7. Design of control Lyapunov functions for "Jurdjevic-Quinn" systems

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Summary.

This paper presents briefly a method to design explicit control Lyapunov functions for control systems that satisfy the so-called "Jurdjevic-Quinn conditions", i.e. posses an "energy-like" function that is naturally non-increasing for the un-forced system. The results with proof will appear in a future paper. The present note rather focuses on the method, and on its application to the model of a mechanical system, the translational oscillator with rotation actuator (TORA) (also known as RTAC).

7.1 Introduction

For differentiable dynamical systems (without control), a Lyapunov function is a convenient tool to analyze the asymptotic stability of an equilibrium. See [12] for instance. It is of course not the only one, and its main drawback is that there is no systematic way to find a Lyapunov function in general, even though converse Lyapunov theorems (see [12, 15]) tell us that existence of a Lyapunov function is equivalent to asymptotic stability.

For control systems and the stabilization problem, Lyapunov functions have also been used extensively. For linear systems, that have the nice property that Lyapunov functions may always be taken quadratic, optimizing a quadratic criteria or assigning a quadratic Lyapunov function are more or less synonymous (see for instance [4]). For nonlinear systems, the so-called "Lyapunov design" (see [2, 6]) consists in designing a Lyapunov function *together with* a control that makes it decrease, i.e. that assigns this function to be a Lyapunov function for the closed-loop system. Artstein's theorem [1] makes this method theoretically consistent by characterizing the functions that may be assigned to be Lyapunov functions via a suitable continuous feedback control. Let us recall it briefly : consider a control system

$$\dot{x} = f_0(x) + \sum_{k=1}^{m} u_k f_k(x) \tag{7.1}$$

with state $x \in I\!R^n$ and control $(u_1, \ldots, u_m) = u \in I\!R^m$, the f_k's being smooth vector fields in $I\!R^n$.

Theorem 1 (Artstein's theorem) *A differentiable function V that is positive definite (zero at the origin, positive elsewhere) and infinite at infinity can be assigned to be a Lyapunov function for the closed-loop system via a continuous feedback in (7.1) if and only if*

1. *it is a* control Lyapunov function *(CLF) :*

$$\forall x \in I\!R^n \backslash \{0\}, \quad \left.\begin{array}{c} L_{f_1} V(x) = 0 \\ \vdots \\ L_{f_m} V(x) = 0 \end{array}\right\} \Longrightarrow L_{f_0} V(x) < 0 \ , \tag{7.2}$$

2. *it satisfies the so called* small control property *(SCP) : for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, for $x \in I\!R^n$,*

$$\left.\begin{array}{r} x \neq 0 \\ \|x\| < \delta \end{array}\right\} \exists u \ \left\{\begin{array}{l} \|u\| < \varepsilon \\ L_{f_0 + \sum_{k=1}^{m} u_k f_k} V(x) < 0 \ . \end{array}\right. \tag{7.3}$$

This is only an existence result, but Sontag gave in [20] an explicit formula that gives a systematic way to obtain a stabilizing control corresponding to a given CLF. On practical examples, there are of course many other possibilities.

Other methods than building a control Lyapunov function exist to design stabilizing control laws. However, it is well know (see for instance [18, 19, 6]) that a control Lyapunov function, when available, is a very convenient tool to analyze stability, and its robustness to perturbations, or even to modify the design to enhance robustness or performances. For these reasons, it is interesting to obtain control Lyapunov functions for systems that may be stabilized by other methods. This is the topic of the present paper, at least for the situation where stabilization has been obtained using the "Jurdjevic-Quinn" method, also called "damping control".

Let us call (7.2)-(7.3) Artstein's equation, and draw a parallel with optimal control and Hamilton-Jacobi-Bellman's equation . Optimal control (for instance minimum time) is a quantitative problem, whose solution is unique, and often very difficult to describe. Asymptotic stabilization, a qualitative problem, whose solution is highly non-unique, the requirement being somehow weaker. Artstein's equations play, for stabilization, the role of Hamilton-Jacobi-Bellman's equation for optimal control : the dynamic programming principle asserts that a solution to the HJB's equation yields (at least when it is differentiable !) the optimal synthesis, and in the same way, Artstein's theorem allows one to derive a stabilizing control law from a solution to Artstein's equations, with some universal formulas available. However it is known that the solutions to HJB's equation are in general non-smooth, and the analysis of this equation has given rise to a lot of mathematical developments to handle this problem. Artstein's equations are much less constrained, and for this reason, they have smooth solutions, far from being unique : for instance one may freely change V outside the origin and the points where its derivative along the control vector fields is zero, at least if the changes are not big enough (in C^1 topology) for the derivative of the deformed function along the control vector fields to vanish. We do not give a bibliography on HJB's equation and dynamic programming, but the reader may find in [11, 9, 18, 19] some investigations on the link between optimality, control Lyapunov functions and robustness. We shall not elaborate more along these line, but we do think that Artstein's equations deserve a deeper analysis. This paper is a contribution to this analysis in a particular case.

This paper is organized as follows. In section 7.2, we present the type of systems we are interested in, and review very briefly the popular method to stabilize them (without obtaining a CLF). The method is outlined in section 7.3 and gives some general results. We then apply, in section 7.4 the method to a mechanical system used as a benchmark in [19], see also [3]. Note that this model does not satisfy the assumptions theoretically needed in section 7.3, but the method still yields a global CLF.

7.2 Jurdjevic-Quinn systems

It was noticed in [13], that a first integral for the drift vector field, plus some controllability conditions allow one to derive smooth asymptotically stabilizing control laws. This method has been generalized in [10, 2, 17] for instance. This is now popular, under the name "Jurdjevic-Quinn method" or "damping control".

Let us be more specific. Consider the affine control system (7.1). The following conditions are a bit more restrictive than these in [17], more general that these in other papers, but the idea is basically the same. To state them, let us define, for a positive and radially unbounded function V_0, the set $\mathcal{W}_L(V_0)$:

$$\mathcal{W}_L(V_0) = \left\{ \begin{array}{c} x \in I\!R^n,\ L_{f_0}V_0(x) = L_{ad^i_{f_0}(f_k)}V_0(x) = 0 \\ k = 1 \ldots m;\ i = 1 \ldots L \end{array} \right\} \tag{7.4}$$

Assumption A ((weak) Jurdjevic-Quinn conditions)

1. *The vector field f_0 has only one equilibrium :*

$$f_0(x) = 0 \iff x = 0 \ . \tag{7.5}$$

2. *A function $V_0 : I\!R^n \to I\!R$ is known and has the following three properties :*
   a) *it is positive definite and radially unbounded,*
   b) *it satisfies :*

$$\forall x \in I\!R^n,\ L_{f_0} V_0(x) \ \leq \ 0 \ , \tag{7.6}$$

   c) *there is an integer L such that*

$$\mathcal{W}_L(V_0) \ = \ \{0\} \ . \tag{7.7}$$

Proposition 1 *Under assumption A, the smooth control law $u_k = -Lf_kV_0$ asymptotically stabilizes the origin for system (7.1).*

This proposition is also correct replacing assumption A by the following one, that obviously implies A, and corresponds to the most common situation :

Assumption A$'$ *Same as assumption A, but instead of (7.6), V_0 is assumed to be a first integral of the vector field f_0 :*

$$\forall x \in I\!R^n,\ L_{f_0} V_0(x) \ = \ 0 \ . \tag{7.8}$$

Let us make our discussion in the simpler case of assumption A$'$. The proof is based on the fact that that these u_k yield a non-positive $\dot{V}_0$:

$$\dot{V}_0 \ \leq \ - \sum_{k=1}^{m} (L_{f_k} V_0)^2 \ .$$

This is enough to prove asymptotic stability through LaSalle's invariance principle (see [16, 12]). However, under assumption A$'$, $\dot{V}_0$ is zero exactly at the points where all the functions Lf_kV_0 vanish, and no control can do better at these points because u has no effect on $\dot{V}_0$. This V_0 is therefore *not* a CLF for system (7.1) : in Artstein's equation (7.2), the strict inequality is replaced by an equality. Although this "weak" Lyapunov function is as good as a "strict" one for the purpose of proving asymptotic stability, the fact that V_0 is not strictly decreasing along the solutions, does not give a "margin" that can be exploited for robustness analysis or enhancement.

7.3 Reshaping Lyapunov functions

Let us present a way to construct a CLF using the known function V_0 described above.

7.3.1 The idea

It is tempting to consider $\dot{V}_0$ as "almost" a CLF, and to design a "deformation" V_λ, with λ a real parameter, such that
- for all λ, V_λ is positive definite and radially unbounded,
- for $\lambda = 0$, V_λ is the above V_0, and
- for positive value of λ, V_λ is a CLF, i.e. instead of $= 0$, one gets the strict inequation (7.2), ideally more and more negative as λ grows.

This idea was already presented in [7]. Let us sum up the obtained results, announced in [7] in a more restrictive form, and presented into details in the forthcoming [8].

The type of deformation considered is the very natural

$$V_\lambda = V_0 \circ \phi_\lambda^G \ , \tag{7.9}$$

where G is a complete vector field, and ϕ_λ^G stands for the flow at time λ of G, i.e. :

$$\frac{\partial}{\partial \lambda}(\phi_\lambda^G(x)) \ = \ G(\phi_\lambda^G(x)) \ , \quad \phi_0^G(x) \ = \ x \ . \tag{7.10}$$

Since ϕ_λ^G is a diffeomorphism for all λ, V_λ is positive definite and radially unbounded for all λ. Also, the condition $G(0) = 0$ is sufficient to ensure $V_\lambda(0) = 0$.

Of course, V_λ being a CLF for $\lambda > 0$ requires further conditions on G. The least is to require that $Lf_0 V_\lambda(x)$ be a decreasing function of λ, for small values of λ and at points where $Lf_k V_0(x) = 0$. It turns out that the following formula holds, for all x under assumption A$'$, and at points x where $L_{f_0} V_0$ vanishes under assumption A :

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \left(L_{f_0} V_\lambda^G\right)(x) = L_{f_0} L_G V_0(x) \tag{7.11}$$

(this is established from standard differential calculus). With in mind the concern of finding explicit formulae, using the flow of a vector field is not the best choice. In the (usual) case where ϕ_λ^G cannot be explicitly computed, we may use instead

$$W_\lambda = V_0 \circ (Id + \lambda G) \ , \tag{7.12}$$

or any W_λ that approaches V_λ up to order 1 at $\lambda = 0$ (in (7.12), this is obtained by using the first order approximation of the flow). Note that (7.9) is an intrinsic definition of V_λ whereas the above (7.12) is coordinate dependent : for instance, in coordinates where G would be a "constant" vector field (impossible near an equilibrium !), the formula (7.12) would be equivalent to (7.12).

7.3.2 The results

The following results can be found in [7] in the particular case when $L_{f_0}V_0$ is zero (assumption A′). They will appear, with proofs, in a future paper [8]. For the definitions of homogeneity and dilations, and related properties, see for instance [14]. We do not insist here on this property.

Theorem 2 *Suppose that the control affine system (7.1) satisfies the assumption A and furthermore, that the vector fields $f_0, \ldots, f_m$ are* homogeneous *with respect to a certain dilation, f_0 having a degree strictly larger than the others, and that the function V_0 is also homogeneous with respect to the same dilation. If G is a homogeneous vector field of degree* 0 *which satisfies, for all x in $\mathbb{R}^n\backslash\{0\}$,*

$$\left.\begin{array}{r} L_{f_0}V_0(x) = 0 \\ L_{f_1}V_0(x) = 0 \\ \vdots \\ L_{f_m}V_0(x) = 0 \end{array}\right\} \Longrightarrow L_{f_0}L_G V_0(x) < 0 \tag{7.13}$$

Then there exists a positive real number λ_0 such that for all λ that satisfies $0 < \lambda < \lambda_0$, V_λ and W_λ are homogeneous CLFs of the same degree as V_0, satisfying the small control property. This would also be true for any formula for W_λ instead of (7.12), provided that the derivative with respect to λ at $\lambda = 0$ is the same, and W_λ is homogeneous.

This result is completed by the following, that gives a "universal" formula for a vector field G, homogeneous of degree 0, that satisfies condition (7.13).

Theorem 3 *Suppose that the control affine system (7.1) satisfies the assumption A and that the vector fields $f_0, \ldots, f_m$ and the function V_0 satisfy the homogeneity assumptions of theorem 2. Define the vector field G by :*

$$G = \sum_{i=0}^{L-1}\sum_{k=1}^{m} \lambda_{i,k}\, ad_{f_0}^i f_k \tag{7.14}$$

with $\lambda_{i,k}$ $(i = 0, \ldots, L-1; k = 1, \ldots, m)$ some functions defined by

$$\begin{cases} \lambda_{i,k} = \displaystyle\sum_{j=i}^{L-1} (-1)^{j-i+1} \frac{L_{ad_{f_0}^{(2j-i+1)}(f_k)} V_0}{(2V_0)^{\alpha_{j,k}}} \\ \alpha_{j,k} = \frac{(2j+1)c_0+2c_k+d}{2} \ . \end{cases} \tag{7.15}$$

This vector field G is homogeneous of degree zero and satisfies the conditions of theorem 2.

7.3.3 Some remarks on using this method to obtain CLFs

To sum up, the proposed method for designing a strict Lyapunov function consists in :
- finding a vector field G that satisfies the condition (7.13),
- computing V_λ according to (7.9) or W_λ according to the explicit formula (7.12), or another one that gives the same $\partial W_\lambda / \partial \lambda$ at $\lambda = 0$.

This yields a function that, if the homogeneity assumptions are satisfied, is guaranteed to be a (global) control Lyapunov function for λ positive and small enough.

On the choice of G : Theorem 3 gives a "universal" formula to obtain a good G, but there are many other choices, and it is important to have some choice (see next remarks). It is usually a good idea, following (7.14), to write G as a linear combination of the vector fields $L^j_{f_0} f_k$, and to write the (linear differential) relations on the coefficients implied by (7.13). A thorough discussion will appear in [8]. This is illustrated on an example in section 7.4.

How large can λ be ? The theorems only say that V_λ or W_λ will be a CLF for *small enough* positive values of λ. It is of course important that λ can can be "reasonably large", since if it is not the case, $\dot{V}_\lambda$ or $\dot{W}_\lambda$ will he negative but very small at the points where the control has no effect on them. One needs to compute on each example to see how big λ can be chosen, and how good a Lyapunov function one obtains, but that usually depends on the choice of G (hence it is important to have some choice) and even on the formula one chooses for $\dot{W}_\lambda$.

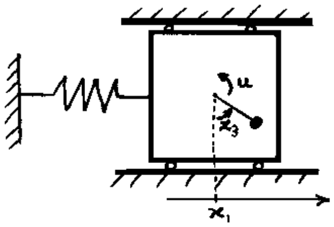
If the homogeneity conditions are not met, this method still yields some positive definite functions V_λ or W_λ. The theorems however do not apply, and on some examples, these functions may fail to be CLFs (problems around the origin, and at infinity), or fail to satisfy the small control property (automatically satisfied in the homogeneous case). However, the experience shows that they often work as CLFs although the theorems do not apply. Again, having some choice on G is crucial here. The following example is an illustration.

7.4 Example: The TORA system

Using our ideas to find some control Lyapunov function for this system was suggested by Rodolphe Sepulchre, from Université de Liège.

Let us consider the mechanical system called TORA (Translational Oscillator with Rotating Actuator) in [19] and RTAC (Rotational/Translational proof-mass Actuator) in [3]. It consists of a platform connected to a fixed frame of reference by a linear spring. The platform can oscillate without friction in the horizontal plane. On the platform, an eccentric rotating mass is actuated by a DC motor. The control of this rotating motion is used to dampen the translational oscillations of the platform. A precise description is given in [5]. After normalization (the time is also normalized), the dimensionless variables (x_1, x_2, x_3, x_4) may be used, where x_1 and x_2 are proportional to the translational displacement (from the equilibrium position) and the velocity of the platform, x_3 and x_4 are the angle (from a direction perpendicular to the spring) and the angular velocity of the eccentric mass. The following equations are obtained, where ε is a parameter, depending on the mass of the cart, the mass, length and inertia of the eccentric mass, and the stiffness of the string. A typical value for ε is 0.1. Note that the picture below is in a horizontal plane, so that gravity is ignored.

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-x_1 + \varepsilon {x_4}^2 \sin x_3}{1 - \varepsilon^2 \cos^2 x_3} + \frac{-\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{\varepsilon \cos x_3 (x_1 - \varepsilon {x_4}^2 \sin x_3) + u}{1 - \varepsilon^2 \cos^2 x_3}
\end{aligned}
$$



In [19], this system is extensively used as an illustrative example, and the papers in [3] also propose various control methods. The following is an illustration of our methods on this example to derive a (global) CLF. We do not propose new or "better" control laws.

Note that the coordinates of the center of mass are $(x_1 + \varepsilon \sin x_3, -\varepsilon \cos x_3)$. It is natural to use also the variables z_1 and z_2, that are the horizontal component of the position and velocity of the center of mass :

$$z_1 \;=\; x_1 + \varepsilon \sin x_3 \ , \quad z_2 \;=\; x_2 + \varepsilon x_4 \cos x_3 \ .$$

As noticed in [19] or some articles in [3], setting $z_3 = x_3$ and $z_4 = x_4$, the state equations take the following simpler form in the coordinates (z_1, z_2, z_3, z_4) :

$$\begin{aligned}\dot z_1 &= z_2\\ \dot z_2 &= -z_1 + \varepsilon \sin z_3\\ \dot z_3 &= z_4\\ \dot z_4 &= \frac{1}{1-\varepsilon^2\cos^2 z_3}[\varepsilon \cos z_3(z_1 - \varepsilon(1+{z_4}^2)\sin z_3) + u]\ .\end{aligned} \tag{7.16}$$

It is also convenient in some occurrences to keep x_1 :

$$\begin{aligned}\dot x_1 &= z_2 - \varepsilon z_4 \cos z_3\\ \dot z_2 &= -x_1\\ \dot z_3 &= z_4\\ \dot z_4 &= \frac{1}{1-\varepsilon^2\cos^2 z_3}[\varepsilon \cos z_3(x_1 - \varepsilon {z_4}^2\sin z_3) + u]\end{aligned} \tag{7.17}$$

The potential energy of the spring is (in dimensionless units) $\frac{1}{2}x_1^2$ and the kinetic energy is $\frac{1}{2}z_2^2 + \frac{1}{2}z_4^2(1+\mu\sin^2 z_3)$ with μ some coefficient. The total is naturally constant along the solutions for $u=0$. However, it is not proper (infinite at infinity), and it may be zero for any value of z_3. It is therefore reasonable to "strengthen" it by adding a term that is proper with respect to z_3, like z_3^2. In fact, following [19], we modify the energy and use the following definite positive and radially unbounded function V_0 :

$$V_0(z) = \frac{1}{2}x_1^2 + \frac{1}{2}{z_2}^2 + \frac{k_1}{2}{z_3}^2 + \frac{1}{2}{z_4}^2 \tag{7.18}$$

$$= \frac{1}{2}\left(z_1 - \varepsilon\sin z_3\right)^2 + \frac{1}{2}z_2^2 + \frac{k_1}{2}{z_3}^2 + \frac{1}{2}{z_4}^2 \tag{7.19}$$

with k_1 some positive constant. Using the following preliminary feedback

$$\begin{aligned}u = &-\varepsilon\cos z_3(x_1 - \varepsilon {z_4}^2 \sin z_3)\\ &+(1-\varepsilon^2\cos^2 z_3)(\varepsilon x_1 \cos z_3 - k_1 z_3 + v)\ ,\end{aligned} \tag{7.20}$$

we obtain a control system

$$\dot x \ = \ f_0(x) \ + \ v\, f_1(x)$$

where f_0 and f_1 are, in the coordinates (x_1, z_2, z_3, z_4),

$$f_0 = \begin{bmatrix} z_2 - \varepsilon z_4\cos z_3\\ -x_1\\ z_4\\ \varepsilon x_1\cos z_3 - k_1 z_3\end{bmatrix} \qquad f_1 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \tag{7.21}$$

A simple computation yields to the following equations :

$$\begin{aligned}L_{f_0}V_0(z) &= 0\\ L_{f_1}V_0(z) &= z_4\\ L_{f_0}L_{f_1}V_0(z) &= \varepsilon\cos z_3(z_1 - \varepsilon\sin z_3) - k_1 z_3\\ L_{f_0}^2 L_{f_1}V_0(z) &= \varepsilon z_2\cos z_3 + z_4(-\varepsilon z_1 \sin z_3 - \varepsilon^2\cos 2z_3 - k_1)\\ L_{f_0}^3 L_{f_1}V_0(z) &= \varepsilon\cos z_3(-z_1 + \varepsilon\sin z_3) + z_4 h(z_1,z_2,z_3,z_4)\end{aligned} \tag{7.22}$$

Hence this system satisfies the Jurdjevic-Quinn conditions (assumption A, and even A′). Precisely, (7.22) implies that $\mathcal{W}_3(V_0) = \{0\}$.

In [19] the authors are able to design a continuous feedback law which makes the origin of the system asymptotically stable by applying some "damping control" (see Proposition 1 above), and them modifying these control laws to get more robustness, but asymptotic stability is obtained via a weak Lyapunov function. Unfortunately this system does not meet the homogeneous assumptions of our theorems. Yet in this particular case we will apply successfully our method to obtain a control Lyapunov function.

The first step consists in finding vector field G that meets the condition (7.13). Let us illustrate the second remark in section 7.3.3, and try to find "all" the good G's instead of using formula (7.14)-(7.15). It is not difficult to see that the second and first component of G in the coordinates (z_1, z_2, z_3, z_4) should vanish when $\cos z_3$ vanishes, so let us take G of the form :

$$G = \begin{pmatrix} \varepsilon \cos z_3 g_1 \\ \varepsilon \cos z_3 g_2 \\ g_3 \\ g_4 \end{pmatrix} = \varepsilon \cos z_3 \left(g_1 \frac{\partial}{\partial z_1} + g_2 \frac{\partial}{\partial z_2} \right) + g_3 \frac{\partial}{\partial z_3} + g_4 \frac{\partial}{\partial z_4}$$

with g_1, g_2, g_3, and g_4 some functions. We have

$$[f_0, G] = \begin{pmatrix} \varepsilon \cos z_3 \, (f_0 g_1 - g_2) - \varepsilon z_4 \sin z_3 \, g_1 \\ \varepsilon \cos z_3 \, (f_0 g_2 + g_1 - g_3) - \varepsilon z_4 \sin z_3 \, g_2 \\ f_0 g_3 - g_4 \\ f_0 g_4 - \varepsilon \cos z_3 g_1 + \left(k_1 + \varepsilon z_1 \sin z_3 + \varepsilon^2 \cos(2 z_3)\right) g_3 \end{pmatrix}$$

and hence, since $L_{f_0} L_G V_0 = L_{[f_0, G]} V_0$,

$$\begin{aligned} L_{f_0} L_G V_0 = {} & (z_1 - \varepsilon \sin z_3)\,(\varepsilon \cos z_3 \, (f_0 g_1 - g_2) - \varepsilon z_4 \sin z_3 \, g_1) \\ & + z_2 \,(\varepsilon \cos z_3 \,(f_0 g_2 + g_1 - g_3) - \varepsilon z_4 \sin z_3 \, g_2) \\ & + (k_1 z_3 - \varepsilon \cos z_3 (z_1 - \varepsilon \sin z_3))\,(f_0 g_3 - g_4) \\ & + z_4 \,(\cdots \cdots) \end{aligned}$$

Finally, we have the following equality at points where $z_4 = 0$:

$$L_{f_0} L_G V_0 = -\,\varepsilon \cos z_3 \,(x_1 \,\alpha + z_2 \,\beta\,) - (k_1 z_3 - \varepsilon x_1 \cos z_3)\,\gamma \tag{7.23}$$

with α, β and γ defined by

$$g_2 - L_{f_0} g_1 = \alpha \tag{7.24}$$

$$g_3 - L_{f_0} g_2 - g_1 = \beta \tag{7.25}$$

$$g_4 - L_{f_0} g_3 = \gamma \; . \tag{7.26}$$

Since $L_{f_0} V_0$ is identically zero and $L_{f_1} V_0$ is equal to z_4, equation (7.13) requires that $L_{f_0} L_G V_0$ be negative when $z_4 = 0$ and $z \neq 0$. All the solutions may be constructed in the following way : first find functions α, β and γ such that, for $z_4 = 0$, $\varepsilon \cos z_3 \,(x_1 \alpha + z_2 \beta) + (k_1 z_3 - \varepsilon x_1 \cos z_3)\gamma$ is positive, and then, solve (7.24)-(7.26) for the functions g_i (in fact, g_1 may be chosen arbitrary and defines g_2, g_3, g_4).

It is convenient to try to have $g_2 = 0$. This is possible if

$$\begin{aligned} g_1 &= \varepsilon \cos z_3 \,\rho(z_2) \\ \alpha &= \varepsilon \cos z_3 \,\rho'(z_2) x_1 + \varepsilon z_4 \,\rho(z_2) \sin z_3 \end{aligned}$$

where ρ is a function of z_2 such that

$$z_2 \neq 0 \quad \Rightarrow \quad z_2 \,\rho(z_2) > 0 \tag{7.27}$$

(we might take $\rho(z_2) = z_2$ here, but it will be necessary to have ρ growing slower). Then, one may chose

$$\begin{aligned} \beta &= \varepsilon \cos z_3 \,\rho(z_2) \;, \\ \gamma &= k_1 z_3 - \varepsilon x_1 \cos z_3 \;, \end{aligned} \tag{7.28}$$

to make (7.23) negative when $z_4 = 0$. This yields, from (7.24)-(7.26),

$$\begin{aligned} & g_2 = 0 \;, \qquad g_3 = 2\varepsilon \cos z_3 \,\rho(z_2) \\ & g_4 = -\varepsilon x_1 \,(1 + 2\rho'(z_2)) \cos z_3 + k_1 z_3 - 2\varepsilon z_4 \,\rho(z_2) \sin z_3 \end{aligned} \tag{7.29}$$

and hence the vector field G from formula (7.4). From now on, it is simpler to express the computations *in the coordinates* (x_1, z_2, z_3, z_4). In these coordinates, G reads :

$$G(x_1, z_2, z_3, z_4) = \begin{bmatrix} -\varepsilon^2 \rho(z_2) \cos^2 z_3 \\ 0 \\ 2\varepsilon \rho(z_2) \cos z_3 \\ -\varepsilon x_1 \,(1 + 2\rho'(z_2)) \cos z_3 + k_1 z_3 - 2\varepsilon z_4 \,\rho(z_2) \sin z_3 \end{bmatrix}$$

and we have, from (7.18) and (7.12),

$$\begin{aligned} W_\lambda = {} & \frac{1}{2}\left(x_1 - \lambda \varepsilon^2 \rho(z_2) \cos^2 z_3\right)^2 + \frac{1}{2} {z_2}^2 + \frac{k_1}{2}\left(z_3 + 2\lambda\varepsilon\rho(z_2) \cos z_3\right)^2 \\ & + \frac{1}{2}\left((1 - 2\lambda\varepsilon\rho(z_2) \sin z_3) z_4 - \lambda\varepsilon x_1 (1 + 2\rho'(z_2)) \cos z_3 + \lambda k_1 \, z_3\right)^2 \end{aligned}$$

Recall that $L_{f_1} W_\lambda$ is simply $\partial W_\lambda / \partial z_4$. Hence

$$L_{f_1} W_\lambda = (1 - 2\lambda\varepsilon\rho(z_2) \sin z_3)^2 \, \xi_4 \;, \tag{7.30}$$

$$\text{with} \quad \xi_4 = z_4 + \lambda \, \frac{k_1 z_3 - \varepsilon x_1 (1 + 2\rho'(z_2)) \cos z_3}{1 - 2\lambda\varepsilon\rho(z_2) \sin z_3} \;. \tag{7.31}$$

This vanishes either when $\xi_4 = 0$ or when $1 - 2\lambda\varepsilon\rho(z_2)\sin z_3 = 0$. It will be necessary to take $\rho(z_2)$ bounded and λ such that $2\lambda\varepsilon\rho(z_2) < 1$. Provided this is satisfied, a careful computation yields :

$$\begin{aligned}
\dot{W}_\lambda = -\lambda & \left[k_1\, z_3 - \varepsilon\, \cos z_3 \left(x_1 - \lambda(k_1 + \frac{\varepsilon^2}{2}\cos^2 z_3)\rho(z_2)\right)\right]^2 \\
& -\lambda\,\varepsilon^2\, \cos^2 z_3\, \rho'(z_2) \left[x_1 - \lambda\frac{\varepsilon^2}{2}\cos^2 z_3 \rho(z_2)\right]^2 \qquad (7.32) \\
& -\lambda\,\varepsilon^2 \cos^2 z_3\, \rho(z_2) \Bigg[z_2 \;-\; \lambda^2 \rho(z_2) \\
& \qquad \left(\frac{\varepsilon^4}{4}(1+\rho'(z_2))\cos^4 z_3 + k_1\varepsilon^2\cos^2 z_3 + k_1^2\right)\Bigg] \\
& + \xi_4 \Bigg[\lambda\, R(\lambda, \varepsilon, x_1, z_2, z_3, z_4) \;+\; k_1 z_3 \;-\; \varepsilon x_1 \cos z_3 \\
& \qquad \frac{(1 - 2\lambda\varepsilon\rho(z_2)\sin z_3)^2}{1 - \varepsilon^2\cos^2 z_3}\left(\varepsilon\cos z_3(x_1 - \varepsilon {z_4}^2 \sin z_3) + u\right)\Bigg]
\end{aligned}$$

where R is some function, whose expression is somewhat lengthy but can be easily handled with a computer algebra system.

The three first terms depend only on x_1, z_2, z_3, and they are a negative definite function of these three variables provided $\rho'(z_2)$ is everywhere positive, bounded as well as ρ, say for instance $|\rho| < 1$ and $0 < |\rho'| < 1$, and λ is such that

$$\lambda^2 \left(\frac{\varepsilon^4}{2} + k_1\varepsilon^2 + k_1^2\right) \;<\; 1\,. \qquad (7.33)$$

Note that the requirements on ρ are met for instance with $\rho(z_2) = \frac{2}{\pi}\mathrm{Arctan} z_2$, and that, $k_1 = 1$ and $\varepsilon = 0.1$, λ only has to be taken slightly less than 1.

If that is satisfied, then clearly W_λ is a control Lyapunov function, and it satisfies the small control property since equation (7.32) allows one to derive very explicitly a continuous stabilizing control by making the last term non-positive, and negative when $\xi_4 \neq 0$.

7.5 Conclusion

Section 7.3 gives a method to construct some functions from a known "energy-like" function (or "weak" Lyapunov function). We proved (Theorems 2 and 3) that these are control Lyapunov functions under some homogeneity assumptions. However, even for non homogeneous systems, this method can often be used, as illustrated here on the TORA system, to obtain global CLFs. We therefore have good hope that this gives in general a powerful guide to find CLFs from energy-like functions in conservative systems.

In fact the methodology should be refined to be proved to work in more general situations.

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References

1. Z. Artstein, *Stabilization with relaxed control*, Nonlinear Analysis TMA, 7 (1983), pp. 1163–1173.

2. A. Bacciotti, *Local stabilizability of nonlinear control systems*, vol. 8 of Series on advances in mathematics for applied sciences, World Scientific, Singapore, River Edge, London, 1992.

3. D. S. Bernstein, ed., *Special Issue: A Nonlinear Benchmark Problem*, Int. J. of Robust & Nonlinear Cont., 8 (1998), No 4–5.

4. R. W. Brockett, *Finite Dimensional Linear Systems*, John Wiley and sons, New York, London, Sydney, Toronto, 1970.

5. R. T. Bupp, D. S. Bernstein, and V. T. Coppola, *A benchmark problem for nonlinear control design*, Int. J. Robust & Nonlinear Cont., 8 (1998), pp. 307–310.

6. J.-M. Coron, L. Praly, and A. R. Teel, *Feedback stabilization of nonlinear system : Sufficient conditions and Lyapunov and input-output techniques*, in Trends in Control, a European Perspective, A. Isidori, ed., Springer-Verlag, 1995.

7. L. Faubourg and J.-B. Pomet, *Strict control Lyapunov functions for homogeneous Jurdjevic-Quinn type systems*, in Nonlinear Control Systems Design Symposium (NOLCOS'98), H. Huijberts, H. Nijmeijer, A. van der Schaft, and J. Scherpen, eds., IFAC, July 1998, pp. 823–829.

8. L. Faubourg and J.-B. Pomet, Under preparation, 1999.

9. R. A. Freeman and P. V. Kokotovic, *Inverse optimality in robust stabilization.*, SIAM J. on Control and Optim., 34 (1996), pp. 1365–1391.

10. J.-P. Gauthier, *Structure des Systèmes non-linéaires*, Éditions du CNRS, Paris, 1984.

11. S. T. Glad, *Robustness of nonlinear state feedback - a survey*, Automatica, 23 (1987), pp. 425–435.

12. W. Hahn, *Stability of Motion*, vol. 138 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, New-York, 1967.

13. V. Jurjevic and J. P. Quinn, *Controllability and stability*, J. of Diff. Equations, 28 (1978), pp. 381–389.

14. M. Kawski, *Homogeneous stabilizing feedback laws*, Control Th. and Adv. Technol., 6 (1990), pp. 497–516.

15. J. Kurzweil, *On the inversion of Ljapunov's second theorem on stability of motion*, A.M.S. Translations, ser. II, 24 (1956), pp. 19–77.

16. J.-P. LaSalle, *Stability theory for ordinary differential equations*, J. of Diff. Equations, 4 (1968), pp. 57–65.

17. R. Outbib and G. Sallet, *Stabilizability of the angular velocity of a rigid body revisited*, Syst. & Control Lett., 18 (1992), pp. 93–98.

18. L. Praly and Y. Wang, *Stabilization in spite of matched un-modeled dynamics and equivalent definition of input-to-state stability.*, Math. of Control, Signals & Systems, 9 (1996), pp. 1–33.

19. R. Sépulchre, M. Janković, and P. V. Kokotović, *Constructive Nonlinear Control*, Comm. and Control Engineering, Springer-Verlag, 1997.

20. E. D. Sontag, *Feedback stabilization of nonlinear systems*, in Robust control of linear systems and nonlinear control (vol. 2 of proceedings of MTNS'89), M. A. Kaashoek, J. H. van Schuppen, and A. Ran, eds., Basel-Boston, 1990, Birkhäuser, pp. 61–81.