A remark on the design of time-varying stabilizing feedback laws for controllable systems without drift

Jean-Michel Coron
Université Paris-Sud
Laboratoire d'Analyse Numérique
Bât. 425
91405 ORSAY cédex (France)

Jean-Baptiste Pomet*
Laboratoire d'Automatique de Nantes
E. C. N.
1, rue de la Noë
44072 NANTES cédex 03 (France)

Presented at: *2nd IFAC Nonlinear Cont. Systems Design Symposium* (NOLCOS), Bordeaux, France, 1992.
Published in the proceedings (M. Fliess editor), pp.413-417.

Abstract : This paper gives an approach to design time-varying feedback laws for controllable systems without drift. This approach is based on a time-varying Lyapunov function.

KEYWORDS : Nonlinear systems, Asymptotic stabilization, Controllability, Time-varying feedback.

*This article was written when this author was with Université Catholique de Louvain, CE.S.A.ME., Bât. Maxwell, Place du levant 3 , 1348 Louvain-la-Neuve (Belgium). This paper presents research results partially obtained, for this author, within the Belgian Programme on Interuniversity Poles of attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility rests with the authors.

1 Introduction

Let $f = (f_1, \ldots, f_m)$ be m vector fields of class $\mathcal{C}^\infty$ on $I\!R^n$. We consider the control system without drift

$$\dot{x} \quad = \quad \sum_{k=1}^{m} u_k f_k \quad . \tag{1}$$

Let $\mathrm{Lie}(f)$ be the Lie algebra of vector fields generated by f. Through all this paper, we assume that, for all x in $I\!R^n \backslash \{0\}$,

$$\mathrm{Lie}(f)\,(x) \quad = \quad \{\, h(x)\,;\, h \in \mathrm{Lie}(f)\,\} \quad = \quad I\!R^n \quad . \tag{2}$$

It is well known that (2) implies (and is in fact equivalent if the vector fields $f_1, \ldots, f_m$ are analytic) that system (1) is completely controllable on $I\!R^n \backslash \{0\}$. However, it has been proven by R. Brockett in [1] that (2), even with $I\!R^n$ instead of $I\!R^n \backslash \{0\}$, does not imply that system (1) can be locally asymptotically stabilized by means of a continuous feedback law $u = u(x)$. In fact, a simple consequence of [1] is (see [7]) :

Proposition 1 *If $m < n$ and*

$$\mathrm{Rank}\,\{\, f_1(0), \ldots, f_m(0)\,\} \quad = \quad m \tag{3}$$

then (1) cannot be locally asymptotically stabilized by means of a continuous feedback law $u = u(x)$, nor can it even be locally asymptotically stabilized by means of a continuous dynamic feedback law $u = u(\xi, x)$, $\dot{\xi} = g(\xi, x)$.

C. Samson has suggested in [8] (see also [9]) to stabilize some systems of the form (1) by means of a continuous time-varying periodic feedback law $u = u(t, x)$. He has proved in particular, with proposing an explicit expression for the feedback, that the system

$$\begin{array}{rcl} \dot{x_1} & = & u_1 \\ \dot{x_2} & = & x_1 u_2 \\ \dot{x_3} & = & u_2 \quad , \end{array} \tag{4}$$

which satisfies (2) but also (3), can be globally asymptotically stabilized by means of a time-varying continuous feedback law, periodic with respect to time. He studied some other systems from nonholonomic robotics, see [9]. This example was also extended by R. Sépulchre in [10].

Note that the interest of time-varying feedback laws for stabilization of systems (with a drift) had been previously shown by E. Sontag and H. Sussmann in [13].

It turns out that these time-varying stabilizing control laws exist in general for systems (1) meeting condition (2), and this is proved in [3]. On the other hand, [7] contains a method to actually design these laws under a more restrictive assumption than (2). The purpose of the present paper is to show how the results contained in [3] may be used to extend the design method proposed in [7] to the general situation (2), thus giving a way to design the control laws whose existence was established in [3].

Next section 2 is an overview of the results presented in [3] and [7]. Section 3 presents the main result of this paper and the resulting design method. Section 4 is a brief conclusion.

2 Two different approaches

The following result, proved in [3], establishes existence of stabilizing time-varying laws under assumption (2) :

Theorem 2 ([3]) *If (2) is met, there exists, for any positive T, a map $u : I\!R \times I\!R^n \to I\!R^m$ such that*

$$u \quad \in \quad \mathcal{C}^\infty(I\!R \times I\!R^n, I\!R^m) \quad , \tag{5}$$

$$u(t, 0) \quad = \quad 0 \quad \forall t \in I\!R \quad , \tag{6}$$

$$u(t + T, x) \quad = \quad u(t, x) \quad \forall t \in I\!R,\ \forall x \in I\!R^n \quad , \tag{7}$$

and

$$\left.\begin{array}{l} 0 \in I\!R^n \text{ is a globally assymptotically stable} \\ \text{equilibrium point of} \quad \dot{x} \;=\; \displaystyle\sum_{k=1}^{m} u_k(t, x) f_k(x) \quad . \end{array}\right\} \tag{8}$$

The proof of this theorem relies on the analysis of the controllability of the linearized equation around some selected trajectories of (1). More precisely, for a given time-varying feedback law $u : I\!R^n \times I\!R \to I\!R^m$, let $\phi(u) : I\!R^n \times I\!R \to I\!R^n$ be defined by

$$\frac{\partial \phi(u)}{\partial t}(t, x) \quad = \quad \sum_{k=1}^{m} u_k(t, \phi(u)(t, x))\, f_k(\phi(u)(t, x)) \tag{9}$$

$$\phi(u)(0, x) \quad = \quad x \quad , \tag{10}$$

i.e. $t \mapsto \phi(u)(t, x)$ is the solution starting from x at time 0 of the closed-loop system obtained by applying the time-varying feedback u to (1). The linearized control system along this trajectory is

$$\dot{y} \quad = \quad \frac{\partial}{\partial x}\left[\sum_{i=1}^{m} u_i(t, \phi(u)(t, x)) f_i(\phi(u)(t, x))\right] y \\ + \sum_{i=1}^{m} w_i f_i(\phi(u)(t, x)) \tag{11}$$

where $w = (w_1, \ldots, w_m)$ is the control and $x \in I\!R^n$ and the law u act as parameters. The main step in [3] is to prove :

Proposition 3 ([3, Sections 2,3,4]) *If system (1) satisfies (2), there exists $\overline{u}$ satisfying (5), (6), (7) such that :*

$$\left\| \sum_{k=1}^{m} \overline{u}_k(t, x) f_k(x) \right\| \quad \leq \quad 1 \qquad \forall\,(t, x) \in I\!R \times I\!R^n \tag{12}$$

$$\phi(\overline{u})(x, T) \quad = \quad x \qquad \forall\, x \in I\!R^n \tag{13}$$

and

$$\left.\begin{array}{l} \text{for } u = \overline{u} \text{ and for all } x \text{ in } I\!R^n \backslash \{0\}, \\ \text{the linear system (11) is controllable} \\ \text{with impulsive controls at time } t = \frac{3T}{4}. \end{array}\right\} \tag{14}$$

For a proper definition of "controllable with impulsive controls", see e.g. T. Kailath [6, p.614]. It follows easily from a theorem by L.M. Silverman and H.E. Meadows [11] (see also [6, p.614]) that (14) is equivalent to

$$\left.\begin{array}{l}\mathrm{Span}\left\{\left[\mathcal{L}(\overline{u})^p \widetilde{f}_k\right](\tfrac{3T}{4}, x)\,;\ \begin{array}{l}p \geq 0\,,\\ k=1,\ldots,m\end{array}\right\}\\ \qquad = \ I\!R^n \quad \forall x \in I\!R^n \backslash \{0\}\end{array}\right\} \tag{15}$$

where $\widetilde{f}_k \in \mathcal{C}^\infty(I\!R \times I\!R^n)$ is defined by :

$$\widetilde{f}_k(t,x) \ = \ f_k(x)$$

and where, for $u \in \mathcal{C}^\infty(I\!R \times I\!R^n, I\!R^m)$, $\mathcal{L}(u) : \mathcal{C}^\infty(I\!R \times I\!R^n, I\!R^n) \to \mathcal{C}^\infty(I\!R \times I\!R^n, I\!R^n)$ is defined by :

$$\mathcal{L}(u)\, X \ = \ \frac{\partial X}{\partial t} + [\, \sum_{k=1}^m u_k f_k \,,\, X\,] \tag{16}$$

where $[\,.\,,\,.\,]$ denotes the classical Lie bracket and, by convention, $\mathcal{L}(u)^0\, X \ = \ X$.

In fact, (15) is proved directly in [3], see [3, Equation (1.15)].

From (13), the control $\overline{u}$ guarantees that all the solutions of the closed-loop system are periodic, and therefore this control does not solve our stabilization problem. However, controllability of (14) implies that for any η in $\mathcal{C}^\infty(I\!R^n, [0,+\infty))$, there exists w satisfying (5), (6) and (7) such that :

$$\left(\frac{\partial \phi}{\partial u}(\overline{u}).w\right)(x,T) \ = \ -\,\eta(x)\,x \ ; \tag{17}$$

hence one may hope that $u = \overline{u} + \varepsilon w$ is a solution of our problem if ε is small enough and η has been chosen positive on $I\!R^n \backslash \{0\}$. This is proven to be true in [3, Section 5].

In [7], following a different approach, theorem 2 is proved under the extra assumption that, for all x in $I\!R^n \backslash \{0\}$, **for all x ds l'eq ?**

$$\mathrm{Rank}\left\{\mathrm{ad}_{f_1}^j f_k(x)\,,\ j \geq 0\,,\ 1 \leq k \leq m\right\} \ = \ n \tag{18}$$

One of the main interests of [7] is that it provides a method to design explicitely the control laws, and leads to very simple expressions in many interesing cases (for example $f_1 = \frac{\partial}{\partial x_1}$). Moreover it has the advantage to provide also a Lyapunov function. This is convenient when the system to be stabilized is not (1), but (1) with pure integrators added, see J. Tsinias [14] or [3, Section 6]. This is also usefull when analysing the robustness of the obtained closed-loop stability.

The method used in [7] is the following. Let α in $\mathcal{C}^\infty(I\!R \times I\!R^n, I\!R^m)$ be such that, for all t in $I\!R$ and x in $I\!R^n$:

$$\alpha(\,t + 2\pi\,,\,x\,) \ = \ \alpha(\,t\,,\,x\,) \tag{19}$$

$$\alpha(\,-t\,,\,x\,) \ = \ -\,\alpha(\,t\,,\,x\,) \tag{20}$$

$$\alpha(\,t\,,\,0\,) \ = \ 0 \tag{21}$$

$$|\alpha(t,x)|\,\|f_1(x)\| \ \leq \ K\,(1 + \|x\|) \tag{22}$$

where K is some positive constant. Let $u^\star(\alpha)$ be defined by

$$u^\star(\alpha)(t,x) \ = \ (\,\alpha(t,x), 0, \ldots, 0\,) \ , \tag{23}$$

and the non-negative function V by :

$$V(t,x) \ = \ \tfrac{1}{2}\left\|\phi(u^\star(\alpha))^{-1}(x,t)\right\|^2 \ . \tag{24}$$

where $\phi(u)^{-1}$ is defined by $\phi(u)\left(\phi(u)^{-1}(x,t), t\right) = x$, and let the time-varying stabilizing control be defined by

$$u_1(t,x) \ = \ \alpha(t,x) \,-\, L_{f_1} V(t,x) \tag{25}$$

$$u_k(t,x) \ = \ -\,L_{f_k} V(t,x) \quad \forall k \in [2,m] \ . \tag{26}$$

Then the following theorem is proved in [7] :

Theorem 4 *If system (1) satisfies (18) and α is chosen such that (19), (20), (21) and (22) are met, as well as (27) :*

$$\left.\begin{array}{rcl}\frac{\partial^j \alpha}{\partial t^j}(t,x) &=& 0 \quad j \geq 0\\ L_{f_k} V(t,x) &=& 0 \quad 1 \leq k \leq m\end{array}\right\} \Rightarrow \ x = 0 \quad , \tag{27}$$

then the map u given by (23)-(24) satisfies (5), (6), (7) and (8). Moreover, the function V is strictly decreasing along the nonzero solutions of $\dot{x} \ = \ \sum_{k=1}^m u_k(t,x) f_k(x)$.

There exists α meeting the required conditions. A possible choice is

$$\alpha(t,x) \ = \ \frac{\|x\|^2}{(1 + \|x\|^2)\,(1 + \|f_1(x)\|^2)}\,\sin t \ . \tag{28}$$

3 Main result

Although the approaches in [7] and [3] are somehow different, it is to be noticed that $u^\star = (\alpha, 0, \ldots, 0)$ in (23) plays exactly the same role as $\overline{u}$. Actually, under the extra assumption (18), one may, in proposition 3, chose $\overline{u} = u^\star$ with $u^\star$ given by (28); more precisely, such a $\overline{u}$ satisfies (12), (13) and (14). Most of the difficulties in [3] (contained in the proof of proposition 3) therefore disappear if (18) is satisfied. Without the extra assumption (18), the appraoch in [7] fails, at least if $u^\star$ is still restricted to be of the form (23), and it is no longer sufficient to choose $\overline{u}$ in proposition 3 with $m-1$ zero entries.

The idea of this paper is to take advantage of the generality [3] and the simplicity of [7]. This is done by applying the design method coming from [7], but with $u^\star$ given by [3], i.e. we allow all the entries of $u^\star$ to be nonzero, and take precisely $u^\star = \overline{u}$ with $\overline{u}$ given by proposition 3. From now on, we no longer assume (18).

Let $\overline{u}$ be as in proposition 3, and let $V : I\!R \times I\!R^n \to [0,+\infty)$ be defined by :

$$V(t,x) \ = \ \tfrac{1}{2}\left\|\phi(\overline{u})^{-1}(x,t)\right\|^2 \tag{29}$$

where $\phi(\overline{u})^{-1}$ is defined by $\phi(\overline{u})(\phi(\overline{u})^{-1}(x,t),t) = x$, as in (24), and let the stabilizing control u be defined by

$$u_k(t,x) \;=\; \overline{u}_k(t,x) \;-\; L_{f_k}V(t,x) \quad \forall k \in [1,m] \;. \tag{30}$$

Our result is :

Theorem 5 *Under assumption (2), the above constructed map u satisfies (5), (6), (7) and (8). Moreover :*

$$V \;\in\; \mathcal{C}^\infty(I\!R \times I\!R^n, [0,+\infty)) \tag{31}$$

$$V(t+T,x) \;=\; V(t,x) \quad \forall (t,x) \in I\!R \times I\!R^n \tag{32}$$

$$V(t,x) \;=\; 0 \;\Leftrightarrow\; x = 0 \tag{33}$$

$$\left.\begin{array}{l} \forall K > 0, \;\; \{\, x \,|\, \exists t,\, V(t,x) \le K \,\} \\ \qquad\qquad \textit{is a bounded set} \end{array}\right\} \tag{34}$$

$$\frac{\partial V}{\partial t} \;+\; \sum_{k=1}^{m} \overline{u}_k \, L_{f_k} V \;=\; 0 \tag{35}$$

$$\left.\begin{array}{l} V \textit{ is non-increasing along the} \\ \textit{solutions of } \;\; \dot{x} \;=\; \sum_{k=1}^m u_k(t,x) f_k(x) \end{array}\right\} \tag{36}$$

$$\left.\begin{array}{r} V(\,T\,,\,\phi(u)(x,T)\,) \;<\; V(\,0\,,\,\phi(u)(x,0)\,)\,, \\ \forall (t,x) \in I\!R \times I\!R^n \backslash \{0\} \end{array}\right\} \tag{37}$$

Proof of theorem 5 : The proof is similar to the one given in [7]. Since $\overline{u}$ satisfies (5), (6), (7), (12) and (13), V given by (29) satisfies (31), (32) and (33), and u satisfies (5), (6) and (7). Property (34) is a consequence of relations (29), (32) and the fact that, thanks to (12), $x \mapsto \phi(\overline{u})(t,x)$ is a homeomorphism from $I\!R^n$ to $I\!R^n$. By (29), we have

$$V(\,t\,,\,\phi(\overline{u})(t,x)\,) \;=\; \tfrac{1}{2}\|x\|^2 \;. \tag{38}$$

Differentiating (38) with respect to t, we get (35). From (30) and (35), we have :

$$\frac{d}{dt} V(t,\phi(u)(x,t)) \;=\; -\sum_{k=1}^{m} \left(L_{f_k} V(t,\phi(u)(x,t))\right)^2 \tag{39}$$

which obviously implies (36). Note that (8) is a consequence of (7), (32), (33), (34), (36) and (37), which imply, for instance, asymptotic stability of the Poincaré map $x \mapsto \phi(u)(x,T)$.

The proof of (37) is an adaptation of V. Jurdjevic and J.-P. Quinn [4] –see also [7]–. Let us fix x in $I\!R^n$; for simplicity, we will write $\phi(t)$ for $\phi(u)(x,t)$ and $\overline{\phi}(t)$ for $\phi(\overline{u})(x,t)$. (39) clearly implies

$$V(\,T\,,\,\phi(u)(x,T)\,) \;\le\; V(\,0\,,\,\phi(u)(x,0)\,) \tag{40}$$

where the inequality is an equality if and only if

$$L_{f_k} V(t,\phi(t)) \;=\; 0 \quad \forall k = 1,\ldots,m,\; \forall t \in [0,T] \;. \tag{41}$$

We now assume that (40) is an equality. By (41) and (30),

$$\phi(t) \;=\; \overline{\phi}(t) \quad \forall t \in [0,T] \;. \tag{42}$$

Let now X be in $\mathcal{C}^\infty(I\!R \times I\!R^n, I\!R^n)$. For any such X, we have

$$\begin{aligned} &\frac{d}{dt}\left(L_X V(t,\overline{\phi}(t))\right) \\ &\quad= \frac{\partial L_X V}{\partial t}(t,\overline{\phi}(t)) \;+\; \left(L_{\sum_{k=1}^m \overline{u}_k f_k} L_X V\right)(t,\overline{\phi}(t)) \\ &\quad= \left(L_{\mathcal{L}(\overline{u})\,X} V\right)(t,\overline{\phi}(t)) \\ &\qquad + L_X \left(\frac{\partial V}{\partial t} + \sum_{k=1}^m \overline{u}_k \, L_{f_k} V\right)(t,\overline{\phi}(t))\,. \end{aligned}$$

Hence, from (35),

$$\frac{d}{dt}\left(L_X V(t,\overline{\phi}(t))\right) \;=\; \left(L_{\mathcal{L}(\overline{u})\,X} V\right)(t,\overline{\phi}(t)) \;\;, \tag{43}$$

and, by induction on p,

$$\frac{d^p}{dt^p}\left(L_X V(t,\overline{\phi}(t))\right) \;=\; \left(L_{\mathcal{L}(\overline{u})^p\,X} V\right)(t,\overline{\phi}(t)) \;. \tag{44}$$

By (44), (42) and (41), we have

$$\left(L_{\mathcal{L}(\overline{u})^p\,f_k} V\right)(\frac{3T}{4},\overline{\phi}(\frac{3T}{4})) \;=\; 0 \qquad \begin{array}{l} k = 1,\ldots,m \\ p \ge 0 \end{array} \;. \tag{45}$$

Hence, by (15),

$$\frac{\partial V}{\partial x}(\frac{3T}{4},\overline{\phi}(\frac{3T}{4})) \;=\; 0 \;. \tag{46}$$

Since $\phi(\overline{u})^{-1}(\frac{3T}{4},.)$ is a diffeomorphism, (29) implies, together with (46), that $\overline{\phi}(\frac{3T}{4}) = 0$ and therefore that $x = 0$. This proves (37) and completes the proof of the theorem. ∎

4 Conclusion

Theorem 5 leads to a design method, made of three separate parts :

1. Find $\overline{u}$ meeting (12), (13) and (14). Existence of such a $\overline{u}$ is established by proposition 3. The proof of proposition 3 actually provides a guide to find a suitable $\overline{u}$.
2. Compute V from $\overline{u}$ according to (29).
3. The control u is then given by (30).

The method proposed in [7] for the case when assumption (18) was met was a particular case of this one. Finding a suitable $\overline{u}$ was however, due to the additionnal assumption, somehow simpler : it may be taken of the form $(\alpha, 0, \ldots, 0)$ with α explicitly given by (28).

The method proposed in [3] for the same general case as here differs from the present one at steps "2" and "3" above. In [3], these two steps are replaced by a construction of u from $\overline{u}$ which relies on the controllability by impulsive controls of the time-varying linear system obtained with $\overline{u}$. This construction leads to much more complicated computations than those of step 2 and 3, even when the f_k's are simple (e.g. when assumption (18) is met).

We have therefore been able to derive here a method keeping most of the advantages of [7], but not restricted to the simpler case of assumption (18). The hard part in the design process remains the choice of a suitable $\overline{u}$. It turns out however that, in many practical cases, not only a suitable $\overline{u}$ may be exhibited, but there is a wide choice of possible solutions, and $\overline{u}$ may be considered as a "design" parameter. An interesting subject for future research is therefore to study the relationship between the choice of $\overline{u}$ and the "performances" of the controller.

Aknowledgements

The authors would like to express there appreciation to Laurent Praly for fruitful discussions on this paper.

References

[1] R. W. Brockett : Asymptotic stability and feedback stabilization, in: R.W. Brockett, R.S. Millman and H. J. Sussmann Eds., *Differential Geometric Control Theory*, Birkäuser Basel-Boston, 1983.

[2] G. Campion, B. d'Andréa-Novel and G. Bastin : Controllability and State-Feedback Stabilization of Non Holonomic Mechanical Systems, *Int. Worshop in Adaptive and Nonlinear Control: Issues in Robotics*, Grenoble, France, Springer-Verlag, 1990.

[3] J.-M. Coron : Global asymptotic stabilization for controllable systems without drift, *preprint*, 1991.

[4] V. Jurdjevic and J.P. Quinn : Controllability and Stability. *Journal of diff. eq.* Vol. 28 (1978) pp.381-389.

[5] J.-P. LaSalle : Stability theory for ordinary differential equations. *Journal of diff. eq.* Vol. 4 (1968) pp.57-65.

[6] T. Kailath : *Linear Systems*. Prentice-Hall, 1980.

[7] J.-B. Pomet : Explicit design of time-varying stabilizing control laws for a class of time-varying controllable systems without drift, 1991. To appear in *Systems & Contol Letters*.

[8] C. Samson : Velocity and torque feedback control a nonholonomic cart. *Int. Worshop in Adaptive and Nonlinear Control: Issues in Robotics*, Grenoble, Springer, 1990.

[9] C. Samson : "Time-varying stabilization of a Nonholonomic Car-like Mobile Robot." *Rapport INRIA* Sophia-Antipolis, France, 1990.

[10] R. Sépulchre : private communication, 1991.

[11] L.M. Silverman and H.E. Meadow : "Controllability and observability in time variable linear systems" SIAM J. of Control **5** (1967), pp. 64-73.

[12] E. D. Sontag : Feedback Stabilization of Nonlinear Systems, in : M.A. Kaashoek & al. Eds., *Robust Control of Linear Systems and Nonlinear Control, Proceedings of MTNS-89*, Birkäuser Basel-Boston, 1990.

[13] E. D. Sontag and H. J. Sussmann : Remarks on continuous feedback. 20th IEEE Conf. on Decision and Control, Albuquerque, 1980, **2** pp. 916-921.

[14] J. Tsinias : Sufficient Lyapunov-like conditions for stabilization. *Math. of Control Signals and Systems* **2** (1989), pp. 343-357.