Global behaviour of polynomial differential systems in the positive orthant

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Abbreviated title: Polynomial systems in the positive orthant.

Abstract. We study polynomial n-dimensional differential systems defined in the positive orthant. By using tools from positivity and functions that decrease along the trajectories, we give sufficient conditions for a regular global behaviour: that is, all the trajectories either converge towards the equilibria or are unbounded. We also give results on global stability or unstability in some cases.

Key words. Global behaviour, polynomial differential systems, positivity, auxiliary decreasing functions.

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1 Introduction

Consider a polynomial n-dimensional differential system (or more exactly a sum of power functions of many variables). We can write it:

$$\dot{x}_i = \sum_{j=1}^{q} a_{ij} v_j(x_1, \ldots, x_n) \quad (i = 1, \ldots, n) \tag{1}$$

where the a_{ij} are real and the $v_j(x_1, \ldots, x_n)$ mononomials of the form $x_1^{\beta_1} \ldots x_n^{\beta_n}$, where the β_i are real; q is the number of distinct monomials in the whole system. If this system is defined for $x \in \mathbf{R}^n$, it is well known it can have a very complicated behaviour. In some particular cases, it is possible to study such systems, but global results are rather rare (cf. [9, 10]).

We are going here to study this system for positive x only, i.e. the system is defined in an open subset of the open positive n-dimensional orthant. Of course, we could study the system in any of the open orthants: the important fact is that we are interested in the behaviour of the system as long as x keeps the same sign. Such situations, for example, arise in chemical or biological modelling (where the variables are constrained to be positive) ([7]). Interestingly, some important tools have been developed, that are deeply founded on the sign properties of the variables: let us cite the work on chemical kinetics of Feinberg, Horn and Jackson ([11, 4, 5, 6]), of Clarke ([2, 3]) and others (see the references in [2]). These works study the existence of steady states and the stability properties of chemical reaction networks. We can also cite the works on monotone flows, that are founded on the sign properties of the jacobian of the differential system (these sign properties often depend on the signs of the variables); these flows have properties of convergence towards steady states or attractors (see [9, 10]) A related domain of application is population dynamics, where one modelize the way species interact together by competition, cooperation ...([12, 10]). Let us precise here that we do not impose to the positive orthant to be invariant by the flow; we are interested only in the solution inside this cone, but the solution can leave the orthant (and possibly reenter later).

The tools used here are mainly results from positivity and theory of positive matrices ([1]) and auxiliary functions that decrease along the trajectories and Lasalle's theorem ([13]. If some decomposition (depending on the a_{ij} and the β_i) is possible, then one of the main results is that all the trajectories either converge towards some attractor set S containing the set of equilibria, or that they cannot remain in any compact set of the positive orthant; they have therefore limit points either at infinity or on the faces of the orthant, or they leave the orthant; of course, they can reenter later, if the orthant is not invariant. If, moreover, the set S is actually the set of equilibria, we deduce that the system cannot have a complex behaviour (such as periodic solutions, recurrent trajectories, chaos, ...) in any compact set of the interior of the positive orthant. In some cases, we can make the description more precise and obtain results on global stability or unstability. In a number of particular cases, this approach is related to the works in chemical kinetics of Horn and Jackson ([11]), Feinberg ([4]), and Clarke ([2]). However, our goal is the study of differential systems with positive variables, not only those issued from chemical kinetics.

Notations: For x in $\mathbf{R}^n$, we write $x > 0$ if $x_i > 0$ $(i = 1, \ldots, n)$ and $x \geq 0$ if $x_i \geq 0$ $(i = 1, \ldots, n)$. The closed positive orthant is $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n; x \geq 0\}$. We will frequently use the open positive orthant $\mathbf{P}^n = \{x \in \mathbf{R}^n; x > 0\}$. Let us denote by ${}^t u$ the transpose of u, by e^x the vector ${}^t(e^{x_1}, \ldots, e^{x_n})$, and similarly for $\ln x$. The Kronecker product between two vectors $x \otimes y$ is the vector with components $(x_i y_i)$. If x is a vector, we denote by x^{-1} the vector of components $(1/x_i)$. $\mathbf{1}$ is the vector ${}^t(1 \ldots 1)$ and diag (x) the diagonal matrix with diagonal x.

If $V \subset \mathbf{R}^n$ is open, $h : V \to \mathbf{R}^n$ is C^1, and $x_0 \in V$, for the differential system $\dot{x} = h(x)$ ($\dot{x}$ is the derivative with respect to time t), we denote by $x(t, x_0)$ or sometimes by $x(t)$ the (maximally defined) solution in V with initial value x_0 for $t = 0$.

2 The system

We will first write the system into a more compact form; we will suppose that x is positive (the system is defined in $\mathbf{P}^n$). We remark that the powers β_j in the monomials can be real negative: the system is always C^∞ in $\mathbf{P}^n$.

It is easy to see we can write (1) into the form :

$$\dot{x} = x \otimes Ae^{(B \ln x)} \tag{2}$$

where A is a $n \times p$ matrix and B a $p \times n$ matrix. The elements of B are nothing but the powers β_j or $\beta_j - 1$ of x_i in the monomials; p is the number of distinct monomials in the system after factorisation of x_i in the ith equation.

We first transform the equation by the change of variables (well defined because $x > 0$):

$$y = \ln x$$

so that:

$$\dot{y} = Ae^{By} \tag{3}$$

We now study the equilibria of this system; let $s = \text{rank } A$. The equilibria are such that:

$$\dot{y} = 0 \Leftrightarrow e^{By} \in \ker A$$

and therefore belong to the intersection, in a p-space, of $\ker A$ (a vector space of dimension $(p-s)$) and of a manifold of dimension $r = \text{rank } B$ ($r \leq (\max(\text{n}, \text{p}))$) and situated in the positive orthant $\mathbf{P}^p$. We don't want here to investigate deeply the problem, but we can make some simple remarks:

- If $\ker A \cap \mathbf{P}^p = \emptyset$, then there is no equilibrium.
- Because $\ker\, {}^tA$ is of dimension $(n-s)$, then the system (3) has $(n-s)$ linear first integrals ${}^tqy = const.$
- In general, the equilibria of (3) are not isolated. But, for a given initial condition, the trajectory will stay on the first integrals, that is on an affine invariant manifold R_0 of dimension $(p-(n-s))$; we will study the equilibria on such a manifold (depending of the initial condition), that is the equilibria that can actually be attained. We begin by reducing the system (for a similar approach, see [3]); without restriction, we can suppose that:

  $$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

  where A_1 is a surjective $(s \times p)$ matrix and A_2 a $((n-s) \times p)$ matrix. There exists a $((n-s) \times s)$ matrix L such that:

  $$A_2 = LA_1$$

  and we can decompose y into $y_1 \in \mathbf{R}^s$ and $y_2 \in \mathbf{R}^{(n-s)}$ with:

  $$\dot{y}_2 = A_2 e^{By} = LA_1 e^{By} = L\dot{y}_1$$

  and by integrating:

  $$y_2(t) = Ly_1(t) + c$$

  where c is a constant $(n-s)$ dimensional vector depending on the initial conditions $y_1(0)$ and $y_2(0)$. We have then:

  $$By = B\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = B\begin{pmatrix} y_1 \\ Ly_1 + c \end{pmatrix} = B_1 y_1 + B_2 c$$

  where B_1 is a $p \times s$ matrix, B_2 a $p \times (n-s)$ matrix. The equilibria on the invariant manifold R_0 (given by $y_2 = Ly_1 + c$) reduce to:

  $$A_1 e^{B_1 y_1 + B_2 c} = 0$$

or

$$A_1 D_0 e^{B_1 y_1} = 0$$

where D_0 is a positive diagonal matrix depending on the initial conditions $(y_1(0), y_2(0))$. The equilibria on R_0 therefore belong to the intersection, in a p-space, of $\ker A_1$ (a vector space of dimension $(p-s)$) and of a manifold of dimension rank $B_1 \leq s$ and situated in the positive orthant $\mathbf{P}^p$.

- If B_1 is of full rank s and if $\ker A_1 \cap \mathbf{P}^p \neq \emptyset$, then we can expect "generically" (by an argument of transversality) that the set of equilibria on R_0 is of dimension zero, because it is the intersection of a manifold of dimension s and of a vector space of dimension $p - s$. The set of equilibria is then a discrete, maybe empty, set of points: we deduce that the equilibria on R_0 are "generically" isolated. If B_1 is not of full rank, the equilibria (if they exist) will not be robust with respect to perturbations in B_1.

- We can illustrate what happens by taking the important case where $\ker A_1$ is one-dimensional (and consequently $\ker A_1 D_0$). We therefore suppose that $\ker A_1 D_0$ is generated by a *positive* vector k. Then:

$$e^{B_1 y_1} = \lambda k \Leftrightarrow B_1 y_1 = (\ln \lambda)\mathbf{1} + \ln k$$

where λ is positive. This last equation has a solution if and only if im B_1 intersects the affine line of vector $\mathbf{1}$ and going through the point $\ln k$. If B_1 is of full rank, im B_1 is an hyperplane, and there will be one single equilibrium if and only if $\mathbf{1} \notin$ im B_1. If B_1 is not of full rank, there can exist an equilibrium, but it will not be robust with respect to perturbations in B_1 (this kind of equilibrium is thus physically or biologically not plausible).

If $\ker A \cap \mathbf{P}^p = \emptyset$, we have already remarked that there is no equilibrium. We now show that, in such a case, all positive orbits have an unbounded closure (cf ([7])) and that, moreover, every solution leaves every compact and never comes back. Indeed, if $\ker A \cap \mathbf{P}^p = \emptyset$, then (see [15]) im ${}^tA \cap \mathbf{R}^p_+ \neq \{0\}$, and we can choose $r = {}^tAq$ in this intersection. Let $V(y) = {}^tqy$, then $\overline{\dot{V(y)}} = {}^tqAe^{By} = {}^tre^{By}$, where r is a nonnegative and non-zero vector. So this last expression never cancels, and Lasalle's theorem ([13]) gives us the first result. Take now any compact C . If a solution has a limit point z in

C, then $\overline{\dot{V(z)}} = 0$; but it is impossible. Therefore every solution definitely leaves C.

It means that, given an initial condition (and if $\ker A \cap \mathbf{P}^p = \emptyset$), the orbit cannot remain in any compact; in particular, there is no complicated behaviour (like periodic or recurrent solutions, chaos...) inside any compact; for the original system (2), it means that, given a positive initial condition, the orbit is unbounded or has points of the face of the orthant as limit points, or leaves the orthant: that is, the positive orbit definitely leaves any compact set of the interior of the positive orthant.

These results heavily depends on the number n and p of variables and distinct monomials; suppose that $n \geq p$ and that $s = p$: then A is injective and the condition $\ker A \cap \mathbf{P}^p = \emptyset$ is trivially verified. This means that, in a polynomial system, if there is more variables than (distinct) monomials, then, "generically" (for almost any choice of the matrix A), all the trajectories of (3) are unbounded, whatever the precise form of the monomials (the matrix B) could be . Therefore the most interesting case is $n \leq p$.

3 The decomposition

We want to find, to study the above system, auxiliary functions of the variables that decrease along the trajectories ([13]); it is a kind of weak version of Lyapunov functions. To construct these functions, we shall use tools from theory of positive matrices (cf. [1]), also related to probabilistic problems ([14]). The function $\phi(p)$ that we use is a generalization of a Lyapunov function often discussed for chemical networks (see [2]). We shall use the following lemmas, demonstrated in ([8]); for the sake of completeness, we give the demonstration in the appendix.

Lemma 1 *Let M be a square off-diagonal non-negative singular matrix such that:*

$$Mk = 0 \quad {}^{t}M\mathbf{1} = 0$$

where k is a positive vector and let $\phi(p) = {}^{t}(\ln p - \ln k)Mp$. Then:

$$\forall p > 0 \quad \phi(p) \leq 0$$

Lemma 2 *Suppose that the (non-oriented) graph of M is connected; then (for $p > 0$)*

$$\phi(p) = 0 \Leftrightarrow p = \lambda k \ \ (\lambda > 0)$$

Let us recall that the non-oriented graph of a square n-matrix M is the graph with n vertices having an edge between vertices i and j if m_{ij} or m_{ji} is non-zero. We can remark that ${}^tM\mathbf{1} = 0$ implies by the Perron-Frobenius theorem that it exists $k \geq 0$ such that $Mk = 0$. In fact, in our case, ${}^tM + I$ is a stochastic matrix without transitory states (see ([1])).

If the graph is not connected, we can consider the set of n points as the union of two or more independant subsets with connected graphs, and study each set independently. From the above lemma we deduce that (see [8]):

Lemma 3 *If the graph of M is made of l disconnected classes, associated with matrices M_j ($j = 1, \dots, l$), then (for $p > 0$):*

$$\phi(p) = 0 \Leftrightarrow p = \sum_{j=1}^{l} \lambda_j k_j$$

where k_j is a positive vector in $\ker M_j$ having the property that the components not corresponding to the vertices of the graph of M_j are zero, the other components being equal to those of k, and the λ_j are real nonnegative.

We can now construct, under good hypotheses, an auxiliary function decreasing along the trajectory; we consider the system (3), and use the same notations than in the above lemmas:

Lemma 4 *Suppose that, for given A and B, there exists a square $(p \times p)$ off-diagonal nonnegative singular matrix M and two vectors $k > 0$ and w, such that ${}^tM\mathbf{1} = 0$, $Mk = 0$, $\ln k = Bw$, and a symmetric square $(n \times n)$ matrix P such that:*

$$PA = {}^tBM$$

Then, if

$$V(y) = \frac{1}{2}\, {}^t(y - w)P(y - w)$$

the function $V(y)$ decreases along the trajectories of system (3) ($\overline{\dot{V}(y(t))} \leq 0$).

Moreover, the derivative vanishes if and only if:

$$e^{By} = \sum_{j=1}^{l} \lambda_j k_j$$

Proof: Indeed, $\dot{\overline{V(y(t))}} = {}^t(y-w)PAe^{By} = {}^t(y-w)^tBMe^{By}$. Let $By = \ln p$ with $p > 0$. Then $\dot{\overline{V(y(t))}} = {}^t(\ln p - \ln k)Mp$ and we can apply the preceeding lemmas.

Of course, we have not yet determined completely k and w; we will choose them in the following, in order to have the set where the derivative vanishes as small as possible. For the time being, we want to make some remarks on this decomposition:

- The decomposition $PA = {}^tBM$, where we must find M and P (symmetric), and ${}^tM\mathbf{1} = 0$, $Mk = 0$, reduces to find the kernel of a linear system with $np + 2p$ equations and $p^2 + n(n+1)/2$ indeterminates (therefore more indeterminates than equations if n and p are large enough, and because we can suppose $p \geq n$, see the end of section 2); but we must also impose the sign of the off-diagonal elements of M. The problem is hence to determine if a vector subspace intersects some positive orthant. This problem is easy to solve numerically by linear programming methods.

- If (A, B) admits a decomposition (P, M), then $(\alpha A, B)$ admits a decomposition $(\alpha^{-1}P, M)$ for all real $\alpha \neq 0$. In particular, $(-A, B)$ admits a decomposition, that means that the differential system in reverse time will have the same properties.

- It is also interesting to know if, given a (P, M) decomposition of (A, B), the new system (AD, B) admits also a decomposition, where D is a positive diagonal matrix of diagonal d (see [2][p.35]). A possible decomposition is (P, MD), but we must also verify that it exists k' and w' such that $\ln k' = w'$. We can take $k' = D^{-1}k$, and we obtain

  $$\ln d = B(w - w')$$

  Therefore we see that we are able to find a decomposition if and only if $\ln d$ belongs to the image of B (this remark is due to one of the reviewer).

Before the study of the behaviour of the system, with the help of these lemmas, is is interesting to see what happens for a linear change of variables in system (3). We show that the preceeding decomposition is invariant: take $z = Uy$, U being a regular square matrix. Then :

$$\dot{z} = UAe^{BU^{-1}z}$$

and a decomposition with P^*, M^* will verify:

$$P^*UA = {}^tU^{-1}\,{}^tBM^* \Leftrightarrow ({}^tUP^*U)A = {}^tBM^*$$

and $({}^tUP^*U)$ is a symmetric matrix associated with the same quadratic form as P^*. If therefore the system (3) admits a decomposition with P and M, then the transformed system will also admit a decomposition with the same quadratic form and the same matrix M.

4 Global behaviour

We will now try to describe the behaviour of the two systems; if, given A and B, we can find an off-diagonal nonnegative and singular matrix M and two vectors $k > 0$ and w, such that ${}^tM\mathbf{1} = 0$, $Mk = 0$ and $\ln k = Bw$, and a symmetric matrix P, with $PA = {}^tBM$, we will summarize these hypotheses by saying the system (A, B) admits a "PM-decomposition".

In the following, we will use Lasalle's theorem ([13]) to obtain the convergence of bounded trajectories towards an invariant set; we try to impose that this set is as small as possible, the minimal set being the set of equilibria.

We first make two remarks: first, we can suppose that $\ker A \cap \mathbf{P}^p$ is non-empty (if not, all the trajectories are unbounded, cf. section 2) so there exists $k > 0$ with $Ak = 0$; secondly, if the system has at least one equilibria y^*, we can choose (in lemma 4) $\ln k = By^*$ $(w = y^*)$ and $Ak = 0$; but if the system does not have equilibria, then we are forced to choose k' and w' such that $\ln k' = Bw'$, k' being not in the positive kernel of A; consequently $PAk' = B^tMk' = 0$ and the set where $\overline{\dot{V}(y(t))} = 0$ will be non-empty, although the set of equilibria is empty.

We can now state a general theorem, that we shall make more precise in the following:

Theorem 4.1 *If A and B admit a PM-decomposition, the trajectories of (3) are unbounded or converge towards the maximal invariant set included in the set of y such that:*

$$e^{By} = \sum_{j=1}^{l} \lambda_j k_j$$

To prove this, it is enough to apply Lasalle's theorem together with the lemma 3.

Corollary 1 *If A and B admit a PM-decomposition with* $\ker P \cap A\mathbf{P}^p = \{0\}$ *(in particular if P is bijective), then the trajectories of (3) are unbounded or converge towards the (non empty) set of equilibria.*

Indeed, because of the preceeding theorem, the bounded trajectories converge towards the points y such that $Me^{By} = 0 \Rightarrow PAe^{By} = 0$, and with the hypothesis of the corollary, it implies that $Ae^{By} = 0$, so y is an equilibrium.

If the hypothesis of the corollary are fulfilled, the differential system has a rather simple behaviour: a trajectory either goes towards the set of equilibria or is unbounded (it can happen that there is no equilibrium on the linear first integrals of the solution, in such a case the trajectory is unbounded). For the original system (1), it means that a trajectory either converges towards the set of equilibria, or has limit points at infinity or on the faces of the orthant, or leaves the orthant. Intuitively, the trajectory converges to equilibrium or "leaves" the interior of the orthant. Of course, such a trajectory can reenter the orthant later, and infinitely often (this behaviour makes no sense when the variables have a biological meaning).

If now we can only verify the hypothesis of the theorem, then the trajectories are unbounded or converge towards some attractor set S: if this set is still not too big, it can still be simple to study the behaviour inside.

We use now the eigenvalues of P to make the description more precise; to simplify the discussion, we keep the supplementary hypothesis of the corollary; we remark also that the trajectories are constrained to stay on the affine space $y(0) + \mathrm{im}\ A$:

Theorem 4.2 *Let us suppose A and B admit a PM-decomposition with* $\ker P \cap A\mathbf{P}^p = \{0\}$*. If w is an isolated equilibrium on $y(0) + \mathrm{im}\ A$, then:*

- *if P restricted to* $\mathrm{im}\ A$ *has only positive eigenvalues, the trajectory starting from $y(0)$ is bounded; moreover, the equilibrium is (locally) asymptotically stable.*
- *if P restricted to* $\mathrm{im}\ A$ *has a negative eigenvalue, the equilibrium is locally unstable.*

Indeed, we can take $e^{Bw} = k \in \ker A$, and $y(t) - w$ stays in $\mathrm{im}\ A$. If P restricted to $\mathrm{im}\ A$ has only positive eigenvalues, the function $V(y)$ is actually a Lyapunov function.

If P restricted to im A has a negative eigenvalue, take $y'(0)$ in a neighbourhood U of w on $y(0) + \text{im } A$ such that $V(y'(0)) < 0$ and $\overline{\dot{V}(y)} < 0$ for $y \in U$ (it is possible because w is an isolated equilibrium and because the set where $\overline{\dot{V}(y)} = 0$ is actually the set of equilibria). Then $y'(t)$ leaves U, and the equilibrium is unstable.

We can give a more detailed description in the important case where $p = n + 1$ and $\ker A$ is one-dimensional positive. We know (see section 2) that if $\mathbf{1} \notin \text{im } B$, and if B is of full rank $p - 1 = n$, then there is one single equilibrium.

Corollary 2 *If A and B admit a PM-decomposition with $\ker P \cap A\mathbf{P}^p = \{0\}$ (with $p = n + 1$), and if $\ker A$ is one-dimensional (positive) and $\mathbf{1} \notin \text{im } B$, then:*

- *If P is positive definite, the single equilibrium is globally asymptotically stable in the whole space.*
- *If P has a negative eigenvalue, the equilibrium is unstable and there exists unbounded trajectories.*

The results are global because of the unicity of the equilibrium.

5 Examples

Let us give the matrix $(p \times n)$ B, and consider the family of polynomial differential systems of the form:

$$\dot{y} = Q^t B M e^{By}$$

where Q is a symmetric bijective matrix and M an off-diagonal nonnegative and singular matrix such that ${}^t M\mathbf{1} = 0$ and $Mk = 0$, k being any positive vector such that there exists w with $\ln k = Bw$. Then this system admits a PM-decomposition with $P = Q^{-1}$ bijective, and we deduce that, for all the systems of the family, the trajectories are unbounded or converge towards the (non empty) set of equilibria. The behaviour is given by the signs of the eigenvalues of Q.

In fact, we can still extend this family by remarking that, because ${}^t M\mathbf{1} = 0$, it changes nothing if we add to each line of the matrix ${}^t B$ a multiple of the vector $\mathbf{1}$. The new family is now:

$$\dot{y} = Q^t B M e^{(B+DJ)y}$$

where D is any diagonal matrix and J is the $(n \times p)$ matrix with all the elements equal to one.

Let us give now a more precise example. The following differential system of dimension four is related to population dynamics, and modelize the way four populations interact two by two, by predation, symbiosis ...(see [12]):

$$\begin{cases} \dot{a} = -3a^2 + 2ab + ad \\ \dot{b} = bc - bd \\ \dot{c} = -2ac + bc - 2c^2 + 3cd \\ \dot{d} = 3ad - bd + 2cd - 4d^2 \end{cases}$$

It is easy to verify that the positive orthant and the faces are invariant. This system admits the set of equilibria $(\alpha, \alpha, \alpha, \alpha)$, where α is positive. We can compute that a decomposition is:

$$P = \begin{pmatrix} 6 & 4 & 4 & 3 \\ 4 & 4 & 3 & 4 \\ 4 & 3 & 5 & 2 \\ 3 & 4 & 2 & 6 \end{pmatrix} \quad M = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

and P is positive definite. We can therefore conclude that the equilibrium (unique on the first integral determined by the initial condition, see section 2) is globally asymptotically stable in the positive orthant (relatively to the first integral). We are not aware of other ways to obtain this result.

6 Conclusion

The PM-decomposition enables us to construct auxiliary decreasing functions and study some aspects of the behaviour of polynomial differential systems in the positive orthant. Of course, this decomposition must be studied more deeply: an interesting question is :"How big is the set of (A, B) admitting a decomposition in the set of the couples (A, B) of fixed dimensions and such that $\ker A \cap \mathbf{P}^p \neq \emptyset$?".

7 Appendix

We give here a short (but computational) proof of the two lemmas 1 and 2 (for a more general result, see [8]):

Let $\phi(p) = {}^t(\ln p - \ln k)Mp$. Then

$$\begin{aligned}
\phi(p) &= \sum_i \sum_j (\ln p_i - \ln k_i) m_{ij} p_j \\
&= \sum_i ((\ln p_i - \ln k_i) m_{ii} p_i + \sum_i \sum_{j \neq i} (\ln p_i - \ln k_i) m_{ij} p_j
\end{aligned}$$

and using ${}^tM\mathbf{1} = 0$

$$\begin{aligned}
\phi(p) &= \sum_i (\ln p_i - \ln k_i)(-\sum_{k \neq i} m_{ki}) p_i + \sum_i \sum_{j \neq i} (\ln p_i - \ln k_i) m_{ij} p_j \\
&= -\sum_j \sum_{i \neq j} (\ln p_j - \ln k_j) m_{ij} p_j + \sum_i \sum_{j \neq i} (\ln p_i - \ln k_i) m_{ij} p_j \\
&= \sum_i \sum_{j \neq i} (\ln \frac{p_i}{k_i} - \ln \frac{p_j}{k_j}) m_{ij} p_j
\end{aligned}$$

and we use the facts that $\ln x \leq x - 1$ (for $x > 0$) and $m_{ij} \geq 0$ if $i \neq j$ to obtain

$$\begin{aligned}
\phi(p) &\leq \sum_i \sum_{j \neq i} (\frac{p_i k_j}{p_j k_i} - 1) m_{ij} p_j \\
&= \sum_i (\sum_{j \neq i} m_{ij} k_j) \frac{p_i}{k_i} - \sum_i \sum_{j \neq i} m_{ij} p_j
\end{aligned}$$

and using $Mk = 0$

$$\begin{aligned}
\phi(p) &\leq -\sum_i m_{ii} p_i - \sum_i \sum_{j \neq i} m_{ij} p_j \\
&= -\sum_i (m_{ii} + (\sum_{j \neq i} m_{ji})) p_i \\
&= 0
\end{aligned}$$

because of ${}^tM\mathbf{1} = 0$. This proves the first lemma.

Let us now examine when $\phi(p) = 0$. We know that $\ln x \leq x - 1$ with equality if and only if $x = 1$. Therefore we must have equality for each term of the above sum, and we deduce

$$m_{ij} \neq 0 \text{ or } m_{ji} \neq 0 \ \Rightarrow \frac{p_i}{k_i} = \frac{p_j}{k_j}$$

If the graph of M is connected, that implies that $p = \mu k$ (with $\mu > 0$). For the converse, it is enough to note that, if an off-diagonal nonnegative singular matrix M has a connected graph, then its kernel is reduced to a positive vector ([1]).

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