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Topological versus Smooth Linearization of Control Systems

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Abstract: This note deals with "Grobman-Hartman like" theorems for control systems (or in other words under-determined systems of ordinary differential equations). The main results (proved elsewhere) is that when a control system is topologically conjugate to a linear controllable one, then it is also "almost" differentiably conjugate. We focus on the meaning of this result, and on an open question resulting from it.

Key-words: Nonlinear control systems, Topological invariants, Feedback linearization, Grobman-Hartman Theorem

Warmly dedicated to Velimir Jurdjevic on his 60th birthday.

This was presented at the Conference "Geometric Control Theory and Applications" organized in his honor in Mexico City, September, 2000.

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Sur la différentiabilité de la linéarisation topologique des systèmes contrôlés

Résumé : Cette note traite des théorèmes de type Grobman-Hartman en automatique, c'est-à-dire de la "ressemblance" entre un système de contrôle non linéaire et son approximation linéaire. Le résultat principal (démontré dans une autre publication) dit qu'un système qui est topologiquement conjugué a un système linéaire commandable lui est forcément aussi "presque" différentiablement conjugué, et il est connu que cela arrive fort peu souvent. On s'intéresse ici surtout au sens de ce résultat, et à une question ouverte qui y est rattachée.

Mots-clés : Automatique non-linéaire, Invariants topologiques, Linéarisation par retour d'état, Théorème de Grobman-Hartman

1. Introduction

In this note, we discuss the *local* behavior of a nonlinear control system

$$\dot{x} = f(x,u)\ , \quad x \in \mathbb{R}^n\ , \quad u \in \mathbb{R}^m\ , \tag{1}$$

say around $(0,0) \in \mathbb{R}^{n+m}$. For general control systems (as opposed e.g. to affine in the control), "local" has to be understood with respect to both state *and* control.

The first reaction when dealing with local properties is to compute the linear approximation of (1). When this linear control system happens to be controllable, all the *local* usual control objectives can be met using linear control, based on the linear approximation. For instance, a linear control that asymptotically stabilizes the linear approximation will also stabilize the nonlinear system, locally; minimizing a quadratic cost can also be achieved up to first order based on the linear approximation only. Hence, the linear approximation is a good enough model for the purpose of designing controllers achieving a desired behavior for small states and controls. We believe that all control engineers or control theorists agree on this statement, arising from practice, although we would welcome some contradiction.

Rephrasing the above statement without reference to control objectives leads to an imprecise statement, grounded mostly on some necessarily subjective intuition, and that should rather be taken as an opening sentence to launch a debate than as a conjecture :

$$\begin{cases} \textit{nothing distinguishes qualitatively the behavior of a nonlinear control} \\ \textit{system from the one of its linear approximation if the latter is controllable.} \end{cases} \tag{2}$$

It is natural to try to formalize this statement, as a prerequisite to any proper theory of nonlinear modeling and identification of control systems, in a very preliminary manner since it only deals with *local* phenomena. A nice way to turn that belief into a sound, and correct, assertion would be to find some equivalence relation between control systems (or models) that preserves at least "qualitative" behavior, and for which these two systems (a nonlinear system and its controllable linear approximation) are in general equivalent.

We assume controllability of the linear approximation. When this fails none of the above is correct, at least in the most common case when the nonlinear system is itself controllable. Indeed, (non-)controllability is a qualitative phenomenon : for instance, feeding a linear non controllable system with "random" inputs, one observes that the state is confined in leafs of positive codimension, while for a controllable system the whole state space is explored.

To enlighten the discussion on local behavior of control systems, let us recall the situation for ordinary differential equations $\dot{x} = F(x)$ (particular case of (1) where the control u has dimension 0) :

- If $F(0) \neq 0$, the "flow-box theorem" (see e.g. [**Arn74**, §7]), gives local coordinates, smooth if F is smooth, in which F is of the form $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.
- If $F(0) = 0$ and the square matrix $F'(0)$ has no pure imaginary eigenvalue (hyperbolic equilibrium), then Grobman-Hartman Theorem [**Har82**, Theorem IX-7.1] tells us that the flow of the differential equation is locally conjugate to the flow

of its linear approximation via a homeomorphism that need not, in general, be smooth if F is smooth (and in fact smooth conjugation requires more assumption, resonances are obstructions to it) (see e.g. [**Arn80**, §22]).

- If $F(0) = 0$ and the square matrix $F'(0)$ has some pure imaginary eigenvalue, then the situation is more intricate even locally, namely the phase portrait of the nonlinear dynamical system $\dot{x} = F(x)$ can be very different locally from the one of a linear system. This case is of high interest in the theory of dynamical systems, but can be considered as "degenerate", in the same way as non controllability of the linear approximation for control systems.

Since conjugation of flows does preserve qualitative phenomena like the overall aspect of the phase portrait, one can indeed assert that, locally around all points except non hyperbolic equilibria, a differentiable dynamical system "behaves like" a linear one, and this is translated by conjugation via a homeomorphism, although conjugation via a smooth diffeomorphism preserves some more subtle local invariants (resonances, etc...).

Coming back to control systems, first of all, the equivalent of conjugation by a smooth change of coordinates is *(smooth) feedback equivalence*, whose study was initiated in [**Bro78**], see a survey in [**Jak90**]. In fact this is conjugation via a smooth diffeomorphism on the state and control, forced to have a triangular structure (see Proposition 2.5 below). The conditions under which a control system (1) is smoothly feedback equivalent to a linear controllable one are well known [**JR80, HSM83**] (and contrary to the case of ordinary differential equations, they are very simple), but they reveal that very few nonlinear systems are locally feedback equivalent to a linear one, even when the linear approximation is controllable. This remark and the review of the situation for ordinary differential equations naturally brings about the question whether for control systems, relaxing the regularity of the conjugating maps, i.e. considering conjugacy by homeomorphisms instead of smooth diffeomorphisms would make more systems equivalent to a linear one.

After recalling some basic facts in section 2, we give in section 3 an essentially negative answer to the question evoked above, based on quoting a result to appear in [**BCP**], that topological conjugacy to a linear controllable system implies conjugacy by "almost" smooth feedback (but the gap is really small). Section 4 recalls, also from [**BCP**], a technical open question that would allow a nicer result and a nicer description of that "almost" smooth conjugacy, and finally section 5 extends the discussion of the results from section 3, their implications, and the questions they raise in nonlinear modeling.

2. Preliminaries on equivalence of control systems

2.1. Definitions. Consider two smooth control systems with state x (resp. z) and input u (resp. v) :

$$\dot{x} = f(x,u)\ ,\quad x \in \mathbb{R}^n\ ,\quad u \in \mathbb{R}^m\ , \tag{3}$$

$$\dot{z} = g(z,v)\ ,\quad z \in \mathbb{R}^{n'}\ ,\quad v \in \mathbb{R}^{m'}\ , \tag{4}$$

or, expanded in coordinates,

$$\dot{x}_i = f_i(x_1,\dots,x_n,u_1,\dots,u_m), \qquad \dot{z}_j = g_j(z_1,\dots,z_{n'},v_1,\dots,v_{m'}),$$

$1 \le i \le n$, $1 \le j \le n'$, with the f_i's and g_i's some smooth (i.e. C^∞) maps.

We assume that f and g are defined respectively on the whole of $\mathbb{R}^n \times \mathbb{R}^m$ and $\mathbb{R}^{n'} \times \mathbb{R}^{m'}$ because it simplifies many of the statements below; this is actually no loss of generality to us for all the results we prove are local with respect to x, u, z, v, so that f and g can be extended using partitions of unity outside some neighborhoods of the arguments under consideration without affecting the results.

Definition 2.1. By *a solution of (3) that remains in an open set* $\Omega \subset \mathbb{R}^{n+m}$, we mean a mapping γ defined on a real interval :

$$\begin{array}{rcccl} \gamma : & I & \to & \Omega & \\ & t & \mapsto & \gamma(t) & = \; (\, \gamma_{\mathrm{I}}(t)\, ,\, \gamma_{\mathrm{II}}(t)\,)\;, \end{array} \tag{5}$$

with $\gamma_{\mathrm{I}}(t) \in \mathbb{R}^n$ and $\gamma_{\mathrm{II}}(t) \in \mathbb{R}^m$, such that γ is measurable, locally bounded, γ_{I} is absolutely continuous and, whenever $[T_1, T_2] \subset I$, we have :

$$\gamma_{\mathrm{I}}(T_2) \;-\; \gamma_{\mathrm{I}}(T_1) \;\; = \;\; \int_{T_1}^{T_2} f(\, \gamma_{\mathrm{I}}(t)\, ,\, \gamma_{\mathrm{II}}(t)\,)\, \mathrm{d}t \;.$$

Solutions of (4) that remain in $\Omega' \subset \mathbb{R}^{n'+m'}$ are likewise defined to be mappings $\gamma' : I \to \Omega'$ having the corresponding properties with respect to g.

We now define the notion of conjugacy for control systems.

Definition 2.2. Let

$$\begin{array}{rcccl} \chi : & \Omega & \to & \Omega' & \\ & (x,u) & \mapsto & \chi(x,u) & = \; (\, \chi_{\mathrm{I}}(x,u)\, ,\, \chi_{\mathrm{II}}(x,u)\,) \end{array} \tag{6}$$

be a bijective mapping between two open subsets of $\mathbb{R}^{n+m}$ and $\mathbb{R}^{n'+m'}$ respectively. We say that χ *conjugates* $\gamma : I \to \Omega$ and $\gamma' : I \to \Omega'$ if and only if $\gamma' = \chi \circ \gamma$.

We say that χ *conjugates* systems (3) and (4) if, for any real interval I, a map $\gamma : I \to \Omega$ is a solution of (3) that remains in Ω if, and only if, $\chi \circ \gamma$ is a solution of (4) that remains in Ω'.

We say that systems (3) and (4) are *locally topologically conjugate at* $(0,0)$ if we can chose Ω and Ω' to be neighborhoods of the origin and χ a homeomorphism. We say that they are *locally smoothly conjugate* if, in addition, χ and χ^{-1} are smooth. Here the word smooth means C^∞.

In case there is no control, so that $m = m' = 0$ and we omit u and χ_{II}, Definition 2.2 coincides with the classical notion of local topological conjugacy for non controlled differential equations, and may serve as a definition in this case too. Let us write more formally the classical local results on ordinary differential equations that we recalled in the introduction :

Theorem 2.3 (Flow-box theorem). *If* $m = 0$ *and* $f(0) \neq 0$, *system (3) is locally smoothly conjugate at* 0 *to the linear system* $\dot{z}_1 = 1, \dot{z}_2 = \cdots = \dot{z}_n = 0$.

Theorem 2.4 (The Grobman-Hartman theorem). *If* $m = 0$, $f(0) = 0$, *and* $f'(0)$ *has no pure imaginary eigenvalue, system (3) is locally topologically conjugate at* 0 *to the linear system* $\dot{z} = f'(0)z$.

From now on, we consider "real" control system, i.e. we assume $m \ge 1$.

2.2. Some properties of conjugating maps. It turns out that conjugating homeomorphisms preserve the dimension of both the state and the control and must have a triangular structure :

PROPOSITION 2.5. *With the notations of Definition 2.2, suppose that (3) and (4) are topologically conjugate via a homeomorphism* $\chi : \Omega \to \Omega'$*. Then* $n = n'$*,* $m = m'$*, and* χ_{I} *depends only on* x*:*

$$\chi(x,u) \;=\; (\, \chi_{\mathrm{I}}(x)\, ,\, \chi_{\mathrm{II}}(x,u)\,)\ . \tag{7}$$

Moreover, $\chi_{\mathrm{I}} : \Omega_{\mathbb{R}^n} \to \Omega'_{\mathbb{R}^n}$ *is a homeomorphism.*

PROOF. Let $\bar{x}$, $\bar{u}$, $\bar{u}'$ be such that $(\bar{x},\bar{u})$ and $(\bar{x},\bar{u}')$ belong to Ω. Let further $x(t)$ be the solution to (3) with $x(0)=\bar{x}$ and $u(t)=\bar{u}$ for $t\leq 0$ and $u(t)=\bar{u}'$ for $t>0$. By conjugacy, $z(t)=\chi_{\mathrm{I}}(x(t),u(t))$ is a solution to (4) with v given by $v(t)=\chi_{\mathrm{II}}(x(t),u(t))$, for $t\in(-\epsilon,\epsilon)$ and some $\epsilon>0$. In particular $\chi_{\mathrm{I}}(x(t),u(t))$ is continuous in t so its values at 0^+ and 0^- are equal. Hence $\chi_{\mathrm{I}}(\bar{x},\bar{u})=\chi_{\mathrm{I}}(\bar{x},\bar{u}')$ so that $\chi_{\mathrm{I}} : \Omega_{\mathbb{R}^n} \to \Omega'_{\mathbb{R}^{n'}}$ is well defined and continuous. Similarly, $\left(\chi^{-1}\right)_{\mathrm{I}}$ induces a continuous inverse $\Omega'_{\mathbb{R}^{n'}} \to \Omega_{\mathbb{R}^n}$. □

In view of Proposition 2.5, we will only consider conjugacy between systems having the same number of states and inputs. Hence the distinction between (n,m) and (n',m') from now on disappears.

Taking into account the triangular structure of χ in Proposition 2.5, one may describe conjugation as the result of changing coordinates in the state-space (by setting $z=\chi_{\mathrm{I}}(x)$) and feeding the system with a function both of the state and of a new control variable v (by setting $u=(\chi^{-1})_{\mathrm{II}}(z,v)$), in such a way that the correspondence $(x,u)\mapsto(z,v)$ is invertible. In the language of control, this is known as a *static feedback transformation*, and two conjugate systems in the sense of Definition 2.2 would be termed *equivalent under static feedback*. This notion has received much attention, although only in the *differentiable* setting (i.e. when the triangular transformation χ is a diffeomorphism), see e.g. [**Bro78, Jak90**].

2.3. Linearization. Recall that f is assumed to be smooth (of class C^∞). Let us make a formal definition of topological and smooth linearizability.

DEFINITION 2.6. The system (3) is said to be *locally topologically linearizable at* $(\bar{x},\bar{u})\in\mathbb{R}^{n+m}$ if it is locally topologically conjugate, in the sense of Definition 2.2, to a linear controllable system $\dot{z}=Az+Bv$.

DEFINITION 2.7. The system (3) is said to be *locally smoothly linearizable at* $(\bar{x},\bar{u})\in\mathbb{R}^{n+m}$ if it is locally smoothly conjugate, in the sense of Definition 2.2, to a linear controllable system $\dot{z}=Az+Bv$.

Explicit necessary and sufficient conditions for a nonlinear system to be locally smoothly linearizable at a point were given in [**JR80, HSM83**], and also in [**vdS84**] (the previous two references dealt with control affine systems only), and is recalled in many nonlinear control textbooks. Without mentioning these conditions, let us simply say that they require a certain number of distributions to be involutive, and that this is a very non-generic property.

3. Main result on topological linearization

Let us now give a —basically negative— answer to the natural question raised at the end of section 1 : for control systems, removing the differentiability requirement on the conjugacy does *not* allow many more control systems to be (topologically) conjugate to a linear controllable system, contrary to the situation of ordinary differential equations (without control), (see section 1 and Theorems 2.3 and 2.4). Recall that f is assumed to be smooth (of class C^∞).

THEOREM 3.1 ([**BCP**]). *System (3) is locally topologically linearizable at* $(0,0)$ *if, and only if there exists an open neighborhood* $\widetilde{\Omega}$ *of* $(0,0)$ *in* $\mathbb{R}^{n+m}$ *and a homeomorphism*

$$\begin{array}{rccl} \widetilde{\chi}: & \widetilde{\Omega} & \to & \widetilde{\Omega}' \\ & (x,u) & \mapsto & \widetilde{\chi}(x,u) \;=\; (\,\widetilde{\chi}_{\mathrm{I}}(x)\,,\,\widetilde{\chi}_{\mathrm{II}}(x,u)\,) \end{array} \tag{8}$$

(possibly different from the homeomorphism defining topological linearizability of the system) such that

(1) $\widetilde{\chi}$ *conjugates system (3) to a linear controllable system* $\dot{z} = Az + Bv$*, in the sense of Definition 2.2,*
(2) $\widetilde{\chi}_{\mathrm{I}} : \widetilde{\Omega}_n \to \widetilde{\Omega}'_n$ *defines a smooth* (C^∞) *diffeomorphism.*

This does not state that topological linearizability implies smooth linearizability for $\widetilde{\chi}$ need not be a diffeomorphism even though $\widetilde{\chi}_{\mathrm{I}}$ is. In [**BCP**], the conclusion of the theorem is called *quasi smooth linearizability.* A thorough discussion as well as the proof of Theorem 3.1 is given there. Let us recall here what is necessary to make this theorem clearer.

PROPOSITION 3.2. *The conclusions of Theorem 3.1 imply that*

(1) $B\widetilde{\chi}_{\mathrm{II}} : \widetilde{\Omega} \to \mathbb{R}^m$ *is smooth,*
(2) *the rank of* B *is the maximum rank of* $\partial f/\partial u$ *in small neighborhoods of the origin.*

PROOF. Computing $\dot{z}$ at the origin of a trajectory starting from $(x,u) \in \Omega$ implies, by the smoothness of $\widetilde{\chi}_{\mathrm{I}}$,

$$\frac{\partial \widetilde{\chi}_{\mathrm{I}}}{\partial x}(x)\, f(x,u) \;=\; A\,\widetilde{\chi}_{\mathrm{I}}(x) \;+\; B\,\widetilde{\chi}_{\mathrm{II}}(x,u)\;. \tag{9}$$

This gives an obviously smooth expression of $B\widetilde{\chi}_{\mathrm{II}}$. The second point is proved using Corollary 3.5[1] at points close to the origin where the rank of $\partial f/\partial u$ is maximum, and hence locally constant. □

If B is left invertible (i.e. has rank m), the first point implies that $\widetilde{\chi}_{\mathrm{II}}$ itself is smooth, and we have the following immediate corollary :

COROLLARY 3.3. *If there are points arbitrarily close to* $(0,0)$ *where the rank of* $\partial f/\partial u$ *is* m *(i.e. where this linear map is injective), then* $\widetilde{\chi}$ *in Theorem 3.1 is a smooth mapping.*

[1] The proof of Corollary 3.5 does not use Proposition 3.2.

Of course if B has rank strictly less that m, $\widetilde{\chi}_{\mathrm{II}}$ need not be smooth. This is discussed in section 4.

Note that the assumption of Corollary 3.3 is very "reasonable": for instance for single input systems, the only case where it is not met is when f does not depend on u in a neighborhood of $(0,0)$, but then the system cannot be topologically conjugate to a *controllable* linear system :

COROLLARY 3.4. *If $m=1$, i.e. if (3) is a single input system, then $\widetilde{\chi}$ in Theorem 3.1 is a smooth mapping.*

This is however still not "smooth linearizability" because even though χ is smooth, its inverse might fail to be differentiable at the point of interest. The simplest example is the system

$$\dot{x} = u^3\,,\ x \in \mathbb{R},\ \ u \in \mathbb{R}\ \ , \tag{10}$$

clearly conjugate by $(z,v) = \chi(x,u) = (x,u^3)$ to the linear controllable system $\dot{z} = v$. Obviously, χ is smooth, χ^{-1} is continuous, χ_{I} is the identity smooth diffeomorphism, but the inverse of χ itself fails to be differentiable at the origin. In fact, no smooth diffeomorphism can conjugate these two systems. This can easily be proved but is also a consequence of the necessity part of the following result that tells us exactly when smooth linearizability is implied by topological linearizability :

COROLLARY 3.5. *When f is of class C^∞, system (3) is locally smoothly linearizable at $(0,0)$ if and only if it is locally topologically linearizable at $(0,0)$ and the rank of $\partial f/\partial u$ is constant around $(0,0)$.*

PROOF. Smooth linearizability is a particular case of topological linearizability, and it implies constant rank of $\partial f/\partial u$ because differentiability of the smooth diffeomorphism and its inverse allow one to get a formula for $\partial f/\partial u(x,u)$.

Let us prove the converse. Suppose that the rank of $\partial f/\partial u$ is $r \leq m$ in a neighborhood of $(0,0)$ and that system (3) is locally topologically linearizable at $(0,0)$. From Theorem 3.1, this implies that there exists a triangular homeomorphism $(x,u) \mapsto (z,v) = \widetilde{\chi}(x,u) = (\widetilde{\chi}_{\mathrm{I}}(x), \widetilde{\chi}_{\mathrm{II}}(x,u))$ that conjugates system (3) to a linear controllable system $\dot{z} = Az + Bv$ with the additional property that $\widetilde{\chi}_{\mathrm{I}}$ defines a *smooth diffeomorphism* from a neighborhood of $0 \in \mathbb{R}^n$ onto its image.

Let $r' \leq m$ be the rank of the matrix B. There are invertible $n \times n$ and $m \times m$ matrices P and Q such that

$$B_c \;=\; PBQ \;=\; \left(\begin{array}{c|c} I_{r'} & 0 \\ \hline 0 & 0 \end{array}\right) \tag{11}$$

where $I_{r'}$ is the $r' \times r'$ identity matrix.

Computing $\dot{z}$ at the origin of a trajectory starting from $(x,u) \in \Omega$ implies (9) by the smoothness of $\widetilde{\chi}_{\mathrm{I}}$. Hence the map $B_c Q^{-1} \widetilde{\chi}_{\mathrm{II}}$ is smooth where it is defined, and differentiating

(9) with respect to u yields :

$$P\frac{\partial\widetilde{\chi}_{\mathrm{I}}}{\partial x}(x)\,\frac{\partial f}{\partial u}(x,u) \;=\; \frac{\partial(B_cQ^{-1}\widetilde{\chi}_{\mathrm{II}})}{\partial u}(x,u)\ .$$

Since $P\frac{\partial\widetilde{\chi}_{\mathrm{I}}}{\partial x}(x)$ is invertible and the rank of $\partial f/\partial u$ is r, both sides have constant rank r. This implies $r \leq r'$. This also implies that the mapping $\mathbb{R}^{n+m} \to \mathbb{R}^{n+r'}$ defined by

$$(x,u) \;\mapsto\; (\,x\,,\,B_cQ^{-1}\widetilde{\chi}_{\mathrm{II}}(x,u)\,)\ ,$$

has a constant rank $n+r$ in a neighborhood of the origin, hence, by the constant rank theorem applied to this mapping, there is a $r\times m$ matrix K of rank r (selects r lines that are independent among the m lines of $\frac{\partial(B_cQ^{-1}\widetilde{\chi}_{\mathrm{II}})}{\partial u}(x,u)$), a neighborhood Ω of $(0,0)$ in $\mathbb{R}^{n+m}$, two smooth mappings $\alpha:\mathbb{R}^{n+r}\to\mathbb{R}^{n+m}$ and $\beta:\mathbb{R}^{n+m}\to\mathbb{R}^{r'-r}$ (in fact they only need to be defined in suitable neighborhoods of the origin) such that $\widetilde{\chi}$ is defined on Ω, and

$$B_cQ^{-1}\widetilde{\chi}_{\mathrm{II}}(x,u) \;=\; \begin{pmatrix} \alpha(\,x\,,\,KB_cQ^{-1}\widetilde{\chi}_{\mathrm{II}}(x,u)\,) \\ 0 \end{pmatrix} \tag{12}$$

for all $(x,u)\in\Omega$ and

$$(x,u) \;\mapsto\; (\,x\,,KB_cQ^{-1}\widetilde{\chi}_{\mathrm{II}}(x,u)\,,\,\beta(x,u)\,)\ , \tag{13}$$

defines a smooth diffeomorphism from Ω onto its image. This implies that $r=r'$ because from (12, $r<r'$ would prevent $\widetilde{\chi}$ from being one-to-one. Hence K can be taken the identity matrix. Define $\widetilde{\widetilde{\chi}}:\Omega\to\mathbb{R}^{n+m}$ by $\widetilde{\widetilde{\chi}}=L\circ\psi$ with

$$\psi(x,u) \;=\; (\,P\widetilde{\chi}_{\mathrm{I}}(x)\,,\,KB_cQ^{-1}\widetilde{\chi}_{\mathrm{II}}(x,u)\,,\,\beta(x,u)\,)$$

and $L(z,v) \;=\; (P^{-1}z, Q^{-1}v)$. ψ is a smooth diffeomorphism because (13) is one, and L is obviously a (linear) smooth diffeomorphism. Setting $(\tilde{z},\tilde{v})=\widetilde{\widetilde{\chi}}(x,u)$ conjugates system (3) to $\dot{\tilde{z}}=A\tilde{z}+B\tilde{v}$. □

4. An open question

It is a reasonable question to ask whether the conclusion of Corollary 3.3 holds in general, namely whether Theorem 3.1 can be strengthened so as to state that $\widetilde{\chi}$ is, on top of its other properties, a smooth mapping (when the rank of $\partial f/\partial u$ is not locally constant, $\widetilde{\chi}$ would fail to be be differentiable, from the necessity part of Corollary 3.5).

Let us examine the case where the assumptions of Corollaries 3.3, 3.4 and 3.5 fail (these three corollaries already state the desired conclusion), namely the case of systems with $m\leq 2$ controls where the rank of $\partial f/\partial u$ is everywhere strictly smaller than m, studied locally around a point where this rank is not constant (i.e. the rank at the point is strictly less than the maximum rank in arbitrary small neighborhoods of this point, itself strictly smaller than m.

The smallest dimensions where this occurs is $n=1$, $m=2$, i.e. systems $\dot{x}=f(x,u_1,u_2)$ with x, u_1 and u_2 scalar. In order to state our open question in the smallest dimension possible, let us drop the dependence on the right-hand side on x and consider systems

$$\dot{x} = a(u_1, u_2)\,,\ x \in \mathbb{R},\ u = (u_1, u_2) \in \mathbb{R}^2 \quad , \tag{14}$$

where $a : \mathbb{R}^2 \to \mathbb{R}$ is smooth. Let us assume that this system is locally topologically linearizable around $(x, u) = (0, 0, 0)$. The only canonical controllable linear system with one state $z \in \mathbb{R}$ and two controls $(v_1, v_2) \in \mathbb{R}^2$ is $\dot{z} = v_1$, hence local topological linearizability means existence of a homeomorphism

$$\chi : (x, u_1, u_2) \mapsto (z, v_1, v_2) = (\chi_1(x)\,, \chi_2(x, u_1, u_2)\,, \chi_3(x, u_1, u_2)) \tag{15}$$

(in the terms of Definition 2.6, χ_{I} is χ_1 and χ_{II} is (χ_2, χ_3)) that conjugates (14) to the linear system $\dot{z} = v_1$. From Theorem 3.1, this implies existence of another homeomorphism $\widetilde{\chi}$ of the same triangular form, that we denote by χ instead of $\widetilde{\chi}$, such that χ_1 is a smooth diffeomorphism (from a real interval containing zero onto an open interval) and χ_2 is a smooth mapping from an open neighborhood of the origin in $\mathbb{R}^3$ to $\mathbb{R}$, while our results do not grant that χ_3 has any more regularity than continuity. In fact the conjugation reads

$$\frac{\partial \chi_1}{\partial x}(x) a(u_1, u_2) = \chi_2(x, u_1, u_2). \tag{16}$$

This implies in particular that χ_2 does not depend on x, and then one can replace $v_2 = \chi_3(x, u_1, u_2)$ with $v_2 = \chi_3(0, u_1, u_2)$ without changing the conjugating property. Composing χ given by (15) with $(z, v_1, v_2) \mapsto (\chi_1^{-1}(z), \frac{\partial \chi_1}{\partial x}(\chi_1^{-1}(z))^{-1} v_1, v_2)$, one finally gets a conjugating homeomorphism of the form

$$(x, u_1, u_2) \mapsto (x\,, a(u_1, u_2)\,, \beta(u_1, u_2)) \tag{17}$$

where $\beta(u_1, u_2) = \chi_3(0, u_1, u_2)$. Hence local topological linearizability amounts to existence of a continuous mapping β from an open neighborhood of the origin in $\mathbb{R}^2$ to $\mathbb{R}$ such that $(u_1, u_2) \mapsto (a(u_1, u_2), \beta(u_1, u_2))$ defines a homeomorphism from a neighborhood of the origin in $\mathbb{R}^2$ onto its image (we just proved it is necessary, but conversely, it makes (17) a local homeomorphism, that obviously conjugates (14) to $\dot{z} = v_1$). Similarly, conjugacy via a homeomorphism that is a smooth map amounts to existence of a *smooth* mapping having the same property. Hence the question whether $\widetilde{\chi}$ can be taken a smooth mapping in Theorem 3.1 reduces to the following

Open Question 4.1. *Let a and β be two mappings $]-\varepsilon, \varepsilon[^2 \to \mathbb{R}$, $\varepsilon > 0$, such that a is* smooth, *β is* continuous, *and $(u_1, u_2) \mapsto (a(u_1, u_2), \beta(u_1, u_2))$ defines a homeomorphism from $]-\varepsilon, \varepsilon[^2$ onto its image. Does there exist a* smooth *mapping $b :]-\varepsilon', \varepsilon'[^2 \to \mathbb{R}$, $0 < \varepsilon' < \varepsilon$, such that $(u_1, u_2) \mapsto (a(u_1, u_2), b(u_1, u_2))$ defines a homeomorphism from $]-\varepsilon', \varepsilon'[^2$ onto its image ?*

This question in differential topology can be posed in higher dimension of course, see below. It is of interest in its own right and seems to have no answer so far, even for $p = q = 1$.

Open Question 4.2. *Let O be a neighborhood of the origin in $\mathbb{R}^{p+q}$ and $F : O \to \mathbb{R}^p$ a smooth map. Suppose there is a* continuous *map $G : O \to \mathbb{R}^q$ such that $F \times G : O \to \mathbb{R}^p \times \mathbb{R}^q$ is a local homeomorphism at* 0.

Does there exist another neighborhood of the origin $O' \subset O$ and a smooth *mapping $H : O' \to \mathbb{R}^q$ such that $F \times H : O' \to \mathbb{R}^p \times \mathbb{R}^q$ is again a local homeomorphism at* 0 *?*

5. Implications in Control Theory

Let us come back to the discussion we started in the Introduction. Consider a control system (1), assume for simplicity that we work around an equilibrium, i.e. $f(0,0) = 0$, and let us write its linear approximation, i.e.

$$f(x,u) \;=\; Ax \;+\; Bu \;+\; F(x,u) \tag{18}$$

$$\text{with} \quad F(0,0) = \frac{\partial F}{\partial x}(0,0) = \frac{\partial F}{\partial u}(0,0) = 0 \;, \tag{19}$$

so that the nonlinear system (1) reads

$$\dot{x} \;=\; Ax \;+\; Bu \;+\; F(x,u) \;. \tag{20}$$

From the remarks made in the introduction, the relevant situation is the one where the linear system

$$\dot{z} \;=\; Az \;+\; Bv \tag{21}$$

is controllable. Let us assume slightly more to rule out the pathologies described in the previous section. The additional assumption is very mild and is, for instance, always true when the constant $n \times m$ matrix B has rank m; it is implied by linear controllability for single input systems.

Assumption 5.1. The pair (A,B) is controllable and the rank of $\dfrac{\partial F}{\partial u}(x,u)$ is equal to the rank of B for small (x,u).

The question raised in the introduction was the one of finding a reasonable equivalence relation that would make the two systems (20) and (21) locally equivalent. Comparing the situation of ordinary differential equations (without control), a candidate was local topological conjugacy as in Definitions 2.2 and 2.6, and if that candidate was successful, we would have a result making precise the vague statement (2).

Corollary 3.5 implies that, for A, B and F satisfying Assumption 5.1, systems (20) and (21) are locally topologically conjugate if and only if they are locally smoothly conjugate, and it is known from [**JR80, HSM83**] that this is false for a generic F, even satisfying (19). This discards topological conjugacy as a candidate for the above mentioned equivalence relation, but this does *not* contradict the basic belief behind statement (2).

A way to contradict that statement would be to find at least one example satisfying the assumption, but where the nonlinear system (20) displays some local "qualitative" phenomenon that do not occur for the linear system (21). In the qualitative theory of dynamical systems (without control), the phase portrait gives a picture of the behavior, on which phenomena like attractors, invariant set, (stable) closed orbits can just be "seen". A control system is more complex : it describes how the behavior of the state (at least in the state space representation) is linked to the control. It is not very clear what a qualitative phenomenon should be for a control system. The least to require is that it be invariant by topological conjugacy as defined here. In the introduction, we pointed out that (non-)controllability *is* a qualitative property, but it is of no help here since (2) only refers to controllable systems.

We do believe that clarifying the status of a statement like (2) is very relevant to control theory and modeling. Our negative results (section 3) say that topological conjugacy is not the right tool to answer this. A looser equivalence could be a way to state (2) properly. It could also be that the intuition behind (2) is totally wrong and that some nonlinearities F allow system (20) to display some qualitative phenomena locally that cannot occur on a linear system (21).

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