IEEE Trans. Automatic Control, vol. 40, 1995, p. 127–132

However, it is straightforward to verify that there exist constants $\sigma_1 > 0$ and $\sigma_2 > 0$ such that we can write $\sigma_1(\|u(\cdot)\|_2^2 + \|x(\cdot)\|_2^2 + \sum_{s=0}^{k} \|\xi_s(\cdot)\|_2^2) \leq (1/2c_1)\|u(\cdot)\|_2^2 + \frac{1}{2}\|x(\cdot)\|_2^2 + \|\xi_0(\cdot)\|_2^2 + \sum_{s=1}^{k} \tau_s\|\xi_s(\cdot)\|_2^2$, and moreover, $c_0[\|x_0\|^2 + d_0 + \sum_{s=1}^{k} \tau_s d_s) \leq \sigma_2[\|x_0\|^2 + \sum_{s=0}^{k} d_s]$. Hence, by combining these inequalities with (4.14), it follows that $\|u(\cdot)\|_2^2 + \|x(\cdot)\|_2^2 + \sum_{s=0}^{k} \|\xi_s(\cdot)\|_2^2 \leq \sigma_2/\sigma_1[\|x_0\|^2 + \sum_{s=0}^{k} d_s]$. That is, condition iii) of Definition 2.2 is satisfied. Conditions ii) and iv) of Definition 2.2 follows directly from the stability of the matrix P. Thus, we can now conclude that the system (2.1), (2.3) is absolutely stabilizable via the control (4.4). □

Remark: The above theorem gives a necessary and sufficient condition for absolute stabilizability in terms of the solution to a corresponding H^∞ control problem. It is well known that such an H^∞ control problem can be solved in terms of two algebraic Riccati equations; e.g., see [10].

The following corollary is an immediate consequence of the above theorem.

Corollary 4.1: If the uncertain system (2.1), (2.3) satisfies Assumptions 4.1)–4.6) and is absolutely stabilizable via nonlinear control, then it will be absolutely stabilizable via a linear controller of the form (4.4).

References

[1] M. A. Rotea and P. P. Khargonekar, "Stabilization of uncertain systems with norm bounded uncertainty—A control Lyapunov approach," *SIAM J. Contr. Optimiz.*, vol. 27, no. 6, pp. 1462–1476, 1989.

[2] P. P. Khargonekar and K. R. Poolla, "Uniformly optimal control of linear time-invariant plants: Nonlinear time-varying controllers," *Syst. Contr. Lett.*, vol. 6, no. 5, pp. 303–309, 1986.

[3] P. P. Khargonekar, T. T. Georgiou, and A. M. Pascoal, "On the robust stabilizability of linear time-invariant plants with unstructured uncertainty," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 201–207, 1987.

[4] K. R. Poolla and T. Ting, "Nonlinear time-varying controllers for robust stabilization," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 195–200, 1987.

[5] V. A. Yakubovich, "Dichotomy and absolute stability of nonlinear systems with periodically nonstationary linear part," *Syst. Contr. Lett.*, vol. 11, no. 3, pp. 221–228, 1988.

[6] ——, "Absolute stability of nonlinear systems with a periodically nonstationary linear part," *Sov. Phys. Doklady*, vol. 32, no. 1, pp. 5–7, 1988.

[7] A. Megretsky and S. Treil, "Power distribution inequalities in optimization and robustness of uncertain systems," *J. Math. Syst., Estimat., Contr.*, vol. 3, no. 3, pp. 301–319, 1993.

[8] J. S. Shamma, "Robustness analysis for time-varying systems," in *Proc. 31st IEEE Conf. Dec. Contr.*, Tuscon, AZ., Dec. 1992.

[9] A. V. Savkin and I. R. Petersen, "Nonlinear versus linear control in the absolute stabilizability of uncertain linear systems with structured uncertainty," in *Proc. 32 IEEE Conf. Dec. Contr.*, San Antonio, TX., Dec. 1993.

[10] T. Basar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach.* Boston: Birkhäuser, 1991.

A Linear Algebraic Framework for Dynamic Feedback Linearization

E. Aranda-Bricaire, C. H. Moog, and J.-B. Pomet

***Abstract*—To any accessible nonlinear system we associate a so-called infinitesimal Brunovsky form. This gives an algebraic criterion for strong accessibility as well as a generalization of Kronecker controllability indices. An output function which defines a right-invertible system without zero dynamics is shown to exist if and only if the basis of the Brunovsky form can be transformed into a system of exact differential forms. This is equivalent to the system being differentially flat and hence constitutes a necessary and sufficient condition for dynamic feedback linearizability.**

I. Introduction

The problem of exact linearization of a nonlinear system using static state feedback was solved in [18] and [22]. It can be shown that this problem is linked with the classification of functions (or exact one-forms) with respect to their relative degree [8]. When the linearization problem can not be solved using static state feedback, it is appealing to try to solve it using dynamic state feedback. This note emphasizes the links between this new problem and the classification of (non necessarily exact) one-forms with respect to their relative degree. The dynamic feedback linearization problem was stated in its full generality for the first time in [6]: given a nonlinear control system

$$\Sigma\colon \dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in \boldsymbol{R}^n$, $u \in \boldsymbol{R}^m$, find a dynamic compensator

$$\mathcal{C}\colon \begin{cases} u = \alpha(x, \xi) + \beta(x, \xi)v \\ \dot{\xi} = \gamma(x, \xi) + \delta(x, \xi)v \end{cases} \tag{2}$$

and an extended set of coordinates $z = \phi(x, \xi)$ in which the extended system reads as a controllable linear one. Relying upon the differential geometric approach, sufficient conditions and necessary conditions have been given in [7]. In some particular cases, necessary and sufficient conditions are given there. A less general formulation of the dynamic linearization problem is as follows. Consider a nonlinear system where the output $y = h(x)$, $y \in \boldsymbol{R}^m$, has been specified. If the system is right-invertible, it is always possible to construct a dynamic compensator in such a way that noninteracting control is achieved [10], [26]. The standard decoupling feedback provides also input–output linearization and if, in addition, the system has the property of having no zero dynamics, then it actually solves the dynamic linearization problem [19], [21]. Thus, the existence of such an output function is a sufficient condition for dynamic linearizability.

In [12], [13], and [24], the notions of linearizing output and endogeneous feedback were introduced. A linearizing output is a system of functions φ_i of $x, u, \dot{u}, \cdots$, which are differentially

Manuscript received February 5, 1993; revised September 15, 1993 and March 1, 1994. The work of E. Aranda-Bricaire was supported by CONACYT and CINVESTAV-IPN, Mexico.

E. Aranda-Bricaire and C. H. Moog are with the Laboratoire d'Automatique de Nantes (URA CNRS 823), Ecole Centrale de Nantes–Université de Nantes, 1 rue de la Noë, 44072 Nantes cedex 03, France.

J.-B. Pomet is with INRIA, B. P. 93, 06902 Sophia-Antipolis cedex, France, Part of this work was performed while J.-B. Pomet was with the Laboratoire d'Automatique de Nantes.

IEEE Log Number 9407025.

0018-9286/95$04.00

independent and such that both x and u can be expressed as functions of the φ_i's and their time derivatives. Existence of a linearizing output is called differential flatness of the system. A dynamic compensator of the form (2) such that ξ can be expressed as a function of $x, u, \dot{u}, \cdots$, is called endogeneous. flatness is equivalent to dynamic linearizability by endogeneous feedback. Furthermore, recent results [14], [15] show that no system is linearizable by nonendogeneous feedback without being flat. An equivalence relation between systems may be defined such that differential flatness is equivalence to a linear system. This equivalence corresponds to transformations by endogeneous feedback.

In [23], the notion of a dynamic equivalence has been defined in terms of D-algebras. A system is equivalent to a controllable linear system if and only if its associated D-algebra is free. The set of generators of this D-algebra plays the role of linearizing output.

The goal of the present note is to characterize the existence of these linearizing outputs depending on $x, u, \dot{u}, \cdots$, defined in a new algebraic approach which can be recovered from the early notions of zero dynamics and infinite zero structure [9], [19], [20], [26]. This characterization is equivalent to the system's property of being differentially flat, and hence constitutes, from the results in [14] and [15], a necessary and sufficient condition for dynamic feedback linearizability.

It will be shown that to any nonlinear system, one can associate a so-called infinitesimal Brunovsky form which may be viewed as a time-varying Brunovsky form of the first-order approximation of Σ [11]. The construction of this Brunovsky form provides an accessibility criterion, as well as a generalization of linear Kronecker indices. This form singles out a family of m elements of the formal vector space of differential forms, and it is shown that a linearizing output exists if, and only if, this family can be transformed into a system of exact one-forms *via* some invertible transformation. A preliminary version of this work was presented in [28], where the infinitesimal Brunovsky form was defined (it was called nonexact instead of infinitesimal), and was shown to provide a tool to characterize linearizing outputs in the sense of [12], [13], [24]. Let us also mention that a more "differential geometric" presentation of the present material may be found in [2] and [29].

Section II is devoted to some preliminaries from [9], and to a problem statement of dynamic linearization in terms of the infinite zero structure, as in [19]. The infinitesimal Brunovsky form is introduced in Section III with an algorithmic construction. An accessibility criterion is given which involves purely algebraic computations. In Section IV, existence of a linearizing output is characterized in terms of the infinitesimal Brunovsky form. The above theory is illustrated in Section V by the study of various particular cases. Concluding remarks are offered in Section VI.

II. Preliminaries

A. The Infinite Zero Structure [9], [26]

Consider the nonlinear control system Σ, where $f(\cdot)$ and the columns of $g(\cdot)$ are meromorphic vector fields. Throughout the note it is assumed that rank $g(x) = m$. Let $\mathcal{K}$ denote the field of meromorphic functions of $x, u, \dot{u}, \cdots$. The time derivative of a function $\varphi \in \mathcal{K}$ is defined by

$$\frac{d}{dt}\varphi = \dot{\varphi} = \frac{\partial \varphi}{\partial x}(f(x) + g(x)u) + \sum_{j \geq 0} \frac{\partial \varphi}{\partial u^{(j)}} u^{(j+1)}.$$

Clearly, $\mathcal{K}$ is closed under time-differentiation. Let $\mathcal{E}$ denote the $\mathcal{K}$-vector space spanned by $dx, du, d\dot{u}, \cdots$. The elements of $\mathcal{E}$ are called differential forms of degree one, or simply one-forms. d/dt induces a derivation on $\mathcal{E}$ in the following way d/dt: $\omega = \sum_j a_j\, dv_j \mapsto \dot{\omega} = \sum_j (\dot{a}_j\, dv_j + a_j\, d\dot{v}_j)$. The relative degree of a one-form $\omega \in \text{span}_{\mathcal{K}}\{dx\}$ is defined as the smallest integer r such that $\omega^{(r)} \notin \text{span}_{\mathcal{K}}\{dx\}$. If such an integer does not exist, set $r = \infty$.

Now, consider the system Σ and suppose that the output function $y = h(x)$, $y \in \boldsymbol{R}^m$, has been specified. Introduce the chain of subspaces $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n$ of $\mathcal{E}$, defined by

$$\mathcal{E}_k := \text{span}_{\mathcal{K}}\{dx, dy, \cdots, dy^{(k)}\}. \tag{3}$$

The number of zeros at infinity of order less than or equal to k, for $1 \leq k \leq n$, is

$$\sigma_k = \dim \frac{\mathcal{E}_k}{\mathcal{E}_{k-1}}. \tag{4}$$

The infinite zero structure can be given either by the list $\{\sigma_k\}$ or by the list $\{n_i'\}$ of the orders of the zeros at infinity. Σ is said to be (right) invertible if $\sigma_n = m$. Following [9], one has the following.

Lemma 2.1: Assume that Σ is invertible. Let $\mathcal{X} := \text{span}_{\mathcal{K}}\{dx\}$, $\mathcal{Y} := \text{span}_{\mathcal{K}}\{dy^{(k)},\ k \geq 0\}$. Then $\dim(\mathcal{X} \cap \mathcal{Y}) = \Sigma_i n_i'$.

Remark 2.2: Note that although $\mathcal{Y}$ is, in general, infinite dimensional, the intersection of subspaces $\mathcal{X} \cap \mathcal{Y}$ is, at most, of dimension n. The subspace $\mathcal{X} \cap \mathcal{Y}$ has been first considered in [5] for studying minimality in dynamic decoupling.

B. Dynamic Feedback Linearization Problem Statement

Any nonlinear system with outputs and which is right-invertible can be fully linearized whenever it has no zero dynamics, in the sense of the dynamics of the reduced inverse system [20]. Thus, the absence of zero dynamics is a sufficient condition for dynamic feedback linearization [19], [21]. This is equivalent to $\Sigma_i n_i' = n$. This yields (more precisely) the following.

Problem Statement 1: Given Σ, find, if possible, an m-dimensional output function $y = h(x)$ such that the system is right-invertible and $\Sigma_i n_i' = n$. ∎

Solvability of this problem is not necessary for dynamic feedback linearizability. A more general approach to the dynamic feedback linearization problem, which originates in [12], [13], [24], where it is instrumental to define differential flatness, consists of allowing the output function to explicitly depend on the input u as well as on a finite number, say, $\nu - 1$, of its time derivatives. From Lemma 2.1, this situation may be stated as the existence of an m-dimensional output function $y = h(x, u, \cdots, u^{(\nu-1)})$ such that the system is right-invertible and $\dim(\mathcal{X}_\nu \cap \mathcal{Y}) = n + m\nu$, where $\mathcal{X}_\nu := \text{span}_{\mathcal{K}}\{dx, du, \cdots, du^{(\nu-1)}\}$. For square invertible systems, one has $\mathcal{X}_\nu + \mathcal{Y} = \mathcal{X} + \mathcal{Y}$ and consequently $\dim(\mathcal{X}_\nu \cap \mathcal{Y}) = \dim(\mathcal{X} \cap \mathcal{Y}) + m\nu$. So, the more general problem is stated as follows.

Problem Statement 2: Given Σ, find, if possible, an integer ν and an m-dimensional output function $y = h(x, u, \cdots, u^{(\nu-1)})$ such that the system is right-invertible and

$$\dim(\mathcal{X} \cap \mathcal{Y}) = n. \tag{5}$$

If such an output exists, it is called a *linearizing output.* ∎

C. Differential Flatness

In [12], [13], and [24], the notions of linearizing output, differential flatness, endogeneous and exogeneous feedback were introduced. In [12] and [13], this is done within a differential algebraic framework, whereas in [24] the analytic case is also considered.

Roughly speaking, a linearizing output [12], [13] is a system of differentially independent functions φ_i of $x, u, \dot{u}, \cdots$, such that x, u

can in turn be expressed as functions of the φ_i's and a finite number of their time derivatives. This definition is equivalent to the one in Problem Statement 2 because condition (5) means that one is able to recover the variables x, u as functions of the outputs y_i and their time derivatives, and, on the other hand, right-invertibility ensures that the outputs y_i are differentially independent in the sense that they do not satisfy any differential equation independent of u. We refer the reader to [16] for an exhaustive discussion of differential flatness and its link with an equivalence relation between systems.

A system Σ is said to be differentially flat (or simply flat) if it admits a system of linearizing outputs. It is proved in [12], [13], and [24] that differential flatness, for a system Σ, is equivalent to linearizability via a dynamic compensator $\mathcal{C}$ which has the property of being endogeneous (this may be defined as the possibility to express the variables ξ as functions of x, u, $\dot{u}, \cdots$). Recent results [14], [15] show that flatness is in fact equivalent to dynamic linearizability without any restriction on the nature of compensators.

III. The Infinitesimal Brunovsky Form

A. A Flag for the Differential Vector Space $\mathcal{E}$

We shall construct a sequence of subspaces of $\mathcal{E}$ in the following manner. Define

$$\mathcal{H}_0 = \operatorname{span}_{\mathcal{K}}\{dx, du\},$$

$$\mathcal{H}_k = \{\omega \in \mathcal{H}_{k-1} | \dot{\omega} \in \mathcal{H}_{k-1}\}, \qquad k > 0 \tag{6}$$

It is clear that $\mathcal{E} \supset \mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots$, and that, at the first step, the above induction yields $\mathcal{H}_1 = \operatorname{span}_{\mathcal{K}}\{dx\}$. Proposition 3.1 is a simple consequence of the construction. Feedback invariance comes from the fact that the relative degree of a one-form is obviously invariant under regular static state feedback. Existence of the integer k^* comes from the fact that each $\mathcal{H}_k$ is a finite-dimensional $\mathcal{K}$-vector space so that, at each step either its dimension decreases or $\mathcal{H}_{k+1} = \mathcal{H}_k$.

Proposition 3.1: $\mathcal{H}_k$ is the space of one-forms which have relative degree greater than or equal to k. Both the subspaces $\mathcal{H}_k$ and the integers $\rho_k = \dim \mathcal{H}_k$ are invariant under regular static state feedback. There exists an integer $k^* > 0$ such that $\mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots = \mathcal{H}_\infty$.

The following algorithm allows us to explicitly construct bases for the subspaces $\mathcal{H}_k$.

Step 1: Take $\{dx_1, \cdots, dx_n, du_1, \cdots, du_m\}$, $\{dx_1, \cdots, dx_n\}$ as bases of $\mathcal{H}_0$ and $\mathcal{H}_1$.

Step $k+1$: Suppose that $\{\eta_1, \cdots, \eta_{\rho_k}, \mu_1, \cdots, \mu_{\rho_{k-1}-\rho_k}\}$, $\{\eta_1, \cdots, \eta_{\rho_k}\}$ are bases, respectively, of $\mathcal{H}_{k-1}$ and $\mathcal{H}_k$, and let us construct a basis for $\mathcal{H}_{k+1}$. The elements of $\mathcal{H}_{k+1}$ are the one-forms $\omega \in \mathcal{H}_k$ such that $\dot{\omega} \in \mathcal{H}_k$. Let $\omega = \Sigma_j \lambda_j \eta_j \in \mathcal{H}_k$, then $\dot{\omega} = \Sigma_j(\dot{\lambda}_j \eta_j + \lambda_j \dot{\eta}_j)$. It is clear that $\dot{\omega} \in \mathcal{H}_k$ if, and only if, $\Sigma_j \lambda_j \dot{\eta}_j \in \mathcal{H}_k$. Now, note that since $\eta_j \in \mathcal{H}_k$, $\dot{\eta}_j$ must be in $\mathcal{H}_{k-1}$, so $\Sigma_j \lambda_j \dot{\eta}_j$ may be written in the following form:

$$\sum_j \lambda_j \dot{\eta}_j = \sum_j \lambda_j \left(\sum_l \sigma_{l,j} \eta_l + \sum_l \tau_{l,j} \mu_l \right).$$

Thus, $\dot{\omega} \in \mathcal{H}_k$ if and only if the coefficients λ_j satisfy the following system of linear equations:

$$\sum_{j=1}^{\rho_k} \tau_{i,j} \lambda_j = 0, \qquad 1 \le i \le \rho_{k-1} - \rho_k. \tag{7}$$

This system of equations has $\rho_k - \rho$ linearly independent solutions, ρ being the rank of the matrix $[\tau_{i,j}]$. Thus, $\dim \mathcal{H}_{k+1} = \rho_{k+1} = \rho_k - \rho$. A basis of $\mathcal{H}_{k+1}$ can be computed as

$$\alpha_j = \sum_{1=1}^{\rho_k} \lambda_i^j \eta_i, \qquad 1 \le j \le \rho_{k+1}$$

where $(\lambda_1^j, \cdots, \lambda_{\rho_k}^j)$ are the ρ_{k+1} independent solutions of (7). The algorithm stops after a finite number k^* of steps when $\rho = 0$. In fact, it is not difficult to show that $k^* \le n - m + 1$.

Remark 3.2: In general the subspaces $\mathcal{H}_k$ may depend on u, $\dot{u}, \cdots$ in the following sense: the elements α_i built using the above algorithm can be written as linear combinations of $dx_1, \cdots, dx_n$, the coefficients being functions of x, u, $\dot{u}, \cdots$. However, a careful inspection of the construction shows that, at the kth step, these coefficients may be chosen to depend, at most, on u and its first $k-2$ time-derivatives.

B. Accessibility Criteria

Proposition 3.3:

1) $\mathcal{H}_\infty$ is the largest subspace of $\mathcal{H}_0 = \operatorname{span}_{\mathcal{K}}\{dx, du\}$ which is invariant under time-differentiation. It is also, for any $K \ge 1$, the largest subspace of $\mathcal{H}_{-K} = \operatorname{span}_{\mathcal{K}}\{dx, du^{(j)}, 0 \le j \le K\}$ which is invariant under time-differentiation.

2) Let $\{\alpha_1, \cdots, \alpha_{\rho_\infty}\}$ be a basis for $\mathcal{H}_\infty$. Then, the Frobenius condition $d\alpha_i \wedge \alpha_1 \wedge \cdots \wedge \alpha_{\rho_\infty} = 0$, $1 \le i \le \rho_\infty$ is satisfied.

Proof: Point 1 is a consequence of the construction (it is clear that, starting from $\mathcal{H}_{-K}$ instead of $\mathcal{H}_0$ in (6), one finds $\mathcal{H}_0$ after K steps). The proof of point 2 is given in the Appendix. ■

In point 2 of Proposition 3.3, $\wedge$ indicates the exterior product of differential forms. These conditions imply integrability of the Pfaffian system $\{\alpha_1, \cdots, \alpha_{\rho_\infty}\}$ around regular points, according to the dual version of Frobenius theorem. The reader who is not familiar with these matters is referred to [1].

Subspace $\mathcal{H}_\infty$ may be interpreted as a codistribution on $\boldsymbol{R}^n \times \boldsymbol{R}^{mK}$, where $K-1$ is the maximum number of input time-derivatives necessary to write a basis of $\mathcal{H}_\infty$. Proposition 3.3 implies that this codistribution is locally integrable around any point where it has constant rank. This implies that $\mathcal{H}_\infty$ is locally spanned by ρ_∞ exact one-forms $d\psi_1, \cdots, d\psi_{\rho_\infty}$ where $\psi_1, \cdots, \psi_{\rho_\infty}$ are functions defined around such regular points. These functions do not depend on u and its time derivatives because $\mathcal{H}_\infty \subset \operatorname{span}_{\mathcal{K}}\{dx\}$, so $\mathcal{H}_\infty$ may be interpreted as a codistribution on $\boldsymbol{R}^n$. Since, for a function $\psi(x)$, $d\psi \in \mathcal{H}_\infty$ is equivalent to ψ being constant along all the vector fields $\operatorname{ad}_f^j g_k(x)$, with $j \ge 0$ and g_k the control vector fields (i.e., to be constant along the strong accessibility distribution) $\mathcal{H}_\infty$ is, around regular points, the annihilator of the strong accessibility distribution. This leads to the following.

Proposition 3.4 (Accessibility Criteria): The following statements are equivalent.

1) System Σ satisfies the strong accessibility rank condition.
2) Any nonzero one-form has finite relative degree.
3) $\mathcal{H}_\infty = \{0\}$.

Proposition 3.4 is one key result of this note since it allows the following construction which we then relate to linearizing outputs, if these exist.

C. The Infinitesimal Brunovsky Form

Theorem 3.5: Suppose $\mathcal{H}_\infty = \{0\}$. There exists a list of integers $\{r_1, \cdots, r_m\}$, invariant under regular static state feedback, and m one-forms $\omega_1, \cdots, \omega_m$ with relative degrees $r_1, \cdots, r_m$ such that

1) $\text{span}_{\mathcal{K}}\{\omega_i^{(j)}, 1 \le i \le m, 0 \le j \le r_i - 1\} = \text{span}_{\mathcal{K}}\{dx\}$;
2) $\text{span}_{\mathcal{K}}\{\omega_i^{(j)}, 1 \le i \le m, 0 \le j \le r_i\} = \text{span}_{\mathcal{K}}\{dx, du\}$;
3) the forms $\{\omega_i^{(j)}, 1 \le i \le m, j \ge 0\}$ are linearly independent. In particular, $\sum_{i=1}^m r_i = n$.

Proof: Let $\mathcal{W}_{k^*}$ be a basis for $\mathcal{H}_{k^*}$. By definition, $\mathcal{W}_{k^*}$ and $\dot{\mathcal{W}}_{k^*}$ are in $\mathcal{H}_{k^*-1}$. Note that $\mathcal{W}_{k^*}$ and $\dot{\mathcal{W}}_{k^*}$ are linearly independent. For, let $\mathcal{W}_{k^*} = \{\eta_1, \cdots, \eta_{\rho_{k^*}}\}$—then $\dot{\mathcal{W}}_{k^*} = \{\dot{\eta}_1, \cdots, \dot{\eta}_{\rho_{k^*}}\}$—and suppose that there exist some coefficients $\{\lambda_i, \mu_i\}$ such that $\Sigma_i(\lambda_i\eta_i + \mu_i\dot{\eta}_i) = 0$. The linear independence of the η_i's implies that not all the μ_i's vanish. Now consider the one-form $\omega = \Sigma_i\mu_i\eta_i$ whose time derivative is $\dot{\omega} = \Sigma_i\dot{\mu}_i\eta_i - \Sigma\lambda_i\eta_i$. This implies that ω is in $\mathcal{H}_{k^*+1}$, which is a contradiction. Hence, it is always possible to choose a set $\mathcal{W}_{k^*-1}$ (possible empty) such that $\mathcal{W}_{k^*}, \dot{\mathcal{W}}_{k^*}, \mathcal{W}_{k^*-1}$ is a basis for $\mathcal{H}_{k^*-1}$. This procedure is repeated k^* times, so the sequence $\{\mathcal{H}_k\}$ is shown to have the following structure:

$$\mathcal{H}_k = \text{span}_{\mathcal{K}}\{\mathcal{W}_i^{(j)}, \quad k \le i \le k^*, \ 0 \le j \le i-k\}, \ 0 \le k \le k^*.$$

$\mathcal{H}_1 = \text{span}_{\mathcal{K}}\{dx\}$ and rank $g(x) = m$ imply that $\mathcal{W}_0 = \emptyset$. It can be proved by induction that, for $0 \le k \le k^*$, the set $\{\mathcal{W}_{k^*}, \cdots, \mathcal{W}_{k^*}^{(k^*-k)}, \cdots, \mathcal{W}_k\}$ is linearly independent. Finally, set $\{\omega_1, \cdots, \omega_m\} = \{\mathcal{W}_{k^*}, \cdots, \mathcal{W}_1\}$. The invariance of the list of relative degrees is rather obvious from the construction. It can also be seen from the fact that the number of r_i's which are equal to k is given by $s_k = \text{card}\,\mathcal{W}_k = \dim \mathcal{H}_k/(\mathcal{H}_{k+1} + \dot{\mathcal{H}}_{k+1})$. ■

The following (straightforward) corollary of Theorem 3.5 is the reason for the name infinitesimal Brunovsky form: the ω_k's provide a basis of one-forms in which the first-order approximation of Σ looks like a linear Brunovsky canonical form [11]. If these forms were integrable, then they would yield a true Brunovsky canonical form for system Σ (for this reason, the term infinitesimal is preferred to "non-exact," used in [28]). In any case, the integers r_i are nice candidates for generalizing to nonlinear systems the notion of linear Kronecker controllability indices.

Corollary 3.6 (The Infinitesimal Brunovsky form): Suppose $\mathcal{H}_\infty = \{0\}$. Then the basis $\{\omega_{i,j}, 1 \le i \le m, 1 \le j \le r_i\}$ of $\text{span}_{\mathcal{K}}\{dx\}$ defined by $\omega_{i,j} = \omega_i^{(j-1)}$ yields

$$\begin{aligned} \dot{\omega}_{i,1} &= \omega_{i,2} \\ &\vdots \qquad\qquad (1 \le i \le m) \\ \dot{\omega}_{i,r_i-1} &= \omega_{i,r_i} \\ \dot{\omega}_{i,r_i} &= \sum_{j=1}^n a_{i,j}dx_j + \sum_{j=1}^m b_{i,j}du_j \end{aligned}$$

where $a_{i,j}, b_{i,j} \in \mathcal{K}$ and $[b_{ij}]$ has an inverse in the ring of $m \times m$ matrices with entries in $\mathcal{K}$.

IV. Main Results

A. Some Preliminaries

An Algebra of Polynomial Operators: Let $\mathcal{K}[d/dt]$ denote the (noncommutative) algebra of polynomials in the operator d/dt with coefficients in $\mathcal{K}$. The addition and external multiplication are the usual ones. The internal multiplication corresponds to operators composition: $(d/dt)(p) = p(d/dt) + \dot{p}$, $\forall p \in \mathcal{K}$. The only invertible elements in $\mathcal{K}[d/dt]$ are the nonzero elements of $\mathcal{K}$ (i.e., nonzero polynomials of degree zero). Let $\mathcal{K}^{m\times m}[d/dt]$ denote the algebra of $m \times m$ matrices with entries in $\mathcal{K}[d/dt]$. Let $\mathcal{E}^m$ be the differential $\mathcal{K}$-vector space spanned by m-tuples of one-forms. Each $P \in \mathcal{K}^{m\times m}[d/dt]$ defines a differential operator in $\mathcal{E}^m$: $P \cdot \Omega = \Sigma_i P_i\Omega^{(i)}$ for all $\Omega = (\omega_1, \cdots, \omega_m)^T \in \mathcal{E}^m$, where $\Omega^{(i)} = (\omega_1^{(i)}, \cdots, \omega_m^{(i)})^T$ and $P := \Sigma_i P_i(d/dt)^i \in \mathcal{K}^{m\times m}[d/dt]$. Invertible elements of $\mathcal{K}^{m\times m}[d/dt]$ play an important role; they are elements $P \in \mathcal{K}^{m\times m}[d/dt]$ such that there exists $Q \in \mathcal{K}^{m\times m}[d/dt]$ such that $P \cdot Q = Q \cdot P = I_m$.

Definition 4.1 (Structure at Infinity for One-Forms): $\Omega = (\omega_1, \cdots, \omega_m)^T$ is said to have $\sigma_k = \dim\left(\text{span}_{\mathcal{K}}\{dx, \Omega, \cdots, \Omega^{(k)}\}/\text{span}_{\mathcal{K}}\{dx, \Omega, \cdots, \Omega^{(k-1)}\}\right)$ zeros at infinity of order less than or equal to k.

For exact one-forms ($\omega_i = dh_i$), Definition 4.1 coincides with (3) and (4), since exterior differentiation and time-differentiation commute, so that the notations of Section II may be adopted verbatim. In particular, Lemma 2.1 is also valid for systems of one-forms.

Proposition 4.2: Consider the system of m one-forms $\Omega := (\omega_1, \cdots, \omega_m)^T$, and the polynomial matrix operator $P \in \mathcal{K}^{m\times m}[d/dt]$. Let $\tilde{\Omega} := P \cdot \Omega$. Then

$$\dim(\mathcal{X}_\nu \cap \text{span}_{\mathcal{K}}\{\tilde{\Omega}^{(k)}, k \ge 0\}) \le \dim(\mathcal{X}_\nu \cap \text{span}_{\mathcal{K}}\{\Omega^{(k)}, k \ge 0\})$$

where ν is an integer large enough such that Ω and $\tilde{\Omega}$ belong to $\mathcal{X}_\nu$.

Proof: Suppose P has degree α. Straightforward computations show that $\tilde{\Omega}^{(k)}$, for $k \ge 0$, can be written as a linear combination of the following form $\tilde{\Omega}^{(k)} = R_0\Omega + R_1\dot{\Omega} + \cdots + R_{k+\alpha}\Omega^{(k+\alpha)}$. Thus, $\text{span}_{\mathcal{K}}\{\tilde{\Omega}^{(k)}, k \ge 0\} \subseteq \text{span}_{\mathcal{K}}\{\Omega^{(k)}, k \ge 0\}$ and the result follows. ■

B. The Results

Our main result is the following. It is an easy consequence of Theorem 3.5 and Proposition 4.2.

Theorem 4.3 (Problem Statement 2): Suppose $\mathcal{H}_\infty = \{0\}$. There exists a system of linearizing outputs if and only if there exists an invertible polynomial operator $P \in \mathcal{K}^{m\times m}[d/dt]$ such that $d(P\Omega) = 0$, where $\Omega = (\omega_1, \cdots, \omega_m)^T$ is a system of one-forms characterized by Theorem 3.5.

Proof:

Necessity: Suppose $y = h(x, u, \cdots, u^{(\nu-1)})$ is a linearizing output. Problem Statement 2 implies that $\mathcal{E} = \mathcal{Y}$. Theorem 3.5 implies that $\mathcal{E} = \text{span}_{\mathcal{K}}\{\Omega^{(k)}, k \ge 0\}$. Thus, there exist polynomial matrix operators P, Q such that $dy = P\Omega$ and $\Omega = Q\,dy$. Clearly, $PQ = QP = I_m$ and hence P is invertible. Moreover, $d(P\Omega) = d(dy) = 0$.

Sufficiency: Let $N = \dim(x_\nu \cap \text{span}_{\mathcal{K}}\{\Omega^{(k)}, k \ge 0\})$, $\tilde{N} = \dim(\mathcal{X}_\nu \cap \text{span}_{\mathcal{K}}\{\tilde{\Omega}^{(k)}, k \ge 0\})$, where $\tilde{\Omega} = P\Omega$. Theorem 3.5 implies that $N = n + m\nu$. Existence of the operator P implies $\tilde{N} \le N$. Invertibility of P implies the existence of an operator Q such that $\Omega = Q\tilde{\Omega}$, i.e., $N \le \tilde{N}$ and hence $\tilde{N} = N$. The result follows because one can assume, without loss of generality, that $\tilde{\Omega} = d\psi(x, u, \cdots, u^{(\nu-1)})$. ψ is a linearizing output. ■

Theorem 4.3 relates linearizing outputs, if they exist, to the set of differential one-forms built in Theorem 3.5 for arbitrary accessible systems. It provides an alternative way to tackle the problem of deciding whether linearizing outputs exist, i.e., whether a given system is linearizable by endogeneous dynamic feedback, by looking for an invertible matrix P meeting the above conditions. This does not provide a practically checkable criterion because the degree (in the operator d/dt) of the matrix P is not known *a priori*, which prevents the condition of the theorem from being finitely checkable. By forcing P to have degree zero (i.e., to be an invertible matrix with entries in $\mathcal{K}$), the problem is made finite, and one obtains the following sufficient condition.

Corollary 4.4: A sufficient condition for the existence of a system of linearizing outputs is that a system of one-forms $\Omega = (\omega_1, \cdots, \omega_m)^T$ satisfying the conditions of Theorem 3.5 satisfy the Frobenius condition

$$d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_m = 0, \qquad 1 \le i \le m. \qquad (8)$$

The relative degrees of $\omega_1, \cdots, \omega_m$ coincide with the orders of the zeros at infinity of the linearizing outputs.

Proof: The Frobenius condition implies that there exists a basis composed of exact one-forms for the codistribution spanned by $\{\omega_1, \cdots, \omega_m\}$, and hence that there exists an invertible matrix (with entries in $\mathcal{K}$) relating this basis to $\{\omega_1, \cdots, \omega_m\}$. ∎

The condition (8) is obviously finitely (and easily) checkable once some ω_i's have been constructed. It is of course not a necessary condition, and it should be noted that, in general, it depends on the choice of the ω_i's: some systems of one-forms statisfying the conditions of Theorem 3.5 may satisfy the Frobenius condition (8), whereas some others do not. Even for a linear system, a wrong choice of the one-forms $\omega_1, \cdots, \omega_m$ prevents condition (8) from being satisfied. However, in many practical cases (see the proofs of the results in Section V or [28, Section 3.2]), it is not difficult to round this difficulty and check whether (8) is met for one of the possible choices of the ω_i's.

A way for bounding the degree of P is to look for linearizing outputs depending on x only, as illustrated by the following result.

Theorem 4.5 (Problem Statement 1): Suppose $\mathcal{H}_\infty = \{0\}$. Then there exists a system of linearizing outputs which depend only on x if and only if the conditions of Theorem 4.3 are satisfied and, in addition, $\deg(P_{ij}(d/dt)) \leq r_j - 1$, $1 \leq i, j \leq m$.

Proof: Sufficiency is obvious. Conversely, suppose that one of the polynomial elements of the matrix $P(d/dt)$, say $P_{ij}(d/dt)$ has degree equal to r_j. Thus, dh_i contains a term which depends on du and that cannot be eliminated by the remaining terms since, by construction, all the $\omega_i^{(k)}$ are linearly independent. This is a contradiction. ∎

Note that it is very easy to write down some similar criteria for the existence of linearizing outputs depending on x, u, and any finite number of time derivatives of u. Such types of conditions as the ones given in Theorem 4.5 can be restated as existence of a finite number of functions—the coefficients of the polynomial entries of P—meeting some differential conditions, namely, $d(P\Omega) = 0$ and P invertible. A possible way to avoid writing the relations on the entries of P for it to be invertible is to write P as a finite product of elementary invertible matrices and taking the coefficients of these elementary matrices as unknowns instead of the entries of P itself; this is exploited in [27]. Checking whether there exist some linearizing outputs depending on x, u, and any finite number of time derivatives of u therefore amounts to checking whether a finite set of PDE's in a finite number of unknown functions has a solution.

This is not new since it is easy (although tedious) to write down the PDE's which have to be satisfied by the linearizing outputs themselves, if they are restricted to depend on x only. This is the underlying idea of the characterizations given in some particular cases (see, for instance, [25]). We, however, believe that looking for the invertible matrix P once the ω_k's have been constructed is more natural and more tractable. This is illustrated by the very short proofs of the theorems of next section, which are known but usually not so natural to prove, and by results like the ones obtained in [28, Section 3.2], [27], [3, Theorem 5.4], which work out some nontrivial particular cases.

V. Particular Cases

In this section we recover some classical results using the infinitesimal Brunovsky form.

Theorem 5.1 (Static State Feedback Linearization): System Σ is linearizable by static state feedback if, and only if, $\mathcal{H}_\infty = \{0\}$ and, for $k = 1, \cdots, k^*$, $\mathcal{H}_k$ is completely integrable.

This is, of course, equivalent to the early characterizations of [18] and [22], or to the more recent one given in [17] and [30]; the infinitesimal Brunovsky form provides a very short proof.

Proof: If each $\mathcal{H}_k$ is completely integrable, one may chose exact forms, say, $\omega_i = d\psi_i(x)$, $i = 1, \cdots, m$ in Theorem 3.5; this yields m functions $\psi_i(x)$ whose relative degrees satisfy $\Sigma_i r_i = n$, and whose decoupling matrix [21] is nonsingular because the ω_i's and all their time derivatives are linearly independent, this completes the if part. The converse is obvious since the $\mathcal{H}_k$'s are invariant by static feedback, and are integrable for a linear system. ∎

Theorem 5.2 (Single-Input Systems): Let Σ be a single-input system and suppose $\mathcal{H}_\infty = \{0\}$. Then there is only one differential form ω_1 in Theorem 3.5, and the following statements are equivalent:

1) Σ is linearizable by static state feedback; 2) Σ is linearizable by dynamic state feedback; 3) $d\omega_1 \wedge \omega_1 = 0$, where ω_1 is such that $\mathcal{H}_n = \operatorname{span}_{\mathcal{K}} \{\omega_1\}$.

This result (equivalence between 1 and 2) was first obtained in [6] and [7]. The infinitesimal Brunovsky form—note that ω_1 is invariant up to a nonzero multiplicative function—allows us to give the simple characterization 3 and the following very simple proof.

Proof: It is obvious from Theorem 5.1 and Theorem 4.3 because the only invertible elements of $\mathcal{K}[d/dt]$ are those having degree 0. In turn, multiplication by a nonzero function does not change the rank of the differential form $d\omega_1 \wedge \omega_1$.

Theorem 5.3 (Systems with $m = n - 1$ inputs [7]): A system Σ with $m = n - 1$ inputs is linearizable by dynamic state feedback if, and only if, $\mathcal{H}_\infty = \{0\}$.

Proof: $\mathcal{H}_2$ is generated by a single nonzero one-form ω_1 which is orthogonal to the distribution spanned by the vector fields $g_1, \cdots, g_{n-1}$, and thus can be chosen independent of u. $\omega_2, \cdots, \omega_{n-1}$ can be chosen arbitrarily, linearly independent of $\{\omega_1, \dot{\omega}_1\}$ and belonging to $\operatorname{span}_{\mathcal{K}} \{dx\}$: they can also be chosen independent of u. Then, the differential forms $d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_m$, $i = 1, \cdots, m$, are zero because they are $(n+1)$-forms in n variables. The converse is obvious. ∎

VI. Conclusion

We have built a so-called infinitesimal Brunovsky form which exhibits m controllable blocks whose dimensions play the role of Kronecker controllability indices in the liner case. This extension to the nonlinear case is innovative since, to our best knowledge, it is the only available one with the property that the sum of these indices equals the dimension of the strong accessibility distribution. This result on nonlinear accessibility was used to derive a necessary and sufficient condition for existence of a linearizing output, and the early results in (either static or dynamic) feedback linearization have been shown to fit naturally in our formalism. Static feedback linearization was shown to be a matter of exact one-forms whereas dynamic feedback linearization is a matter of possibly nonexact one-forms.

Appendix
Proof of Proposition 3.3, Point 2

Define $\mathcal{E}^*$ to be the dual vector space of $\mathcal{E}$ (topology induced by $\|\Sigma a_j\, dv_j\|^2 = \Sigma a_j^2$, dv_j's taken among the dx_i's or the $du_k^{(j)}$'s, both sums are finite), whose elements—"vector fields"—are of the form

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{k=1}^{m} \sum_{j \geq 0} b_{j,k} \frac{\partial}{\partial u_k^{(j)}}$$

where a_i, $b_{j,k}$ are in $\mathcal{K}$ and $\partial/\partial x_i$, $\partial/\partial u_k^{(j)}$ are defined by $\langle \partial/\partial x_i, dx_{i'} \rangle = \delta_{i,i'}$, $\langle \partial/\partial u_k^{(j)}, du_{k'}^{(j')} \rangle = \delta_{j,j'}\delta_{k,k'}$, $\Big\langle \partial/\partial u_k^{(j)},$

$\mathrm{d}x_i\rangle = \langle \partial/\partial x_i, \mathrm{d}u_k^{(j)}\rangle = 0$. Note that the above linear combination need not be finite, but does define an element of $\mathcal{E}^*$ because, for all $\omega \in \mathcal{E}$, $\langle X, \omega\rangle$ may be written as a sum with only finitely many nonzero terms. Define the "time-derivative" of $X \in \mathcal{E}^*$ by

$$\langle \dot{X}, \omega\rangle = \dot{\overbrace{\langle X, \omega\rangle}} - \langle X, \dot{\omega}\rangle. \tag{9}$$

The interior product (or hook) $i(X)$ is defined, for a differential form of degree two $\mu = \sum_{i,j} a_{i,j}\,\mathrm{d}v_i \wedge \mathrm{d}v_j$, by $i(X)\mu = \sum_{i,j} a_{i,j}(\langle X, \mathrm{d}v_i\rangle \mathrm{d}v_j - \langle X, \mathrm{d}v_j\rangle \mathrm{d}v_i)$. Clearly, taking the time-derivative of both sides in this identity yields, from (9),

$$\dot{\overbrace{i(X)\mu}} = i(\dot{X})\mu + i(X)\dot{\mu}. \tag{10}$$

Let $\alpha_1, \cdots, \alpha_{\rho_\infty}$ be a basis of $\mathcal{H}_\infty$, and define the following subspace of $\mathcal{E}^*$

$$\mathcal{G}_\infty = \{X \in \mathcal{E}^* \mid \quad \forall \omega \in \mathcal{H}_\infty, \langle X, \omega\rangle = 0 \quad \text{and} \quad \alpha_1 \wedge \cdots \wedge \alpha_{\rho_\infty} \wedge i(X)\,\mathrm{d}\omega = 0\}$$

whose elements are sometimes called [4] the "Cauchy characteristic vector fields" of $\mathcal{H}_\infty$. The "characteristic system" (or "retracting space" according to [4]) of $\mathcal{H}_\infty$ is defined as its annihilator: $\mathcal{C}(\mathcal{H}_\infty) = \mathcal{G}_\infty^\perp$. For a certain $K \geq 0$, the elements $\alpha_1, \cdots, \alpha_{\rho_\infty}$ of the chosen basis for $\mathcal{H}_\infty$ can be written as linear combinations of $\mathrm{d}x_1, \cdots, \mathrm{d}x_n$ with coefficients function of $x, u, \dot{u}, \cdots, u^{(K)}$ only, hence all the $\mathrm{d}\alpha_i$'s are linear combinations of elements of the form $\mathrm{d}x_j \wedge \mathrm{d}x_l$ or $\mathrm{d}x_j \wedge \mathrm{d}u_k^{(l)}$ with $l \leq K$, so that all the elements $\partial/\partial u_k^{(j)}$ are in $\mathcal{G}_\infty$ for $j \geq K+1$, and finally that $\mathcal{C}(\mathcal{H}_\infty)$ is a subspace of $\mathrm{span}_\mathcal{K}\{\mathrm{d}x, \mathrm{d}u, \mathrm{d}\dot{u}, \cdots, \mathrm{d}u^{(K)}\}$. Furthermore, we have the following.

Lemma: $\mathcal{C}(\mathcal{H}_\infty)$ is invariant by time-differentiation.

From point 1 of Proposition 3.3, this implies $\mathcal{C}(\mathcal{H}_\infty) \subset \mathcal{H}_\infty$, and hence, from standard exterior algebra, $\mathrm{d}\alpha_1 \wedge \alpha_1 \wedge \cdots \wedge \alpha_{\rho_\infty} = 0$ for $i = 1, \cdots, \rho_\infty$.

Proof of the Lemma: Consider $X \in \mathcal{G}_\infty$, $\omega \in \mathcal{H}_\infty$. We have [compare (9) and (10)]

$$\langle \dot{X}, \omega\rangle = \dot{\overbrace{\langle X, \omega\rangle}} - \langle X, \dot{\omega}\rangle$$

$$\begin{aligned}\alpha_1 \wedge \cdots \wedge \alpha_{\rho_\infty} \wedge i(\dot{X})\,\mathrm{d}\omega \\ = \dot{\overbrace{\alpha_1 \wedge \cdots \wedge \alpha_{\rho_\infty} \wedge i(X)\,\mathrm{d}\omega}} - \dot{\alpha}_1 \wedge \cdots \wedge \alpha_{\rho_\infty} \wedge i(X)\,\mathrm{d}\omega - \cdots \\ - \alpha_1 \wedge \cdots \wedge \dot{\alpha}_{\rho_\infty} \wedge i(X)\,\mathrm{d}\omega - \alpha_1 \wedge \cdots \wedge \alpha_{\rho_\infty} \wedge i(X)\,\mathrm{d}\dot{\omega}.\end{aligned}$$

Since $\langle X, \omega\rangle$ and $\alpha_1 \wedge \cdots \wedge \alpha_{\rho_\infty} \wedge i(X)\,\mathrm{d}\omega$ are identically zero, and $\dot{\alpha}_1, \cdots, \dot{\alpha}_{\rho_\infty}$ and $\dot{\omega}$ are in $\mathcal{H}_\infty$ (because $\mathcal{H}_\infty$ is invariant by time-differentiation), all the terms on the right-hand sides above are zero, which implies $\langle \dot{X}, \omega\rangle = 0$ and $\alpha_1 \wedge \cdots \wedge \alpha_{\rho_\infty} \wedge i(\dot{X})\,\mathrm{d}\omega = 0$. Hence, $\dot{X}$ is in $\mathcal{G}_\infty$ because this is true for all $\omega \in \mathcal{H}_\infty$. Now, take η in $\mathcal{C}(\mathcal{H}_\infty)$. For all $X \in \mathcal{G}_\infty$, $\langle X, \dot{\eta}\rangle = \dot{\overbrace{\langle X, \eta\rangle}} - \langle \dot{X}, \eta\rangle = 0$ where $\dot{\overbrace{\langle X, \nu\rangle}}$ is zero because $\langle X, \eta\rangle$ is identically zero, and $\langle \dot{X}, \eta\rangle$ is zero because $\dot{X}$ is in $\mathcal{G}_\infty$. This proves $\eta \in \mathcal{C}(\mathcal{H}_\infty) \Rightarrow \dot{\eta} \in \mathcal{C}(\mathcal{H}_\infty)$, and therefore the lemma. ∎

References

[1] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis and Applications*, 2nd. ed., Applied Mathematical Sciences, vol. 75. New York: Springer, 1988.

[2] E. Aranda-Bricaire, C. H. Moog, and J.-B. Pomet, "Infinitesimal Brunovsky form for nonlinear systems with applications to dynamic linearization," presented at the Ext. Workshop Geometry Nonlinear Control, Warsaw, June 1993; *Proceedings*, Banach Center Publications, to appear.

[3] ——, "Feedback linearization: A linear algebraic approach," in *Proc. 32nd IEEE Conf. Decision Contr.*, 1993, pp. 3441–3446.

[4] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmitt, and P. A. Griffiths, *Exterior Differential Systems*, Mathematical Sciences Research Institute Publications, vol. 18, New York: Springer, 1991.

[5] L. Cao and Y. F. Zheng, "On minimal compensators for decoupling control," *Syst. Contr. Lett.*, vol. 18, pp. 121–128, 1992.

[6] B. Charlet, J. Lévine, and R. Marino, "On dynamic feedback linearization," *Syst. Contr. Lett.*, vol. 13, pp. 143–151, 1989.

[7] ——, "Sufficient conditions for dynamic feedback linearization," *SIAM J. Contr. Opt.*, vol. 29, pp. 38–57, 1991.

[8] D. Claude, "Everything you always wanted to know about linearization," in *Algebraic and Geometric Methods in Nonlinear Control Theory*, M. Fliess and M. Hazewinkel, Eds. Dordrecht, The Netherlands: Reidel, 1986, pp. 181–220.

[9] M. D. Di Benedetto, J. Grizzle, and C. H. Moog, "Rank invariants of nonlinear systems," *SIAM J. Contr. Opt.*, vol. 27, pp. 658–672, 1989.

[10] M. Fliess, "A new approach to the noninteracting control problem in nonlinear system theory," in *Proc. 23rd Allerton Conf.*, Monticello, 1985, pp. 123–129.

[11] M. Fliess, "Some remarks on the Brunovsky canonical form," *Kybernetika*, vol. 29, pp. 417–422, 1993.

[12] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "On differentially flat nonlinear systems," in *Proc. 2nd IFAC NOLCOS Symp.*, M. Fliess, Ed., Bordeaux, 1992, pp. 408–412.

[13] ——, "Sur les systèmes non linéaires différentiellement plats," *C. R. Acad. Sci. Paris*, vol. 315-I, pp. 619–624, 1992.

[14] ——, "Linéarisation par bouclage dynamique et transformations de Lie-Bäcklund," *C. R. Acad. Sci. Paris*, vol. 317-I, pp. 981–986, 1993.

[15] ——, "Towards a new differential geometric setting in nonlinear control," presented at the *Int. Geometrical Colloq.*, Moscow, May 1993; to appear in the *Proceedings*.

[16] ——, "Flatness and defect of nonlinear systems: Introductory theory and examples," *Int. J. Contr.*, to appear.

[17] R. B. Gardner and W. F. Shadwick, "The GS algorithm for exact linearization to Brunovsky normal form," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 224–230, 1992.

[18] L. R. Hunt, R. Su, and G. Meyer, "Design for multi-input nonlinear systems," in *Differential Geometric Control Theory*, R. Brockett, R. Millmann, and H. Sussmann, Eds. Boston: Birkhäuser, 1983, pp. 268–298.

[19] A. Isidori, C. H. Moog, and A. De Luca, "A sufficient condition for full linearization via dynamic state feedback," in *Proc. 25th IEEE Conf. Dec. Contr.*, 1986, pp. 203–208.

[20] A. Isidori and C. H. Moog, "On the nonlinear equivalent of the notion of transmission zeros," in *Modeling and Adaptive Control*, Lecture Notes in Control and Information Science, vol. 105, C. I. Byrnes, and A. Kurzhanski, Eds. New York: Springer-Verlag, 1988, pp. 146–158.

[21] A. Isidori, *Nonlinear Control Systems*, 2nd ed., Communications and Control Engineering Series. New York: Springer-Verlag, 1989.

[22] B. Jacubczyk and W. Respondek, "On linearization of control systems," *Bull. Acad. Pol. Sci., Ser. Sci. Math.*, vol. 18, pp. 517–522, 1980.

[23] B. Jacubczyk, "Remarks on equivalence and linearization of nonlinear systems," in *Proc. 2nd IFAC NOLCOS Symp.*, M. Fliess, Ed., Bordeaux, 1992, pp. 393–397.

[24] P. Martin, "Contribution àl'étude des systèmes non linéaires différentiellement plats," Thèse de Doctorat, Ecole des Mines de Paris, 1992.

[25] ——, "A geometric sufficient condition for flatness of systems with m inputs and m+1 states," in *Proc. 32 IEEE Conf. Dec. Contr.*, pp. 3431–3436, 1993.

[26] C. H. Moog, "Nonlinear decoupling and structure at infinity," *Math. Contr. Signals Syst.*, vol. 1, pp. 257–268, 1988.

[27] J.-B. Pomet, "On dynamic feedback linearization of four-dimensional affine control systems with two inputs," preprint, 1993.

[28] J. B. Pomet, C. H. Moog, and E. Aranda, "A non-exact Brunovsky form and dynamic feedback linearization," in *Proc. 31st IEEE Conf. Dec. Contr.*, pp. 2012–2017, 1992.

[29] J.-B. Pomet, "A differential geometric setting for dynamic equivalence and dynamic linearization," presented at the Ext. Workshop Geometry Nonlinear Contr., Warsaw, June 1993; *Proceedings*, Banach Center Publications, to appear.

[30] W. M. Sluis, "Absolute equivalence and its applications to control theory," Ph.D. dissertation, Univ. Waterloo, 1992.