAN in hybrid architecture is given by:

$$\begin{aligned} V^{hybrid}_{fwlan} = & \frac{\rho\lambda\left(2\bar{b}^{(2)} + 2\bar{b}^2 + b_b^2 + 4b_b\bar{b}\right)}{2(1-\rho)} + \frac{|\mathcal{C}|\alpha^{-1}\left(2\rho + 2\lambda^2\bar{b}^2 - \lambda^2 b_b^2\right)}{4(1-\rho)} \\ & + \frac{\lambda|\mathcal{C}|\alpha^{-1}(b_b + \bar{b})}{1-\rho} + \lambda\alpha^{-1}\left(E_\Psi[Qb(Q)] - (\bar{b} + b_b)E_\Psi[Q]\right) \\ & - \frac{2\lambda^2|\mathcal{C}|\alpha^{-1}(b_b + \bar{b})E_\Psi[\hat{b}(Q)]}{1-\rho}. \end{aligned}$$

When $b_b = 0$, i.e., when the base station transfers the control orders to ferry using negligible information,

$$\begin{aligned} V^{hybrid}_{fwlan} = & \frac{\rho\lambda\left(\bar{b}^{(2)} + \bar{b}^2\right)}{(1-\rho)} + \frac{|\mathcal{C}|\alpha^{-1}\left(2\rho + \lambda^2\bar{b}^2\right)}{2(1-\rho)} \\ & + \lambda\alpha^{-1}\left(E_\Psi[Qb(Q)] - \bar{b}E_\Psi[Q] - \frac{\rho}{1-\rho}|\mathcal{C}|E_\Psi[\hat{b}(Q)]\right). \end{aligned} \quad (20)$$

This can be ensured by placing the base station on the ferry path.

Numerical Examples

We continue with the example of Figure 4 for the hybrid architecture in Figure 5. We notice that the minimizers behave in a way very similar to that in autonomous architecture. In fact, they are very close to the ones in autonomous architecture either when ferry speeds are large or when D_2 is large. From the two figures we notice that the two optimizers differ (that too not significantly) only when the ferry speeds are moderate and D_2 is close to D_1. This could easily be expected observing the two workloads (18) and (20). The two workloads close towards each other for large values of ferry speeds. Thus we make the following third observation:

O.3) The optimal path remains more or less the same irrespective of whether the ferry facilitates the local communication completely on its own or with the aid of the base station.

In this paper we studied some interesting examples and configurations of the FWLAN. One can use this analysis for more interesting case studies (for example zig zag paths, shadowing effects etc.).

7 Conclusions

We study a continuous polling system in which the users can be rerouted to a new independent position to await another service, after completing a service. We obtain an expression for the expected stationary workload. We obtain this result under more general conditions than the usual symmetric conditions. Further the position of arrivals are modeled by a distribution that can be a mixture of discrete and continuous probability measures. We come up with a way of discretization such that the available Pseudo conservation laws of discrete polling systems can be utilized for obtaining the results for the continuous counterparts. We expressed the expected workload as a parametrized function of moment fixed points. The later are some stationary moments obtained as fixed points of a function defined on spaces of left continuous functions with right limits equipped with supremum norm and which are further parametrized by the number of discretization levels. We show the required convergence via the continuous dependence of the fixed points on the parameter. This way we obtained a common expression, which represents the expected virtual workload for continuous as well as the discretized polling systems, at different values of the parameter. We then showed the continuity of the expected virtual workload with respect to the parameter and hence obtained the expected virtual workload for the continuous system as the limit of the expected virtual workloads of the discrete systems, when the levels of discretization tend to infinity.

We applied these results to a wireless LAN in which a ferry assists data transfer among the users of the network as well as the users and the gateway to the external world. We also consider a hybrid architecture in which the ferry facilitates local communication with the help of the base station. We make the following observations with the aid of numerical examples. The ferry path optimizing the first moment of service times approximately optimizes the workload, whenever the ferry moves with significant speeds. The optimal ferry rectangular path divides the breadth of rectangular area into equal partitions as the length of the area increases and or the ferry speed increases, while the optimal circular path in annular areas (with large ferry speeds) is above the middle path of the annular area. The optimal ferry path does not change with the mode in which the ferry assists local communication: i.e., with or without the aid of the base station.

Appendix A

Proof of Lemma 1: The first 3 terms of (6) are independent of σ and simplify as the first 3 terms of (2). With $F_{R_j}(q) := P_{R_j}([0,q]$ for all j,

$$P_{R_0}(I_i)b_{\{i,0\}} = b_{r_0}(i^\sigma)(F_{R_0}(i^\sigma + |\mathcal{C}|/\sigma) - F_{R_0}(i^\sigma)) \tag{21}$$
$$+E_{R_0}\left[(b_{r_0}(R_0) - b_{r_0}(i^\sigma))1_{\{R_0 \in I_i\}}\right] + E\left[b_{r_0}(R_0)\left[P_{R_0}(\{i^\sigma\}) - P_{R_0}(\{i^\sigma + \mathcal{C}|/\sigma\})\right]\right]$$

In similar lines the fourth term of (6) can be approximated[5] by:

$$\frac{\alpha^{-1}}{1-\rho}\sum_{i=1}^{\sigma}\frac{|\mathcal{C}|}{\sigma}\sum_{l=1}^{\sigma}\lambda_{\{l,0\}}\tilde{b}_{\{l,0\}}\sum_{k=Nl}^{Ni}\rho_k \approx \frac{\lambda\alpha^{-1}}{1-\rho}\sum_{i=1}^{\sigma}\frac{|\mathcal{C}|}{\sigma}\sum_{l=1}^{\sigma}\left(b_{r_0}(l^\sigma) + \sum_{j=1}^{N-1}\hat{\epsilon}_j\bar{b}_{r_j}\right)$$
$$\left(F_{R_0}\left(l^\sigma + \frac{|\mathcal{C}|}{\sigma}\right) - F_{R_0}(l^\sigma)\right)\sum_{k=Nl}^{Ni}\rho_k. \tag{22}$$

Circular sum (see [6]), approximately (error again converges to 0) equals:

$$\begin{aligned}\sum_{k=Nl}^{Ni}\rho_k &= 1_{\{i\geq l\}}\sum_{k=Nl}^{Ni}\rho_k + 1_{\{i<l\}}\left(\sum_{k=Nl}^{N\sigma}\rho_k + \sum_{k=1}^{Ni}\rho_k\right)\\ &\approx 1_{\{i\geq l\}}\rho\left([l^\sigma, i^\sigma)\right) + 1_{\{i<l\}}\left[\rho - \rho\left([i^\sigma, l^\sigma)\right)\right]\\ &= \rho([0,i^\sigma]) - \rho([0,l^\sigma]) + 1_{\{i<l\}}\rho = \hat{\rho}(i^\sigma) - \hat{\rho}(l^\sigma) + 1_{\{i<l\}}\rho.\end{aligned}$$

Substituting the above in (22) results in a Riemann-Stieltjes sum which converges and hence the fourth term of (6) converges to:

$$\frac{\lambda\alpha^{-1}}{1-\rho}\int_0^{|\mathcal{C}|}\int_0^{|\mathcal{C}|}\left(b_{r_0}(q) + \sum_{j=1}^{N-1}\hat{\epsilon}_j\bar{b}_{r_j}\right)\left(\hat{\rho}(y) - \hat{\rho}(q) + 1_{\{y<q\}}\rho\right)P_{R_0}(dq)dy.$$

[5] Let $\varrho_{i,l} := \sum_{Nl}^{Ni}\rho_k$ and $\bar{\varrho}_{i^\sigma,l^\sigma} := \varrho_{i,l}$ and note $\bar{\varrho}_{i^\sigma,l^\sigma} < \rho$. Also, $b_{r_0}(\delta^\sigma(q)) - b_{r_0}(q)) \to 0$ as $\sigma \to \infty$ for every q and $\left(P_{R_0}(\{\delta^\sigma(q)\}) - P_{R_0}\left(\left\{\delta^\sigma(q) + \frac{|\mathcal{C}|}{\sigma}\right\}\right)\right) \to 0$ almost for all q by Lemma 7. By bounded convergence theorem (BCT) (since $||b_{r_0}||_\infty < \infty$) as $\sigma \to \infty$,

$$\sum_{i=0}^{\sigma}\frac{1}{\sigma}\sum_{l=0}^{\sigma}\left(\lambda b_{r_0}(l^\sigma)\left[F_{R_0}\left(l^\sigma + \frac{|\mathcal{C}|}{\sigma}\right) - F_{R_0}(l^\sigma)\right] - \lambda_{\{l,0\}}b_{\{l,0\}}\right)\varrho_{i,l}$$
$$= \lambda\int_0^{|\mathcal{C}|}\int_0^{|\mathcal{C}|}\left(b_{r_0}(\delta^\sigma(q)) - b_{r_0}(q)\right)\bar{\varrho}_{\delta^\sigma(q),\delta^\sigma(y)}P_{R_0}(dq)dl$$
$$+\lambda\int_0^{|\mathcal{C}|}\int_0^{|\mathcal{C}|}b_{r_0}(\delta^\sigma(q))\left(P_{R_0}(\{\delta^\sigma(q)\}) - P_{R_0}\left(\left\{\delta^\sigma(q) + \frac{|\mathcal{C}|}{\sigma}\right\}\right)\right)\bar{\varrho}_{\delta^\sigma(q),\delta^\sigma(y)}dqdl \to 0.$$

Similarly, every j-th term in the fifth term of (6) is approximated by:

$$\frac{\hat{\epsilon}_{j+1}\lambda\alpha^{-1}}{1-\rho}\sum_{i=1}^{\sigma}\left[F_{R_j}\left(i^\sigma+\frac{|\mathcal{C}|}{\sigma}\right)-F_{R_j}\left(i^\sigma\right)\right]\sum_{l=1}^{\sigma}\left(b_{r_{j+1}}(l^\sigma)+\sum_{k=j+2}\check{\epsilon}_{j+2}^{k}\bar{b}_{r_k}\right)$$
$$\left[F_{R_{j+1}}\left(l^\sigma+\tfrac{|\mathcal{C}|}{\sigma}\right)-F_{R_{j+1}}\left(l^\sigma\right)\right]\left(\tfrac{l-i}{\sigma}|\mathcal{C}|1_{\{i<l\}}+\tfrac{\sigma-(i-l)}{\sigma}|\mathcal{C}|1_{\{i\geq l\}}\right)$$

and hence it converges to (note $(i-1)|\mathcal{C}|/\sigma = i^\sigma$):

$$\frac{\hat{\epsilon}_{j+1}\lambda\alpha^{-1}}{1-\rho}\int_0^{|\mathcal{C}|}\int_0^{|\mathcal{C}|}\left(b_{r_{j+1}}(q')+\sum_{k=j+2}^{N-1}\check{\epsilon}_{j+2}^{k}\bar{b}_{r_k}\right)(q'-q+|\mathcal{C}|1_{\{q>q'\}})P_{R_{j+1}}(dq')P_{R_j}(dq).$$

Proof of Lemma 3 : Let N represent the number of users that caused the workload $\mathcal{T}_{r_0}([0,|\mathcal{C}|],T)$. Note that $N=\hat{\mathcal{T}}_{r_0}([0,|\mathcal{C}|],T)$, where $\hat{\mathcal{T}}_{r_0}([0,|\mathcal{C}|],T)$ represents workload due to the arrivals which request unit service time (i.e., $\hat{\mathcal{T}}$ is with $B_{r_0}\equiv 1$, so $b_{r_0}(q)=1$ for all q). Hence by Lemma 2

$$E[N]=\lambda\int_0^{|\mathcal{C}|} f_{R_0}(q)\tau(q)dq \text{ where } \tau(q)=E[T(q)].$$

Let N_{r_j} represent the number of j-rerouted users among these N. Then (by independence of rerouted locations)

$$N_{r_j}=\sum_{i=1}^{N}1_{\{\text{user } i \text{ } j\text{-rerouted to } [a,c]\}}. \tag{23}$$

The user after every service reroutes himself independent of everything else and hence by applying Wald's lemma to (23):

$$E[N_{r_j}]=E[N]E\left[1_{\{j\text{-reroute to } [a,c]\}}\right]=E[N]\Pi_{i=1}^{j}\epsilon_i P_{R_j}([a,c]).$$

The service time requirements in the $(j+1)$-service B_{r_j} is independent of every other process and its average conditioned that the arrival is in interval $[a,c]$ is $E[B_{r_j}|R_j\in[a,c]]$ and hence by applying Wald's lemma again:

$$E[\mathcal{T}_{r_j}([a,c],T)]=\hat{\epsilon}_j P_{R_j}([a,c])E[B_{r_j}|R_j\in[a,c]]E[N]. \qquad \square$$

Proof of Theorem 3: Continuity properties: We consider function Θ defined over the Banach space, $\top\in(\mathcal{D}^{(N+1)})^{N+1}$, where $\mathcal{D}^{(N+1)}:=\mathcal{D}\left([0,|\mathcal{C}|]^{N+1}\right)$, is the space of left continuous functions with right limits on $[0,|\mathcal{C}|]^{N+1}$ equipped with sup norm, and parametrized by $\sigma\in\mathbf{N}$. It is easy to see by boundedness that Θ is continuous and linear in variable $\top$. The continuity of Ω function (12) with respect to σ (the only limit point is at ∞, i.e., as $\sigma\to\infty$) is readily seen by inspection itself while the continuity of the function $\sigma\mapsto\Upsilon(\tau^*_{\delta^\sigma})$ as $\sigma\to\infty$ in sup norm is given by Theorem 2. Thus the function Θ defined in (10) is continuous at $\sigma=\infty$.

Contraction: From equations (10), for any σ

$$||\Omega(\top_1;\sigma)-\Omega(\top_2;\sigma)||_\infty\leq 3\rho\,||\Delta_1-\Delta_2||_\infty.$$

$$\text{Let } \lambda_0 := \frac{1}{3\left(\int_0^{|\mathcal{C}|} b_{r_0}(q) P_{R_0}(dq) + \sum_{j=1}^{N-1} \hat{\epsilon}_j \int_0^{|\mathcal{C}|} b_{r_j}(q) P_{R_j}(dq)\right)}. \tag{24}$$

For all $\lambda < \lambda_0$, $3\rho < 1$ and so Θ is a contraction for all σ. Thus by Contraction Mapping Theorem (Corollary 3.1.4, page 112, [2]) we obtain: 1) the existence of unique fixed points for all σ; and 2) the continuous dependence upon the parameter, σ and hence the theorem is proved. □

Appendix B

Lemma 6 *For every q and j (with $\hat{N}_j^\sigma(y,q)$ defined recursively via (26), (27) and (28)),*

$$\begin{aligned}\nu_j^\sigma(q) &= b_{r_j}(q) E_{R_{(j-1)}}\left[\hat{N}_j^\sigma(R_{(j-1)}, q)\right] \\ &+ \lambda \tau_*^\sigma(|\mathcal{C}|)\left(\hat{\epsilon}_j \rho_{r_j}\left(\left(q, \delta^\sigma(q) + \frac{|\mathcal{C}|}{\sigma}\right)\right) + \sum_{j'=1}^{j-1} \hat{\epsilon}_{j'} \rho_{r_{j'}}\left(\left[\delta^\sigma(q), \delta^\sigma(q) + \frac{|\mathcal{C}|}{\sigma}\right)\right)\right)\end{aligned} \tag{25}$$

Further, $E_{R_j}\left[\nu_j^\sigma(R_j) b_{r_j}(R_j)\right] \to E_{R_j}\left[\nu_j^\infty(R_j) b_{r_j}(R_j)\right]$.

Proof : We refer the user under consideration (who is j-rerouted to q and for whom the time difference, $\nu_j^\sigma(q)$, is being calculated) as the tagged user. The time $\nu_j^\sigma(q)$ is the difference between the instance the external service for users belonging to $[\delta^\sigma(q), \delta^\sigma(q) + |\mathcal{C}|/\sigma)$ starts and the instance the tagged user's service started (which happened before the 'external' service). Thus we will need to consider the time taken by the users that were served after the tagged user and before the external users and so the time difference $\nu_j^\sigma(q)$ is due to the time taken to serve:

1) the j-rerouted users belonging to strip $(q, \delta^\sigma(q) + |\mathcal{C}|/\sigma)$ (arrival position order service);

2) all the j'-rerouted users of strip $[\delta^\sigma(q), \delta^\sigma(q) + |\mathcal{C}|/\sigma)$ with $1 \le j' < j$ (users with maximum completed services is served first and so on); and

3) j-rerouted users that arrived exactly at q and after the tagged user (FIFO).

All 3 points contribute in case of a discrete system while the continuous system has contribution only due to points 3 and 2. Note that, in case of a continuous system, ν_j^σ is non zero only at point masses of P_{R_j}, i.e., at $\{q_{j,i}\}_{i \le M_j}$ (see (1)).

By Lemma 4, time taken to serve the j-rerouted users belonging to strip $(q, \delta^\sigma(q) + |\mathcal{C}|/\sigma)$ and all the j'-rerouted users with $1 \le j' < j$, is given by the last two terms of ν_j^σ given by (25).

We are now left with the service time due to j-rerouted users of point 3. We calculate this, conditioned on location (say y) at which j-th service was received (whose distribution equals $P_{R_{(j-1)}}$ by independence) by the tagged user. Let $\hat{N}_j^\sigma(y,q)$ represent the total number of users j-rerouted to point q

after the tagged user, who received his j-th service at y. These will be among the ones that were $(j-1)$-rerouted to[6]:

i) segment $(y, \delta^\sigma(q))$ (if $y < \delta^\sigma(q)$);

ii) segment $(y, |\mathcal{C}|) \cup [0, \delta^\sigma(q))$ (if $y \geq \delta^\sigma(q)$);

iii) point $\{y\}$ but after the tagged user.

The average number of $(j-1)$-rerouted users of case (i) (i.e., when $y < \delta^\sigma(q)$), equals $E[\hat{\mathcal{T}}_{r_{(j-1)}}((y, \delta^\sigma(q)), C)$ where $\hat{\mathcal{T}}_{r_{(j-1)}}$ is similar to $\mathcal{T}_{r_{(j-1)}}$ except that it is due to unit service times (as done in the proof of Lemma 3). By Lemma 3, the average of the sum due to cases (i) and (ii) will be[7]:

$$\bar{N}_j^\sigma(y, q) := \lambda \tau_*^\sigma(|\mathcal{C}|)\hat{\epsilon}_{j-1} \left(P_{r_{(j-1)}}((y, \delta^\sigma(q)))1_{\{y<\delta^\sigma(q)\}} \right. \\ \left. + P_{r_{(j-1)}}((\delta^\sigma(q), y))1_{\{\delta^\sigma(q)\leq y\}} \right). \tag{26}$$

The average number of users of case (iii) has to be calculated by induction. The users j-rerouted to q from the same point y after the tagged user, because of FIFO service, must be the among the users that were $(j-1)$-rerouted to y after the tagged user in the previous cycle. Hence,

$$\hat{N}_j^\sigma(y, q) = \begin{cases} \epsilon_j P_{R_j}(\{q\}) \left(\bar{N}_j^\sigma(y, q) + E_{R_{(j-2)}} \left[\hat{N}_{(j-1)}^\sigma(R_{(j-2)}, y) \right] \right) & \text{if } j \geq 2 \\ \epsilon_1 P_{R_1}(\{q\}) \left(\bar{N}_1^\sigma(y, q) + \hat{N}_0^\sigma(y) \right) & \text{if } j = 1 \end{cases} \tag{27}$$

with $\hat{N}_0^\sigma(q)$ representing the external users originated at point q before the tagged user, i.e., in his residual cycle time and is obtained as in (14):

$$\hat{N}_0^\sigma(q) = \sum_{i=1}^{M_0} p_{0,i} 1_{\{q_{0,i}\}}(q) E[C_R^\sigma(q)] \text{ for all } y. \tag{28}$$

By un-conditioning using $P_{R_{(j-1)}}$ we obtain the average number of j-rerouted users to point q after the tagged user and then applying Wald's lemma (by independence) gives the time required to service all of them and hence the first part of the lemma follows. Note that when $\sigma = \infty$ the same quantities define $\{\nu_j^\infty\}$ for continuous systems. Here the terms with empty sets are deleted, and some sets are replaced by singletons etc. (like the third term is deleted in (25) and the fourth term instead contains $\rho_{r_{j'}}(q)$).

The convergence part of the lemma follows by Lemma 7, Theorems 2, 3 and BCT recursively starting with $j = 1$ and all the way up to $j = N - 1$. □

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[6] $\epsilon_j P_{R_j}(\{q\})$ fraction of these users form $\hat{N}_j^\sigma(y, q)$. Note that the users of points (i)-(iii) are the ones that were $(j-1)$-served after the tagged user and if they are j-rerouted to the same point q then by FIFO order will receive the $(j+1)$-service also after the tagged user.

[7] In case (ii) this number results due to users rerouted from 2 adjacent cycles. But by stationarity, the two average cycles equal and hence the expressions.

Lemma 7 *Let g be a nonnegative function with $||g||_\infty < \infty$ and define $f(I) := \int_I g(y)P_{R_j}(dy)$ for every subset, $I \subset [0, |\mathcal{C}|]$. Then, for any decreasing sequence $\{I_\sigma\}$, if*

i)$\cap_\sigma I_\sigma = \{q\}$, *then* $f(I_\sigma) \to P_{R_j}(\{q\})g(q)$; **ii)** $\cap_\sigma I_\sigma = \emptyset$, *then* $f(I_\sigma) \to 0$
iii) *general, then* $f(I_\sigma) \to f(\cap_\sigma I_\sigma)$.

Proof : The new measure $g(y)P_{R_j}(dy)$ is again a finite measure and this lemma follows by continuity of probability measures, i.e., that $Prob(\cap_n A_n) = \lim_n Prob(A_n)$. □

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