Fair assignment of base stations in cellular networks

Sreenath Ramanath[1], Eitan Altman[1], Vinod Kumar[2], Veeraruna Kavitha[1],Thomas Laurent[2]
[1]INRIA, Sophia Antipolis, FRANCE. [2]Alcatel-Lucent Bell Labs, Paris, FRANCE.

***Abstract*—We address the problem of fair assignment of base station locations in a cellular network. We use the generalized α-fairness criterion, which encompasses the different notions of fairness: that of global, proportional, harmonic or max-min fairness in our study. We derive explicit expression for α-fair BS locations under 'large population' limits in the case of simple 1D models. We show analytically that as α increases asymptotically, the optimal location for a single BS converges to the center of the cell. We validate our analysis via numerical examples. We further study throughput achievable as a function of α-fair BS placement, path-loss factor β and noise variance σ^2 via numerical examples. We also briefly address the problem of optimal placement of two base stations and obtain similar conclusions.**

***Index Terms*—Cellular network, Throughput, Fairness.**

I. Introduction

In a cellular network, models used to derive analytic expressions for capacity, coverage, etc, often assume base station (BS) locations to be at the center of the cell. Such a model brings in a regular geometry to the problem being addressed and many a times results in closed-form analytic expressions for metrics of interest.

While, this indeed facilitates analysis, the actual throughput achievable at the BS, tends to vary significantly, depending on the BS placement and cell geometry. The regular geometric model with a centrally located BS is a good model, when one assumes uniform density of users. But, today's cellular networks have concentration of users, for example hot-spots or indoor-outdoor partitions that offer various levels of attenuation to radio signals, not to mention the ever present channel fading and shadowing effects above this.

The goal of our research is to place the BS in a manner which is optimal for any general fairness criterion; that of α-fairness [1], [3], which addresses popular fairness criterion like global, proportional, harmonic and max-min fairness. We show that the α-fair BS location varies continuously with fairness parameter α and moves close to the center of the cell as α increases asymptotically. This implies that, the regular geometric models (which place BS at the center) have max-min fair BS placement.

We further observe (via some numerical examples) that α-fair BS location varies significantly based on system parameters like path-loss, noise variance. However we show that the max-min fair BS placement is close to the center of the cell irrespective of the system parameters.

To bring in the importance of BS placement, we consider cells which are completely outdoors or which have indoor-outdoor partitions (Split cells). We consider cases where user density can be uniform or tend to increase along the cell (a simplistic model for a hot spot) [6]. We consider cases where adjacent cells can use the same frequency or different frequencies. For the later case, we look at the problem of fair assignment of two base stations (BS), where the cell gets divided into sub-cells from the users' perspective based on SINR association criteria. We limit our study to free space path-loss and the analysis with fading and shadowing would be our subsequent focus.

We derive simplified expressions for α-fair objective functions using large population limits, i.e., as the number of users become large. We use Strong Law of Large Numbers (SLLN) to replace summation of large number of terms in the objective function with appropriate expected value almost surely (AS). The expected value is expressed as integrals. Using these large population limits, we obtain both theoretical and numerical results of this paper. The α-fair BS locations obtained are optimal for almost all realizations of the users locations.

We begin our study by introducing our model and review the generalized α-fair fairness criterion in Section II. In Section III, we derive the α-fair placement criterion under large population limits. In Section IV, we analyze α-fair placement of base stations as α increases asymptotically and come across some interesting insights. In Section V, we study the α-fair BS locations for the case of a) an outdoor cell and b) a mixed partition cell (split-cell) via some numerical examples. Next, in Section VI, we derive the α-fair BS locations for an outdoor cell which has two BS. We conclude our study in Section VII.

II. Our model and assumptions

Our focus is on communication in the uplink (UL) direction. Large number, N, of users are located on i.i.d. locations on the line segment $[-D, D]$. The line segment is divided into cells of length L and one or more base stations, each of unit height, are placed in every cell.

The placement of the BS(s) is the issue that we address in this paper. One is usually interested in maximizing global throughput (the sum throughput due to all users) at each BS, i.e., place the BS(s) such that the global throughput is maximized. However, maximizing the global throughput can result in starving the users at a far away location, which in turn can reduce the network efficiency. Hence, several fairness criterion have been suggested and implemented in various network architectures ([1], [3]).

In [1] it is shown that all these fairness criterion are special cases of a generalized fairness concept: the α-fairness. Given a positive constant $\alpha \neq 1$, consider for example the problem of determining z so as to maximize

$$\max_{z\in[0,L]} \sum_{x_i\in[0,L]} \frac{\theta(x_i,z)^{1-\alpha}}{1-\alpha} \tag{1}$$

where, $\theta(x_i, z)$ is the throughput at the BS located at z from a user located at x_i. Note that the above objective function is defined over the convex set $[0, L]$. Further, when the objective function is concave (we will show in later sections that this is the case most of the times) and the constraints are linear, this defines a unique allocation which we call the α-fair allocation. It turns out that α-fairness gives global optimum for $\alpha \to 0$, proportional fairness when $\alpha \to 1$, harmonic (delay minimization) fairness for $\alpha = 2$ and max-min fairness when $\alpha \to \infty$.

We begin our study by first deriving explicit expressions for power, throughput and α-fair placement criterion under large population limits (as the number of users become large). Throughout the paper, we use large population limits for analytical purposes. The idea is similar to fluid limits (see [4]), where summation of large number of terms is approximated by appropriate integrals.

III. Large population limits and problem statement

Large number, N, of users are located at $\{X_i\}_{i\leq N}$, where the locations X_i of the users are i.i.d., according to some probability measure $P(dx) = \lambda(x)dx$. *We assume that each user uses the same power for transmission.* Without loss of generality, the total power in the system equals 1 and hence the power used by each user is $1/N$.

We first consider the case of a single BS in the cell and compute the total power received, throughput achievable and the α-fair placement for the BS under large population limits. The case of two base stations follows in a similar way and is addressed in Appendix.

A. Power computation :

The power received at a BS located at z from a user at X_i is given by,

$$P(X_i, z) = \frac{1}{N}(1 + (z - X_i)^2)^{\frac{-\beta}{2}}.$$

Thus the total power received at the BS is

$$P_{tot}(z) = \sum_{i=1}^{N} P(X_i, z) = \frac{1}{N}\sum_{i=1}^{N}(1 + (z - X_i)^2)^{\frac{-\beta}{2}}.$$

This is a random power. By the Strong Law of Large Number (SLLN) this converges P-a.s. to a constant limit

$$\begin{aligned}\lim_{N\to\infty} P_{tot}(z) &= E\left[(1 + (z - X_i)^2)^{\frac{-\beta}{2}}\right] \\ &= \int_{-D}^{D} (1 + (z - x)^2)^{\frac{-\beta}{2}}\lambda(x)dx. \qquad (2)\end{aligned}$$

Hence for large values of N one can approximate $P_{tot}(z)$ almost surely with the above integral.

B. Throughput computation:

The signal to interference noise ratio (SINR) at the BS located at z from a user at X_i is

$$SINR(X_i, z) = \frac{P(X_i, z)}{\sigma^2 + P_{tot}(z) - P(X_i, z)},$$

where σ^2 is the noise variance. In the above, $P_{tot}(z)$ is approximated $P-$almost surely by a constant value, i.e., by the integral of (2). However $SINR(X_i, z)$ is still random because of the term $P(X_i, z)$. The Shannon capacity or throughput achievable at the BS located at z from a user at X_i is

$$\theta(X_i, z) = \log(1 + SINR(X_i, z))$$

Considering a receiver with an adapted filter and using the approximation $\log(1 + x) \approx x$ (for smaller values of x), the throughput achievable is

$$\theta(X_i, z) = \frac{P(X_i, z)}{\sigma^2 + P_{tot}(z) - P(X_i, z)}$$

The total (global) throughput achievable at the BS from all the users in the cell of interest is:

$$\begin{aligned}f(z) &= \sum_{i=1}^{N} 1_{\{X_i\in[0,L]\}}\theta(X_i, z) \\ &= \frac{1}{N}\sum_{i=1}^{N} 1_{\{X_i\in[0,L]\}}\frac{(1 + (z - X_i)^2)^{\frac{-\beta}{2}}}{\sigma^2 + P_{tot}(z) - P(X_i, z)} \\ &\approx \frac{1}{N}\sum_{i=1}^{N} 1_{\{X_i\in[0,L]\}}\frac{(1 + (z - X_i)^2)^{\frac{-\beta}{2}}}{\sigma^2 + P_{tot}(z)},\end{aligned}$$

as for large values of N, $P(X_i, z)$ is negligible in comparison with $P_{tot}(z)$. Again, the above Random sum can be approximated using Strong Law of Large Numbers whenever the number of users inside the cell is large P-almost surely, giving rise to the following large population approximation:

$$\begin{aligned}f(z) &\approx E\left[I_{\{X_1\in[0,L]\}}\psi(X_1, z)\right] \\ &= \int_0^L \psi(x, z)\lambda(x)dx \text{ with} \qquad (3) \\ \psi(x, z) &:= \frac{(1 + (z - x)^2)^{\frac{-\beta}{2}}}{\sigma^2 + P_{tot}(z)}.\end{aligned}$$

C. α-fair placement criterion :

The α-fair objective function of (1) in a similar way can be approximated almost surely under large population limits by:

$$\tilde{f}_\alpha(z) := N^\alpha \frac{1}{1-\alpha}\int_0^L \psi(x, z)^{(1-\alpha)}\lambda(x)dx$$

Thus α-fair placement of the BS is given by,

$$\begin{aligned} z^*(\alpha) &= \arg\max_z \tilde{f}_\alpha(z) \\ &= \arg\max_z f_\alpha(z) \text{ where} \qquad (4) \\ f_\alpha(z) &:= \frac{1}{1-\alpha}\int_0^L \psi(x,z)^{(1-\alpha)}\lambda(x)dx. \end{aligned}$$

Important point to note here is that, *for almost all realizations of the locations of the users the objective function is approximated by the constant integral and hence $z^*(\alpha)$ is optimal α-fair location for almost all users locations.*

D. Problem statement

Now with this background, we pose the following problems:

1. Find BS location z so as to maximize global throughput $f(z)$. See large population limit (3).

2. Find the α-fair BS location z^* which maximizes $f_\alpha(z)$ for various fairness criterion. See large population limit (4)

In subsequent sections, we analyze and apply the α-fair placement criterion to obtain BS locations which are both optimal and fair in various cellular environments considered.

IV. Analysis : Single BS Placement

We notice that both the global throughput (large population limit (3)) and the α-fair placement (large population limit (4)) of the BS is dependent on the total power received $P_{tot}(z)$ at the BS, which in-turn depends on its location z. In many cases, the total-power received can be assumed independent of the location of the BS, whenever the cell size is small (which is typical of pico cells). This for example is true for cells with user density $\lambda(x)$ being symmetric about $\frac{L}{2}$ (uniform being the trivial case) and completely located outdoors.

The above assumption simplifies analysis to a good extent and is considered in the first subsection, while an approximate analysis is given in the following subsection without this assumption.

We consider asymptotic analysis in this section and hence consider only the cases with $\alpha > 1$. For notational simplicities, we redefine $f_\alpha(z)$ of equation (4) after dropping the division by $(1-\alpha)$ factor and now,

$$z^*(\alpha) := \arg\max_{z\in[0,L]} \left(-f_\alpha(z)\right).$$

A. $P_{tot}(z)$ is independent of BS location z :

As $P_{tot}(z)$ is independent of z, the α-fair location is obtained by minimizing the function,

$$\bar{f}_\alpha(z) := \int_0^L \left(1+(z-x)^2\right)^{-\frac{\beta}{2}(1-\alpha)}\lambda(x)dx$$

We can easily show that $\bar{f}_\alpha(z)$ is concave in z. We also have joint continuity in (α, z) by Bounded Convergence theorem. Hence, by maximum theorem [5] under convexity, we get

Lemma *1:* The function $z^*(\alpha)$ is continuous in α. ■

By differentiability of $\bar{f}_\alpha(.)$,

$$g(\alpha, z^*(\alpha)) = 0.$$

where with $\gamma := \frac{\beta}{2}(\alpha-1)-1$ (for some appropriate $c \neq 0$),

$$\begin{aligned} g(\alpha,z) &:= c\frac{\partial \bar{f}_\alpha(z)}{\partial z} \\ &= \int_0^L (z-x)\left(1+(z-x)^2\right)^\gamma \lambda(x)dx. \end{aligned}$$

If $z < \frac{L}{2}$ then,

$$\begin{aligned} &\left(1+\left(\frac{L}{2}\right)^2\right)^{-\gamma} g(\alpha,z) \\ &= \int_0^{\frac{L}{2}+z}(z-x)\left(\frac{1+(z-x)^2}{1+(\frac{L}{2})^2}\right)^\gamma \lambda(x)dx \\ &\quad + \int_{\frac{L}{2}+z}^L (z-x)\left(\frac{1+(z-x)^2}{1+(\frac{L}{2})^2}\right)^\gamma \lambda(x)dx \end{aligned}$$

tends to $-\infty$ as $\alpha \uparrow \infty$, because the first term tends to zero while the later tends to $-\infty$ (by bounded convergence theorem). Therefore there exists $\alpha_0 > 0$ such that, $g(\alpha,z) < 0$ and hence such that,

$$g(\alpha,z) \neq 0 \text{ for all } \alpha > \alpha_0.$$

Similarly if $z > \frac{L}{2}$ then, $\left(1+\left(\frac{L}{2}\right)^2\right)^{-\gamma} g(\alpha,z)$ tends to ∞ as $\alpha \uparrow \infty$ and hence we have,

$$g(\alpha,z) \neq 0 \text{ for all } \alpha > \alpha_0(z) \text{ whenever } z \neq \frac{L}{2}.$$

However $\left(1+\left(\frac{L}{2}\right)^2\right)^{-\gamma} g\left(\alpha, \frac{L}{2}\right) \to 0$ as $\alpha \uparrow \infty$.

In fact, we have (by monotonicity arguments) for all $z_0 < \frac{L}{2}$:

$$g(\alpha,z) \neq 0 \text{ for all } \alpha > \alpha_0(z_0), z \in [0,z_0] \cup [L-z_0, L],$$

and hence the optimizer lies in a smaller interval around $\frac{L}{2}$ for all larger values of α and thus we get the following:

Lemma *2:* For every $\epsilon < \frac{L}{2}$ there exists an $\alpha_0(\epsilon)$ (depending upon ϵ), such that for all $\alpha > \alpha_0(\epsilon)$

$$z^*(\alpha) \in \left[\frac{L}{2}-\epsilon, \frac{L}{2}+\epsilon\right].$$

i.e, the optimizer lies in a smaller interval around $\frac{L}{2}$ for all larger values of α. That is, $z^*_\alpha(z) \to \frac{L}{2}$ as $\alpha \to \infty$. ■

Whenever the density $\lambda(x)$ is symmetric about $\frac{L}{2}$ within the cell $[0,L]$, using similar derivative arguments one can get,

Lemma *3:* The partial derivatives under symmetric conditions, for all α

$$\left.\frac{\partial \bar{f}_\alpha(z)}{\partial \alpha}\right|_{z=z_0} = 0$$

and hence optimal locations for all α are at $\frac{L}{2}$. ■

Summary of the results :

1. α-fair location is continuous in α (by Lemma 1).

2. When the density function λ is symmetric about $\frac{L}{2}$ then by Lemma 3 all the α-fair locations are at the center of the cell.

3. If density is not symmetric about $\frac{L}{2}$ then, by Lemma 2, the α-fair locations tend to $\frac{L}{2}$ as α tends to infinity.

4. Lemma 2, 3 are correct as long as the support of measure λ contains both the end points, i.e., $\{0, L\} \subset supp(\lambda)$. If not, the same results are true with $\frac{L}{2}$ replaced with $length(supp(\lambda))/2$.

B. $P_{tot}(z)$ is dependent on BS location z :

Next, we consider cases when the total power $P_{tot}(z)$ is dependent on base station location z. This is true for cases with non-symmetric user densities, cells with partitions, etc.

Let $h(.;z)$ represent the following parametrized function :

$$h(x;z) := \left(\sigma^2 + P_{tot}(z)\right)\left(1 + (x-z)^2\right)^{\frac{\beta}{2}}.$$

and let $||h(.;z)||_p$ represent its L_p norm with respect to the probability measure $\frac{\lambda(x)dx}{\int_0^L \lambda(x)dx}$.

With the above definitions, for $\alpha > 1$, we can equivalently write the optimal α-fair location as,

$$z^*(\alpha) = \arg\min_{z\in[0,L]} ||h(.;z)||_{\alpha-1}.$$

As $\alpha \to \infty$, $||h(.;z)||_{\alpha-1} \to ||h(.;z)||_\infty$ and one can show that,

$$\lim_{\alpha\to\infty} z^*(\alpha) \approx \arg\min_{z\in[0,L]} ||h(.;z)||_\infty.$$

Since,

$$\begin{aligned} ||h(.;z)||_\infty &= \left(\sigma^2 + P_{tot}(z)\right)\sup_{x\in[0,L]}\left(1+(x-z)^2\right)^{\frac{\beta}{2}} \\ &= \left(\sigma^2 + P_{tot}(z)\right)\left(1 + (\max\{z, L-z\})^2\right)^{\frac{\beta}{2}} \\ &= \left(\sigma^2 + P_{tot}(z)\right)\max\{(1+z^2)^{\frac{\beta}{2}}, (1+(L-z)^2)^{\frac{\beta}{2}}\}, \end{aligned}$$

the asymptotic α-fair location approximately equals:

$$\lim_{\alpha\to\infty} z^*(\alpha) \approx \arg\min_{z\in[0,L]} \left(\sigma^2 + P_{tot}(z)\right)\max\{z, L-z\}.$$

and note that,

$$\max\{z, L-z\} = 1_{\left\{z\geq\frac{L}{2}\right\}}(z) + 1_{\left\{z<\frac{L}{2}\right\}}(L-z).$$

Clearly if $P_{tot}(z)$ was independent of z, asymptotic α-fair location would be at $\frac{L}{2}$.

In the subsequent sections, we consider some interesting examples and show the validity of the results of this Section. We also derive many more interesting conclusions using the large population limits of Section III for those examples.

V. Optimal and fair placement of a single BS

In this section, we consider two cases. In the first, we consider an outdoor cell, while in the second, we consider a cell which spans over both indoor and outdoor environment (split-cell).

A. Outdoor cell

An outdoor cell is typically characterized by a cell placed in open environment/free space, i.e., the signals from the users are attenuated only due to path-loss. We assume a cellular deployment which uses the same frequency throughout. i.e., the power received from the entire line segment $[-D, D]$ will interfere with the power received from the user under consideration.

By Lemma 3, the α-fair solution for uniform user density, $\lambda(x) \equiv 1/2D$, is trivial (all the α-fair locations are at the center of the cell $\frac{L}{2}$).

Next, we consider another interesting case where user density $\lambda(x) = x$; to mimic a simplistic hot-spot (i.e, the user density proportionally increases towards the hot-spot, which is located around L). Figure 1 depicts the scenario. We want to place the BS such that the locations are optimal and fair.

Optimal base station placement – Outdoor cell

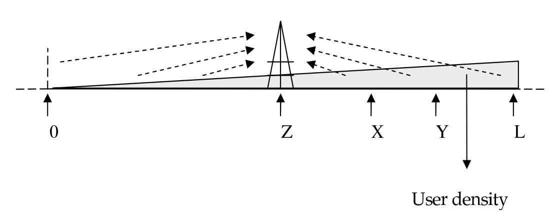

Fig. 1. Open-cell: BS located at z, user density $\lambda(x) = x$

By Lemma 1, the α-fair location varies continuously w.r.t. α. Also, by Lemma 2 and discussions in Section IV-B, the α-fair locations should tend to $\frac{L}{2}$ as α increases to infinity. We will indeed show that this is the case in the following numerical example. We further make some more interesting observations.

Numerical example: We evaluate equation (4) for some typical cases: for $\alpha = 0$ (global), $\alpha = 0.99$ (proportional), $\alpha = 2$ (harmonic) and $\alpha = 128$ (max-min). The example considers cell length $L = 10$, noise variance $\sigma^2 = 1$ and path-loss exponent $\beta = 2, 4$. Fig 2 - 4 shows example plots for the α-fair objective functions $f(z)$ (equation (3) corresponding to $\alpha = 0$), $f_\alpha(z)$ (equation (4)) for $\alpha = 0.99, 2$. Note that the case of global fairness ($\alpha = 0$) is also the case which maximizes sum throughput. Also, note that Fig 2 gives the global throughput as function of BS location z.

We compute the α-fair BS placement for increasing values of α. In Figure 5, we plot the α-fair BS location as a function of α. As given by Lemma 1 the α-fair location is continuous in α. We further, observe that the BS location shifts rapidly going from optimally fair to proportionally fair and finally tends to $\frac{L}{2}$ for being max-min fair.

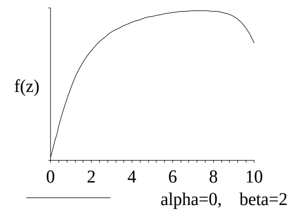

Fig. 2. Open cell: Global throughput (3) as a function of the BS location. User density $\lambda(x) = x$ and $\beta = 2$)

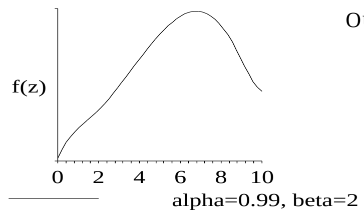

Fig. 3. Proportional fair objective function f_α given by (4) with $\alpha \approx 1$, as function of BS location z. User density $\lambda(x) = x$, $\beta = 2$)

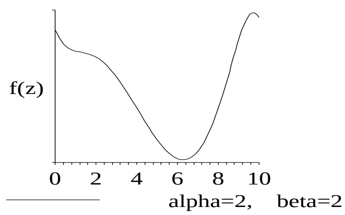

Fig. 4. Harmonic fair objective function f_α given by (4) with $\alpha = 2$, as function of BS location z. User density $\lambda(x) = x$, $\beta = 2$)

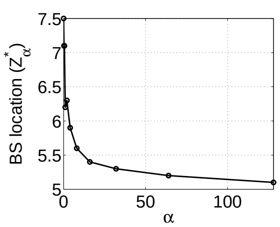

Fig. 5. α-fair BS location, $z^*(\alpha)$ as a function of α.

We tabulate normalized throughput (ratio of the global throughput with BS at α-fair location, $z^*(\alpha)$ to the maximum achievable global throughput, i.e., the total throughput achieved when BS is placed at $z^*(0)$) achievable for these α-fair BS locations in Table I.

We show the impact of path-loss factor β and noise variance σ^2 on the optimally-fair BS placement in Table II and Table III, respectively. In those tables $f(z;\beta,\sigma^2)$ represents the global throughput when BS is placed at z and with path-loss factor β and noise variance σ^2.

TABLE I
OUTDOOR CELL: THE α-FAIR BS LOCATIONS AND NORMALIZED THROUGHPUT. USER DENSITY $\lambda(x) = x$, $L = 10$ AND PATH-LOSS $\beta = 2$

α-fairness	BS lox	Normalized throughput
global ($\alpha = 0$)	7.4	1.000
proportional ($\alpha = 0.99$)	6.8	0.998
harmonic ($\alpha = 2$)	6.3	0.995
max-min ($\alpha = 128$)	5.0	0.981

TABLE II
OUTDOOR CELL: BS PLACEMENT FOR GLOBALLY-FAIR THROUGHPUT FOR VARIOUS PATH-LOSS β. USER DENSITY $\lambda(x) = x$ AND $L = 10$

Path-loss β	BS lox	Throughput ratio $f(z^*(0);\beta,1)/f(z^*(0);2,1)$ (w.r.t $\beta = 2$)
2	7.4	1.00
4	8.2	0.99
6	8.8	0.98

TABLE III
OUTDOOR CELL: BS PLACEMENT FOR GLOBALLY-FAIR THROUGHPUT FOR VARIOUS NOISE-VARIANCE σ^2. USER DENSITY $\lambda(x) = x$, PATH-LOSS $\beta = 2$ AND $L = 10$

Noise variance σ^2	BS lox	Throughput ratio $f(z^*(0);2,\sigma^2)/f(z^*(0);2,1)$ (w.r.t $\sigma^2 = 1$)
$\frac{1}{4}$	6.9	1.05
1	7.4	1.00
4	7.9	0.58

Observations:

a. We observe that the placement of BS affects the throughput achievable in case of an outdoor cell, modeling a hot-spot.

b. The BS location shifts rapidly going from globally fair to proportionally fair and finally settles close to $\frac{L}{2}$ for being max-min fair (Refer Figure 5). This is an interesting observation which implies that the regular geometric models, which assume centrally placed BS are actually positioned to be max-min fair. But, such assumptions does not seem to impact the throughput achievable as seen in this case. The max-min fair throughput is just about 2% below the maximum achievable global throughput. For the other fair locations, the reduction in throughput is quite negligible (Refer Table I).

c. The optimal throughput does not seem to be sensitive to path-loss (Refer Table II)

d. The achievable optimal throughput is very sensitive to noise variance σ^2. A four fold increase in noise variance degrades the throughput by 40%.

B. *Indoor-outdoor cell (Split-cell)*

In this section, we consider a cell which covers both indoor and outdoor environments, partitioned by solid structures like walls etc. We consider a cell which has a single partition or wall, located at y within the cell $[0, L]$ and offers an attenuation of η dB. Here again, a single BS of unit height is located at z. The scenario is depicted in figure 6. We want to find BS locations which optimize various fairness criterion.

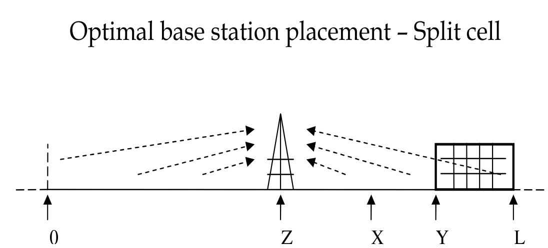

Fig. 6. Split-cell: BS located at z, wall located at y

The total power, throughput and $\alpha-$ fair objective function for this case can be derived exactly in the same way as before to obtain the following large population limits :

$$P_{tot}(z) = \int_{[-D,y)\cup[L,D)} (1+x^2)^{\frac{-\beta}{2}}\lambda(x)dx + \eta\int_y^L (1+x^2)^{\frac{-\beta}{2}}\lambda(x)dx \quad (5)$$

$$f(z) = \int_0^y \psi(x,z)\lambda(x)dx + \eta\int_y^L \psi(x,z)\lambda(x)dx \quad (6)$$

$$f_\alpha(z) = \frac{1}{1-\alpha}\left[\int_0^y \psi(x,z)^{1-\alpha}\lambda dx + \eta\int_y^L \psi(x,z)^{1-\alpha}\lambda(x)dx\right] \quad (7)$$

The equations (5), (6) and (7) are similar respectively to (2), (3) and (4) if $\lambda(x)$ is replaced by (an appropriate constant multiple of) $\lambda(x)(1_{\{x\notin[y,L]} + \eta 1_{\{x\in[y,L]\}})$. Hence the results of Section IV hold good here also. From Lemma 1, Lemma 2 and discussions in Section IV-B, we would expect the α-fair location to vary continuously w.r.t. α and tend close to $\frac{L}{2}$ as α increases asymptotically. We shall validate this via the following numerical example for uniform user density, i.e., for $\lambda(x) \equiv 1/2D$.

Numerical example: We evaluate equation (7) for some typical cases: for global ($\alpha = 0$), proportional ($\alpha = 0.99$) and harmonic ($\alpha = 2$) fairness with path-loss exponent $\beta = 2, 4$, noise variance $\sigma^2 = 1$,wall attenuation $\eta = 12dB$ and wall located at $y = 0.75L$. The results are presented in Fig 7 - 12. Note that the case of global fairness ($\alpha = 0$) is also the case which maximizes global throughput (Refer Figure 7).

In Fig 10, we plot the α-fair BS location as a function of α. We observe that the BS location shifts rapidly going from globally fair to proportionally fair and finally converges close to $\frac{L}{2}$ for max-min fair.

Next, we plot global throughput as a function of BS location z and wall location y in Figure 11. In Fig 12, we show global throughput as a function of BS location z and attenuation η.

Table IV tabulates the normalized throughput achievable for various α-fairness criterion along with the α-fair BS locations, While, Tables V and VI tabulate the BS placement for globally-fair throughput for various path-loss factors β and noise variance σ^2, respectively.

In the tables, Normalized throughput, $f(z;\beta,\sigma^2)$ represent similar terms as in the previous section.

TABLE IV
SPLIT-CELL: THE α-FAIR BS LOCATION AND NORMALIZED THROUGHPUT. USER DENSITY $\lambda(x) \equiv 1/2D$, $L = 10$, $y = 0.75L$, PATH-LOSS $\beta = 2$ AND WALL ATTENUATION $\eta = 12dB$

α-fairness	BS lox	Normalized throughput
global ($\alpha = 0$)	4.35	1.0000
proportional ($\alpha = 0.99$)	3.90	0.9983
harmonic ($\alpha = 2$)	3.88	0.9983
max-min ($\alpha = 128$)	5.00	0.9981

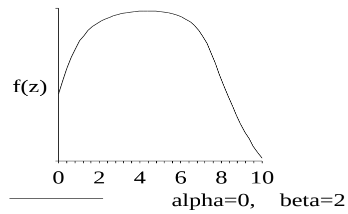

Fig. 7. Split-cell: Global throughput (Objective function $f_\alpha(z)$ (6) with $\alpha = 0$) as a function of BS location z.

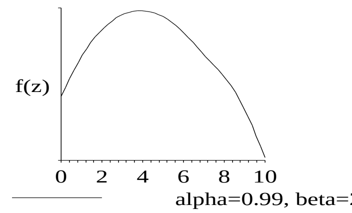

Fig. 8. Split-cell: Objective function $f_\alpha(z)$ (7) for proportional fairness ($\alpha = 0.99$) as a function of BS location z.

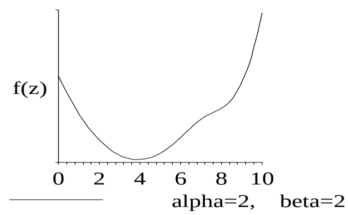

Fig. 9. Split-cell: Objective function $f_\alpha(z)$ (7) for harmonic fairness ($\alpha = 2$) as a function of BS location z.

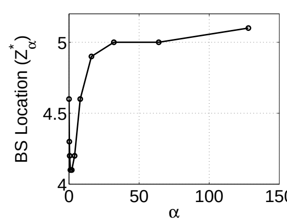

Fig. 10. Split-cell: α-fair BS location $z^*(\alpha)$ as a function of α.

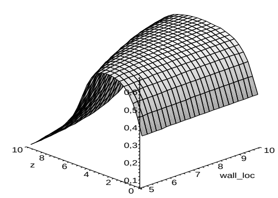

Fig. 11. Split-cell: Global throughput (6) as a function of BS location z and wall location y.

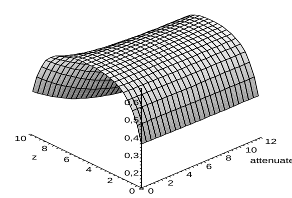

Fig. 12. Split-cell: Global throughput (6) as a function of BS location z and attenuation η.

TABLE V
SPLIT CELL: BS PLACEMENT FOR GLOBALLY-FAIR THROUGHPUT FOR VARIOUS PATH-LOSS β. USER DENSITY $\lambda(x) \equiv 1/2D$ AND $L = 10$

Path-loss β	BS lox	Throughput ratio $f(z^*(0);\beta,1)/f(z^*(0);2,1)$ (w.r.t $\beta = 2$)
2	4.35	1.00
4	4.15	0.93
6	4.00	0.83

Observations:

a. We observe that the BS location shifts rapidly going from globally fair to proportionally fair and finally settles at $\frac{L}{2}$ for being max-min fair (Refer Figure 10).

b. Further, we observe that the placement of BS does not seem to affect the throughput achievable in case of an indoor-outdoor cell.

c. The price in throughput is negligible and the deployment can satisfy various fairness criterion (Refer Table IV).

d. The reduction in globally-fair throughput is quite signif-

TABLE VI
SPLIT CELL: BS PLACEMENT FOR GLOBALLY-FAIR THROUGHPUT FOR VARIOUS NOISE-VARIANCE σ^2. USER DENSITY $\lambda(x) \equiv 1/2D$, PATH-LOSS $\beta = 2$ AND $L = 10$

Noise variance σ^2	BS lox	Throughput ratio $f(z^*(0); 2, \sigma^2)/f(z^*(0); 2, 1)$ (w.r.t $\sigma^2 = 1$)
$\frac{1}{4}$	4.65	1.30
1	4.35	1.00
4	3.90	0.21

icant (as much as 20%) with an increase in path-loss factor β (Refer Table V)

e. As the indoor portion of the split-cell reduces, the globally-fair throughput response tends to become flat. (Refer Figure 11)

f. Wall attenuation does not seem to alter the globally-fair BS placement much, though one can observe a significant reduction in throughput initially (Refer Figure 12)

g. The reduction in globally-fair throughput is quite drastic (as much as 80%) with an increase in noise variance σ^2 (Refer Table VI)

VI. OPTIMAL AND FAIR PLACEMENT OF TWO BS IN AN OUTDOOR CELL

In this section we consider optimal placement of two BS in a single cell for various α-fair criterion. We consider a new scenario in this section, that of a single isolated cell (i.e., no interference from the other cells). One can easily study a single BS problem with this new scenario and vice versa using the tools of this paper. This new scenario is considered for covering all varieties of the settings/scenarios.

Users are located on this segment with density $\lambda(x), x \in [-L, L]$. Assume BS_1 and BS_2 are located at z_1 and z_2, respectively and uses the same frequency and cooperate with each other. Further, we assume that the neighboring cells do not use the same frequency. The users associate themselves with one of the two base stations which maximize their SINR.

Under these assumptions, we first calculate the global (sum) throughput from all the users associated with a particular BS. Under cooperative setting, the sum of these two global throughputs would be the appropriate criteria for optimization. In Appendix A, we derived simplified expressions $f(z_1, z_2)$, for this sum of global throughputs, under large population limits. In a similar way, a general simplified α-fair objective function $f_\alpha(z_1, z_2)$ is also derived in the same Appendix. We now re-state the problem of Section III for the two BS case :

1. Find location (z_1, z_2) so as to maximize global throughput $f(z_1, z_2)$. See large population limit (10) (same as $f_\alpha(z_1, z_2)$ with $\alpha = 0$) of Appendix A.

2. Find the α-fair location (z_1^*, z_2^*) which maximizes $f_\alpha(z_1, z_2)$ for various fairness criterion. See large population limit (13) of Appendix A.

We reproduce from Appendix A, the α-fair location as given by

$$(z_{1\alpha}^*, z_{2\alpha}^*) = \arg\max_{z_1, z_2} f_\alpha(z_1, z_2).$$

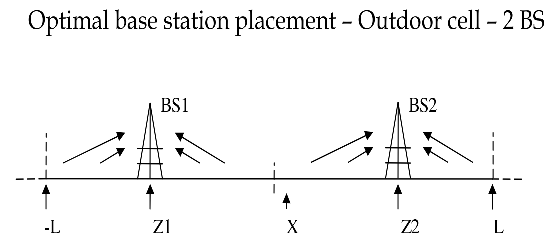

Fig. 13. Open-cell: BS_1 located at $z1$, BS_2 located at $z2$, user density $\lambda(x) \equiv 1/2D$

where,

$$f_\alpha(z_1, z_2) = \frac{\int_{C(z_1,z_2)} \left(1 + (x - z_1)^2\right)^{1-\alpha} \lambda(x) dx}{\left(\sigma^2 + P_{tot}(z_1)\right)^{1-\alpha}} + \frac{\int_{C(z_1,z_2)^c} \left(1 + (x - z_2)^2\right)^{1-\alpha} \lambda(x) dx}{\left(\sigma^2 + P_{tot}(z_2)\right)^{1-\alpha}}$$

Numerical example: We evaluate equation (13) for some typical cases: for global ($\alpha = 0$), proportional ($\alpha = 0.99$) and harmonic ($\alpha = 2$) fairness with path-loss exponent $\beta = 2$ and noise variance $\sigma^2 = 1$.

For the numerical analysis we have assumed that the BS are located symmetrically about the origin to ease the SINR based user association criteria (See Appendix A).

The results are presented in Fig 14 - 17. Note that the case of global fairness ($\alpha = 0$) is also the case of sum global throughput (Refer Figure 14). From the plots, we observe that the BS locations for global fairness is $(-6.5, 6.5)$.

In Figure 17, we plot the α-fair BS_2 location as a function of α. We observe that the BS location shifts rapidly going from globally fair to proportionally fair and finally settles at $L/2$ for being max-min fair. In a similar way, the BS_1 tends to $-L/2$ as α increases to infinity. In fact, we observe that the BS location exhibits max-min fair placement for values of $\alpha = 8$ and beyond.

Table VII tabulates the normalized throughput achievable for various α-fairness criterion along with the α-fair BS locations, while, Table VIII tabulates the BS placement for globally-fair throughput for various path-loss factors β.

TABLE VII
THE α-FAIR BS LOCATION(S) AND NORMALIZED THROUGHPUT FOR OUTDOOR CELL WITH TWO BS, USER DENSITY $\lambda(x) = 1/2L$, $L = 10$, PATH-LOSS $\beta = 2$

α-fairness	BS_1 lox	BS_2 lox	Normalized
global ($\alpha = 0$)	-6.45	6.45	1.0000
proportional ($\alpha = 0.99$)	-5.15	5.15	0.9970
harmonic ($\alpha = 2$)	-5.10	5.10	0.9950
max-min ($\alpha = 128$)	-5.05	5.05	0.9940

Observations:

a. We observe that the BS locations shift rapidly going from globally fair to proportionally fair and finally settles at $(-L/2, L/2)$ for being max-min fair. In fact, the BS location

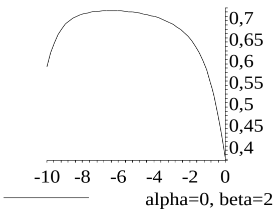

Fig. 14. Outdoor cell, two BS: Global throughput (objective function $f_\alpha(z_1, z_2)$ with $\alpha = 0$) as a function of BS_1 location ($z_2 = -z_1$)).

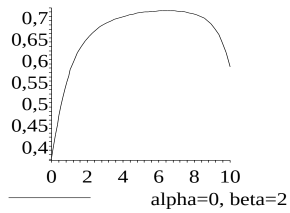

Fig. 15. Outdoor cell, two BS: Global throughput (objective function $f_\alpha(z_1, z_2)$ with $\alpha = 0$) as a function of BS_2 location($z_1 = -z_2$)).

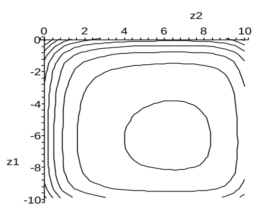

Fig. 16. Outdoor cell, two BS: 3-D contour plot of global throughput (objective function $f_\alpha(z_1, z_2)$ with $\alpha = 0$) as a function of BS locations (z_1, z_2)

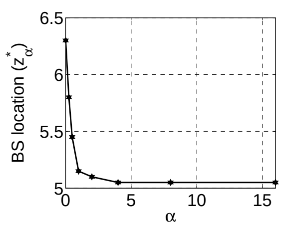

Fig. 17. Outdoor cell, two BS: α-fair BS location $z_2^*(\alpha)$ as a function of α (Placement of BS_2 shown here).

TABLE VIII
OUTDOOR CELL WITH TWO BS: BS PLACEMENT FOR GLOBALLY-FAIR THROUGHPUT FOR VARIOUS PATH-LOSS β. USER DENSITY $\lambda(x) = 1/2L$ AND $L = 10$

Path-loss β	BS_1 lox	Throughput ratio $f(z^*(0); \beta, 1)/f(z^*(0); 2, 1)$ (w.r.t $\beta = 2$)
2	-6.45	1.00
4	-5.55	0.85
6	-5.35	0.76

exhibits max-min fair placement for values of $\alpha = 8$ onwards (Refer Figure 17).

b. Further, we observe that the placement of BS does not seem to affect the throughput achievable in case of an outdoor cell with two BS.

c. The price in throughput is negligible and the deployment can satisfy various fairness criterion.

d. The globally-fair throughput reduces by 25% with an increase in path-loss exponent β from 2 to 6 (Refer Table VIII).

e. We observe that for a uniform distribution of users, when placing fairly two base stations on the segment $[-L, L]$, the distance between the stations decrease as α increases (Refer Table VII). In particular, we note that a model similar to this has been already studied in [2] where the equilibrium location was computed in a non-cooperative context (each base station tries to maximize its own throughput) instead of the fair location. As in the fair placement case that we study here, it was shown there that the equilibrium distance is also closer than the distance corresponding to the globally fair location. As an example, the equilibrium location of the BS that corresponds to the data of Fig 17 here is 5.5 in Table 1 of [2] (the globally fair being around 6.4). This means that the non-cooperative equilibrium location is fairer than the globally fair one - it corresponds to the α fair placement where α is seen from Fig 17 to be around 0.5.

Further, our work can be extended to find the α-fair BS locations when multiple BS are to be located on a line segment or on a 2D grid. This is a step towards optimal BS placement to satisfy various fairness criteria when a macro-cell is divided into a number of small cells. For example, the optimal placement of BS in pico-cell networks.

VII. CONCLUSIONS

We studied the problem of optimal BS placement, optimal for various α-fair criterion in cellular networks. We considered simple 1D models which characterize both indoor and outdoor cellular environments with mixed partitions. We derived explicit expressions for α-fair criterion under large population limits. These limits were used to obtain the theoretical asymptotic analysis of the α-fair locations. We show that the α-fair locations converge close to center of the cell as α increases to infinity (which basically represents the max-min fair location).

The large population limits were also used to numerically compute BS locations which satisfy global, proportional, harmonic and max-min fairness. For the models considered, we presented results via plots and tables to show the variations in achievable throughput for the different fairness criterion. We also confirmed, via numerical examples, that the α-fair locations converge to the center of the cell as α tends to infinity.

We next considered a two base station optimal placement problem again for various α-fair criterion. We obtained large population limits under cooperative setting and using this we showed, via numerical examples, that the α-fair BS locations converge to a pair of locations which divide the cell once again into equal regions.

VIII. APPENDIX A : LARGE POPULATION LIMITS - POWER, THROUGHPUT AND α-FAIR PLACEMENT OF TWO BASE STATIONS:

In Section III, we derived power, throughput and α-fair placement expressions for a single BS located in the cell. In this appendix section, we derive the same for two BS. For simplicity, we consider the cell of interest to span $[-L, L]$. Also, in this case, we assume that neighboring cells use different frequencies (i.e, there is no frequency reuse)

As before, the power from a user located at X_i received at BS_1 located at z_1 is

$$P(X_i, z_1) = \frac{1}{N}(1 + (z_1 - X_i)^2)^{\frac{-\beta}{2}}.$$

The total power received at BS_1 under large population limits is

$$P_{tot}(z_1) = \int_{-L}^{L} (1+(z_1-x)^2)^{\frac{-\beta}{2}} \lambda(x)dx,$$

assuming no frequency re-use.

The throughput (which is approximately equal to the SINR in case of an adaptive filter) at BS_1 is

$$\theta(X_i, z_1) \approx SINR(X_i, z_1) = \frac{P(X_i, z_1)}{\sigma^2 + P_{tot}(z_1)}.$$

Similarly throughput at BS_2 is,

$$\theta(X_i, z_2) \approx SINR(X_i, z_2) = \frac{P(X_i, z_2)}{\sigma^2 + P_{tot}(z_2)}.$$

The user at X_i will associate itself with BS_1 if

$$SINR(X_i, z_1) > SINR(X_i, z_2). \quad (8)$$

Let

$$C(z_1, z_2) := \{x : SINR(x, z_1) \geq SINR(x, z_2)\} \quad (9)$$

represent the set of users which associate themselves with BS_1.

Under cooperative setting, the total sum throughput received at both the base stations is,

$$\begin{aligned} f(z_1, z_2) \quad := \quad & \frac{1}{(1-\alpha)} \sum_{i=1}^{N} \left[\theta(X_i, z_1) 1_{\{X_i \in C(z_1,z_2)\}} \right. \\ & \left. + \theta(X_i, z_2) 1_{\{X_i \in C(z_1,z_2)^c\}} \right] \quad (10) \end{aligned}$$

The α-fair solution in this case is given by the BS location pair (z_1^*, z_2^*) which maximizes f_α where,

$$\begin{aligned} \tilde{f}_\alpha(z_1, z_2) \quad := \quad & \frac{1}{(1-\alpha)} \sum_{i=1}^{N} \left[\theta(X_i, z_1) 1_{\{\in C(z_1,z_2)\}} \right. \\ & \left. + \theta(X_i, z_2) 1_{\{X_i \in C(z_1,z_2)^c\}} \right]^{1-\alpha} \\ = \quad & \frac{1}{(1-\alpha)} \sum_{i=1}^{N} \left[\theta(X_i, z_1)^{1-\alpha} 1_{\{X_i \in C(z_1,z_2)\}} \right. \\ & \left. + \theta(X_i, z_2)^{1-\alpha} 1_{\{X_i \in C(z_1,z_2)^c\}} \right] \end{aligned}$$

which under large population limits is approximated by,

$$\begin{aligned} & \tilde{f}_\alpha(z_1, z_2) \quad (11) \\ \approx \quad & \frac{N^\alpha}{(1-\alpha)} \left[\frac{\int_{C(z_1,z_2)} \left(1+(x-z_1)^2\right)^{1-\alpha} \lambda(x)dx}{(\sigma^2 + P_{tot}(z_1))^{1-\alpha}} \right. \\ & \left. + \frac{\int_{C(z_1,z_2)^c} \left(1+(x-z_2)^2\right)^{1-\alpha} \lambda(x)dx}{(\sigma^2 + P_{tot}(z_2))^{1-\alpha}} \right] \end{aligned} \quad (12)$$

Thus α fair placement of the two BS is given by,

$$\begin{aligned} (z_{1\alpha}^*, z_{2\alpha}^*) \quad = \quad & \arg\max_{z_1,z_2} f_\alpha(z_1, z_2) \text{ where} \\ f_\alpha(z_1, z_2) \quad = \quad & \\ & (-1)^{1_{\{\alpha>1\}}} \left[\frac{\int_{C(z_1,z_2)} \left(1+(x-z_1)^2\right)^{1-\alpha} \lambda(x)dx}{(\sigma^2 + P_{tot}(z_1))^{1-\alpha}} \right. \\ & \left. + \frac{\int_{C(z_1,z_2)^c} \left(1+(x-z_2)^2\right)^{1-\alpha} \lambda(x)dx}{(\sigma^2 + P_{tot}(z_2))^{1-\alpha}} \right] \end{aligned} \quad (13)$$

ACKNOWLEDGMENT

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