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par

Jean-Baptiste Pomet

Sujet :

Équivalence et linéarisation des systèmes de contrôle

Manuscrit examiné par les rapporteurs suivants :

MM. Robert BRYANT,
Witold RESPONDEK,
Pierre ROUCHON.

Soutenue le 23 octobre 2009 devant le jury composé de :

MM. Bernard BONNARD,
Ivan KUPKA (Président),
Claude LOBRY,
Laurent PRALY,
Witold RESPONDEK,
Pierre ROUCHON,
Michel SORINE.

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Sophia Antipolis, le 15 octobre 2009.

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Avant-propos

Pour garder une certaine unité à ce mémoire, on s'est cantonné à un seul sujet : les transformations et l'équivalence des systèmes de contrôle.

Une liste et une brève notice scientifique de l'ensemble des travaux de l'auteur et candidat ont tout de même été insérées en annexe au chapitre 1 (page 21), totalement indépendante du reste du mémoire ; la bibliographie de fin de mémoire ne reprend que les publications nécessaires au texte.

Huit articles publiés sont reproduits aux chapitres 2 à 9, avec seulement quelques modifications pour améliorer la manière dont ils se référencent les uns les autres. Ils sont précédés d'un chapitre 1 – le seul original – dont le but est de présenter de manière unifiée les résultats et la problématique de ces articles.

Chapitre 1

Un tour d'horizon des problèmes d'équivalence en contrôle

On cherche ici à énoncer de manière unifiée et synthétique les résultats contenus dans les articles reproduits aux chapitres suivants. C'est aussi le prétexte à une présentation –en partie originale– des différentes notions d'équivalence des systèmes de contrôle, qui peut être instructive pour le lecteur, mais qui ne prétend pas à l'exhaustivité : ce chapitre n'est pas un « survey[1] *» des résultats sur le sujet ; la bibliographie est très partielle.*

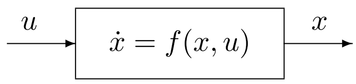
Les résultats donnés comme théorèmes sont des énoncés simplifiés de résultats contenus dans les articles reproduits aux chapitres 2 à 9 ; la référence est à chaque fois indiquée.

1.1 Introduction

On s'intéresse aux systèmes donnés par une équation de la forme

$$\dot{x} = f(x,u)\ , \quad x \in \mathbb{R}^n\ ,\ u \in \mathbb{R}^m\ . \tag{1.1}$$

Le point désigne la dérivée par rapport à une variable indépendante t que l'on appelle le temps ; x est l'*état* et u le *contrôle* ou la *commande*, ou l'*entrée*. C'est l'équation d'un système de contrôle en temps continu et à m contrôles scalaires $u = (u_1, \ldots, u_m)$ en représentation d'état de dimension finie n. Le diagramme



(1.2)

indique que l'entrée est une quantité libre, décidée par un utilisateur ou un dispositif de commande, et que le système produit un comportement, c'est-à-dire une évolution des variables d'état, qui dépend de cette entrée.

Le problème qui nous intéresse ici est de savoir déterminer si deux systèmes de ce type se ressemblent. Cela est bien vague et doit être précisé. Bien sûr l'idéal serait une classification pour une relation d'équivalence pertinente, et le plus souvent, une telle relation d'équivalence est définie par une classe de transformations sur les systèmes, deux systèmes étant équivalents si ils sont transformés l'un en l'autre par une transformation de la classe.

On s'intéresse aux différentes notions d'équivalence, et on fait quelques contributions sur leurs significations pour le contrôle, les conditions d'équivalence, et en particulier les conditions de linéarisation (équivalence à un système linéaire).

[1] Le lecteur aura noté que ce chapitre est rédigé en français, et il se rendra compte par la suite que les autres le sont en anglais. Les mots anglais dans le texte français n'ont d'autre but que de l'accoutumer un peu.

Motivations. Il est tout d'abord très naturel, une fois définie un classe d'objets, de vouloir les « classifier ». Ce n'est pas seulement une manie de mathématicien, c'est aussi une appréhension de la connaissance.

Point de vue du contrôle. Savoir qu'un système est équivalent à un autre, plus simple ou pour lequel un problème a déjà été résolu, peut permettre de transposer la solution du problème pour l'un en solution pour l'autre, qu'il s'agisse de stabiliser un point d'équilibre, de rejeter des perturbations ou de commande optimale. On ne s'intéresse ici à aucun de ces problèmes de contrôle par eux-mêmes, mais exclusivement à leurs transformations.

Point de vue modélisation / identification. On a appelé (1.1) un système, mais c'est plutôt un modèle mathématique qu'un système physique. C'est l'équation dont on espère qu'elle reproduit les comportements du système physique, et sur la base duquel on va chercher à construire des lois de contrôle. Il serait plus juste de dire qu'un classification des équations de type (1.1) est une *classification des modèles*. Ce point de vue est très important, et une telle classification, même sommaire, sous-tend toute théorie de « l'identification non-linéaire » cohérente, si cela peut exister un jour. Le succès de l'identification linéaire, véritable corps de doctrine, est bâti sur une solide compréhension de la structure des systèmes linéaires.

Qu'est-ce qu'un système ? Il y a de très nombreuses réponses à cette question. Le point de vue entrée-sortie a longtemps prévalu en contrôle linéaire et correspond à une réalité (domaine fréquentiel, fonction de transfert). L'équation (1.1) est déjà une représentation d'état et on pourrait objecter (surtout au vu du « point de vue modélisation / identification ») que ce n'est peut-être pas le bon choix ; en tout cas il y en aurait sûrement d'autres.

Pour se cantonner à l'équation (1.1), on peut encore la voir de plusieurs manières.

- C'est, pour chaque $t \mapsto u(t)$ donné, une équation différentielle instationnaire, qui produit une solution $t \mapsto x(t)$ dès que l'on se donne une condition initiale $x(0)$. Sorte de point de vue entrée-sortie, ou plutôt entrée-état, adopté parfois en contrôle optimal ou commandabilité.
- C'est un poly-système dynamique, ou une famille de champs de vecteurs, obtenus par exemple en prenant différentes valeurs de u constant. Point de vue fécond pour la commandabilité, à cause du sens géométrique des champs et de leurs flots.
- C'est une équation différentielle sous-déterminée qui lie les fonctions du temps $t \mapsto (x(t), u(t))$ (n équations scalaires liant $n + m$ fonctions du temps).
- C'est l'ensemble des solutions de cette équation différentielle sous-déterminée.

Nous adoptons les deux derniers points de vue. Le point de vue de J. Willems [109] est le dernier puisque, dans sa « behavioral approach », il définit l'ensemble des solutions par une série d'axiomes, sans recourir a priori à une équation ; le point de vue de M. Fliess [33], qui utilise l'algèbre différentielle pour définir et étudier cela, est également centré sur les deux derniers.

Poursuivons maintenant, en développant un point de vue forcément partial, et des résultats obtenus ces dernières années.

1.2 Systèmes

1.2.1 Le cas « sans contrôle »

Si $m = 0$, l'équation (1.1) devient $\dot{x} = f(x)$, équation différentielle ordinaire déterminée, ou système dynamique différentiable. Une solution est une application $t \mapsto x(t)$ qui satisfait l'équation pour (presque) tout temps.

Il est connu que la solution est entièrement déterminée par sa condition initiale, et qu'être solution force une certaine régularité : même si l'on admet a priori comme solution des objets peu différentiables en prenant la dérivée en un sens généralisé, une solution est forcément aussi différentiable que f elle-même. On peut définir un *flot* à chaque temps t qui est un difféomorphisme, associant à une condition initiale la valeur au temps t de la solution correspondante.

Les véritables systèmes de contrôle sont ceux pour lesquels m n'est pas nul ; on suppose maintenant $m \geq 1$.

1.2.2 Systèmes de contrôle et équations différentielles sous-déterminées.

Revenons à l'équation (1.1), qui représente les systèmes que nous étudions ici. On peut voir (1.1) comme un système de n équations différentielles liant $n+m$ fonctions du temps, les coordonnées de $t \mapsto (x(t), u(t))$. Par *solution* de (1.1) on entend une application $I \to \mathbb{R}^{n+m}$ (I un intervalle de temps)

$$t \mapsto (\, x(t)\, ,\, u(t)\,)\ , \tag{1.3}$$

qui satisfasse l'équation (1.1) pour tout temps, ou presque tout temps. C'est un système d'équations différentielles **sous-déterminé** car sa « solution générale » dépend d'au moins une fonction arbitraire du temps, ici m fonctions arbitraires du temps. Avec cette définition –imprécise à plus d'un titre– des solutions,

$$\text{on appelle } \mathcal{B} \text{ l'ensemble des solutions de (1.1).} \tag{1.4}$$

Notons que $\mathcal{B}$ est la première lettre de *behavior*, en référence sans doute à [109] ; le lecteur soucieux de la préservation et du rayonnement de la langue française sera rassuré en trouvant en (1.11) un second système dont l'ensemble des solutions s'appellera $\mathcal{C}$, comme « comportement ».

Si l'on choisit m fonctions du temps et n constantes, c'est-à-dire $t \mapsto u(t)$ mesurable, voire lisse (voir section 1.2.2.2) pour la régularité et $x(0)$, on définit une unique solution $x(t)$. L'ensemble des solutions est donc beaucoup plus « gros » que pour les systèmes sans contrôle, où il était de dimension finie.

On n'a précisé ni la régularité des solutions, ni leur intervalle de définition. Les articles reproduits plus loin sont beaucoup plus rigoureux sur ces points. Les remarques ci-dessous reviennent sur ces deux points, et d'autres ; leur but est surtout de nous débarrasser de tout scrupule et de pouvoir énoncer les résultats sans donner tous les détails.

1.2.2.1 Régularité et caractère local pour le système

Vu que l'on ne s'intéresse en général qu'à des propriétés locales, il n'y a pas de raisons de s'embarrasser de variétés différentiables : tout sera énoncé dans $\mathbb{R}^n$.

Dans tout ce chapitre, f est supposée analytique réelle ($\mathbf{C}^\omega$) pour simplifier et ne plus se soucier de sa régularité. L'application f pourrait n'être définie que sur un voisinage du point $(\bar{x}, \bar{u})$ autour duquel est menée l'étude locale, et en tout état de cause, seule sa restriction à un tel voisinage arbitrairement petit importera ; on pourrait parler de *germes* de systèmes [43].

1.2.2.2 Remarques sur la régularité des solutions

Il faut bien sûr suffisamment de régularité pour donner un sens à $x(t)$, $\dot{x}(t)$ et $u(t)$, au moins presque partout, dans l'équation (1.1). Rappelons que, pour ce qui est de l'équation elle-même, on a supposé f lisse. Le minimum habituellement requis est d'exiger que $u(.)$ soit mesurable en donnant à l'équation (1.1) le sens suivant : pour tous t_1, t_2 dans I,

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} f(x(\tau), u(\tau))\mathrm{d}\tau\, ; \tag{1.5}$$

$x(.)$ est alors forcément absolument continue. On est loin du cas « sans contrôle » de la section 1.2.1 où toutes les solutions sont au moins aussi différentiables que l'équation ; ici aucune régularité de $u(.)$ n'est imposée par l'équation, on choisit seulement un classe qui permette de donner un sens à l'équation, et $x(.)$ est forcément un peu plus différentiable que $u(.)$.

Dans la suite, on ne demande pas toujours la même régularité aux éléments de $\mathcal{B}$. f étant supposée lisse, on peut par exemple ne s'intéresser qu'aux solutions lisses, c'est-à-dire celles pour lesquelles $t \mapsto (x(t), u(t))$ est infiniment différentiable ; c'est le cas à la section 1.5.

On ne s'intéresse absolument pas à la régularité minimum pour donner un sens à l'équation, mais plutôt à mettre dans $\mathcal{B}$ une classe assez riche pour caractériser le système et pour être stable par les transformations envisagées plus tard. Par exemple, si on choisit que $\mathcal{B}$ ne contienne que les solutions lisses, on laisse clairement de côté un grand nombre de contrôles possibles, ce qui serait préjudiciable si $\mathcal{B}$ devait être utilisé comme un vivier dans lequel puiser la solution d'un problème de contrôle, mais nous sera tout-à-fait suffisant, pour peu que les transformations envisagées préservent cette lissité, car deux systèmes lisses qui ont les mêmes solutions lisses sont identiques, par exemple parce que les contrôles lisses approchent suffisamment bien les contrôles mesurables. On précisera à chaque fois, implicitement ou explicitement, la régularité des « solutions » que l'on admet dans $\mathcal{B}$.

1.2.2.3 Remarques sur l'intervalle de définition des solutions.

Une autre question est celle de l'intervalle de définition (en temps) des solutions. On ne peut pas toujours prendre $\mathbb{R}$ tout entier car certaines solutions explosent en temps fini. En toute rigueur un élément de $\mathcal{B}$ devrait être défini par l'intervalle I *et* l'application $(x(.), u(.))$. Il n'y a pas de raison de considérer la restriction d'une solution à un sous-intervalle comme une solution différente ; on pourrait donc au moins se contenter des solutions sur leur intervalle de définition maximal... Se souvenant que l'on cherche surtout à mettre dans $\mathcal{B}$ une classe qui « représente » le système, on se contentera presque toujours de solutions sur des « petits » intervalles autour de zéro (comme nos systèmes ne dépendent pas explicitement du temps, on peut toujours appeller 0 l'instant initial).

Au Chapitre 9, on formalise cela en considérant que $\mathcal{B}$ est formé de *germes en* $t = 0$ *de solutions* ; cela consiste à identifier deux solutions qui coïncident sur un voisinage de 0, voir par exemple [43] pour une définitions précise. Au Chapitre 2, au contraire, on considère des solutions définies sur tout $\mathbb{R}$, mais avec l'habituel subterfuge d'annuler la dynamique en dehors d'un compact.

Les intervalles ne sont donc pas précisés dans ce chapitre, pour rendre l'exposé plus fluide ; les articles reproduits aux chapitres 2 à 7 contiennent à chaque fois les précisions nécessaires.

1.2.2.4 Le contrôle fait partie de la solution

On peut considérer, par exemple au vu du diagramme (1.2), que la commande est un signal exogène et que c'est la variable d'état x qui décrit le système... on est tenté alors de dire qu'une « solution » est plutôt un $x(t)$ possible qu'un $(x(t), u(t))$ possible comme on l'a fait en (1.3). L'ensemble des solutions serait alors la projection de l'ensemble $\mathcal{B}$ défini plus haut sur la composante x. Cela est discuté plus avant dans la sous-section 3.3.5.2.

Pour la plupart des systèmes, la commande produisant un $t \mapsto x(t)$ possible est unique (inversibilité) et on peut « reconstituer » $u(.)$ à partir de $x(.)$ si bien que la projection de $\mathcal{B}$ est aussi riche que $\mathcal{B}$ lui-même, et les deux choix s'équivalent. Pour les systèmes ne possédant pas cette propriété d'inversibilité, il n'en est pas ainsi ; par exemple, pour le système $\dot{x} = u_1 + u_2$ $(x, u_1, u_2$ scalaires), le points de vue (1.3) considère comme deux solutions distinctes (constantes)

$(x, u_1, u_2) = (0, 0, 0)$ et $(x, u_1, u_2) = (0, 1, -1)$ alors que l'autre point de vue ne les distingue pas ; de ce même autre point de vue qui « oublie la commande », l'ensemble des solutions de ce système est le même que celui du système à un seul contrôle $\dot{x} = u$, ce que nous voulons éviter.

Insistons donc sur le fait que $\mathcal{B}$ est fait de fonctions $t \mapsto (x(t), u(t))$ et non $t \mapsto x(t)$. Un système est l'ensemble des évolutions commande-état possibles.

1.2.2.5 Systèmes différentiels sous-déterminés plus généraux.

Bien sûr, (1.1) n'est pas la forme la plus générale d'équation différentielle sous-déterminée.

Le fait qu'elle soit d'ordre 1 ne nuit pas à la généralité : on se ramène de l'ordre $k > 1$ à l'ordre 1 en rajoutant de nouvelles variables, les dérivées d'ordre 1 à $k - 1$, et de nouvelles équations qui disent que ces variables sont les dérivées les unes des autres.

En revanche, la forme (1.1) a ceci de particulier qu'elle est **explicite** : elle exprime les dérivées de certaines variables (x) en fonction des variables non dérivées. Un système d'équations différentielles général $F(\dot{X}(t), X(t)) = 0$ se ramène, localement et en dehors de certaines singularités (là où le rang du Jacobien partiel de F par rapport aux variables $\dot{X}$ est égal au nombre d'équations), à la forme (1.1) grâce au théorème des fonctions implicites, et par un changement de coordonnées sur X suivi d'une partition des nouvelles variables en deux groupes x et u.

Toutefois, autour d'un point où le rang du dit jacobien chute, on peut avoir affaire à un système singulier qui ne se ramène *pas* à (1.1), et si le rang de ce jacobien est constant mais inférieur au nombre d'équations, le système $F(\dot{X}(t), X(t)) = 0$ dissimule des équations non différentielles et ne se ramène pas non plus, ou pas immédiatement, à un système explicite de type (1.1). Ceci est abordé à la section 5.3.1, page 116. Dans la suite, on ne considère que des système de contrôle (1.1).

1.2.3 Systèmes linéaires

Le système (1.1) est dit linéaire si il est de la forme

$$\dot{x} = Ax + Bu \tag{1.6}$$

où A est une matrice $n \times n$, B une matrice $n \times m$, u un vecteur colonne de taille m. Bien sûr la structure linéaire de $\mathbb{R}^n$ importe ici ; la propriété d'être un système linéaire n'est pas préservée par changement de coordonnées arbitraire.

Donnons un bref aperçu de classification des systèmes linéaires (formes normales de Brunovský [17]) ; on trouvera plus de détails à la section 3.4 ou dans des manuels comme [61]. On définit le rang de commandabilité

$$r \;=\; \operatorname{rank}\{B, AB, \ldots, A^{n-1}B\} \tag{1.7}$$

(rang de la collection des colonnes de ces matrices). Considérons d'abord le cas « mono-entrée » où, dans (1.6), u est scalaire (et B est un vecteur colonne de taille n) :

Proposition ($m = 1$)**.** *Il existe un changement de coordonnées linéaire $z = Px$ et un changement de contrôle $v = Kx + qu$, avec K un vecteur ligne, q un scalaire non nul et P une matrice inversible, tel que le système linéaire mono-entrée (1.6) s'écrive*

$$\begin{array}{rcl} \dot{x}_1 &=& x_2 \\ &\vdots& \\ \dot{x}_{r-1} &=& x_r \\ \dot{x}_r &=& v \\ \dot{Z}_{\mathrm{II}} &=& \widetilde{A} Z_{\mathrm{II}} \end{array} \qquad \textit{où}\;\; Z_{\mathrm{II}} = \begin{pmatrix} z_{r+1} \\ \vdots \\ z_n \end{pmatrix}. \tag{1.8}$$

avec $\widetilde{A}$ *une matrice constante* $(n-r)\times(n-r)$*.*

Si $r < n$, le système contient une partie qui est en elle-même une équation différentielle autonome, dont l'évolution est indépendante du contrôle ; on dit qu'il n'est pas commandable. Si $r = n$, il n'y a pas de partie « autonome » et le système dit simplement que les coordonnées de z sont les dérivées les unes des autres et que $(z_1)^{(n)} = v$.

Pour les systèmes à plus d'une entrée, on a la même partie « autonome » de dimension $n-r$ que ci-dessus si $r < n$; on suppose $r = n$ (commandabilité) pour simplifier l'enoncé suivant :

Proposition. *Pour un système linéaire (1.6) à* m *entrées avec* $r = n$*, il existe des entiers* $r_1, \ldots, r_m$*, un changement de coordonnées linéaire* $z = Px$ *et un changement de contrôle* $v = Kx + Qu$*, avec* K *une matrice* $n \times m$ *et* P *et* Q *des matrices carrées inversibles, tels que le système s'écrive*

$$\begin{array}{rcl} \dot{z}_{s_i+1} & = & z_{s_i+2} \\ & \vdots & \\ \dot{z}_{s_i+r_i-1} & = & z_{s_i+r_1} \\ \dot{z}_{s_i+r_i} & = & v_i \end{array} \qquad 1 \le i \le m \tag{1.9}$$

où $s_1 = 0,\ s_2 = r_1,\ \ldots,\ s_i = r_1 + \cdots + r_{i-1}$.

Trivialité des systèmes linéaires commandables. On vient de voir que, sauf si il contient une partie non commandable, un système linéaire est « trivial » : après un changement de variables linéaire, il dit simplement que les variables en présence sont les dérivées les unes des autres ; les solutions au sens de (1.3) sont faciles à décrire : les variables x_{s_i+1} sont des fonctions arbitraires du temps et les autres variables en sont des dérivées.

On peut considérer que les auteurs de [37], en introduisant la notion de platitude, ont reconnu cette « trivialité » des systèmes linéaires commandables comme une propriété plus importante que la linéarité elle-même. Voir un peu plus loin, section 1.5.2.

1.3 Équivalence

1.3.1 Cadre général

Dans [19], Élie Cartan se pose la question de la notion la plus générale ou naturelle d'équivalence entre deux systèmes d'équations différentielles sous-déterminés[2] et écrit (page 17) :

> **La première idée qui vient à l'esprit, et qu'il s'agira de préciser, est la suivante :** ***deux systèmes seront dits « absolument équivalents » lorsqu'on pourra établir une correspondance univoque*** **(au moins dans un champ fonctionnel suffisamment petit)** ***entre les solutions de ces deux systèmes.*** (1.10)

Adoptons ce point de vue, et discutons des différents champs fonctionnels dans lesquels on peut choisir cette correspondance. Pour être plus spécifiques, considérons deux systèmes Σ et Σ'

$$\begin{array}{lll} (\Sigma) & \dot{x} = f(x,u)\ , & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, \\ (\Sigma') & \dot{z} = g(z,v)\ , & z \in \mathbb{R}^{n'},\ v \in \mathbb{R}^{m'}, \end{array} \tag{1.11}$$

candidats à être «équivalents».

Comme en (1.4), appelons $\mathcal{B}$ et $\mathcal{C}$ l'ensemble des solutions de Σ et Σ' respectivement, voir à la section 1.2.2 des remarques sur la définition de cet ensemble de solutions. Pour que Σ et Σ' soient équivalents, il faut donc qu'il existe, au minimum, une bijection Φ :

$$\mathcal{B} \xrightarrow{\Phi} \mathcal{C}\ . \tag{1.12}$$

Si l'on se contente d'une bijection, sans autre propriété, son existence équivaut à ce que $\mathcal{B}$ et $\mathcal{C}$ aient le même cardinal. A cette aune, tous les systèmes avec contrôle ($m > 0$) sont équivalents, mais les équations différentielles ordinaires ($m = 0$) ne sont pas équivalentes aux systèmes avec contrôle ; en effet, dans le cas sans contrôle, l'ensemble des solutions a le cardinal de $\mathbb{R}^n$, et dans le cas avec contrôle le cardinal de $(\mathbb{R}^m)^{\mathbb{R}^n}$.

Cette remarque n'est pas très profonde. Elle nous convainc que les systèmes dynamiques « déterminés » ne sont pas des cas particuliers de systèmes de contrôle, mais sont d'une autre nature, et qu'en tout cas dans n'importe quelle étude d'équivalence, ils doivent être étudiés à part. Elle nous convainc aussi qu'il est effectivement judicieux de restreindre quelque peu le « champ fonctionnel » dans (1.10) si l'on veut sortir des trivialités.

Un grand champ fonctionnel (on vient d'essayer le plus grand possible) donne de grandes classes d'équivalences et l'espoir d'une classification simple, mais a l'inconvénient que deux systèmes déclarés équivalents risquent de peu se ressembler ; le paragraphe précédent illustre cela jusqu'à la caricature. A l'inverse, il est clair qu'un champ fonctionnel petit et constitué de transformations très bien identifiées fera que deux systèmes équivalents se ressemblent réellement, et que les transformations permettent sans doute de traduire un loi de contrôle élaborée pour l'un en une loi de contrôle pour l'autre, mais aura l'inconvénient d'un très grand nombre de classes d'équivalence, donc d'invariants.

[2] Élie Cartan ne parle pas de contrôle, et représente les systèmes différentiels sous-déterminés par des systèmes de Pfaff, les solution étant les courbes qui annulent ce système de Pfaff. On a vu à la section 1.2.2.5 que les systèmes de contrôle sont des systèmes sous-déterminés un peu particuliers en ce qu'ils sont « explicites », c'est-à-dire que l'on particularise des variables dont la dérivée est exprimée par l'équation. Le système $F(X, \dot{X}) = 0$ est plus général car les variables dépendantes sont indifférenciées, mais on a fixé la variable indépendante, qu'on appelle le temps, alors que dans le point de vue des systèmes de Pfaff, dans [19], elle n'est pas fixée a priori. L'ensemble des solutions est donc plus riche dans [19] qu'ici, mais le principe demeure, voir la prochaine note de bas de page.

On va donc exiger que Φ soit beaucoup plus qu'une bijection au sens ensembliste. En particulier, on aimerait que cette transformation, a priori définie sur des ensembles de fonctions, dérive en réalité de transformations sur des ensembles plus petit, idéalement d'une transformation ponctuelle sur un espace de dimension finie.

Ces remarques sont sans objet pour les systèmes sans contrôle, dont l'ensemble de solutions est déjà de dimension finie.

1.3.2 Cas des systèmes sans contrôle

On vient de voir que les systèmes sans contrôle ne sont jamais équivalents aux systèmes avec contrôle, tout-au-moins pas pour une notion d'équivalence conforme à (1.10) et à la définition de solution que nous avons donnée. Cela justifie de les traiter à part !

Habituellement, la conjugaison entre deux systèmes dynamiques (Σ et Σ' avec $m = m' = 0$) provient d'une transformation ponctuelle $z = \phi(x)$ et on dit que deux systèmes sont conjugués par ϕ si les solutions $t \mapsto z(t)$ de Σ' sont exactement données par $z(t) = \phi(x(t))$ où $x(.)$ décrit toutes les solutions de Σ, et vice-versa. Dans le langage de (1.10)-(1.12), on cherche Φ de la forme (naturelle) d'une composition à gauche par une transformation $\phi : \mathbb{R}^n \to \mathbb{R}^{n'}$ (notons que si ϕ est bi-continue, ce qui est le moins que l'on exige, on doit avoir $n = n'$) :

$$\Phi(\, x(.)\,) \;=\; \phi \;\circ\; x(.)\;.$$

Autrement dit Φ « dérive » d'une transformation ponctuelle $\phi : \mathbb{R}^n \to \mathbb{R}^{n'}$; plus savamment, le diagramme suivant commute

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Phi} & \mathcal{C} \\ \pi_t \downarrow & & \downarrow \pi_t \\ \mathbb{R}^n & \xrightarrow{\phi} & \mathbb{R}^{n'} \end{array} \quad , \tag{1.13}$$

dans lequel les projections π_t consistent à prendre la valeur au temps t d'une solution. Le diagramme commute pour tout t avec le même ϕ, bien sûr.

C'est une notion d'équivalence très naturelle : si un tel ϕ existe, le portrait de phase d'un système est l'image du portrait de phase de l'autre par ϕ, et il est clair que le comportement qualitatif des deux systèmes est similaire. Si ϕ est différentiable, la conjugaison se traduit par la formule : $g(\phi(x)) = \phi'(x)f(x)$.

Cette classification des systèmes dynamique a été très étudiée. On en donne section 3.2 un résumé très bref et orienté : elle ne concerne que la classification locale et prépare le terrain pour l'étude des systèmes de contrôle. En bref :

- Localement autour de points où f et g ne s'annulent pas, il existe toujours un ϕ, difféomorphisme de la même classe de différentiabilité que les systèmes, qui les conjugue.
- Autour d'un point d'équilibre, tout difféomorphisme préserve le spectre du linéarisé ; on se pose la question de l'équivalence entre le système et son linéarisé ; cette question, qui a occupé de grands esprits depuis le début du vingtième siècle, est très difficile et subtile (résonances, convergence des séries formelles qui conjuguent...), mais le théorème de Grobman-Hartman –voir Théorème 2.1.1– donne une réponse simple si l'on accepte de relâcher la différentiabilité : tous les systèmes sont conjugués leur linéarisé aux point hyperboliques quand on se contente d'un homéomorphisme, et d'ailleurs les spectres ne sont pas conservés mais seulement le signe des parties réelles des valeurs propres (c'est-à-dire le nombre de directions stables et instables autour du point d'équilibre hyperbolique.

Ces deux items sont disjoints car ϕ préserve les points d'équilibre c'est-à-dire doit envoyer un zéro de f sur un zéro de g.

1.3.3 Différentes équivalences pour les systèmes de contrôle

Donnons ici différentes classes possibles de « correspondances » (*cf.* (1.10))[3], qui donnent lieu à des équivalences très différentes. Ces classes ne se distinguent pas pour les équations différentielles sans contrôle du fait de la dimension finie des ensembles de solutions. Les résultats concernant ces équivalences sont énoncés ensuite, sections 1.4 et 1.5.

1.3.3.1 Transformations ponctuelles naturelles, feedback statique.

Si l'on reprend la même idée que dans le cas sans contrôle, et que l'on recopie la notion d'équivalence très naturelle dont on vient de parler, on est amené à faire « dériver » les transformations Φ du schéma (1.10)-(1.12) d'une transformations $\phi : \mathbb{R}^{n+m} \to \mathbb{R}^{n'+m'}$ telle que le même diagramme commute pour tout temps t :

$$\begin{array}{rcl} \mathcal{B} & \xrightarrow{\Phi} & \mathcal{C} \\ \pi_t \downarrow & & \downarrow \pi_t \\ \mathbb{R}^{n+m} & \xrightarrow{\phi} & \mathbb{R}^{n'+m'} \end{array} \quad , \tag{1.14}$$

où les projections π_t sont toujours celles qui associent une solution sa valeur au temps t : $\pi_t(\,(x(.), u(.))\,) = (x(t), u(t))$.

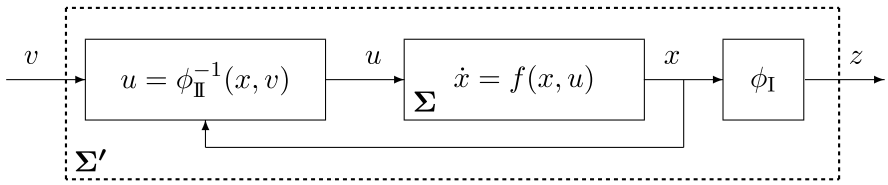
En d'autres termes, les systèmes Σ et Σ' sont équivalents si il existe une transformation $\phi : \mathbb{R}^{n+m} \to \mathbb{R}^{n'+m'}$ tel que l'ensemble des solutions $t \mapsto (z(t), v(t))$ soit exactement donné par $(z(t), v(t)) = \phi(x(t), u(t))$ où $(x(.), u(.))$ décrit l'ensemble des solutions de Σ et vice-versa.
On dit qu'ils sont conjugués par ϕ.

On requiert au minimum que ϕ soit bi-continue –un homéomorphisme– et on dit que les systèmes sont topologiquement équivalents s'il existe un tel ϕ. On peut requérir que ϕ soit un difféomorphisme $\mathbf{C}^\infty$ ou analytique, les systèmes sont alors différentiablement équivalents.

Théorème 1.1 (Proposition 3.3.6)**.** *Si ϕ est un homéomorphisme qui conjugue Σ à Σ', alors*
- $n = n'$, $m = m'$ *et*
- ϕ *a nécessairement une structure triangulaire, c'est dire que $(z, v) = \phi(x, u)$ s'écrit*

$$z = \phi_{\mathrm{I}}(x)\,, \quad v = \phi_{\mathrm{II}}(x, u). \tag{1.15}$$

Cette structure triangulaire fait de ϕ ce que l'on appelle habituellement une transformation par **feedback statique**, illustrée sur le schéma suivant.



(1.16)

Du point de vue du contrôle, si les systèmes sont conjugués par un tel ϕ, on peut obtenir l'un en appliquant un pré-compensateur –feedback prenant une nouvelle entrée v et la mesure de l'état pour construire u– à l'autre.

Dès que ϕ est différentiable, la conjugaison se traduit par la formule

$$\phi_{\mathrm{I}}{}'(x) f(x, u) \;=\; g(\,\phi_{\mathrm{I}}(x)\,,\,\phi_{\mathrm{II}}(x, u)\,)\;.$$

[3] Dans [19], la variable indépendante n'est pas spécifiée à l'avance, voir la note 2 au bas de la page 9, et les transformations sont aussi plus générales qu'ici ; ceci est par exemple étudié dans [104]. Il n'est pas difficile de voir qu'on se restreint ici, par rapport à [19], aux transformations « qui préservent la variable indépendante », c'est-à-dire le temps.

L'étude de cette équivalence a suscité de très nombreux travaux en contrôle, que nous n'avons pas l'ambition de passer en revue ici. Notons que les invariants sont de grande dimension (la « codimension » des classes d'équivalence est très grande, en tout cas infinie la plupart du temps), et notons aussi que la classe des systèmes équivalents à des systèmes linéaires commandables est parfaitement identifiée depuis [57, 50, 103], voir la section 3.5.2.

Nos résultats concernant cette notion d'équivalence sont donnés à la section 1.4 ci-dessous.

1.3.3.2 Transformations fonctionnelles.

On vient de dire que « peu » de systèmes sont équivalents au sens ci-dessus... d'où l'envie d'élargir la classe des transformations Φ. On s'est déjà prêté au jeu des transformations très générales juste après (1.12) en demandant seulement que Φ soit une bijection. On peut et doit bien sûr être un peu plus exigeant.

On a a priori affaire à des transformations fonctionnelles, c'est-à-dire telles que l'image $\Phi(\,(x(.),u(.))\,)$ de la solution $(x(.),u(.))$ de Σ dépende des valeurs passées et futures de $(x(.),u(.))$. $\mathcal{B}$ et $\mathcal{C}$ sont des ensembles de fonctions (dont on peut préciser la nature, voir section 1.2.2.2), des parties d'espaces fonctionnels sur lesquels il y a au moins une topologie, parfois une notion de différentiabilité. La continuité ne suffit pas à obtenir une relation d'équivalence très discriminante, comme on le voit sur l'exemple suivant.

Exemple[4]**.** On peut *toujours* bâtir un Φ bi-continu qui conjugue Σ et Σ' dès que $n = n'$ et $m = m'$: il suffit d'associer une solution $(x(.),u(.))$ de Σ l'unique solution $(z(.),v(.))$ de Σ' telle que $v(t) = u(t)$ pour tout t et $z(0) = x(0)$.

Ici, on n'a à nouveau pas assez restreint le champ fonctionnel évoqué en (1.10) !

On voit qu'il est difficile de manipuler ce type de transformations sans tomber dans des tautologies. Il y a peut-être des choses intéressantes de ce type à faire, mais, comme le disait E. Cartan il y a bientôt un siècle dans le même article [19], un peu plus bas que (1.10), cela reste une « *notion très large qu'il est difficile de soumettre telle quelle aux recherches dans l'état actuel de l'Analyse* »...

1.3.3.3 Transformations fonctionnelles restreintes

On peut revenir à un schéma du type (1.14), qui avait l'avantage de décrire les transformations Φ au travers de simples transformations ponctuelles en dimension finie, et chercher à le « généraliser » :

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Phi} & \mathcal{C} \\ \Pi_t \downarrow & & \downarrow \Pi_t \\ \mathfrak{X} & \xrightarrow{\phi} & \mathfrak{Y} \end{array} \quad . \tag{1.17}$$

Ici, les Π_t projettent une information « au temps t » sur la solution, plus générale que sa simple valeur au temps t, et $\mathfrak{X}$ et $\mathfrak{Y}$ sont des espaces de dimension plus petite que $\mathcal{B}$ et $\mathcal{C}$, idéalement de dimension finie. On va voir deux occurrences de ceci.

A la section 1.4.2, les $\mathfrak{X}$ et $\mathfrak{Y}$ ne sont pas de dimension finie, mais sont un espace sur lequel le système de contrôle donne lieu à un flot, voir (1.20), et on a établi (article reproduit au chapitre 2) une sorte de théorème de Grobman-Hartman dans ce contexte.

A la section 1.5, on définit des transformations –celles utilisées par E. Cartan dans [19], à la restriction près des notes 2 et 3 au bas des page 9 et 11– qui sont réellement descriptibles avec des $\mathfrak{X}$ et $\mathfrak{Y}$ de dimension finies et sont en même temps plus générales que (1.14) : les $\mathfrak{X}$ et $\mathfrak{Y}$

[4] Ceci est aussi valable pour les systèmes sans contrôle et revient à composer un flot par l'inverse de l'autre ; tous les flots de la même dimension sont alors conjugués par de telles transformations (qui dépendent du temps).

sont des espaces de jets, et les projections retiennent un certain nombre de dérivées à l'instant t, mais on ne peut pas réaliser Φ et Φ^{-1} avec les mêmes ϕ, $\mathfrak{X}$ et $\mathfrak{Y}$; voir (1.21).

1.4 Autour d'un théorème de Grobman-Hartman pour les systèmes de contrôle

On expose ici les résultats des articles reproduits aux chapitres 3 et 2. Celui reproduit au chapitre 4 en est une brève discussion.

On suppose toujours les deux systèmes Σ et Σ' dans (1.11) analytiques ; le cas $\mathbf{C}^\infty$ est aussi traité dans les articles.

La motivation et le point de départ était une question de modélisation, ou d'identification non-linéaire, exposée à la section 4.1. Si l'on considère un système de contrôle (1.1), par exemple autour d'un point d'équilibre ($f(0,0) = 0$) :

$$\dot{x} = f(x,u) = Ax + Bu + \rho(x,u) \tag{1.18}$$

ou la fonction ρ est d'ordre 2 en zéro, et si l'on suppose que les matrices A et B sont telles que le système linéaire

$$\dot{z} = Az + Bv \tag{1.19}$$

soit commandable, ce dernier système linéaire est-il un modèle suffisant de (1.18) *pour tout ce qui est local autour de* $(0,0)$?

La réponse semble être positive du point de vue de la conception des lois de contrôle, qui se fondent sur le modèle (1.19) s'il est commandable, si bien que les systèmes (1.18) et (1.19) se ressemblent énormément, et on est tenté de croire à l'affirmation (4.2). On est aussi tenté de formaliser cette croyance, et d'essayer d'établir une conjugaison entre les deux ; on sait que la conjugaison différentiable est rare –on connaît bien les conditions sur f pour qu'il y ait conjugaison différentiable, voir section 1.3.3 (ou 3.5.2 pour plus de détails)– mais il est tentant d'essayer d'établir un résultat « mou » comme le théorème de Grobman-Hartman pour les équations différentielles sans contrôle.

1.4.1 Linéarisation par transformations ponctuelles

On s'intéresse ici à une conjugaison possible par les transformations décrites à la section 1.3.3.1.

Les systèmes (1.18) et (1.19) sont-ils toujours conjugués au moins par un homéomorphisme ϕ au sens de (1.14) dès que le second est contrôlable ? Les auteurs de [9, 10] ont posé la question autour d'eux, en 1998 ou 1999, et ont obtenu davantage d'avis « évidement oui » que « évidement non »[5]. La réponse est en réalité négative d'après le Théorème 3.5.2, et très fortement puisque, contrairement au cas des équations sans contrôle, on ne gagne a peu près rien à relâcher la différentiabilité de la transformation qui conjugue.

Théorème 1.2 (Théorème 3.5.2). *Supposons que (1.18) et (1.19) soient topologiquement équivalents localement autour de l'origine. Si le rang de $\partial f/\partial u$ est localement constant, les systèmes sont aussi conjugués par un difféomorphisme analytique ; sinon, ils sont conjugués par un ϕ tel que ϕ_{I} (voir (1.15)) soit un difféomorphisme analytique.*

[5]Sondage non officiel réalisé informellement auprès d'un échantillon partial et non représentatif. Peut-être l'auteur s'est-il compté trois fois ?...

La cas où le rang de $\partial f/\partial u$ n'est pas localement constant est entièrement précisé dans le Théorème 3.5.2 énoncé au chapitre 3, et discuté abondamment à la section 3.5 ainsi que dans le chapitre 4. Savoir si l'on peut toujours prendre ϕ analytique, voire $\mathbf{C}^\infty$, pose une question ouverte intéressante de topologie différentielle, voir la section 3.5.1 page 74 et la section 4.4.

Ce résultat ne clôt pas la question posée autour de (1.18) et (1.19), et laisse ouverte la question de phénomènes *qualitatifs* et *locaux autour de l'origine* qui distingueraient ces deux systèmes.

Par ailleurs, la preuve de ce théorème exploite beaucoup la structure linéaire contrôlable du second système. Est-il vrai que si deux systèmes quelconques[6] sont topologiquement équivalents, ils sont aussi conjugués par un difféomorphisme analytique, ou au moins un homéomorphisme ϕ tel que ϕ_{I} soit un difféomorphisme analytique ? Cela reste ouvert.

1.4.2 Associer un flot à un système de contrôle...

Le théorème de Grobman-Hartman est un théorème de conjugaison de *flots*, et toute tentative de « généraliser » sa preuve aux systèmes de contrôle est vaine si l'on ne dispose pas de ce flot. On peut imaginer deux cadres qui associent un flot à un système de contrôle.

1. Si le contrôle est généré par un système dynamique de dimension finie (2.4) : $\dot{\zeta} = g(\zeta)$, $u = h(\zeta)$ ($\zeta \in \mathbb{R}^q$), le système combiné génère évidemment un flot dans $\mathbb{R}^{n+q}$. Bien sûr on a un peu triché en restreignant énormément les contrôles possibles : l'ensemble des solutions de ce système est beaucoup plus petit que l'ensemble $\mathcal{B}$ initial.

   Ce générateur de contrôle pourrait être un oscillateur ou un autre générateur de signaux tests pour une sorte d'identification fréquentielle généralisée.

2. Dans [25] ou [26, section 4.3], Colonius et Kliemann associent à un système de contrôle (1.1) un flot sur le produit $\mathbb{R}^n \times \mathcal{U}$ où $\mathcal{U}$ est l'espace fonctionnel des contrôles, par exemple l'ensemble des fonctions mesurables $\mathbb{R} \to \mathbb{R}^m$. Le flot au temps t associe à $x(0)$ et un contrôle (fonction du temps définie sur tout $\mathbb{R}$), la valeur de $x(t)$ si l'on applique ce contrôle et le même contrôle avec un argument (temps) décalé de t (la dynamique sur $\mathcal{U}$ est le « time-shift »). Ceci est précisé à la section 2.3.2.

On établit dans l'article reproduit au chapitre 2 des « théorèmes de Grobman-Hartman » dans chacun de ces deux cas. Dans le premier, Le Théorème 2.3.1 nous dit que l'on peut conjuguer les deux équation différentielles $\dot{x} = f(x, h(\zeta))$, $\dot{\zeta} = g(\zeta)$ et $\dot{x} = Ax + Bh(\zeta))$, $\dot{\zeta} = g(\zeta)$ par un homéomorphisme qui préserve la variable ζ et l'équation qui génère le contrôle.

Développons davantage le second cas, car il ne restreint pas les solutions et rentre dans le programme (1.10)-(1.12). On cherche précisément, pour deux systèmes (1.11) –on oublie pour un moment que l'un des deux est le linéarisé de l'autre– et leurs flots associés sur $\mathbb{R}^n \times \mathcal{U}$ et $\mathbb{R}^{n'} \times \mathcal{V}$, à les conjuguer par une transformation $\phi : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^{n'} \times \mathcal{V}$, c'est-à-dire que l'on cherche une transformation Φ qui fasse commuter le diagramme

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Phi} & \mathcal{C} \\ \pi_t \downarrow & & \downarrow \pi_t \\ \mathbb{R}^n \times \mathcal{U} & \xrightarrow{\phi} & \mathbb{R}^{n'} \times \mathcal{V} \end{array} , \qquad (1.20)$$

où les projections π_t consistent à prendre la valeur de $x(.)$ au temps t et à retenir tout le contrôle $u(.)$, mais en décalant le temps de telle façon que le temps t en haut corresponde au temps 0

[6]On a tout de même besoin de commandabilité : si les deux systèmes ont des parties non commandables, ces parties auront, autour d'un point d'équilibre, des invariants différentiels (valeurs propres, résonances...) qui ne se voient pas topologiquement à cause du théorème de Grobman-Hartman classique.

en bas. Pour ce qui nous intéresse, $n = n'$, $\mathcal{U} = \mathcal{V}$, et le second système est le linéarisé du premier. On prend $\mathcal{U} = L^p(\mathbb{R}, \mathbb{R}^m)$, $p \in [1, \infty]$. On a le théorème suivant, énoncé de manière approximative, voir le Chapitre 2 pour la version techniquement correcte.

Théorème 1.3 (Théorème 2.3.7)**.** *si A est hyperbolique, il existe un homéomorphisme ϕ : $\mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n \times \mathcal{U}$ qui conjugue (1.18) et (1.19) localement. De plus il préserve la composante sur $\mathcal{U}$, c'est-à-dire que $\phi(x, u(.)) = (H(x, u(.)), u(.))$ pour une certaine application $H : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$.*

Cette conjugaison dit-elle que les deux systèmes se ressemblent ? Il est un peu hasardeux de l'affirmer, ce d'autant plus que l'on peut aussi, par ce type de transformation, faire disparaître le terme Bu et conjuguer (1.18) à $\dot{x} = Ax$: l'effet du contrôle est alors tout entier rejeté dans les transformations. Le résultat n'est pas pour autant une tautologie et a une incidence sur la théorie des systèmes telle que vue par [26].

1.5 Résultats sur l'équivalence et la linéarisation dynamique

1.5.1 Les transformations dynamiques

Les transformations dynamiques, ou par feedback dynamique endogène, sont à mi-chemin entre les transformations « statiques » de la section 1.3.3.1, dont on a vu que même en ne demandant pas de différentiabilité, elles donnent une équivalence très rigide, et les transformations fonctionnelles très générales qui donnent une équivalence difficilement exploitable.

Continuons ce que l'on a ébauché en (1.17). Les transformations que l'on va définir consistent effectivement à définir la correspondance Φ en projetant à chaque instant t une information de dimension finie et en faisant porter la transformation –stationnaire– sur cette donnée de dimension finie. Il n'est cependant pas vrai que l'on ait, comme suggéré en (1.17), une transformation inversible en dimension finie ϕ. Le véritable schéma est le suivant ($\mathcal{J}^K$ et $\mathcal{J}^{K'}$ sont notés $J^K(M)$ et $J^{K'}(M')$ dans l'article reproduit au chapitre 5) :

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Phi} & \mathcal{C} \\ \Pi_t^K \downarrow & & \downarrow \pi_t \\ \mathcal{J}^K & \xrightarrow{\phi} & \mathbb{R}^{n'} \times \mathbb{R}^{m'} \end{array} \quad , \qquad \begin{array}{ccc} \mathcal{B} & \xleftarrow{\Phi^{-1}} & \mathcal{C} \\ \pi_t \downarrow & & \downarrow \Pi_t^{K'} \\ \mathbb{R}^n \times \mathbb{R}^m & \xleftarrow{\psi} & \mathcal{J}'^{K'} \end{array} \quad , \tag{1.21}$$

où ϕ et ψ ne sont pas inversibles en général. Les projections envoient les solutions dans des espaces de jets d'ordre K ou K' en prenant simplement les K ou K' premières dérivées au temps t (les dérivées de x ou z ne sont pas nécessaires car elles sont exprimées par (1.11)). Cette notion d'équivalence est exactement celle de la Définition 5.3.8 du chapitre 5, à ceci près que
- la définition 5.3.8 est plus précise et donne une notion locale,
- les notations sont différentes : ce qui est noté ici $\boldsymbol{\phi}$ est noté $\boldsymbol{\Phi}$ au chapitre 5, et ce qui est noté ici $\boldsymbol{\Phi}$ est noté $\boldsymbol{\mathcal{D}_\Phi^K}$ au chapitre 5.

Ces transformations demandent de dériver les solutions un nombre de fois non défini à l'avance, et il faut donc, ici, nécessairement exiger que les élément de $\mathcal{B}$ et $\mathcal{C}$, « les solutions » par définition, soient infiniment différentiables. Cela n'est pas une difficulté puisque, comme on l'a remarqué à la section 1.2.2.2, un système est entièrement caractérisé par ses solutions lisses.

Bien sûr, les entiers K et K' ne sont pas connus, ou majorés, à l'avance, si bien que ces schémas sont en fait potentiellement infinis. Un point de vue très légèrement différent sur ces transformations, peut-être formellement plus élégant mais rigoureusement équivalent, consiste à utiliser des espaces de jets infinis, au lieu des espaces finis ci-dessus. C'est ce que l'on expose dans l'article reproduit au chapitre 6. C'est le point de vue des « diffiétés », pour reprendre un terme introduit en contrôle dans [38, 41]. On a alors un schéma du type

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Phi} & \mathcal{C} \\ \Pi_t^\infty \downarrow & & \downarrow \Pi_t^\infty \\ \mathcal{J}^\infty & \xrightarrow{\phi^\infty} & \mathcal{J}'^\infty \end{array} \quad , \tag{1.22}$$

au lieu de (1.21). L'application ϕ^∞ est inversible, c'est même un difféomorphisme, entre espaces de jets infinis ; ces espaces ont une structure de variété « de dimension infinie » qui fait que les applications différentiables dépendent nécessairement d'un nombre fini de variable, ce qui est exactement traduit par les deux diagrammes (1.21).

1.5.2 Feedback dynamique, linéarisation, platitude et paramétrisabilité

Historiquement [53, 22], la notion de linéarisation dynamique consistait à généraliser le schéma (1.16) en s'autorisant, au lieu de la boite « statique » qui donne u en fonction du nouveau contrôle v et de l'état x par une simple formule, un système dynamique –ayant son propre état– avec ces mêmes entrées, et en autorisant la boite de sortie à utiliser aussi cet état du pré-compensateur.

Le traduire en termes de transformations utilisant des dérivées revient à [68, 37]. En réalité, cette notion de linéarisation par feedback dynamique n'entraîne pas l'existence d'une transformation Φ *inversible*, mais d'une application Φ du type suivant :

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\Phi} & \mathcal{B} \\ \Pi_t^K \downarrow & & \downarrow \pi_t \\ \mathcal{J}^K & \xrightarrow{\phi} & \mathbb{R}^n \times \mathbb{R}^m \end{array} \quad , \tag{1.23}$$

où $\mathcal{L}$ est l'ensemble des solutions d'un système linéaire commandable Λ, mais Φ n'est pas forcément inversible ; on demande seulement qu'elle soit surjective (elle atteint toutes les solutions de Σ) et que les fibres soient de dimension finies, c'est-à-dire que fixer une solution de Σ et chercher ses antécédents revient à résoudre une équation différentielle dont la solution générale ne dépend que d'un nombre fini de constantes. On appelle aussi ϕ une « paramétrisation de Monge » ; pour une définition très précise, voir la Définition 9.2.2 dans l'article reproduit au chapitre 9 ; cette définition est donnée seulement pour les systèmes à deux états et trois entrées ($n = 3, m = 2$ dans (1.1)) mais n'est pas spécifique à ces dimensions.

Parmi ces feedbacks dynamiques, ou paramétrisations de Monge, on distingue, à la suite de [37, 68, 40], ceux qui se traduisent par une application (1.23) qui soit en réalité inversible comme en (1.21) en les qualifiant d'*endogènes*. Un système conjugué à un système linéaire commandable par une telle transformation est dit *linéarisable par feedback dynamique endogène* ou encore *plat*, ou simplement *dynamiquement linéarisable*.

Il est à noter que les feedback non nécessairement endogènes ne définissent ni une classe de transformations ni une notion d'équivalence convenable sur les systèmes.

Ordre d'une paramétrisation. Dans (1.23), on peut supposer le système linéaire « trivial » de gauche sous forme canonique –voir section 1.2.3– et on appellera **ordre de la paramétrisation** ϕ le nombre de fois que les fonctions arbitraires du temps (les coordonnées numéro $s_i + 1$, $1 \leq i \leq m$, dans la forme (1.9)) sont dérivées après avoir tout exprimé en fonction de ces dernières ; ce n'est pas forcément le K du schéma car les variables du système linéaire peuvent déjà contenir des dérivées de ces fonctions arbitraires du temps. Au chapitre 9, on raffine cet ordre en indiquant par un m-uplet d'entiers combien de fois *chaque* fonctions arbitraires est dérivé, et pas seulement le maximum.

Sorties plates. Quand un système est plat, c'est-à-dire si Φ dans (1.23) peut être inversé comme dans (1.21), l'inverse de Φ est donné par un ψ dont les coordonnées numéro $s_i + 1$, $1 \leq i \leq m$ –on suppose à nouveau le système linéaire sous la forme (1.9))– forment un m-uplet de fonctions de $x, u, \dot{u}, \ldots, u^{(K')}$ que l'on appelle une **sortie plate** (dans l'article reproduit au chapitre 8, on disait linéarisantes au lieu de plates). Il se peut qu'elles dépendent de moins de dérivées que cela. On dit que le système est ***x*-plat** si il existe une sorties plates qui ne dépend que de x, **(x, u)-plat** si il existe une sortie plate qui ne dépend que de x et u... et ainsi de suite.

1.5.3 Conditions de platitude, de paramétrisabilité ou d'équivalence dynamique

Étant donné un système, comment décider si il est plat ? si il admet une paramétrisation de Monge ?

Étant donnés deux systèmes, comment décider si ils sont dynamiquement équivalents ?

L'article reproduit au chapitre 7 donne une re-formulation de la première question ci-dessus : un système est plat si et seulement si un certain système de formes différentielles, dont la construction est tout-à-fait explicite, peut être rendu exact à l'aide d'un « facteur intégrant » qui n'est pas une simple matrice inversible à coefficients fonctions scalaires mais un opérateur différentiel inversible, dont l'ordre joue le même rôle que les entiers K et K' de la section précédente. Insistons sur le fait qu'il ne s'agit pas d'une solution du problème de platitude, mais d'une simple re-formulation, utilisée dans l'article reproduit au chapitre 8, et discuté en section 1.5.5.

Revenons aux deux question initiales.

- Exhiber la transformation ϕ qui remplit les conditions prouve évidement que les systèmes sont plats, ou équivalents, et cette information, ainsi que la transformation explicite, peut se révéler très utile pour résoudre des problèmes de contrôle.
- En revanche, prouver qu'un système n'est pas plat, ou que deux systèmes ne sont pas équivalents est une tâche très difficile. Par exemple nul ne sait si le système (9.14), qui s'écrit aussi, pour ressembler davantage à (1.1) :

$$\begin{array}{rcl} \dot{x}_1 &=& u_1 \\ \dot{x}_2 &=& u_2 + x_3\, u_1 \\ \dot{x}_3 &=& x_2 + u_1\, {u_2}^2 \end{array} \tag{1.24}$$

  est plat ou non. La difficulté est précisément, pour cet exemple comme en général, que les entiers K ou K' ne sont pas connus à l'avance. On peut en principe prouver qu'il n'existe pas transformations avec K fixé, mais tant que l'on n'a pas de moyen de majorer K a priori, on n'a pas de moyen de preuve fini.

1.5.4 Conditions nécessaires pour l'équivalence dynamique

Un premier invariant est le nombre d'entrées m ; cela peut se prouver de bien des manières ; ce nombre est par exemple, dans le langage de [33], le degré de transcendance différentielle. Le résultat est prouvé de manière élémentaire au chapitre 6.

Théorème 1.4 (Théorème 6.2). *Si les deux systèmes (1.11) sont dynamiquement équivalents, alors $m = m'$.*

Systèmes mono-entrée ($m = m' = 1$). Il est prouvé dans [23] qu'un système à une seule entrée qui est linéarisable par feedback dynamique (ou admet une paramétrisation de Monge, *cf.* section 1.5.2) est nécessairement linéarisable par feedback statique. Cela se généralise à l'équivalence de systèmes quelconques, si bien que, pour les systèmes mono-entrée, l'équivalence dynamique n'est pas plus riche que l'équivalence statique :

Théorème 1.5 (Théorème 6.3)**.** *Si $m = m' = 1$ et si les deux systèmes (1.11) sont dynamiquement équivalents, alors ils sont statiquement équivalents.*

Critère de variété réglée. La seule condition nécessaire générale de platitude connue à l'heure actuelle est la suivante : *un système plat est réglé* (Théorème 5.4.1).
La condition est aussi nécessaire pour l'existence d'une paramétrisation de Monge. Le résultat est prouvé par Rouchon dans [88] (où il est aussi utilisé pour montrer que la platitude n'est pas générique, ce qui montre sa généralité) et indépendamment, mais avec une notion plus restrictive de la platitude (la classe des transformations Φ est réduite), par Sluis dans [96].

Rappelons qu'un système (1.1) est **réglé** si, pour tout x, l'ensemble décrit en faisant varier u est une sous-variété réglée de l'espace affine $T_x\mathbb{R}^n$ (qui n'est autre que $\mathbb{R}^n$ bien sur, mais on note $T_x\mathbb{R}^n$ pour souligner que sa structure d'espace affine (et même linéaire) est préservée par changement de coordonnées et demeure si on travaille sur une variété).

La condition suivante, prouvée au chapitre 5, est en quelque sorte une extension de ce critère à l'équivalence entre systèmes généraux.

Théorème 1.6 (Théorème 5.4.2)**.** *Si les deux systèmes (1.11), supposés analytiques, sont dynamiquement équivalents, alors*

- si $n < n'$, *Σ' est réglé,*

- si $n = n'$, *soit Σ et Σ' sont réglés tous les deux soit ils sont statiquement équivalents.*

1.5.5 Des résultats pour les systèmes de petite dimension

On a étudié la plus petite dimension où le problème de déterminer quels systèmes sont plats, ou paramétrables, ne soit pas résolue simplement. Il faut pour cela au moins deux entrées d'après le Théorème 1.5 ci-dessus, et un état de plus pour que le système ne soit pas trivial. On est donc ramené aux systèmes à trois états et deux entrées que nous étudions dans l'article reproduit au chapitre 9 ; le Théorème 1.6 indique que seuls les systèmes réglés sont susceptibles d'être plats, et cette propriété permet de se ramener aux systèmes définis par trois champs de vecteurs dans $\mathbb{R}^4$, c'est-à-dire les systèmes à quatre états et deux entrées qui ont la particularité que la dépendance en u du membre de gauche dans (1.1) est affine ; ces systèmes sont l'objet de l'article reproduit au chapitre 8.

On passe d'un système réglé à trois états à un système affine à quatre états en « ajoutant un intégrateur » ; il y a en revanche plusieurs façons de « couper un intégrateur » pour passer d'un système affine à quatre états à un système réglé à trois états. Cela est bien expliqué, par exemple, dans la thèse [7].

L'article du chapitre 8 est le plus ancien. Il utilise la re-formulation du chapitre 7 dont on parle plus haut, et une décomposition des opérateurs différentiels inversibles en opérateurs élémentaires. Le résultat principal caractérise les systèmes « (x, u)-plats » (voir la section 1.5.2) de ces dimension, ce qui est déjà très difficile et calculatoire. L'article est long et les preuves très techniques ; par exemple, certaines simplifications sont obtenues sur des formes normales en coordonnées en utilisant *Maple*. Comme on l'a indiqué à la section 1.5.3, il est très difficile de montrer qu'un système n'est pas plat. L'article montre que toute une classe de systèmes de cette dimension ne sont pas « (x, u)-plats ». Mais certains d'entre eux sont-ils par exemple $(x, u, \dot{u})$-plats ? Aucune réponse n'est donnée, mais on est amené à faire la conjecture suivante.

Conjecture. *Aucun des systèmes dont on montre au chapitre 8 qu'ils ne sont pas (x, u)-plats n'est plat.*

L'étude du chapitre 9 est plus récente, et n'utilise pas du tout les mêmes outils. Elle retrouve les mêmes résultats, et en montre de plus généraux pour trois raisons :

- On y étudie la paramétrisabilité plutôt que la platitude, ce qui est a priori plus général.
- On va un peu plus loin en termes d'ordre de dérivation, quoi que les comparaisons ne soient pas aisées vues les représentations différentes ; les relations entre les deux sont décrites à la section 9.7. Les systèmes (x, u)-plats du chapitre 8 admettent, au chapitre 9 une paramétrisation d'ordre (1,2), et on peut montrer que les autres n'admettent ni paramétrisation d'ordre $(2, k)$ pour k arbitraire, ni paramétrisation d'ordre (3,3).
- Le chapitre 9 ne reste pas muet sur les ordres supérieurs et donne, pour toute paire (k, ℓ) un système d'équations aux dérivées partielles qui a des solutions si et seulement si le système admet une paramétrisation d'ordre (k, ℓ).

De plus, les techniques du chapitre sont beaucoup plus élémentaires et les preuves lisibles.

Ce chapitre ne sait toutefois ni infirmer ni confirmer la conjecture suivante, Conjecture 9.5.3 au chapitre 9, et qui n'est pas très loin de la précédente :

Conjecture. *Aucun des systèmes dont on montre au chapitre 9 qu'ils n'admettent pas de paramétrisation d'ordre (1,2) n'admet de paramétrisation de quelque ordre que ce soit.*

Annexe : Liste de publications du candidat et brève notice scientifique

Ces quelques pages donnent une liste des publications importantes du candidat et un résumé de ces travaux. Comme indiqué dans l'avant-propos, le mémoire, par soucis d'unité, n'en reprend qu'une partie.

Ici, les références bibliographiques ne renvoient pas à la bibliographie en fin de mémoire mais à la « liste de publication » comprise dans cette annexe.

Encadrement de thèses

Le candidat a participé de très près à l'encadrement des deux thèses suivantes.

- *Eduardo Aranda-Bricaire*, “Linéarisation par bouclage des systèmes non linéaire” , soutenue en 1993 (École Centrale de Nantes). Directeur de thèse : Claude Moog. Publications qui en ont résulté : [10], [11] et [vii] dans la liste qui suit.
- *Pascal Morin*, “sur la stabilisation par retour d'état instationnaire”, soutenue en 1994 (École des Mines). Directeur de thèse : Claude Samson. Publications qui en ont résulté : [13], [14] et [ix] dans la liste qui suit.

Il a co-encadré (encadrant principal de fait) (directeur de thèse : Laurent Baratchart)
les trois thèse suivantes.

- *Ludovic Faubourg*, “Construction de fonctions de Lyapunov contrôlées et stabilisation non linéaire”, soutenue en décembre 2001 (Université de Nice - Sophia Antipolis).
- *David Avanessoff*, “Linéarisation dynamique des systèmes non linéaires et paramétrage de l'ensemble des solutions”, soutenue en juin 2005 (Université de Nice - Sophia Antipolis).
- *Alex Bombrun*, “Les transferts orbitaux à faible poussée : optimalité et stabilisation”, soutenue en mars 2007 (École des Mines).

Il co-encadre actuellement avec Ludovic Rifford la thèse d'*Ahed Hindawi* portant sur le transport optimal avec contraintes différentielles (Université de Nice - Sophia Antipolis).

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[vii] E. Aranda-Bricaire, J.-B. Pomet, « Some Explicit Conditions for a Control System to be Feedback Equivalent to Extended Goursat Form », *In : 3rd. IFAC Conference on Nonlinear Control Design (NOLCOS)*, p. 535–540, Lake Tahoe (U.S.A.), 1995.

[viii] J.-B. Pomet, « On the Use of Computer Algebra in the Proof of Conditions for Dynamic Feedback Linearization of Low Dimensional Nonlinear Systems », *In : 13th IFAC World congress*, San Francisco (USA), 1996.

[ix] P. Morin, J.-B. Pomet, C. Samson, « Developments in Time-varying Feedback Stabilization of Nonlinear Systems », *In : IFAC Nonlinear Control Systems Design Symposium NOLCOS'98*, H. Huijberts, H. Nijmeijer, A. van der Schaft, J. Scherpen (éditeurs), p. 587–594, 1998.

[xi] L. Faubourg, J.-B. Pomet, « Nonsmooth functions and uniform limits of control Lyapunov functions », *In : 41st IEEE Conf. on Decision and Control*, Las Vegas (USA), 2002.

[xii] A. Bombrun, J.-B. Pomet, « On the Average Control System », *In : 17th Int. Sympos. on Mathematical Theory of Networks and Systems (MTNS'2006)*, Kyoto (Japan), p. 2912–2917, 2006.

Rapports de recherche et publications internes

[xiii] D. Avanessoff, L. Baratchart, J.-B. Pomet, « Sur l'intégrabilité (très) formelle d'une partie des équations de la platitude des systèmes de contrôle », *Rapp. de recherche* INRIA, 2003, `http://www.inria.fr/rrrt/rr-5045.html`.

[xiv] L. Baratchart, J.-B. Pomet, « On local linearization of control systems », preprint, 2006, `http://fr.arxiv.org/abs/math/0607524`.

Notice des travaux scientifiques

Travail antérieur à la thèse. La publication **[1]** étudie la solution radiale d'une EDP elliptique sur un anneau fin et donne une estimation –en fonction de l'épaisseur de cet anneau– de son indice en tant que point critique d'une certaine fonctionnelle. La motivation de cette estimation était de distinguer cette solution radiale évidente d'autres solutions mises en évidence par des méthodes variationnelles.
La suite de mes travaux n'a pas de rapport direct avec cette contribution à l'analyse des EDP.

Rappel succinct du sujet de thèse et des résultats obtenus.
Titre : *Sur la commande adaptative des systèmes non-linéaires*. Soutenue en septembre 1989.

Son sujet était, en bref : si on a une famille de systèmes contrôlés $\dot{x} = f(p, x, u)$, où p est un paramètre de dimension finie (en pratique on se trouve devant l'un de ces systèmes mais la valeur de p n'est pas connue), et si l'on connaît, pour chaque valeur du paramètre, une loi de commande stabilisante, peut-on (et comment) construire *un* contrôleur stabilisant valable pour *tous les* systèmes de la famille. Ce problème avait déjà été étudié lorsque chaque système peut être rendu linéaire par un feedback bien choisi, puis stabilisé par des méthodes linéaires ; on est sorti de ce cadre en ne faisant pas d'hypothèse a priori sur le type de contrôle utilisé « à paramètres connus ». On suppose en général une dépendance affine en les paramètres.

Les résultats sont publies dans les articles [2, 3, i, ii, 4, 5] ; ils sont décrits et mis en perspective à la section 1.1 ci-dessous et dans une moindre mesure à la section 2.1.

1 Problèmes de contrôle, feedback, stabilisation

On considère des systèmes contrôlés $\dot{x} = f(x, u)$, où x est l'état, ici de dimension finie, et u le contrôle, la fonction f décrivant la dynamique supposée suffisamment lisse.

Cette fonction f peut être mal connue, elle est en elle-même un paramètre du problème ; on peut mettre en évidence une incertitude de plus petite dimension en écrivant $\dot{x} = f(p, x, u)$, où p est un paramètre (constant, en général de dimension finie).

Si une partie seulement de l'état est mesurée, on écrit $y = h(x)$, où y représente la sortie mesurée.

Le problème de contrôle auquel on s'intéresse est essentiellement la stabilisation, c'est-à-dire à la construction de contrôles, dépendant de l'état entier, ou de la sortie, qui rendent stable un point d'équilibre ou une trajectoire, et permettent de les rallier asymptotiquement, ou en temps fini. Le problème est l'existence et la construction de contrôleurs, qui peuvent être eux-mêmes des systèmes dynamiques, réalisant cet objectif.

1.1 Commande adaptative

On se demande ici si il est possible de construire de tels contrôleurs sans une connaissance totale de la dynamique f, c'est-à-dire un contrôleur indépendant du paramètre p qui fonctionne pour "toutes" les valeurs de p. On suppose pour cela donnée –sauf dans le premier paragraphe– une famille de contrôleurs, dépendant de p, telle que chaque contrôleur stabilise le système correspondant.

Le travail **[3]** concerne plutot la commande adaptative "linéaire", c'est-à dire que les systèmes sont linéaires (en les états et les entrées) ; les contrôleurs adaptatifs pour de tels cas sont assez bien connus, mais non-linéaires en général. Dans cet article, on n'est pas tout-à fait dans le cadre ci-dessus en ce sens que l'on applique ce contrôleur à un système qui n'appartient *pas* à la famille paramétrée par p : on a étudié l'effet d'un contrôleur adaptatif linéaire appliqué à un système d'ordre plus grand que le contrôleur ne le prévoit. Le système bouclé est alors un système dynamique non linéaire qui, en général, peut avoir des comportements à peu près quelconques, cf. une littérature assez abondante. On montre ici que l'on peut malgré tout prédire, si l'on désire suivre un signal périodique en sortie, l'existence d'une solution périodique stable pour le système bouclé. L'article **[2]**, lui, étudie la (non)-robustesse de la propriété « toute trajectoire est bornée et rentre en temps fini dans un compact ».

Ces deux résultats sont des préliminaires qui ne figurent pas explicitement dans la thèse.

On décrit dans **[4]** un apport important de cette thèse, qui consiste à utiliser explicitement dans l'algorithme d'estimation des paramètres une fonction de Lyapunov correspondant aux systèmes bouclés « à paramètres connus ». On montre que cela fournit une pondération des l'estimation très adaptée à la stabilisation et fournit, sous certaines conditions, des résultats globaux. On obtient ainsi « aussi bien » que les résultats précédemment connus quand on peut employer des commandes linéarisantes. Cette publication contient aussi une méthode de projection des paramètres qui a beaucoup été utilisée par la suite. L'article **[5]** est un survey qui reprend et développe toute cette partie de la thèse.

On a aussi étudié la structure des familles de systèmes paramétrées ; on peut trouver dans [4], et surtout dans **[ii]**, des résultats beaucoup plus forts lorsque l'on a des restrictions, en quelque sorte, sur les directions dans lesquelles la dynamique dépend des paramètres, ce qui permet de compenser ces effets par un contrôle. La formulation de ces restrictions est toutefois très implicite a priori ; une étude structurelle des familles de systèmes a permis une caractérisation géométrique, [7, iii], sur lesquels nous reviendrons à la section 2.1.

Contrôleur universel. Dans **[iv]**, on s'intéresse à la question de l'information suffisante sur un système pour le stabiliser, ou le problème du "contrôleur universel". Existe-t-il un contrôleur qui stabilise "tous" les systèmes ? Cette référence donne une construction de recherche dense, déja utilisée en contrôle linéaire, qui n'a pas d'intêret pratique mais pousse à se demander si il y a vraiment des conditions nécessaires pour qu'une famille de système soit stabilisable par un même contrôleur sans restriction sur celui-ci.

1.2 Contrôle de sortie / observateurs

Avec des techniques proches, de [4] et de la thèse, j'ai obtenu des résultats de stabilisation par retour de sortie, où l'on ne dispose que de mesures partielles **[6]**.

1.3 Stabilisation instationnaire

Certains systèmes non linéaires, disons $\dot{x} = f(x, u)$ avec $f(0, 0) = 0$, quoique contrôlables, ont des points d'équilibre (ici $x = 0$) qui ne peuvent être stabilisés par un contrôle qui dépende *continûment* de l'état ; c'est-à-dire qu'il y a une obstruction de nature topologique[1] à l'existence de α continu tel que le point d'équilibre 0 pour $\dot{x} = f(x, \alpha(x))$ soit asymptotiquement stable.

De nombreux systèmes mécaniques contrôlés, en robotique mobile (contraintes non-holonomes), sont dans ce cas ; celà avait été noté, et il avait été proposé[2], pour deux cas de tels systèmes mécaniques, d'utiliser des contrôles qui dépendent non seulement de l'état, mais aussi du temps (périodiquement). Dans **[8]**, il est montré que ce moyen de contourner l'obstruction à la stabilisation par une retour continu d'état «pur» fonctionne pour une classe générale de systèmes présentant ladite obstruction, et une construction très explicite est donnée. Dans **[v]**, on a donné un exemple de synthèse de telles lois sur un système simple de robotique mobile déjà étudié dans la référence mentionnéee ci-dessus, mais où l'on laisse plus de libertés dans la loi de commande pour satisfaire, en plus, certains critères.

[1] R. W. Brockett, «Asymptotic Stability and Feedback Stabilization», in *Differential geometric control theory*, Birkäuser, 1983, p. 181-191.

[2] C. Samson, «Velocity and torque feedback control of a nonholonomic cart», in *Proc. in Advanced Robot Control*, vol. 162, Springer-Verlag, 1991.

L'article [8] fut, avec [*J.-M. Coron, 1992*][3], le départ d'une nombreuse littérature sur la stabilisation instationnaire : stabilisation de systèmes généraux, application à des systèmes concrets, études de robustesse, méthodes explicites de synthèse. Ce fut l'objet d'un exposé plénier **[ix]** à la Conférence Européenne de Contrôle en 1998. Dans **[13]**, on donne une loi stabilisante pour le contrôle de l'attitude d'un satellite en mode dégradé. Ce système n'appartient pas à la classe de systèmes traités dans **[8]** et la construction du contrôle fait appel à une technique différente. Dans **[14]**, on donne une méthode constructive (mais complexe) très générale utilisant l'idée classique d'approximer un déplacement selon le crochet de Lie de 2 champs de vecteurs par une oscillation entre les deux champs.

1.4 Control Lyapunov functions, Robustesse

Les fonctions de Lyapunov contrôlées (**CLF**) sont non seulement un outil d'analyse, comme les fonctions de Lyapunov pour la stabilité des équations différentielles, mais aussi un puissant outil de *synthèse* de contrôleurs pour la stabilisation des systèmes contrôlés. Elles sont utilisées dans la plupart des travaux décrits ci-dessus, mais un sujet d'intérêt[4] est de systématiser la construction de CLF. Nous avons proposé **[16, 15, xi]**, [thèse de L. Faubourg] des méthodes en ce sens. [16] construit des CLF (voir §1.b) pour des systèmes qui sont stabilisables par la méthode «de Jurdjevic-Quinn» (aussi appelée *damping control*). Voir aussi [15], où l'on donne un grand nombre d'exemples.

1.5 Contrôle en poussée faible, moyennation

Pour un système conservatif avec un "petit" contrôle, il y a naturellement des variables lentes (les intégrales premières de la dérive) et des variables rapides. C'est le cas par exemple d'un satellite en orbite autour d'un corps central muni d'un moteur à poussée faible. Il y a longtemps que des techniques de moyennation (averaging en anglais) sont utilisées dans ce genre de situations, y compris en contrôle optimal en moyennant les équations données par le Principe du Maximum de Pontryagin.

L'originalité de **[xii]** est de proposer un *système de contrôle* moyen, c'est-à-dire de faire la moyennation *avant* de décider du type de contrôle utilisé. Cela a un intérêt conceptuel certain, et permet aussi, par exemple, de montrer une conjecture sur le développement asymptotique du temps minimum de transfert en fonction de la très faible poussée, voir **[23]**.

2 Étude structurelle, transformations, classification

D'un point de vue mathématique, les systèmes contrôlés sont des systèmes sous-déterminés d'équations différentielles ordinaires, c'est-à-dire dont la solution générale dépend non seulement d'un certain nombre de conditions initiales mais aussi de fonctions arbitraires du temps. La notions d'équivalence entre deux tels systèmes peut résulter, comme pour les équations différentielles déterminées de la conjugaison des solutions par une transformation sur un espace de dimension finie, mais aussi, vue la présence d'au moins une fonction du temps arbitraire, par une transformation dépendant fonctionnellement des solutions, par exemple via un certain nombre de dérivées.

Du point de vue du contrôle, les transformation du premier type sont les transformations par "feedback statique", qui outre un changement de coordonnées sur l'espace d'état, re-paramètrent les contrôles via une transformation ponctuelle dépendant aussi de l'état, et celles du second type, lorsque la dépendance fonctionnelle se fait au travers d'un nombre fini de dérivées, sont les transformations par "feedback dynamique", où ce ce re-paramétrage se fait au travers d'un système dynamique, qui fait partie de la transformation.

[3] *Math. of Control, Signals and Systems*, vol. 5 (1992), p.295-312. Voir **[vi]** pour une comparaison entre ce travail et [8], et une manière de marier les deux techniques.

[4] Ceci est développé dans l'introduction de [15], par exemple

On s'intéresse à l'action sur les systèmes contrôlés de ces diverses transformations. L'intérêt est double : d'une part les feedbacks statiques ou dynamiques peuvent être "implémentés", c'est-à-dire que la solution d'un problème de contrôle pour un système produit automatiquement une solution pour le système transformé, donc une stratégie de contrôle peut être étendue d'un système à tous les systèmes équivalents ; d'autre part, si l'on voit —et c'est le bon point de vue— les système mathématiques comme des *modèles* de vrais systèmes physiques, cette classification des modèles est presque un pré-requis à une théorie de la modélisation qui serait fondée non pas sur des modèles de connaissance mais sur des observations.

2.1 Équivalence par feedback statique

Familles paramétrées Considérant une famille paramétrée de systèmes contrôlés, **[7]** donne une condition géométrique pour qu'ils soient tous équivalents par feedback statique, via une transformation qui, bien sûr, dépend du paramètre. On caractérise aussi (et même beaucoup plus explicitement) les familles telles que ces transformations satisfassent des conditions sous lesquelles les algorithmes de stabilisation adaptative donnent les meilleurs résultats, *cf.* section 1.1. Dans **[iii]**, on donne des conditions sous lesquelles ces transformations s'étendent globalement.

Forme de contact partielle, forme chaînée On trouve dans **[vii]** une caractérisation de l'équivalence par feedback statique à une forme particulière de système : «chain form» étendue (motivation : robotique mobile), qui revient à la conjugaison d'un système de Pfaff à un système de contact "partiellement prolongé". La caractérisation a été obtenue indépendamment par d'autres auteurs[5].

Linéarisation topologique. On s'est posé la question suivante : y a-t-il, pour les systèmes contrôlés, un équivalent du théorème de Grobman-Hartman ? Est-il vrai que, localement autour d'un point d'équilibre pas trop dégénéré, un système contrôlé soit conjugué à un système linéaire, par exemple son approximation linéaire ?

Il se trouve que le théorème de Grobman-Hartman démontre une conjugaison de *flots*, et c'est donc dans un contexte où un système contrôlé admet un flot qu'il est naturel —cela parait tout-au-moins naturel *a posteriori* !— de le généraliser : dans **[22]** on montre, via un théorème de point fixe, que le flot engendré par un système de contrôle sur un produit de l'espace d'état par l'espace fonctionnel des contrôles, avec le décalage en temps comme dynamique, est bien localement conjugué au flot d'un système linéaire, via une transformation sur un espace fonctionnel. Cela ne revêt peut-être qu'une signification pratique assez limitée, vu que la transformation dépend de tout l'histoire passée et future.

Si en revanche on cherche à conjuguer un système contrôlé lisse (resp. analytique) à un système linéaire commandable via un homéomorphisme sur un espace de dimension finie, alors on montre dans **[xiv, 18]** que la conjugaison via un tel homéomorphisme implique la conjugaison via un difféomorphisme lisse (resp. analytique), ce qui est connu pour être très rare.

2.2 Équivalence par feedback dynamique, platitude

Deux systèmes peuvent être équivalents pas feedback dynamique sans l'être par feedback statique ; les transformations par feedback statique sont un cas particulier de transformations par feedback dynamique. La notion d'équivalence elle-même n'est pas évidente dans le cas du feedback dynamique car le fait d'appliquer un compensateur dynamique donne un système bouclé de dimension strictement plus grande (sauf s'il est statique).

[5] W. Pasillas-Lépine, W. Respondek, «Contact systems and corank one involutive subdistributions», *Acta Appl. Math.*, vol. 69 (2001), pp. 105–128.

On a proposé dans **[9]** un cadre géométrique qui fait apparaître ces transformations comme des sortes de difféomorphismes dans des espaces de jets infinis. Ces transformations sont aussi appelées Lie-Bäcklund dans le cas des EDP. On retrouve une approche similaire indépendante par d'autres auteurs[6]. Deux systèmes sont équivalents si ils sont conjugués par une transformation ou les nouveaux états et contrôle s'expriment en fonction des anciens et d'un nombre fini K de leurs dérivées par rapport au temps et réciproquement. Pour décider si deux systèmes donnés sont équivalents, la grande difficulté est que l'on ne connaît pas de borne a priori pour cet entier K, c'est-à-dire le nombre de dérivées qui interviennent dans la transformation.

Intéressons-nous désormais à la platitude, c'est-à-dire la propriété pour un système d'être équivalent à un système linéaire commandable. Décider si un système est plat présente bien sûr la même difficulté. C'est une version théorie des systèmes du "problème de Monge", posé il y a plus d'un siècle, qui consiste à décider si la solution générale d'un système différentiel donné peut s'exprimer en fonction d'un certain nombre de fonctions du temps arbitraires et de leur dérivées jusqu'à un certain ordre. La platitude demande de surcroît que ces fonctions arbitraires correspondent chacune à une et une seule solution.

On a proposé dans **[10, 11]** une reformulation du problème, qui consiste à rechercher une sorte d'opérateur différentiel inversible intégrant (par analogie à "facteur intégrant") pour un système de formes différentielles que l'on peut construire explicitement. Cela n'évite pas la majoration de l'entier K mentionné ci-dessus ; il est en quelque sorte remplacé par l'ordre de l'opérateur différentiel en question.

Les systèmes à 4 états et 2 contrôles, affines en ces contrôles, ou 3 états et 2 contrôles non affines, sont les plus «petits» pour lesquels le problème de la platitude ne soit pas résolu. L'article **[12]** donne une description très explicite des systèmes qui ont cette propriété (platitude), mais en bornant artificiellement K. En jargon : on ne donne pas une condition pour que ces systèmes soient plats, mais pour qu'ils soient «(x, u)-plats», alors qu'un système plat général serait $(x, u, \dot{u}, \ddot{u}, \ldots, u^{(L)})$-plat pour un certain L non nécessairement nul (les entiers L et K ne sont pas les mêmes mais sont une fonction croissante l'un de l'autre). On conjecture que tous les systèmes plats de cette dimension sont en réalité (x, u)-plats. L'article [12], qui utilise la formulation donnéee dans [10, 11], est long (80 pages) et technique, et la preuve du résultat principal a nécessité le recours au calcul formel, ceci est détaillé dans **[viii]**.

Plus récemment, la thèse de D. Avanessoff a porté sur ce difficile problème. Dans **[21]**, on regarde à nouveau les systèmes de la même petite dimension, mais on étudie le problème de Monge stricto sensu, et des résultats plus forts que dans [12] sont donnés, avec des preuves beaucoup plus élémentaires (en tout cas lisibles par un humain sans recourir au calcul formel !). Mentionnons enfin une tentative de s'affranchir de la connaissance a priori de l'entier K en essayant de mettre sur pied un théorie de l'intégrabilité formelle "à une infinité de variables" adaptée à ce problème ; cela est relaté dans **[xiii]**.

2.3 Structure of trajectories, controllability

Une étude topologique de l'espace des trajectoires **[17]** caractérise les courbes qui peuvent être approchées par des trajectoires admissibles d'un ysstème affine. Cela a des répercussions en trajectographie, et en calcul stochastique via le théorème du support (Strook et Varadhan) d'une diffusion.

[6]M. Fliess, J. Lévine, P. Martin, P. Rouchon, «A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems», *IEEE Trans. Automat. Control, vol. 44*, 1999, p. 922–937.

Deuxième partie

REPRODUCTION D'ARTICLES

Chapitre 2

Reproduction de l'article:

L. Baratchart, M. Chyba et J.-B. Pomet, "A Grobman-Hartman theorem for control systems"

J. Dynamics & Differ. Equ., vol. 19, pp. 75–107, 2007.

Abstract. We consider the problem of locally linearizing a control system via topological transformations. According to chapter 3, a generic control system cannot be linearized using *pointwise* transformations on the state and the control values. Allowing the transformations to depend on the control at a *functional* level, we prove a version of the Grobman-Hartman theorem for control systems.

2.1 Introduction

Our point of departure will be a brief review of the classical Grobman-Hartman theorem, following for instance [47]. Consider the differential equation

$$\dot{x}(t) = f(x(t)), \tag{2.1}$$

where $f \in C^1(U, \mathbb{R}^n)$ and U is an open subset of $\mathbb{R}^n$. Assume that $x_0 \in U$ is an equilibrium, i.e. $f(x_0) = 0$. The linearized system associated to (2.1) near x_0 is

$$\dot{x}(t) = Ax(t) - Ax_0 \tag{2.2}$$

where $A = Df(x_0)$ is the derivative of f at x_0. The equilibrium x_0 is said to be *hyperbolic* if the matrix A has no purely imaginary eigenvalue. Systems (2.1) and (2.2) are called *topologically conjugate* at x_0 if there exist neighborhoods V, W of x_0 in U and a homeomorphism $h : V \to W$ mapping the trajectories of (2.1) in V onto the trajectories of (2.2) in W in a time-preserving manner : for each $x \in V$, we should have

$$h \circ \phi_t(x) = e^{At}\big(h(x) - h(x_0)\big) + h(x_0) \tag{2.3}$$

provided that $\phi_\rho(x) \in V$ for $0 \leq \rho \leq t$, where ϕ_t denotes the flow of (2.1). The Grobman-Hartman theorem now goes as follows [47] :

Theorem 2.1.1 (Grobman-Hartman). *If x_0 is an hyperbolic equilibrium point, then* (2.1) *is topologically conjugate to* (2.2) *at x_0.*

More general versions and a precise study of C^k-linearizability with finite k can be found in [16]. See also [89] for a discussion of infinitely differentiable and analytic linearizability. This theorem entails that the only invariant under local topological conjugacy around a hyperbolic equilibrium is the number of eigenvalues with positive real part in the Jacobian matrix, counting multiplicity. Indeed, it is well-known (*cf* [5]) that the linear system $\dot{x} = Ax$ where A has no pure imaginary eigenvalue is topologically conjugate to the linear system $\dot{x} = DX$ where D is diagonal with diagonal entries ± 1, the number of occurrences of $+1$ being the number of eigenvalues of A with positive real part, counting multiplicity.

When trying to extend this result to a control systems $\dot{x} = f(x, u)$, with state $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}^m$, one has first to decide what the meaning of "topologically conjugate" should be, i.e. what kind of map should play the role of the homeomorphism h in (2.3). The simplest idea is to ask for a pointwise transformation on the $n+m$ variables x, u, i.e. a local homeomorphism of $\mathbb{R}^{n+m}$; this is investigated in Chapter 3 where it is shown that no extension of Theorem 2.1.1 to control systems may hold with this kind of conjugating homeomorphisms : unpredictability of future control values forces a rather rigid triangular structure on conjugating homeomorphisms that ultimately results in their "almost" smoothness, and hence they would preserve too many special features of linear control systems, that are highly non-generic among control systems.

Theorem 2.1.1 is about conjugating *flows* while, since the control is an arbitrary function of time whose future values are not determined by past ones, a control system does not generate a flow on $\mathbb{R}^{n+m}$ or any finite dimensional manifold. The present paper is devoted to notions of local linearization of control systems that do amount to conjugating "flows". Analogs to the Grobman-Hartman theorem for control systems are derived in two different contexts : either (section 2.3.1) when the control is generated by a finite dimensional dynamical system or (section 2.3.2) when one associates to a control system a flow on a suitable functional space in the style of [25]. These results do not contradict these of Chapter 3 because the notion of conjugacy is here much weaker ; they are consequences of an abstract principle, established in Section 2.2, saying that if the controls are generated by a one parameter group of homeomorphism (flow) on some general topological space, then, under hyperbolicity assumptions, the system can be linearized *via* transformations that are continuously parameterized by elements of this topological space.

Related bibliography. In [76], dynamics on general abstract spaces are studied to derive results on the dependence upon various data of solutions of integral equations ; the dynamical issues addressed there are very related to our section 2.2. Let us also mention [27], that states a Hartman Grobman for "random dynamical systems" where, roughly speaking, the role of the topological space mentioned above is played by a probability space. As for the view on control system that we adopt in Section 2.3, it is very inspired by [25] (see also the monograph [26]) : to a control system, one associates naturally a flow on a "skew product" ; see these references for many properties of this flow ; we allow a slightly more general picture by not requiring continuity of that flow, thus allowing L^∞ topology for the controls even for non-affine systems.

2.2 An abstract Grobman-Hartman Theorem

We shall prove an abstract result on the linearization of dynamical systems which implies the local linearizability properties of control systems stated in sections 2.3.1 and 2.3.2. The proof closely follows that of the classical Grobman-Hartman theorem for ODEs as given by Hartman in [47, chap. IX, sect. 4, 7, 8, 9], and we tried to stick to his notations as much as possible. Nevertheless, we provide a detailed argument because the modifications needed to handle the dynamics of the control are not straightforward. Like [47], we state Theorem 2.2.1 below as a *global* linearizability property for a linear equation perturbed by a suitably normalized additive term. In sections 2.3.1 and 2.3.2, we shall use this result to derive local linearizability results for systems that locally coincide with a normalized one.

Let us mention in passing that the Grobman-Hartman Theorem for "random dynamical systems" given in [27] is similar in spirit to Theorem 2.2.1 : there, the set $\mathcal{E}$ of control parameters is a probability space instead of a topological space, and the conjugating transformation H is only required to be measurable with respect to $\zeta \in \mathcal{E}$ but need not be continuous. Both can be viewed as Grobman-Hartman Theorem "with parameters".

Consider a topological space $\mathcal{E}$ endowed with a one-parameter group of homeomorphisms $(\mathcal{S}_\tau)_{\tau\in\mathbb{R}}$. The space $\mathcal{E}$ is to be regarded as an abstract collection of input-producing events for a control system, these events being themselves subject to the dynamics of the flow $\mathcal{S}_\tau$. To describe the action of such an event on the system we simply let ζ enter as a parameter in the differential equation describing the evolution of the state variable x :

$$\dot{x} \;=\; Ax \;+\; G(x,\zeta,t)\;, \tag{2.4}$$

where the linear term at the origin Ax was singled out for convenience (but without loss of generality). Here, $G : \mathbb{R}^n \times \mathcal{E} \times \mathbb{R} \to \mathbb{R}^n$ is assumed to be measurable with respect to t for fixed x, ζ, and of class C^1 with respect to x for fixed ζ, t. To ensure the compatibility between the dynamics of ζ and that of x (see (2.7) below), we also require the condition

$$G(x, \mathcal{S}_\tau(\zeta), t) = G(x, \zeta, t+\tau) \tag{2.5}$$

to hold for all $(x,\zeta,\tau,t) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R} \times \mathbb{R}$. Now, if we suppose that to each $(x,\zeta) \in \mathbb{R}^n \times \mathcal{E}$ there is a locally integrable function $\phi_{x,\zeta} : \mathbb{R} \to \mathbb{R}^+$ satisfying $G(x,\zeta,t) \leq \phi_{x,\zeta}(t)$ for all $t \in \mathbb{R}$, and that to each $\zeta \in \mathcal{E}$ there is a locally integrable function $\psi_\zeta : \mathbb{R} \to \mathbb{R}^+$ satisfying $\partial G/\partial x\,(x,\zeta,t) \leq \psi_\zeta(t)$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, then for each $\zeta \in \mathcal{E}$ the solution to (2.4) with initial condition $x(0) = x_0 \in \mathbb{R}^n$ uniquely exists for all $t \in \mathbb{R}$, *cf.* [98, Theorem 54, Proposition C.3.4, Proposition C.3.8]. Subsequently, denoting by

$$\widehat{x}(\tau, x_0, \zeta) \tag{2.6}$$

the value of this solution at time $t = \tau$, it follows from (2.5) that

$$\widehat{x}(t+\tau, x_0, \zeta) = \widehat{x}(t, \widehat{x}(\tau, x_0, \zeta), \mathcal{S}_\tau(\zeta)) \tag{2.7}$$

and thus

$$\widehat{\Phi}_t(x_0,\zeta) \;=\; (\,\widehat{x}(t,x_0,\zeta)\,,\,\mathcal{S}_t(\zeta)\,) \tag{2.8}$$

defines a flow on $\mathbb{R}^n \times \mathcal{E}$, the group property being a consequence of (2.7) and of the group property of $\mathcal{S}_\tau$. We call $(\widehat{\Phi}_t)_{t\in\mathbb{R}}$ the flow of system (2.4).

We also define the partially linear flow L_t by the formula :

$$L_t(x_0,\zeta) \;=\; (\,e^{tA}x_0\,,\,\mathcal{S}_t(\zeta)\,)\;; \tag{2.9}$$

it is the flow of (2.4) when $G = 0$, and the whole point in this subsection is to give conditions on G for $\widehat{\Phi}_t$ and L_t to be topologically conjugate over $\mathbb{R}^n \times \mathcal{E}$.

We will assume throughout that the $n \times n$ matrix A is hyperbolic, hence it is similar to a block diagonal one :

$$A \sim \begin{pmatrix} A_e & 0 \\ 0 & A_l \end{pmatrix} \tag{2.10}$$

where A_e and A_l are $e \times e$ and $l \times l$ real matrices, with $e + l = n$, whose eigenvalues have strictly negative and strictly positive real parts respectively. Now, there exist Euclidean norms on $\mathbb{R}^e$ and $\mathbb{R}^l$ for which e^{A_e} and e^{-A_l} are strict contractions, because their eigenvalues have modulus strictly less than 1 and any square complex matrix is similar to an upper triangular one having the eigenvalues of the original matrix as diagonal entries while the remaining entries are arbitrarily small, see *e.g.* [5, ch.3, sec.22.4, Lemma 4]. Therefore, combining (2.10) with a suitable linear change of variable on each factor in $\mathbb{R}^n = \mathbb{R}^e \times \mathbb{R}^l$, we can write

$$A = E^{-1}\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} E, \tag{2.11}$$

where E is some nonsingular $n \times n$ real matrix while P and Q are $e \times e$ and $l \times l$ real matrices such that e^P and e^{-Q} are strict contractions for the standard Euclidean norm :

$$c \stackrel{\Delta}{=} \|e^P\|_{\mathrm{O}} < 1 \quad \text{and} \quad \frac{1}{d} \stackrel{\Delta}{=} \|e^{-Q}\|_{\mathrm{O}} < 1\ , \tag{2.12}$$

where $\|.\|_{\mathrm{O}}$ designates the familiar operator norm of a matrix. Subsequently, we define the real numbers

$$b_1 \stackrel{\Delta}{=} \|e^{-P}\|_{\mathrm{O}} + \|e^{-Q}\|_{\mathrm{O}} = \frac{1}{d} + \|e^{-P}\|_{\mathrm{O}}, \tag{2.13}$$

$$c_1 \stackrel{\Delta}{=} \left\|EAE^{-1}\right\|_{\mathrm{O}} = \max\{\, \|P\|_{\mathrm{O}}\,, \|Q\|_{\mathrm{O}}\}. \tag{2.14}$$

Besides the operator norm, we shall make use of another norm on real matrices, namely the Frobenius norm $\|.\|_{\mathrm{F}}$ which is the square root of the sum of the squares of the entries. Let us record the elementary inequalities, valid for any two real square matrices M, N :

$$\|M\|_{\mathrm{O}} \le \|M\|_{\mathrm{F}}\ ,\ \|MN\|_{\mathrm{F}} \le \min\{\|M\|_{\mathrm{O}}\|N\|_{\mathrm{F}}, \|M\|_{\mathrm{F}}\|N\|_{\mathrm{O}}\}. \tag{2.15}$$

As usual, we keep the symbol $\|.\|$ to indicate the standard Euclidean norm on $\mathbb{R}^j$ irrespectively of j. Now, our main result is the following :

Theorem 2.2.1. *Let the hyperbolic matrix A and the numbers c, d, b_1 and c_1 be as in* (2.11), (2.12), (2.13) *and* (2.14). *Assume that the topological space $\mathcal{E}$, its one-parameter group of homeomorphisms $(\mathcal{S}_\tau)$, and the map $G : \mathbb{R}^n \times \mathcal{E} \times \mathbb{R} \to \mathbb{R}^n$ satisfy the following conditions :*

- *Equation* (2.5) *holds for all $(x, \zeta, \tau, t) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R} \times \mathbb{R}$.*
- *For fixed $\zeta \in \mathcal{E}$, the map $\tau \mapsto \mathcal{S}_\tau(\zeta)$ is Borel measurable $\mathbb{R} \to \mathcal{E}$, that is to say the inverse image of an open subset of $\mathcal{E}$ is measurable in $\mathbb{R}$.*
- *The map $x \mapsto G(x, \zeta, t)$ is continuously differentiable $\mathbb{R}^n \to \mathbb{R}^n$ for fixed $(\zeta, t) \in \mathcal{E} \times \mathbb{R}$, the map $t \mapsto G(x, \zeta, t)$ is measurable $\mathbb{R} \to \mathbb{R}^n$ for fixed $(x, \zeta) \in \mathbb{R}^n \times \mathcal{E}$, and to each $\zeta \in \mathcal{E}$ there are locally integrable functions $\phi_\zeta, \psi_\zeta : \mathbb{R} \to \mathbb{R}^+$ such that, for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, one has :*

$$\|G(x, \zeta, t)\| \le \phi_\zeta(t)\ , \qquad \|\frac{\partial G}{\partial x}(x, \zeta, t)\|_{\mathrm{F}} \le \psi_\zeta(t)\ . \tag{2.16}$$

- *Defining the flow $\widehat{x}$ of* (2.4) *as in* (2.6), *the map $(x_0, \zeta) \mapsto \widehat{x}(t, x_0, \zeta)$ is continuous $\mathbb{R}^n \times \mathcal{E} \to \mathbb{R}^n$ for fixed $t \in \mathbb{R}$.*
- *There are real numbers $M > 0$ and $\eta > 0$ such that*

$$\forall \zeta \in \mathcal{E}\ ,\ \|\phi_\zeta\|_{L^1([0,1])} \le M\ , \tag{2.17}$$

$$\|\psi_\zeta\|_{L^1([0,1])} \le \eta\ ; \tag{2.18}$$

Moreover, the number η in (2.18) *is so small that, putting*

$$\theta \stackrel{\Delta}{=} \eta\, \|E\|_{\mathrm{O}}\, \|E^{-1}\|_{\mathrm{O}}$$

and then

$$\alpha_1 \stackrel{\Delta}{=} \theta e^{c_1}\left(1 + e^{\theta + c_1}\,(\theta + c_1)\right),$$

one has

$$0 < b_1\alpha_1 < 1 \quad \textit{and} \quad \alpha_1(1 + 1/d) + \max(c, 1/d) < 1\ . \tag{2.19}$$

Then, there exists a homeomorphism

$$\mathcal{H}: \mathbb{R}^n \times \mathcal{E} \to \mathbb{R}^n \times \mathcal{E}$$

of the form

$$(x,\zeta) \quad \mapsto \quad \mathcal{H}(x,\zeta) = (H(x,\zeta),\zeta),$$

that conjugates $\widehat{\Phi}_t$ *defined in* (2.8) *to the partially linear flow* (2.9)*, namely* $\mathcal{H}\circ\widehat{\Phi}_t = L_t \circ \mathcal{H}$ *or, equivalently,*

$$H(\,\widehat{\Phi}_t(x,\zeta)\,) \quad = \quad e^{tA}H(x,\zeta) \tag{2.20}$$

for all $(t,x,\zeta) \in \mathbb{R}\times\mathbb{R}^n\times\mathcal{E}$.

To establish Theorem 2.2.1, we shall rely on two lemmas. The first one runs parallel to [47, chap. IX, lemma 8.3], and gives us sufficient conditions for perturbations of a map $(x,\zeta) \mapsto (Lx, \mathcal{S}_\tau(\zeta))$ to be topologically conjugate on $\mathbb{R}^n\times\mathcal{E}$, when τ is fixed and the linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ is the product of a dilation and a contraction. This lemma is the mainspring of the proof, in that it will provide us with the desired conjugating $\mathcal{H}$ when applied to the flows (2.8) and (2.9) evaluated at $t = 1$ (this arbitrary value comes from the normalization of the constants c and d through (2.12)). The proof of the lemma is similar to that of [47, chap. IX, lemma 8.3], except that we need to keep track more carefully of uniqueness and continuity issues here ; it uses the shrinking lemma on Lipschitz-small perturbations of hyperbolic linear maps, a classical device to build conjugating homeomorphisms that has many other applications, see [47, chap. IX, notes]. The reader will notice that the statement of the lemma redefines the constants c, d, b_1, and α_1 that were already fixed in the statement of Theorem 2.2.1. We allow ourself this minor incorrection, because we feel it helps following the argument since the lemma will be applied precisely with the previously defined constants.

Lemma 2.2.2. *Let us be given a homeomorphism* $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ *and two non-singular real matrices* C, D *of size* $e\times e$ *and* $l \times l$ *respectively, such that* $c = \|C\| < 1$ *and* $\frac{1}{d} = \|D^{-1}\| < 1$.

For $i = 1, 2$*, let* $Y_i : \mathbb{R}^e\times\mathbb{R}^l\times\mathcal{E} \to \mathbb{R}^e$ *and* $Z_i : \mathbb{R}^e\times\mathbb{R}^l\times\mathcal{E} \to \mathbb{R}^l$ *be two pairs of bounded continuous functions satisfying*

$$\max\{\|\Delta Y_i\|, \|\Delta Z_i\|\} \le \alpha_1(\|\Delta y\| + \|\Delta z\|), \tag{2.21}$$

where ΔY_i *and* ΔZ_i *stand respectively for* $Y_i(y+\Delta y, z+\Delta z,\zeta) - Y_i(y,z,\zeta)$ *and* $Z_i(y+\Delta y, z+\Delta z,\zeta) - Z_i(y,z,\zeta)$*, and where* α_1 *is a constant such that, if we put* $a = \|C^{-1}\|$ *and* $b_1 = a + 1/d$*, then* $0 < b_1\alpha_1 < 1$ *and* $\alpha_1(1 + 1/d) + \max(c, 1/d) < 1$*. If we define for* $i = 1, 2$ *the maps*

$$\begin{array}{rccl} T_i: & \mathbb{R}^e\times\mathbb{R}^l\times\mathcal{E} & \to & \mathbb{R}^e\times\mathbb{R}^l\times\mathcal{E} \\ & (y,z,\zeta) & \mapsto & (Cy + Y_i(y,z,\zeta), Dz + Z_i(y,z,\zeta), \mathcal{T}(\zeta)), \end{array}$$

then there exists a unique map $R_0 : \mathbb{R}^e\times\mathbb{R}^l\times\mathcal{E} \to \mathbb{R}^e\times\mathbb{R}^l\times\mathcal{E}$ *of the form*

$$R_0(y,z,\zeta) = (H_0(y,z,\zeta),\zeta) \tag{2.22}$$

such that :

– $H_0(y,z,\zeta) - (y,z)$ *is bounded on* $\mathbb{R}^e\times\mathbb{R}^l\times\mathcal{E}$,

– *one has the commuting relation :*

$$R_0T_1 = T_2R_0. \tag{2.23}$$

Moreover, R_0 *is then necessarily a homeomorphism of* $\mathbb{R}^e\times\mathbb{R}^l\times\mathcal{E}$.

The second lemma that we need in order to prove Theorem 2.2.1 is of technical nature and ensures that, under the hypotheses stated in that proposition, we can indeed apply Lemma 2.2.2 to the flow (2.8) evaluated at $t = 1$. Recalling from (2.6) the definition of $\widehat{x}$, it will be convenient to define a map $\Xi : \mathbb{R} \times \mathbb{R}^n \times \mathcal{E} \to \mathbb{R}^n$ by the equation :

$$\widehat{x}(t, x_0, \zeta) \;=\; \exp(tA)\, x_0 + \Xi(t, x_0, \zeta)\;. \tag{2.24}$$

Thus the map Ξ capsulizes the deviation of the flow of (2.4) from the flow of the linearized equation $\dot{x} = Ax$.

Lemma 2.2.3. *Under the assumptions of Theorem 2.2.1, the map Ξ defined by* (2.24) *is bounded on $[0,1]\times\mathbb{R}^n\times\mathcal{E}$, it is of class C^1 with respect to x_0 for fixed t, ζ, and it satisfies, for all (t, x_0, ζ) in $[0,1]\times\mathbb{R}^n\times\mathcal{E}$, the inequality :*

$$\|\frac{\partial \Xi}{\partial x_0}(t, x_0, \zeta)\|_{\mathrm{F}} \;\leq\; \eta\, e^{\|A\|_{\mathrm{O}}} \left(1 + e^{\eta + \|A\|_{\mathrm{O}}}\,(\eta + \|A\|_{\mathrm{O}})\right). \tag{2.25}$$

Assuming Lemma 2.2.2 and Lemma 2.2.3 for a while, let us proceed immediately with the proof of Theorem 2.2.1.

Proof. Proof of Theorem 2.2.1. Performing on $\mathbb{R}^n$ the change of variables $x \mapsto E\,x$ and taking (2.15) into account, we may assume upon replacing M by $M\|E\|_{\mathrm{O}}$ in (2.17) and η by θ in (2.18) that $E = I_n$, the identity matrix of size n. Then $c_1 = \|A\|_{\mathrm{O}}$ and the right-hand side of (2.25) is just α_1. Moreover (2.11) expresses that A assumes a block-diagonal form, according to which we block-decompose the flow $\widehat{\Phi}_t(x_0, \zeta)$ defined by (2.8) into

$$\begin{pmatrix} y_0 \\ z_0 \\ \zeta \end{pmatrix} \;\mapsto\; \begin{pmatrix} e^{tP} y_0 + Y(t, y_0, z_0, \zeta) \\ e^{tQ} z_0 + Z(t, y_0, z_0, \zeta) \\ \mathcal{S}_t(\zeta) \end{pmatrix} \tag{2.26}$$

where $(y_0^T, z_0^T)^T$ is the natural partition of $x_0 \in \mathbb{R}^n \sim \mathbb{R}^e \times \mathbb{R}^l$, and where Y and Z are respectively the first e and the last l components of the map Ξ defined in (2.24). Still taking into account the block decomposition induced by (2.11) where $E = I_n$, the partially linear flow L_t defined by (2.9) in turn splits into

$$\begin{array}{rccc} L_t : & \mathbb{R}^e \times \mathbb{R}^d \times \mathcal{E} & \to & \mathbb{R}^e \times \mathbb{R}^d \times \mathcal{E} \\ & (y_0, z_0, \zeta) & \mapsto & (\exp(Pt)\, y_0, \exp(Qt)\, z_0, \mathcal{S}_t(\zeta)). \end{array}$$

We shall apply Lemma 2.2.2 with $\mathcal{T} = \mathcal{S}_1$ to $T_1 = \widehat{\Phi}_1$ and $T_2 = L_1$, that is to say we choose $C = e^P$, $D = e^Q$, $Y_2 = 0$, $Z_2 = 0$, and we define Y_1 and Z_1 by $Y_1(y, z, \zeta) = Y(1, y, z, \zeta)$ and $Z_1(y, z, \zeta) = Z(1, y, z, \zeta)$ where Y, Z are as in (2.26). The hypotheses on C and D are satisfied by (2.12), while the hypotheses on Y_2 and Z_2 are trivially met. As to Y_1 and Z_1, we observe that :
- their continuity, i.e. the continuity of $(x_0, \zeta) \mapsto \Xi(1, x_0, \zeta)$, follows *via* (2.24) from the continuity of $(x_0, \zeta) \mapsto \widehat{x}(1, x_0, \zeta)$ which is part of the hypotheses (see point 4 in the statement of the proposition) ;
- their boundedness, i.e. the boundedness of $(x_0, \zeta) \mapsto \Xi(1, x_0, \zeta)$, follows from Lemma 2.2.3 ;
- the inequalities on the Lipschitz constants of Y_1 and Z_1 required in Lemma 2.2.2 follow from the mean-value theorem and Lemma 2.2.3, equation (2.25), granted (2.19), (2.15), and the triangle inequality.

Therefore Lemma 2.2.2 does apply, providing us with a homeomorphism of $\mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E} = \mathbb{R}^n \times \mathcal{E}$ of the form $R_0 = H_0 \times \mathrm{id}$, which is such that $H_0(x, \zeta) - x$ is bounded on $\mathbb{R}^n \times \mathcal{E}$ and, in addition, such that

$$R_0 \circ \widehat{\Phi}_1 \;=\; L_1 \circ R_0. \tag{2.27}$$

Equation (2.27) expresses that H_0 conjugates the flow $\widehat{\Phi}_t(x,\zeta))$ to the partially linear flow L_t at time $t=1$, whereas we want these flows to be conjugate at any time t. For this, we use the same averaging trick (originally due to S. Sternberg) as in [47, chap. IX, sec. 9], namely we define $H:\mathbb{R}^n\times\mathcal{E}\to\mathbb{R}^n$ by the integral formula :

$$H(x,\zeta)=\int_0^1 e^{-rA}H_0(\widehat{\Phi}_r(x,\zeta))\,dr \tag{2.28}$$

where H_0, being the first factor of R_0, satisfies by virtue of (2.27) :

$$H_0(\widehat{\Phi}_1(x,\zeta)) \;=\; e^A H_0(x,\zeta). \tag{2.29}$$

We need of course show that (2.28) is well-defined. Firstly, let us check that the integrand is a measurable function of r. As H_0 is continuous $\mathbb{R}^n\times\mathcal{E}\to\mathbb{R}^n$, this reduces to showing that the map

$$r\mapsto\widehat{\Phi}_r(x,\zeta)=(\,\widehat{x}(r,x,\zeta)\,,\,\mathcal{S}_r(\zeta)\,) \tag{2.30}$$

is measurable $\mathbb{R}\mapsto\mathbb{R}^n\times\mathcal{E}$. Now, the map $r\mapsto\widehat{x}(r,x,\zeta)$ is *a fortiori* measurable since it is absolutely continuous, and the map $r\mapsto\mathcal{S}_r(\zeta)$ is also measurable by assumption (see point 2 in the statement of the proposition). Hence the inverse image under (2.30) of an open rectangle is measurable in $\mathbb{R}$. But any open subset of $\mathbb{R}^n\times\mathcal{E}$ is a countable union of open rectangles because $\mathbb{R}^n$ has a countable basis of open neighborhoods, and this establishes the measurability of (2.30). Secondly, the integrand in (2.28) is bounded, for $\|H_0(\widehat{\Phi}_r(x,\zeta))-\widehat{x}(r,x,\zeta)\|$ is majorized uniformly with respect to r, x, and ζ since $H_0(x,\zeta)-x$ is bounded on $\mathbb{R}^n\times\mathcal{E}$ by the properties of R_0, while the continuous function $r\mapsto\widehat{x}(r,x,\zeta)$ is bounded for fixed x and ζ on the compact set $[0,1]$. Therefore, the integral on the right-hand side of (2.28) indeed exists.
Observe now that $H(x,\zeta)-x$ is also bounded on $\mathbb{R}^n\times\mathcal{E}$. Indeed, by definition of $\widehat{\Phi}_r$ *via* (2.8) and of Ξ *via* (2.24), we can write

$$\begin{aligned} H(x,\zeta)-x \;&=\; \int_0^1 e^{-rA}\big(H_0(\,\widehat{x}(r,x,\zeta)\,,\,\mathcal{S}_r(\zeta)\,)-\widehat{x}(r,x,\zeta)\,\big)\,dr\\ &+\;\int_0^1 e^{-rA}\,\Xi(r,x,\zeta)\,dr, \end{aligned} \tag{2.31}$$

and since both integrals on the right-hand side are bounded (the first because $H_0(x,\zeta)-x$ is bounded on $\mathbb{R}^n\times\mathcal{E}$ and the second because Ξ is bounded on $[0,1]\times\mathbb{R}^n\times\mathcal{E}$ by Lemma 2.2.3), we get the desired boundedness of $H(x,\zeta)-x$. Next, *we claim that* (2.20) *holds*, and once we have proved this the proposition will follow because, specializing (2.20) to $t=1$, we shall conclude by the uniqueness part of Lemma 2.2.2 that $H\times\mathrm{id}=R_0$ and therefore that R_0, which is a homeomorphism of $\mathbb{R}^n\times\mathcal{E}$ with the desired form, will meet $R_0\circ\widehat{\Phi}_t=L_t\circ R_0$, not just for $t=1$ as we knew already but in fact for all t. Thus it will be possible to take $\mathcal{H}=R_0$.

To establish the claim, we use the group property of the flow to write

$$e^{-tA}H(\,\widehat{\Phi}_t(x,\zeta)\,) \;=\; \int_0^1 e^{-(t+r)A}H_0(\widehat{\Phi}_{t+r}(x,\zeta)\,)\,dr,$$

and we set $t+r=\tau$ to convert the above integral into

$$\int_t^{t+1} e^{-\tau A}H_0(\widehat{\Phi}_\tau(x,\zeta)\,)\,\mathrm{d}\tau \;=\; \int_t^1 \ldots\,\mathrm{d}\tau+\int_1^{t+1}\ldots\,\mathrm{d}\tau, \tag{2.32}$$

where the dots indicate that the integrand is repeated in each integral. Now, putting $\lambda=\tau-1$, the last integral in the right-hand side becomes

$$\int_0^t e^{-(\lambda+1)A}H_0(\widehat{\Phi}_{\lambda+1}(x,\zeta)\,)\,d\lambda \;=\; \int_0^t e^{-\lambda A}H_0(\widehat{\Phi}_\lambda(x,\zeta)\,)\,d\lambda,$$

where we have used the group property of the flow again together with (2.29). Plugging this into (2.32), we recover back $\int_0^1 e^{-tA}H_0(\widehat{\Phi}_t(x,\zeta)\,)dt$ on the right-hand side, so that finally $e^{-tA}H\circ\widehat{\Phi}_t = H$ *as claimed.* □ □

Let us now tie the loose ends in the proof of Theorem 2.2.1 by establishing Lemma 2.2.3 and Lemma 2.2.2.

Proof. Proof of Lemma 2.2.3 From (2.4) and (2.24), we see that $t\mapsto \Xi(t,x_0,\zeta)$ is the solution to

$$\dot{\xi}(t) = A\xi(t) + G\left(\xi(t) + e^{tA}x_0\,,\,\zeta\,,\,t\right)$$

with initial condition $\xi(0)=0$. Since $\|G(x,\zeta,t)\|$ is bounded by $\phi_\zeta(t)$ with $\|\phi_\zeta\|_{L^1([0,1])}\le M$ by (2.16) and (2.17), we get

$$\|\xi(t)\| \le M + \int_0^t \|A\|_{\mathrm{O}}\|\xi(s)\|\,ds, \qquad t\in[0,1],$$

and finally, by the Bellman-Gronwall lemma, $\|\xi(t)\|\le M\,e^{t\|A\|_{\mathrm{O}}}$. This entails that ξ is bounded on $[0,1]$; hence Ξ is bounded on $[0,1]\times\mathbb{R}^n\times\mathcal{E}$.

To prove (2.25), we consider for fixed x_0,ζ the matrix-valued function $R(t)=\frac{\partial\widehat{x}}{\partial x_0}(t,x_0,\zeta)$, whose existence and continuity with respect to x_0 for fixed t,ζ depend on (2.16), (2.17) and (2.18) (*cf* Lemma 2.4.3), inducing in turn the existence and continuity with respect to x_0 of $Q(t)=\frac{\partial\Xi}{\partial x_0}(t,x_0,\zeta)$ *via* (2.24). The variational equation for $\frac{\partial\widehat{x}}{\partial x_0}$ (see again Lemma 2.4.3) yields :

$$\dot{R}(t) \;=\; \left[A + \frac{\partial G}{\partial x}\left(\widehat{x}(t,x_0,\zeta),\zeta,t\right)\right]R(t)\;, \quad R(0)\;=\;I_n\;,$$

and, since $R(t)=Q(t)+e^{tA}$ by (2.24), we have that

$$\dot{Q}(t) = \left[A + \frac{\partial G}{\partial x}\left(\widehat{x}(t,x_0,\zeta),\zeta,t\right)\right]Q(t) + \frac{\partial G}{\partial x}\left(\widehat{x}(t,x_0,\zeta),\zeta,t\right)e^{tA}\;,\; Q(0)=0.$$

Put $\rho(t)=\|Q(t)\|_{\mathrm{F}}$. Due to the definition of the Frobenius norm, $\rho(t)$ is locally absolutely continuous and, by the Cauchy-Schwartz inequality, one has $\dot{\rho}(t)\le\|\dot{Q}(t)\|_{\mathrm{F}}$. Thus, the differential equation satisfied by $Q(t)$ together with (2.16) yield :

$$\dot{\rho}\;\le\;(\psi_\zeta(t)+\|A\|_{\mathrm{O}})\,\rho(t)\;+\;\psi_\zeta(t)\,e^{\|A\|_{\mathrm{O}}}, \qquad \rho(0)=0,$$

where we have used (2.15) and the elementary fact that $\|e^{tA}\|_{\mathrm{O}}\le e^{\|A\|_{\mathrm{O}}}$ for all $t\in[0,1]$. Integrating this inequality and applying the Bellman-Gronwall lemma yields, taking (2.18) into account, that ρ is bounded on $[0,1]$ by $\eta e^{2\|A\|_{\mathrm{O}}+\eta}$. This implies (2.25) by definition of ρ. □ □

Proof. Proof of Lemma 2.2.2 If we endow $\mathbb{R}^e\times\mathbb{R}^l$ with the norm $\|(y,z)\|=\|y\|+\|z\|$, it follows from (2.21) that, for fixed $(y,z,\zeta)\in\mathbb{R}^e\times\mathbb{R}^l\times\mathcal{E}$, the map $T_{y,z,\zeta}:\mathbb{R}^e\times\mathbb{R}^l\to\mathbb{R}^e\times\mathbb{R}^l$ defined by

$$T_{y,z,\zeta}(y',z') = (C^{-1}y, D^{-1}z) - \left(C^{-1}Y_1(y',z',\zeta)\,,\,D^{-1}Z_1(y',z',\zeta)\right)$$

is a shrinking map with shrinking constant $b_1\alpha_1<1$, whose fixed point is the unique $(\bar{y},\bar{z})\in\mathbb{R}^e\times\mathbb{R}^l$ satisfying $T_1(\bar{y},\bar{z},\zeta)=(y,z,\mathcal{T}(\zeta))$. In addition, it holds that $(\bar{y},\bar{z})=\lim_{k\to\infty}T^k_{y,z,\zeta}(y',z')$ for any (y',z'), and this classically implies that $(\bar{y},\bar{z})$ is continuous with respect to y, z, and ζ. Indeed, the continuity of Y_1 and Z_1 entails that $T_{y,z,\zeta}(y',z')$ is continuous with respect to y, z and ζ for fixed y',z'. Therefore, if we write $\bar{y}(y,z,\zeta)$, $\bar{z}(y,z,\zeta)$ to emphasize the functional dependence, and if we choose y_0, z_0, ζ_0 together with $\varepsilon > 0$, there is a neighborhood $\mathcal{V}_0$ of (y_0, z_0, ζ_0) in $\mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E}$ such that $(y, z, \zeta) \in \mathcal{V}_0$ implies :

$$\begin{aligned} &\|T_{y,z,\zeta}(\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0)) - (\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0))\| = \\ &\|T_{y,z,\zeta}(\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0)) - T_{y_0,z_0,\zeta_0}((\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0))\| \\ &< \varepsilon. \end{aligned}$$

Consequently, for $(y, z, \zeta) \in \mathcal{V}_0$, we have by the shrinking property that

$$\begin{aligned} &\big\|\big(\bar{y}(y, z, \zeta), \bar{z}(y, z, \zeta)\big) - \big(\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0)\big)\big\| \\ &= \|\lim_{k\to\infty} T^k_{y,z,\zeta}((\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0)) - (\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0))\| \\ &\leq \sum_{k=0}^{\infty} \|T^{k+1}_{y,z,\zeta}((\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0)) - T^k_{y,z,\zeta}((\bar{y}(y_0, z_0, \zeta_0), \bar{z}(y_0, z_0, \zeta_0))\| \\ &\leq \frac{\varepsilon}{1 - b_1\alpha_1} \end{aligned}$$

which implies the desired continuity. Then, $(x, y) \mapsto (\bar{y}(y, z, \zeta), \bar{z}(y, z, \zeta))$ is, for fixed ζ, the inverse of the concatenation of the first two components of T_1, and it is continuous with respect to (x, y), and to ζ. Moreover, we see from the definition of $T_{y,z,\zeta}$ and the fixed point property of $\bar{y}, \bar{z}$ that

$$(\bar{y}, \bar{z}) = (C^{-1}y, D^{-1}z) - (C^{-1}Y_1(\bar{y}, \bar{z}, \zeta)\,,\, D^{-1}Z_1(\bar{y}, \bar{z}, \zeta))$$

and, since Y_1 and Z_1 are continuous and bounded, this makes for a relation of the form

$$(\bar{y}(y, z, \zeta), \bar{z}(y, z, \zeta)) = (C^{-1}y + \hat{Y}_1(y, z, \zeta), D^{-1}z + \hat{Z}_1(y, z, \zeta))$$

where $\hat{Y}_1$, $\hat{Z}_1$ are in turn continuous and bounded on $\mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E}$ with values in $\mathbb{R}^e$ and $\mathbb{R}^l$ respectively. All this yields the existence of an inverse for the map T_1 itself, namely

$$T_1^{-1}(y, z, \zeta) = (C^{-1}y + \hat{Y}_1(y, z, \mathcal{T}^{-1}(\zeta)), D^{-1}z + \hat{Z}_1(y, z, \mathcal{T}^{-1}(\zeta)), \mathcal{T}^{-1}(\zeta)). \tag{2.33}$$

Let us now seek the map H_0 in (2.22) in the prescribed form, namely

$$H_0(y, z, \zeta) = (\, y + \Lambda(y, z, \zeta)\,,\, z + \Theta(y, z, \zeta)\,), \tag{2.34}$$

where the unknowns are bounded maps Λ and Θ with values in $\mathbb{R}^e$ and $\mathbb{R}^l$ respectively. Using (2.33), one checks easily that (2.23) is equivalent to the following pair of equations :

$$\begin{aligned} \Lambda \;=\; & C\left[\hat{Y}_1 + \Lambda(T_1^{-1})\right] \\ & + Y_2\left(C^{-1}y + \hat{Y}_1 + \Lambda(T_1^{-1}), D^{-1}z + \hat{Z}_1 + \Theta(T_1^{-1}), \mathcal{T}^{-1}(\zeta)\right), \end{aligned} \tag{2.35}$$

$$\Theta = D^{-1}[Z_1 + \Theta(Cy + Y_1, Dz + Z_1, \mathcal{T}(\zeta)) - Z_2(y + \Lambda, z + \Theta, \zeta)], \tag{2.36}$$

where the argument of $\Lambda, \Theta, Y_i, Z_i, \hat{Y}_i, \hat{Z}_i, T_1^{-1}$, when omitted, is always (y, z, ζ). The existence of Λ and Θ will follow from another application of the shrinking lemma, this time in the space $\mathcal{B}$ of bounded functions $\mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E} \to \mathbb{R}^e \times \mathbb{R}^l$ endowed with a suitable norm. More precisely, letting (Λ_1, Θ_1) denote an arbitrary member of $\mathcal{B}$ acting coordinate-wise as $(y, z, \zeta) \mapsto (\Lambda_1(y, z, \zeta), \Theta_1(y, z, \zeta))$ where Λ_1 and Θ_1 are bounded $\mathbb{R}^e$ and $\mathbb{R}^l$-valued functions respectively, we define its norm to be

$$|||(\Lambda_1, \Theta_1)|||_+ = |||\Lambda_1||| + |||\Theta_1|||,$$

where $|||.|||$ indicates the *sup* norm of a map $\mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E} \to \mathbb{R}^k$, irrespectively of k ; this makes $(\mathcal{B}, |||.|||_+)$ into a a Banach space. Now, to each $(\Lambda_1, \Theta_1) \in \mathcal{B}$, we can associate another member (Λ_2, Θ_2) of $\mathcal{B}$ where $\Lambda_2 : \mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E} \to \mathbb{R}^e$ and $\Theta_2 : \mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E} \to \mathbb{R}^l$ are defined by

$$\begin{aligned} \Lambda_2 &= C\left[\hat{Y}_1 + \Lambda_1(T_1^{-1})\right] \\ &\quad + Y_2\Big(C^{-1}y + \hat{Y}_1 + \Lambda_1(T_1^{-1}), D^{-1}z + \hat{Z}_1 + \Theta_1(T_1^{-1}), \mathcal{T}^{-1}(\zeta)\Big), \end{aligned} \tag{2.37}$$

$$\Theta_2 = D^{-1}\Big[Z_1 + \Theta_1(Cy + Y_1, Dz + Z_1, \mathcal{T}(\zeta)) - Z_2(y + \Lambda_1, z + \Theta_1, \zeta)\Big], \tag{2.38}$$

the argument (y, z, ζ) being omitted again for simplicity. The fact that (Λ_2, Θ_2) is indeed well-defined and belongs to $\mathcal{B}$ is a consequence of the preceding part of the proof. Consistently designating by a subscript 2 the effect of the right hand-side of (2.37) an (2.38) on some initial map, itself denoted with a subscript 1, we see from (2.21)) by inspection on (2.37) and (2.38) that, if (Λ_1, Θ_1) and (Λ_1', Θ_1') are two members of $\mathcal{B}$, then

$$|||\Lambda_2 - \Lambda_2'||| \leq c\, |||\Lambda_1 - \Lambda_1'||| + \alpha_1 |||(\Lambda_1 - \Lambda_1', \Theta_1 - \Theta_1')|||_+, \tag{2.39}$$

$$|||\Theta_2 - \Theta_2'||| \leq \frac{1}{d}\left(|||\Theta_1 - \Theta_1'||| + \alpha_1 |||(\Lambda_1 - \Lambda_1', \Theta_1 - \Theta_1')|||_+\right). \tag{2.40}$$

Adding up (2.39) and (2.40), we obtain

$$\begin{aligned} &|||(\Lambda_2 - \Lambda_2', \Theta_2 - \Theta_2')|||_+ \\ &\quad \leq \; [\alpha_1(1 + 1/d) + \max(c, 1/d)]\; |||(\Lambda_1 - \Lambda_1', \Theta_1 - \Theta_1')|||_+ \\ &\quad = \; \alpha\, |||(\Lambda_1 - \Lambda_1', \Theta_1 - \Theta_1')|||_+ \end{aligned}$$

where by assumption $\alpha < 1$. This means that $(\Lambda_1, \Theta_1) \mapsto (\Lambda_2, \Theta_2)$ is a shrinking map on $\mathcal{B}$ whose fixed point (Λ, Θ) provides us with the unique bounded solution to (2.35) and (2.36). Equivalently, if H_0 is defined through (2.34) and R_0 through (2.22), then R_0 is the unique map $\mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E} \to \mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E}$ of the form (H, id), where id is the identity map on $\mathcal{E}$, such that $H - (y, z) \in \mathcal{B}$ and such that the commuting relation (2.23) holds. It remains for us to show that R_0 is a homeomorphism. For this, notice first that R_0 is continuous, because H_0 turns out to be continuous : indeed, iterating the formulas (2.37) and (2.38) starting from any initial pair (Λ_1, Θ_1) yields a sequence of maps converging to (Λ, Θ) in $\mathcal{B}$, and if the initial pair is continuous (we may for instance choose the zero map) so is every member of the sequence hence also the limit since $|||.|||_+$ induces on $\mathcal{B}$ the topology of uniform convergence. Next, if we switch the roles of T_1 and T_2, the above argument provides us with a continuous map $R_0' : \mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E} \to \mathbb{R}^e \times \mathbb{R}^l \times \mathcal{E}$ of the form (H', id)) with $H' - (y, z) \in \mathcal{B}$, satisfying $R_0' T_2 = T_1 R_0'$. Then, the composed map $R = R_0' R_0$ satisfies $RT_1 = T_1 R$, and since it is again of the form (H'', id)) with $H'' - (y, z) \in \mathcal{B}$, we get $R = \mathrm{id}$ by the uniqueness part of the previous proof. Similarly $R_0 R_0' = \mathrm{id}$, so that finally R_0 is invertible with continuous inverse R_0' hence a homeomorphism. □ □

2.3 Grobman-Hartman theorems for control systems

We consider a control system of the form :

$$\dot{x} = f(x, u)\ , \quad x \in \mathbb{R}^n\ , \quad u \in \mathbb{R}^m\ , \tag{2.1}$$

and we suppose that $f(0, 0) = 0$, *i.e.* we work around an equilibrium point that we choose to be the origin without loss of generality. We assume that f is continuous, and throughout we also make the hypothesis that $\partial f/\partial x(x,u)$ exists and is jointly continuous with respect to (x,u). Subsequently, we single out the linear part of f by consistently setting $A = \frac{\partial f}{\partial x}(0,0)$, so that (2.1) can be rewritten as

$$\begin{array}{rcl} \dot{x} & = & Ax \; + \; P(x,u) \\ \text{with} \quad P(0,0) & = & \dfrac{\partial P}{\partial x}(0,0) \; = \; 0 \; . \end{array} \tag{2.2}$$

If in addition f happens to be continuously differentiable with respect to u as well, we set $B = \frac{\partial f}{\partial u}(0,0)$ and we further expand (2.2) into

$$\begin{array}{rcl} \dot{x} & = & Ax \; + \; Bu \; + \; F(x,u) \\ \text{with} \quad F(0,0) & = & \dfrac{\partial F}{\partial x}(0,0) \; = \; \dfrac{\partial F}{\partial u}(0,0) \; = \; 0 \; . \end{array} \tag{2.3}$$

Since (2.3) is derived under the stronger hypothesis that f is of class C^1 with respect to both x and u, one would expect stronger results to hold in this case. We want to stress that, deceptively enough, local linearization of (2.3) will turn out to be a consequence of local linearization of (2.2) although the latter was derived without differentiability requirement with respect to u. This is due to the – even more surprising – fact that (2.2) will be locally conjugate to the *non controlled* system $\dot{x} = Ax$, that is to say the influence of the control can be entirely assigned to the linearizing homeomorphism. Compare Theorems 2.3.1 and 2.3.3, and see also Remark 2.3.8.

2.3.1 Prescribed dynamics for the control

We investigate in this subsection the situation where, in system (2.1), the control function $u(t)$ is itself the output of a dynamical system of the form :

$$\begin{array}{rcl} \dot{\zeta} & = & g(\zeta), \\ u & = & h(\zeta), \end{array} \tag{2.4}$$

where $\zeta(t) \in \mathbb{R}^q$, while $g : \mathbb{R}^q \to \mathbb{R}^q$ is locally Lipschitz continuous and $h : \mathbb{R}^q \to \mathbb{R}^m$ is continuous with, say, $h(0) = 0$. In particular, $u(t)$ is entirely determined by the finite-dimensional data $\zeta(0)$ and, from the control viewpoint, this is a particular instance of feed-forward on system (2.1) by system (2.4) where the input may only consist of Dirac delta functions.

Assume first that f is of class C^1 with respect to x and u so that (2.3) holds. Plugging (2.4) into the latter yields an ordinary differential equation in $\mathbb{R}^{n+q}$:

$$\begin{array}{rcl} \dot{x} & = & Ax \; + \; Bh(\zeta) \; + \; F(x, h(\zeta)), \\ \dot{\zeta} & = & g(\zeta). \end{array} \tag{2.5}$$

To motivate the developments to come, observe that if g is continuously differentiable with $g(0) = 0$, if A and $\partial g/\partial\zeta(0)$ are hyperbolic, and if h is continuously differentiable, then we can apply the standard Grobman-Hartman theorem on ordinary differential equations to conclude that the flow of (2.5) is topologically conjugate, *via* a local homeomorphism $(x,\zeta) \mapsto (z,\xi)$ around $(0,0)$, to that of

$$\begin{pmatrix} \dot{z} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A & B\frac{\partial h}{\partial \zeta}(0) \\ 0 & \frac{\partial g}{\partial \zeta}(0) \end{pmatrix} \begin{pmatrix} z \\ \xi \end{pmatrix} .$$

However, the hyperbolicity requirement on $\partial g/\partial\zeta(0)$ is more stringent than it seems. Indeed, it is often desirable to study non-trivial steady behaviors, which usually entail oscillatory controls. This is why we rather seek a transformation of the form $(x,\zeta) \mapsto (H(x,\zeta),\zeta)$ that linearizes the first equation in (2.5) but preserves the second one. This can be done, as asserted by the following result which does not require hyperbolicity nor even continuous differentiability on g.

Theorem 2.3.1. *Suppose in system* (2.5) *that* $g : \mathbb{R}^q \to \mathbb{R}^q$ *is locally Lipschitz continuous, that* $h : \mathbb{R}^q \to \mathbb{R}^m$ *is continuous with* $h(0) = 0$*, that* $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *is continuously differentiable with* $F(0,0) = \partial F/\partial x(0,0) = 0$*, and that* A *is hyperbolic. Then, there exist two neighborhoods* V *and* W *of 0 in* $\mathbb{R}^n$ *and* $\mathbb{R}^q$ *respectively, and a map* $H : V \times W \to \mathbb{R}^n$ *with* $H(0,0) = 0$*, such that*

$$\begin{array}{rccc} H \times Id: & V \times W & \to & \mathbb{R}^n \times W \\ & (x,\zeta) & \mapsto & (H(x,\zeta),\zeta) \end{array}$$

is a homeomorphism from $V \times W$ *onto its image that conjugates* (2.5) *to*

$$\begin{array}{rcl} \dot{z} & = & Az + Bh(\zeta), \\ \dot{\zeta} & = & g(\zeta). \end{array} \tag{2.6}$$

Remark 2.3.2. *In Theorem 2.3.1 (resp. Theorem 2.3.3 to come), we assume for convenience that all the functions involved, namely* F *(resp.* P*),* g*, and* h*, are globally defined. However, since the conclusion is local with respect to* x *and* ζ*, the same holds when these functions are only defined locally on a neighborhood of the origin, as a partition of unity argument immediately reduces the local version to the present one.*

Although it looks natural, the above theorem deserves one word of caution for the homeomorphism H depends heavily on g and h, and in a rather intricate manner. In fact, it is possible to entirely incorporate the influence of the control into the change of variables, so as to obtain a statement in which the term $Bh(\zeta)$ does not even appear in the transformed system. This will follow from Theorem 2.3.3 to come, for which we no longer assume in (2.1) that f is differentiable with respect to u. Accordingly, we plug (2.4) into (2.2) rather than (2.3), and we obtain instead of (2.5) the following ordinary differential equation in $\mathbb{R}^{n+q}$:

$$\begin{array}{rcl} \dot{x} & = & Ax + P(x, h(\zeta)), \\ \dot{\zeta} & = & g(\zeta), \end{array} \tag{2.7}$$

whose flow will be denoted by $(t, x_0, \zeta_0) \mapsto (x(t, x_0, \zeta_0), \zeta(t, \zeta_0))$.

Theorem 2.3.3. *Suppose in system* (2.7) *that* $g : \mathbb{R}^q \to \mathbb{R}^q$ *is locally Lipschitz continuous, that* $h : \mathbb{R}^q \to \mathbb{R}^m$ *is continuous with* $h(0) = 0$*, that* $P(x,u)$ *is continuous* $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *with* $P(0,0) = 0$*, that* $\partial P/\partial x$ *exists and is continuous* $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$ *with* $\partial P/\partial x(0,0) = 0$*, and that* A *is hyperbolic. Then, there exist two neighborhoods* V *and* W *of 0 in* $\mathbb{R}^n$ *and* $\mathbb{R}^q$ *respectively, and a map* $H : V \times W \to \mathbb{R}^n$ *with* $H(0,0) = 0$*, such that*

$$\begin{array}{rccc} H \times Id: & V \times W & \to & \mathbb{R}^n \times W \\ & (x,\zeta) & \mapsto & (H(x,\zeta),\zeta) \end{array}$$

is a homeomorphism from $V \times W$ *onto its image that conjugates* (2.7) *to*

$$\begin{array}{rcl} \dot{z} & = & Az, \\ \dot{\zeta} & = & g(\zeta), \end{array} \tag{2.8}$$

i.e. for all t, x_0, ζ_0 *such that* $(x(\tau, x_0, \zeta_0), \zeta(\tau, \zeta_0)) \in V \times W$ *for all* $\tau \in [0,t]$ *(or* $[t,0]$ *if* $t < 0$*), one has*

$$H(x(t, x_0, \zeta_0), \zeta(t, \zeta_0)) = e^{tA} H(x_0, \zeta_0).$$

Theorem 2.3.1 is a consequence of Theorem 2.3.3 because the latter implies that (2.5) and (2.6) are both conjugate to (2.8). As to Theorem 2.3.3 itself, we will show that it is a consequence of Theorem 2.2.1. This will require an elementary lemma enabling us to normalize the original control system. To state the lemma, we fix, once and for all, a smooth function $\rho : [0,+\infty) \to [0,1]$ such that

$$\left.\begin{array}{lll} \forall t, & & |\dot\rho(t)| < 3, \\ 0 \le t \le \frac{1}{2} & \Rightarrow & \rho(t) = 1, \\ \frac{1}{2} < t < 1 & \Rightarrow & 0 < \rho(t) < 1, \\ 1 \le t & \Rightarrow & \rho(t) = 0, \end{array}\right\} \tag{2.9}$$

and we associate to any map $\beta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ a family of functions $G_s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, indexed by a real number $s > 0$, using the formula :

$$G_s(x,u) \triangleq \rho\left(\frac{\|x\|^2}{s^2}\right)\beta(x,u). \tag{2.10}$$

Since the context will always make clear which β is involved, our notation does not explicitly indicate the dependency of G_s on the map β. The symbol $\|.\|$, in the statement of the lemma, denotes the norm, not only of a vector, but also of a matrix; the result does not depend on a specific choice of this norm. Also, $B(x,r)$ stands for the open ball of radius r, centered at x, in any Euclidean space.

Lemma 2.3.4. *Let $\beta(x,u)$ be continuous $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\partial\beta/\partial x$ continuously exist $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$, with $\beta(0,0) = \partial\beta/\partial x(0,0) = 0$. Then $G_s(x,u)$ defined by (2.10) is in turn continuous and continuously differentiable with respect to x for every $s > 0$, and to each $\eta > 0$ there exist $\sigma > 0$ and $\theta > 0$ such that*

$$\forall (x,u) \in \mathbb{R}^n \times B(0,\theta)\,, \quad \|\frac{\partial G_\sigma}{\partial x}(x,u)\| \;\le\; \eta\;. \tag{2.11}$$

Proof. For the proof, we use the standard Euclidean norm on $\mathbb{R}^n$, $\mathbb{R}^m$, and the familiar operator norm on matrices. Clearly G_s is continuous and continuously differentiable with respect to x for every $s > 0$, and we have :

$$\frac{\partial G_s}{\partial x}(x,u) \;=\; \rho\left(\frac{\|x\|^2}{s^2}\right)\frac{\partial\beta}{\partial x}(x,u) \;+\; \frac{2}{s^2}\,\rho'\left(\frac{\|x\|^2}{s^2}\right)\;\beta(x,u)\;x^T, \tag{2.12}$$

where x^T is the transpose of x. Since β is continuously differentiable and $\partial\beta/\partial x\,(0,0) = 0$, we get for $s > 0$ small enough that $\|\partial\beta/\partial x\,(x,u)\| < \eta/14$ as soon as $\|x\|, \|u\| < s$. Let σ be an s with this property. Since β is continuous with $\beta(0,0) = 0$, we can in turn pick θ with $0 < \theta \le \sigma$ such that $\|\beta(0,u)\| < \eta\sigma/12$ whenever $\|u\| < \theta$. Altogether, we get that

$$\left.\begin{array}{l} \|x\| < \sigma \\ \|u\| < \theta \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \|\frac{\partial\beta}{\partial x}(x,u)\| < \frac{\eta}{14}\,, \\ \|\beta(0,u)\| < \frac{\eta\sigma}{12}\,. \end{array}\right. \tag{2.13}$$

Now, we need only check (2.11) when $\|x\| < \sigma$ for otherwise G_σ is identically zero; therefore we restrict ourselves to pairs (x,u) where $\|x\| < \sigma$ and $\|u\| < \theta$. On this domain, we get from (2.13) and the mean value theorem that

$$\|\beta(x,u)\| \;\le\; \frac{\eta}{14}\sigma \;+\; \frac{\eta\sigma}{12} \;=\; \frac{13\eta\sigma}{84}.$$

Using this together with (2.13) and the inequalities $|\rho| \le 1$, $\|\rho'\| \le 3$, as well as $\|x^T\| < \sigma$, formula (2.12) with $s = \sigma$ yields :

$$\|\frac{\partial G_\sigma}{\partial x}(x,u)\| \;\le\; \frac{\eta}{14} \;+\; \frac{6}{\sigma^2}\,\frac{13\eta\sigma}{84}\,\sigma = \eta. \quad \square$$

□

Proof. Proof of Theorems 2.3.1 and 2.3.3. We already mentioned that Theorem 2.3.1 is a consequence of Theorem 2.3.3. To establish the latter, consider the following "renormalized" version of (2.7) :

$$\begin{array}{rcl} \dot{x} &=& Ax + \rho\left(\frac{\|x\|^2}{\sigma^2}\right)\, P\left(x, \rho\left(\frac{\|h(\zeta)\|}{\theta}\right) h(\zeta)\right)\right), \\ \dot{\zeta} &=& \rho(\|\zeta\|)\, g(\zeta), \end{array} \tag{2.14}$$

where ρ is as in (2.9) and where σ, θ are strictly positive real numbers to be adjusted shortly. Because the flows of (2.14) and (2.7) do coincide as long as $\|x\| < \sigma/\sqrt{2}$, $\|\zeta\| < 1/2$, $\|h(\zeta)\| < \theta/2$, and since these inequalities define a neighborhood $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^m$ by the continuity of h and the fact that $h(0) = 0$, it is enough to prove the theorem when (2.7) *gets replaced by* (2.14) for *some pair* of strictly positive σ, θ. To this effect, we shall apply Theorem 2.2.1 with $\mathcal{E} = \mathbb{R}^q$ endowed with the flow of $\rho(\|\zeta\|)\, g(\zeta)$, namely $\mathcal{S}_\tau(\zeta_0)$ is the value at $t = \tau$ of the solution to the second equation in (2.14) whose value at $t = 0$ is ζ_0, and with

$$G(x, \zeta, t) = \rho\left(\frac{\|x\|^2}{\sigma^2}\right)\, P\left(x, \rho\left(\frac{\|h(\mathcal{S}_t(\zeta))\|}{\theta}\right) h(\mathcal{S}_t(\zeta))\right).$$

We now proceed to check that the assumptions of Theorem 2.2.1 are fulfilled if σ and θ are properly chosen. Firstly, since g is locally Lipschitz continuous while ρ is smooth with compact support on $[0, +\infty)$, we see that $\zeta \mapsto \rho(\|\zeta\|)\, g(\zeta)$ is a bounded Lipschitz continuous vector field on $\mathbb{R}^q$ hence it has a globally defined flow, which is continuous by Lemma 2.4.1. This tells us that $(\tau, \zeta) \mapsto \mathcal{S}_\tau(\zeta)$ is continuous $\mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$, so $\mathcal{S}_\tau$ is indeed a one-parameter group of homeomorphisms on $\mathbb{R}^q$ and $\tau \mapsto \mathcal{S}_\tau(\zeta)$ is certainly Borel measurable since it is even continuous. The continuity of $(\tau, \zeta) \mapsto \mathcal{S}_\tau(\zeta)$ also makes it clear that $G(x, \zeta, t)$ is continuous and continuously differentiable with respect to x granted the continuity of h, the smoothness of ρ, and the fact that P itself is continuous and continuously differentiable with respect to the first variable. *A fortiori* then, $x \mapsto G(x, \zeta, t)$ is continuously differentiable and $t \mapsto G(x, \zeta, t)$ is measurable.

Secondly, observe since ρ is bounded by 1 and vanishes outside $[0, 1]$ that $\|\rho(\theta^{-1}\|u\|)u\| < \theta$ for all $u \in \mathbb{R}^m$, consequently G takes values in the smallest ball centered at 0 that contains $P(B(0, \sigma), B(0, \theta))$; this last set is relatively compact by the continuity of P hence G is bounded. The same argument shows that $\partial G/\partial x$ is also bounded, in other words we can choose ϕ_ζ and ψ_ζ to be suitable constant functions in (2.16), independently of ζ. In particular, (2.17) and (2.18) will hold. Moreover, if we set $\beta(x, u) = P(x, u)$, we have with the notations of (2.10) that

$$G(x, \zeta, t) = G_\sigma\left(x, \rho\left(\frac{\|h(\mathcal{S}_t(\zeta))\|}{\theta}\right) h(\mathcal{S}_t(\zeta))\right). \tag{2.15}$$

Since $\rho(\theta^{-1}\|h(v)\|)h(v)$ lies in $B(0, \theta)$ for all $v \in \mathbb{R}^q$ so in particular for $v = \mathcal{S}_t(\zeta)$, we deduce from (2.15) and Lemma 2.3.4 that $\partial G/\partial x$ can be made uniformly small for suitable σ and θ. That is to say, the number η in (2.18) can be made arbitrarily small upon choosing σ and θ adequately, in particular we can meet (2.19).

Thirdly, the condition (2.16) that we just proved to hold (actually with constant functions ϕ_ζ and ψ_ζ independent of ζ) entails that the first equation in (2.14) has a unique solution given initial conditions $x(0)$ and $\zeta(0)$ (*cf* for instance [98, Theorem 54, Proposition C.3.4, Proposition C.3.8]) and, since the same holds true for the second equation as was pointed out when we defined $\mathcal{S}_\tau(\zeta)$, we conclude that the whole vector field in the right hand-side of (2.14) has a flow on $\mathbb{R}^{n+q} = \mathbb{R}^n \times \mathbb{R}^q$, which is continuous by Lemma 2.4.1. As $\widehat{x}$, defined in (2.6), is nothing but the projection of this flow onto the first factor $\mathbb{R}^n$, we conclude that $(\tau, x_0, \zeta) \mapsto \widehat{x}(\tau, x_0, \zeta)$ is continuous. Finally, notice that (2.5) is immediate from the group property of $\mathcal{S}_\tau$. Having verified all the hypotheses of Theorem 2.2.1, we apply the latter to conclude the proof of Theorem 2.3.3. □ □

2.3.2 Control systems viewed as flows

In [25], a general way of associating a flow to a control system is proposed, based on the action of the time shift on some functional space of inputs. Before giving the proper framework for our results, let us first carry out a few measure-theoretic preliminaries.

For arbitrary exponents $p \in [1, \infty]$, we denote by $\mathcal{L}^p(\mathbb{R}, \mathbb{R}^m)$, or simply by $\mathcal{L}^p$ for short, the space of measurable functions $\Upsilon : \mathbb{R} \to \mathbb{R}^m$ such that

$$\begin{aligned} \|\Upsilon\|_p &= \left(\int_{\mathbb{R}} \|\Upsilon(t)\|^p \, dt\right)^{1/p} < \infty \qquad \text{if } p < \infty, \\ \|\Upsilon\|_\infty &= \operatorname{ess.\,sup}_{t \in \mathbb{R}} . \, \|\Upsilon(t)\| \quad < \infty \qquad \text{if } p = \infty. \end{aligned}$$

In the above, measurability and summability were implicitly understood with respect to Lebesgue measure. The same definitions can of course be made for any positive measure. We only consider measures defined on the same σ-algebra as Lebesgue measure (namely the completion of the Borel σ-algebra with respect to sets of Lebesgue measure zero). We explicitly indicate the dependence on the measure μ of the corresponding functional spaces and norms by writing $\mathcal{L}^{p,\mu}$ and $\|.\|_{p,\mu}$.

Remark 2.3.5. *If μ is a positive measure on $\mathbb{R}$ as above, and if μ and Lebesgue measure are mutually absolutely continuous, then for any Lebesgue measurable (hence also μ-measurable) function Υ it holds that $\|\Upsilon\|_\infty = \|\Upsilon\|_{\infty,\mu}$. Indeed, we have that $\|\Upsilon\|_\infty \leq \alpha$ if, and only if, the set E_α of those $x \in \mathbb{R}$ for which $\|\Upsilon\|(x) > \alpha$ has Lebesgue measure zero. Since the latter holds if, and only if, $\mu(E_\alpha) = 0$, it is equivalent to require that $\|\Upsilon\|_{\infty,\mu} \leq \alpha$ as announced.*

For any $p \in [1, \infty]$ and $\tau \in \mathbb{R}$, we define the time shift $\Theta_\tau : \mathcal{L}^p \to \mathcal{L}^p$ by

$$\Theta_\tau(\Upsilon)(t) = \Upsilon(\tau + t) \; . \tag{2.16}$$

It is well known that, for fixed $\Upsilon \in \mathcal{L}^p$, the map $\tau \mapsto \Theta_\tau(\Upsilon)$ is continuous $\mathbb{R} \to \mathcal{L}^p$ if $1 \leq p < \infty$ [90, Theorem 9.5]. When $p = \infty$ it is no longer so, but the map is at least Borel measurable :

Lemma 2.3.6. *For fixed $\Upsilon \in \mathcal{L}^\infty$, consider the map $T_\Upsilon : \mathbb{R} \to \mathcal{L}^\infty$ defined by $T_\Upsilon(\tau) = \Theta_\tau(\Upsilon)$. If V is open in $\mathcal{L}^p$, then $T_\Upsilon^{-1}(V)$ is measurable in $\mathbb{R}$.*

Proof. Set for simplicity $T_\Upsilon(\tau) = \Upsilon_\tau$, and fix arbitrarily $v \in \mathcal{L}^\infty$ together with $\varepsilon > 0$. It is enough to show that the set

$$E = \{\tau \in \mathbb{R}; \;\; \|\Upsilon_\tau - v\|_\infty > \varepsilon\}$$

is measurable. Let μ be the measure on $\mathbb{R}$ such that $d\mu(t) = dt/(1 + t^2)$. In view of Remark 2.3.5, we can replace $\|.\|_\infty$ by $\|.\|_{\infty,\mu}$ in the definition of E. Now, since μ is finite, the functions Υ_τ and v belong to $\mathcal{L}^{1,\mu}$, which is to the effect that

$$\lim_{p \to \infty} \|\Upsilon_\tau - v\|_{p,\mu} = \|\Upsilon_\tau - v\|_{\infty,\mu}, \tag{2.17}$$

see *e.g.* [90, Chap. 3, Ex.4]. In particular, if we let

$$E_{p,\mu} = \{\tau \in \mathbb{R}; \;\; \|\Upsilon_\tau - v\|_{p,\mu} > \varepsilon\},$$

we deduce from (2.17) that

$$E = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_{j,\mu}$$

where k and j assume integral values, so we are left to prove that $E_{j,\mu}$ is measurable. But since translating the argument is a continuous operation $\mathbb{R} \to \mathcal{L}^{p,\mu}$ when $p < \infty$ [90, Theorem 9.5][1], each $E_{j,\mu}$ is in fact open in $\mathbb{R}$ thereby proving the lemma. □ □

Endowed with $\|.\|_p$-balls as neighborhoods of 0, the set $\mathcal{L}^p$ is a topological vector space but it is not Hausdorff; identifying functions that agree almost everywhere, we obtain the familiar Lebesgue space L^p of equivalence classes of $\mathcal{L}^p$-functions; it is a Banach space, whose norm, still denoted by $\|.\|_p$, is induced by $\|.\|_p$ defined in $\mathcal{L}^p$, and whose topology coincides with the quotient topology arising from the canonical map $\mathcal{L}^p \to L^p$. The time shift $\Theta_\tau : \mathcal{L}^p \to \mathcal{L}^p$ defined by (2.16) induces a well defined map $\Theta_\tau : L^p \to L^p$. In what follows, results are stated in terms of L^p, but we do make use of $\mathcal{L}^p$ for the proof because point-wise evaluation makes no sense in L^p.

Let us now come back to our control system, namely (2.2), which is obtained from (2.1) by singling out the linear term in x around the equilibrium $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$. This time, however, we emphasize the functional dependence on the control by writing

$$\dot{x} \;=\; Ax \;+\; P(x, \Upsilon(t)), \tag{2.18}$$

where, as in the preceding subsection, $P : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and has continuous derivative with respect to the first argument $\frac{\partial P}{\partial x} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$. We fix some $p \in [1, \infty]$ and we consider controls $\Upsilon \in L^p(\mathbb{R}, \mathbb{R}^m)$. Thus, when $p < \infty$, we shall have to handle unbounded values for $\Upsilon(t)$, and this will necessitate an extra assumption. Namely, if $1 \le p < \infty$, we assume that to each compact set $K \subset \mathbb{R}^n$, there are positive constants $c_1(K)$, $c_2(K)$ such that

$$\|P(x,u)\| + \|\frac{\partial P}{\partial x}(x,u)\| \le c_1(K) + c_2(K)\,\|u\|^p, \quad (x,u) \in K \times \mathbb{R}^m, \tag{2.19}$$

where we agree, for definiteness, that the norm of a matrix is the operator norm. Classical results imply (see *e.g.* [98, Theorem 54, Proposition C.3.4]) that the solution to (2.18) uniquely exists on some maximal time interval once $x(0) = x_0$ and $\Upsilon \in L^p$ are chosen. This solution we denote by

$$t \mapsto x(t, x_0, \Upsilon)\ .$$

This allows one to define a flow on $\mathbb{R}^n \times \mathcal{L}^p$, or on $\mathbb{R}^n \times L^p$, the flow at time τ being given by

$$(x_0, \Upsilon) \;\mapsto\; (\,x(\tau, x_0, \Upsilon)\,,\, \Theta_\tau(\Upsilon)\,)\ . \tag{2.20}$$

The main result in this subsection is the theorem below. It is of purely *open loop* character, that is to say the linearizing transformation $(x, \Upsilon) \mapsto (z, \Upsilon)$ operates at a functional level where z depends not only on x, but also on the whole input function $\Upsilon : \mathbb{R} \mapsto \mathbb{R}^m$. That type of linearization is intriguing in the authors' opinion, but its usefulness in control is not clear unless the structure of the transformation is thoroughly understood. Unfortunately our method of proof does not reveal much in this direction, which may deserve further study.

Theorem 2.3.7. *Suppose in* (2.18) *that $P(x,u)$ is continuous $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ with $P(0,0) = 0$, that $\partial P/\partial x$ exists and is continuous $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$ with $\partial P/\partial x(0,0) = 0$, and that A is hyperbolic. Let $p \in [1,\infty]$, and,if $p < \infty$, assume that, to each compact set $K \subset \mathbb{R}^n$, there are positive constants $c_1(K)$, $c_2(K)$ such that* (2.19) *holds. Then, there exist two neighborhoods V*

[1] The proof is given there for Lebesgue measure only, but it does carry over *mutatis mutandis* to any complete regular Borel measure on $\mathbb{R}$, hence in particular to μ.

and W of 0 in $\mathbb{R}^n$ and $L^p(\mathbb{R},\mathbb{R}^m)$ respectively, and a map $H : V\times W \to \mathbb{R}^n$ with $H(0,0)=0$, such that

$$\begin{array}{rccc} H\times Id: & V\times W & \to & \mathbb{R}^n\times W \\ & (x,\Upsilon) & \mapsto & (H(x,\Upsilon),\Upsilon) \end{array} \tag{2.21}$$

is a homeomorphism from $V\times W$ onto its image that conjugates (2.18) to

$$\dot z = Az, \tag{2.22}$$

i.e. for all $(t,x_0,\Upsilon)\in\mathbb{R}\times\mathbb{R}^n\times L^p(\mathbb{R},\mathbb{R}^m)$ such that $(x(\tau,x_0,\Upsilon),\Upsilon)\in V\times W$ for all $\tau\in[0,t]$ (or $[t,0]$ if $t<0$) one has

$$H(x(t,x_0,\Upsilon)) = e^{tA}H(x_0,\Upsilon). \tag{2.23}$$

Remark 2.3.8. *The above theorem parallels Theorem 2.3.3 of section 2.3.1, in that we initially wrote $\dot x = f(x,u)$ in the form (2.2), assuming that f is continuously differentiable with respect to x, to finally conclude, under suitable hypotheses, that (2.18) is locally conjugate in some appropriate sense to the non-controlled linear system (2.22). We might as well have stated an analog to Theorem 2.3.1 where, assuming this time that f is of class C^1, we write $\dot x = f(x,u)$ in the form (2.3) with hyperbolic A, assuming in addition if $p<\infty$ that for any compact $K\subset\mathbb{R}^n$ one has*

$$\|F(x,u)\| + \|\frac{\partial F}{\partial x}(x,u)\| \le c_1(K) + c_2(K)\,\|u\|^p, \quad (x,u)\in K\times\mathbb{R}^m, \tag{2.24}$$

to conclude that $\dot x = Ax + B\Upsilon(t) + F(x,\Upsilon(t))$ is conjugate via $z = H(x,\Upsilon)$ to $\dot z = Az + B\Upsilon(t)$, where $H\times Id$ is a local homeomorphism at 0×0 of $\mathbb{R}^n\times L^p$. Again, although the presence of the control term $B\Upsilon(t)$ in the linearized equation makes it look more natural, the result we just sketched is a logical consequence of Theorem 2.3.7 just like Theorem 2.3.1 was a consequence of Theorem 2.3.3.

To prove Theorem 2.3.7 we shall again apply Theorem 2.2.1 to a suitably normalized version of (2.18), the normalization step depending on the following lemma which stands analogous to Lemma 2.3.4 in the $\mathcal{L}^p$ context. For convenience, we denote below by $B_{\mathcal{L}^p}(v,r)$ the ball centered at v of radius r in $\mathcal{L}^p$, and by $\mathcal{L}^1_{loc}(\mathbb{R},\mathbb{R}^m)$ (or simply $\mathcal{L}^1_{loc}$ if no confusion can arise) the space of *locally integrable functions*, namely those whose restriction to any compact $K\subset\mathbb{R}$ belongs to $\mathcal{L}^1(K,\mathbb{R}^m)$.

Lemma 2.3.9. *Let $\beta(x,u)$ be continuous $\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ and $\partial\beta/\partial x$ continuously exist $\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^{n\times n}$, with $\beta(0,0)=\partial\beta/\partial x(0,0)=0$. Assume for some $p\in[1,\infty)$ that, to each compact set $K\subset\mathbb{R}^n$, there are positive constants $c_1(K)$, $c_2(K)$ such that*

$$\|\beta(x,u)\| + \|\frac{\partial\beta}{\partial x}(x,u)\| \le c_1(K) + c_2(K)\,\|u\|^p, \quad (x,u)\in K\times\mathbb{R}^m. \tag{2.25}$$

Then, G_s being as in (2.10), it holds that for every $s>0$ and any $\Upsilon\in\mathcal{L}^p(\mathbb{R},\mathbb{R}^m)$ we have $G_s(x,\Upsilon)\in\mathcal{L}^1_{loc}(\mathbb{R},\mathbb{R}^n)$ and $\partial G_s/\partial x(x,\Upsilon)\in\mathcal{L}^1_{loc}(\mathbb{R},\mathbb{R}^{n\times n})$ for fixed $x\in\mathbb{R}$. Moreover, to each $\eta>0$ there exist $\sigma>0$ and $\theta>0$ such that G_σ satisfies :

$$\begin{array}{ll} \forall\Upsilon\in B_{\mathcal{L}^p}(0,\theta)\,, & \textit{there exists } \psi_\Upsilon\in\mathcal{L}^1_{loc}(\mathbb{R},\mathbb{R}) \textit{ such that} \\ \|\psi_\Upsilon\|_{L^1[0,1]}\le\eta & \textit{and, } \forall x\in\mathbb{R}^n\,,\ \ \|\frac{\partial G_\sigma}{\partial x}(x,\Upsilon)\| \le \psi_\Upsilon. \end{array} \tag{2.26}$$

Proof. For fixed $x\in\mathbb{R}$, it is clear from (2.25) that both $G_s(x,\Upsilon)$ and $\partial G_s/\partial x(x,\Upsilon)$ belong to $\mathcal{L}^1_{loc}(\mathbb{R},\mathbb{R}^n)$ when $\Upsilon\in\mathcal{L}^p(\mathbb{R},\mathbb{R}^m)$, measurability being ensured by the continuity of G_s and $\partial G_s/\partial x$. To prove (2.26), first apply Lemma 2.3.4 to find $\sigma>0$ and $\theta_0>0$ such that

$$\forall(x,u)\in\mathbb{R}^n\times B(0,\theta_0)\,,\quad \|\frac{\partial G_\sigma}{\partial x}(x,u)\| \ \le\ \eta/2\ . \tag{2.27}$$

Next, let $c_1 = c_1(\overline{B}(0,\sigma))$ and $c_2 = c_2(\overline{B}(0,\sigma))$ be defined after (2.25), and observe that

$$\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m, \quad \|\partial G_\sigma/\partial x\,(x,u)\| \le (1+6/\sigma)(c_1 + c_2\|u\|^p) \tag{2.28}$$

because when $\|x\| < \sigma$ this follows from (2.12), (2.25) and the fact that $|\rho'| < 3$, whereas G_σ vanishes anyway when $\|x\| \ge \sigma$. Introduce now the set

$$E_{\Upsilon,\theta_0} = \{t \in [0,1], \quad \|\Upsilon\| < \theta_0\}. \tag{2.29}$$

Letting $\psi_\Upsilon(t) = \eta/2$ for $t \in E_{\Upsilon,\theta_0}$ and $\psi_\Upsilon(t) = (1+6/\sigma)(c_1 + c_2\|\Upsilon(t)\|^p)$ otherwise, it is clear that $\psi_\Upsilon \in \mathcal{L}^1_{loc}(\mathbb{R},\mathbb{R})$ and it follows from (2.29), (2.27), and (2.28) that $\|\partial G_{\sigma_0}/\partial x(x,\Upsilon)\| \le \psi_\Upsilon$ for any $x \in \mathbb{R}^n$. In another connection, let ν be the measure on $\mathbb{R}$ given by $d\nu(t) = |\Upsilon(t)|^p dt$. By absolute continuity of ν with respect to Lebesgue measure, there is $\varepsilon > 0$ such that

$$\int_E \|\Upsilon\|^p\, dt < \frac{\eta}{4c_2(1+6/\sigma)} \qquad \text{as soon as } |E| < \varepsilon, \tag{2.30}$$

where $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}$ [90, Theorem 6.11]. Pick $\theta > 0$ so small that

$$\frac{\theta}{\theta_0} < \max\left\{\varepsilon, \frac{\eta}{4(1+6/\sigma)c_1}\right\}. \tag{2.31}$$

Then, if $\|\Upsilon\|_p < \theta$, the set $[0,1]\setminus E_{\Upsilon,\theta_0}$ has measure at most θ/θ_0 hence, by definition of ψ_Υ, we get in view of (2.30) and (2.31) the estimate :

$$\|\psi_\Upsilon\|_{L^1[0,1]} \le \frac{\eta}{2} + \frac{\theta}{\theta_0}(1+6/\sigma)c_1 + \frac{\eta}{4}$$

which is less that $\eta/2 + \eta/4 + \eta/4 = \eta$ by (2.31) again, as desired. □ □

We are now in position to establish Theorem 2.3.7.

Proof. Proof of Theorem 2.3.7 For the proof we can replace L^p by $\mathcal{L}^p$, because if we find a local homeomorphism of $\mathbb{R}^n \times \mathcal{L}^p$ at 0×0, of the form $\widetilde{H} \times Id$, that conjugates (2.18) to (2.22), the fact that $x(\tau, x_0, \Upsilon)$ depends only on the equivalence class of Υ in L^p implies that the same holds true for $\widetilde{H}(x_0,\Upsilon)$, and therefore $\widetilde{H} \times Id$ will induce a quotient map $H \times Id$ around 0×0 in $\mathbb{R}^n \times L^p$ that is still a local homeomorphism by definition of the quotient topology. To prove the $\mathcal{L}^p$ version, we consider the following “re-normalization” of (2.18) :

$$\dot{x} \;=\; Ax \;+\; \rho\left(\frac{\|x\|^2}{\sigma^2}\right)\; P\left(x, \rho\left(\frac{\|\Upsilon\|_p}{\theta}\right)\Upsilon\right), \tag{2.32}$$

where ρ is as in (2.9) and σ, θ are strictly positive real numbers to be fixed. Because the right-hand sides of (2.32) and (2.18) agree as long as $\|x\| < \sigma/\sqrt{2}$ and $\|\Upsilon\|_p < \theta/2$ which defines a neighborhood $(0,0)$ in $\mathbb{R}^n \times \mathcal{L}^p$, it is enough to prove the theorem when (2.18) *gets replaced by* (2.32) for *some pair* σ, θ. To this effect, we shall apply Theorem 2.2.1 with $\mathcal{E} = \mathcal{L}^p$, endowed with the one-parameter group of transformations $\mathcal{S}_\tau = \Theta_\tau$ defined by (2.16), and

$$G(x,\zeta,t) = \rho\left(\frac{\|x\|^2}{\sigma^2}\right)\; P\left(x, \rho\left(\frac{\|\zeta\|_p}{\theta}\right)\zeta(t)\right).$$

Let us check that the assumptions of Theorem 2.2.1 are met if σ and θ are suitably chosen.

Firstly, it is obvious that $\mathcal{S}_\tau$ is continuous (hence a homeomorphism since $\mathcal{S}_\tau^{-1} = \mathcal{S}_{-\tau}$) because it is a linear isometry of $\mathcal{L}^p$. In addition, $\tau \mapsto \mathcal{S}_\tau(\zeta)$ is certainly Borel measurable, because it is even continuous when $p < \infty$ [90, Theorem 9.5] while Lemma 2.3.6 applies if $p = \infty$.

Secondly, it follows immediately from the assumptions on P and the smoothness of ρ that $G(x,\zeta,t)$ is continuously differentiable with respect to x for fixed ζ and t, while the measurability of $t \mapsto G(x,\zeta,t)$ follows from the continuity of P and the measurability of ζ. To prove the existence of ϕ_ζ and ψ_ζ in (2.16), we distinguish between $p<\infty$ and $p=\infty$. If $p<\infty$, by (2.19) and the fact that ρ is bounded by 1 and vanishes outside $[0,1]$, a valid choice for ϕ_ζ is

$$\phi_\zeta(t) = c_1(\,\overline{B}(0,\sigma)\,) + c_2(\,\overline{B}(0,\sigma)\,)\ \rho^p\left(\frac{\|\zeta\|_p}{\theta}\right)\|\zeta(t)\|^p$$

and, since by the properties of ρ we have that

$$\left\|\rho\left(\frac{\|\zeta\|_p}{\theta}\right)\ \zeta\right\|_p \le \theta \qquad \forall \zeta\in\mathcal{L}^p,\quad 1\le p\le\infty, \tag{2.33}$$

it follows that (2.17) is met with

$$M = c_1(\,\overline{B}(0,\sigma)\,) + c_2(\,\overline{B}(0,\sigma)\,)\ \theta^p.$$

As to ψ_ζ, observe if we set $\beta(x,u)=P(x,u)$ that, with the notations of (2.10), one has

$$G(x,\zeta,t) = G_\sigma\left(x,\rho\left(\frac{\|\zeta\|_p}{\theta}\right)\zeta(t)\right), \tag{2.34}$$

so Lemma 2.3.9 ensures the existence of ψ_ζ and also that the number η in (2.18) can be made arbitrarily small upon choosing σ and θ adequately ; in particular we can meet (2.19). If $p=\infty$, we let

$$\phi_\zeta(t) = \sup_{x\in\overline{B}(0,\sigma)} \left\|P\left(x,\rho\left(\frac{\|\zeta\|_p}{\theta}\right)\zeta(t)\right)\right\|$$

so that the first half of (2.16) holds by the properties of ρ. By (2.33) we also have that

$$\|\phi_\zeta\|_\infty \ \le \sup_{(x,u)\in\overline{B}(0,\sigma)\times\overline{B}(0,\theta)} \|P(x,u)\|, \tag{2.35}$$

so that $\phi_\zeta\in\mathcal{L}^\infty(\mathbb{R},\mathbb{R})$ hence it is locally summable, and the right-hand side of (2.35) may serve as M in (2.17). As to ψ_ζ, observe that (2.34) still holds for $p=\infty$, again with $\beta(x,u)=P(x,u)$, so we can set

$$\psi_\zeta(t) = \sup_{x\in\overline{B}(0,\sigma)} \left\|\frac{\partial G_\sigma}{\partial x}\left(x,\rho\left(\frac{\|\zeta\|_\infty}{\theta}\right)\zeta(t)\right)\right\|,$$

and using (2.33) once more we get

$$\|\psi_\zeta\|_\infty \le \sup_{(x,u)\in\overline{B}(0,\sigma)\times\overline{B}(0,\theta)} \left\|\frac{G_\sigma}{\partial x}(x,u)\right\|. \tag{2.36}$$

Thus $\psi_\zeta\in\mathcal{L}^\infty(\mathbb{R},\mathbb{R})$ hence it is locally summable, and applying Lemma 2.3.4 to the right-hand side of (2.36) shows that $\|\psi_\zeta\|_\infty$ can be made arbitrarily small upon choosing σ and θ adequately. Consequently η in (2.18) can be as small as we wish and in particular we can meet (2.19).

Thirdly, $t\mapsto\widehat{x}(t,x_0,\zeta)$ defined in (2.6) is just the solution to (2.32) corresponding to $\Upsilon=\zeta$ and $x(0)=x_0$, which uniquely exists for all t by (2.16), see *e.g.* [98, Theorem 54, Proposition C.3.4, Proposition C.3.8]. The continuity $\mathbb{R}^n\times\mathcal{L}^p\to\mathbb{R}^n$ of $(x_0,\zeta)\mapsto\widehat{x}(t,x_0,\zeta)$ is now ascertained by Proposition 2.4.4, once it is observed that $F(x,u)=Ax+\rho(\|x\|^2/\sigma^2)P(x,u)$ satisfies the hypotheses of that proposition by (2.19) and the properties of ρ, and also that $Ax+G(x,\zeta,t)$

is the composition of F with the continuous map on $\mathbb{R}^n \times \mathcal{L}^p$ given by $(x, \zeta) \mapsto (x, \rho(\|\zeta\|_p/\theta)\zeta)$ (Proposition 2.4.4 was actually proved for L^p controls, but nothing is to be changed if we work in $\mathcal{L}^p$).

Finally, notice that (2.5) is immediate by the very definition of Θ_τ. Thus we can apply Theorem 2.2.1 to conclude the proof of Theorem 2.3.7. □ □

Remark 2.3.10. *It should be noted that, unlike Theorems 2.3.1 and 2.3.3, Theorem 2.3.7 cannot be localized with respect to u when $p < \infty$. However, using a partition of unity argument, the result carries over to the case where, in* (2.18)*, the map P is only defined on $\mathcal{V} \times \mathbb{R}^m$ where $\mathcal{V}$ is a neighborhood of* 0 *in $\mathbb{R}^n$.*

In [25], particular attention is payed to the weak-* topology on $\mathcal{L}^\infty$ for the control space, because it makes the flow $\tau \mapsto \Theta_\tau(\Upsilon)$ continuous for fixed Υ. Subsequently, this reference focuses on systems that are affine in the control : $\dot{x} = X_0(x) + C(x)u$, where X_0 is a C^1 vector field on $\mathbb{R}^n$ and $C : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ a C^1 matrix-valued function; the reason for this affine restriction is that it ensures, in the weak-* context, the sequential continuity of $(x_0, \Upsilon) \mapsto x(\tau, x_0, \Upsilon)$ for fixed τ, whenever the flow makes sense : this is easily deduced from the Ascoli-Arzela theorem and the fact that weak-* convergent sequences are norm-bounded [91, Theorem 2.5]. Although the continuity of the flow Θ was never a concern to us (only Borel measurability was required), it is natural in this connection to ask what happens with Theorem 2.3.7 if we endow L^∞ with the weak-* topology inherited from the (L^1, L^∞) duality. On the one hand, *in case one restricts his attention, as is done in [25], to a balanced,* weak-* compact *time-shift invariant subset of L^∞ containing* 0*, e.g. a ball $\bar{B}_{L^\infty}(0, r)$, then the conclusions of the theorem still hold if we equip the subset in question with the weak-* topology.* Indeed, the weak-* topology is metrizable on any compact set E because L^1 is separable [91, Theorems 3.16] and, since weak-* convergent sequences are norm- bounded, it follows if E is balanced that one can find a neighborhood of 0 in E which is included in $\bar{B}_{L^\infty}(0, \theta)$ for arbitrary small θ. In particular we can embed this neighborhood in W of Theorem 2.3.7, and then it only remains to show that (2.21) remains continuous if W is equipped with the weak-* topology; this in turn reduces *via* (2.23) to the already mentioned fact that $(x_0, \Upsilon) \mapsto x(\tau, x_0, \Upsilon)$ is sequentially continuous for fixed τ when the topology on Υ is the weak-* one. On the other hand, working weak-* with unrestricted controls in L^∞ raises serious difficulties, for no weak-* neighborhood in L^∞ can be norm-bounded. This results in the fact that, although Θ is now continuous, the domain of definition of the flow (2.20) may fail to be open : for instance the equation $\dot{x} = x + x^2\Upsilon(t)$ with initial condition $x(0) = x_0$, where x and Υ are real-valued, cannot have a solution on a fixed interval $[0, t]$ for every $(x_0, \Upsilon) \in B(0, r) \times \mathcal{W}_0$ if $\mathcal{W}_0$ is a weak-* neighborhood of 0 in $L^\infty(\mathbb{R}, \mathbb{R})$. Therefore it is hopeless to build a local homeomorphism by integrating the flow as is done in the proof of Theorem 2.2.1, and the authors do not know what analog to Theorem 2.3.7 could be carried out in this context.

Remark 2.3.11. *The paper [21] considers transformations $\mathbb{R}^n \times L^\infty \to \mathbb{R}^n \times L^\infty$, using for the input space a topology on L^∞ which is intermediate between the weak-* and the strong one. There the structure of conjugating homeomorphisms is not* (2.21) *but rather a triangular form :*

$$(x, \Upsilon) \quad \mapsto \quad (\,H(x)\,,\, F(x, \Upsilon)\,)$$

that combines what is called in this reference "topological static state feedback equivalence" and "topological state equivalence"[21, Definition 5]. We refer the interested reader to the original paper for a result on topological linearization of systems with two states and one control, using this type of transformation, under some global hypotheses.

2.4 Appendix on the flow of ODEs

Continuity without Lipschitz assumption. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^d$. We say that a continuous vector field $X : \mathcal{U} \to \mathbb{R}^d$ *has a flow* if the Cauchy problem $\dot{x}(t) = X(x(t))$ with initial condition $x(0) = x_0$ has a unique solution, defined for $t \in (-\varepsilon, \varepsilon)$ with $\varepsilon = \varepsilon(x_0) > 0$. The flow of X at time t is denoted by X_t, in other words, with the preceding notations, $X_t(x_0) = x(t)$. It is easy to see that the domain of definition of $(t, x) \mapsto X_t(x)$ is open in $\mathbb{R} \times \mathcal{U}$. Although there is no clear characterization of the vector fields, or ODEs, that have a flow, they enjoy special properties, even without assuming the well known sufficient Lipshitz conditions on the right-hand side. Lemma 2.4.1 below is a special case of [24, Chapter 2, Theorem 4.3] or [47, chap. V, Theorem 2.1]. Lemma 2.4.2 is an application of [24, chapter 2, Theorem 4.1] (that theorem refers to a continuous parameter μ instead of the integer k : apply it to $X(\mu, x)$ piecewise affine with respect to μ and such that $X(0, x) = X(x)$ and $X(\pm\frac{1}{k}, x) = X^k(x)$).

Lemma 2.4.1. *If $X : \mathcal{U} \to \mathbb{R}^d$ is a continuous vector field that has a flow, the map $(t, x) \mapsto X_t(x)$ is continuous on the open subset of $\mathbb{R} \times \mathcal{U}$ where it is defined.*

Lemma 2.4.2. *Assume that the sequence of continuous vector fields $X^k : \mathcal{U} \to \mathbb{R}^d$ converges to X, uniformly on compact subsets of $\mathcal{U}$, and that all the X^k as well as X itself have a flow. Suppose that $X_t(x)$ is defined for all $(t, x) \in [0, T] \times K$ with $T > 0$ and $K \subset \mathcal{U}$ compact. Then $X_t^k(x)$ is also defined on $[0, T] \times K$ for k large enough, and the sequence of mappings $(t, x) \mapsto X_t^k(x)$ converges to $(t, x) \mapsto X_t(x)$, uniformly on $[0, T] \times K$.*

Differentiability with measurable dependence on time. Consider a differential equation

$$\dot{x} = X(x, t) \tag{2.37}$$

where the time-dependent vector field $X : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfies the following properties :

(i) for fixed $t \in \mathbb{R}$, the map $x \to X(x, t)$ is continuously differentiable $\mathbb{R}^n \to \mathbb{R}^n$;

(ii) for fixed $x \in \mathbb{R}^n$, the map $t \to X(x, t)$ is measurable $\mathbb{R} \to \mathbb{R}$;

(iii) for some $x_1 \in \mathbb{R}^n$ there is a measurable and locally integrable function $\alpha_{x_1} : \mathbb{R} \to \mathbb{R}^+$ such that

$$\|X(x_1, t)\| \le \alpha_{x_1}(t), \quad \text{for all } t \in \mathbb{R};$$

(iv) there is a measurable and locally integrable function $\psi : \mathbb{R} \to \mathbb{R}^+$ satisfying

$$\left\| \frac{\partial X}{\partial x}(x, t) \right\|_{\mathrm{O}} \le \psi(t), \quad \text{for all } (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

where $\| \ \|_{\mathrm{O}}$ denotes the familiar operator norm on $n \times n$ real matrices.

The choice of the operator norm in **(iv)** is only for definiteness since all norms are equivalent on $\mathbb{R}^{n \times n}$. Note also that, using **(iv)** and the mean-value theorem, property **(iii)** immediately strengthens to :

(iii)' to each $x \in \mathbb{R}^n$ there is a measurable and locally integrable function $\alpha_x : \mathbb{R} \to \mathbb{R}^+$ such that

$$\|X(x, t)\| \le \alpha_x(t), \quad \text{for all } t \in \mathbb{R}.$$

Measurability with respect to time and local Lipshitz continuity with respect to the state ("Caratheodory conditions") are known to be sufficient for local existence and uniqueness of solution. Let us state a differentiability result in the not-so-classical case where the dependence on time is L^1 but possibly unbounded. It is proved in [74, chapter III], where one can find comprehensive results assuming assuming only measurability with respect to time.

By **(i)**, **(ii)**, **(iii)'**, and **(iv)**, the solution to (2.37) with arbitrary initial condition $x(0) = x_0 \in \mathbb{R}^n$ uniquely exists for all $t \in \mathbb{R}$ (see [74, Theorem 2.1, Chapter III]), in the sense that there is a unique locally absolutely continuous function $x : \mathbb{R} \to \mathbb{R}^n$ satisfying (2.37) for almost every t and such that $x(0) = x_0$. We shall denote by $\widehat{x}(\tau, x_0)$ the value of this solution at time $t = \tau$, in other words we let $(t, x_0) \mapsto \widehat{x}(\tau, x_0)$ designate the flow of (2.37). By definition, the *variational equation* of (2.37) along the trajectory $t \mapsto \widehat{x}(t, x_0)$ is the linear differential equation :

$$\dot{R} = \frac{\partial X}{\partial x}\left(\widehat{x}(t, x_0), t\right) R \tag{2.38}$$

in the unknown matrix-valued function $R : \mathbb{R} \to \mathbb{R}^{n\times n}$.

Lemma 2.4.3 (Theorem 6.1, Chapter III in [74])**.** *If $X : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfies properties **(i)**-**(iv)** above, and if $\widehat{x}$ is the flow of* (2.37)*, then $\widehat{x}(t, x)$ is continuously differentiable with respect to x and*

$$t \;\mapsto\; \frac{\partial \widehat{x}}{\partial x}(t, x) \tag{2.39}$$

is the unique solution of (2.38) *with initial condition $R(0) = I_n$, where I_n is the identity matrix of size n.*

Continuity with L^p controls. Consider a differential equation of the form

$$\dot{x} = F(x, \Upsilon(t)) \tag{2.40}$$

where $x \in \mathbb{R}^n$ while Υ belongs to $L^p(\mathbb{R}, \mathbb{R}^m)$, the familiar Lebesgue space of (equivalence classes of) functions $\mathbb{R} \to \mathbb{R}^m$ whose p-th power is integrable in case $p < \infty$ and whose norm is essentially bounded if $p = \infty$; we endow L^p with the usual norm, namely $\|\Upsilon\|_p = (\int_{\mathbb{R}} \|\Upsilon\|^p dt)^{1/p}$ if $p < \infty$ and $\|\Upsilon\|_\infty = \text{ess.sup.}_{\mathbb{R}}\|\Upsilon\|$, where $\|.\|$ denotes the Euclidean norm. Of course, a solution to the differential equation is understood here in the sense that $x(t)$ is absolutely continuous, and that its derivative is a locally summable function whose value is given by the right-hand side of (2.40) for almost every t. Classically, even if $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is very smooth, the existence of solutions to (2.40) when $1 \leq p < \infty$ requires some restrictions on the growth of F at infinity. The following continuity property of the solution with respect to both the initial condition and $\Upsilon \in L^p$ holds :

Lemma 2.4.4 (Theorem 3 in [75], with $a = 1$)**.** *Let $F(x, u)$ be continuous $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and the partial derivative $\partial F/\partial x$ exist continuously $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$. Let $p \in [1, \infty]$ and assume if $p < \infty$ that, to each compact $K \subset \mathbb{R}^n$, there are constants $c_1(K)$, $c_2(K)$, such that :*

$$\|F(x, u)\| + \|\frac{\partial F}{\partial x}(x, u)\| \leq c_1(K) + c_2(K)\, \|u\|^p, \quad (x, u) \in K \times \mathbb{R}^m. \tag{2.41}$$

Then, for any $\Upsilon \in L^p(\mathbb{R}, \mathbb{R}^m)$, the solution $t \mapsto x(t, x_0, \Upsilon)$ to (2.40) *with initial condition $x(0) = x_0$ uniquely exists on some maximal time interval $\mathcal{I}_{x_0,\Upsilon}$ containing* 0*. Moreover, if $\mathcal{K}$ is a compact subinterval of $\mathcal{I}_{x_0,\Upsilon}$, there is a neighborhood $\mathcal{V}$ of (x_0, Υ) in $\mathbb{R}^n \times L^p(\mathbb{R}, \mathbb{R}^m)$ such that $\mathcal{K} \subset \mathcal{I}_{x'_0,\Upsilon'}$ whenever $(x'_0, \Upsilon') \in \mathcal{V}$; within this neighborhood, it further holds that*

$$\lim_{(x'_0,\Upsilon')\to(x_0,\Upsilon)} x(t, x'_0, \Upsilon') = x(t, x_0, \Upsilon), \tag{2.42}$$

uniformly with respect to $t \in \mathcal{K}$.

Chapitre 3

Reproduction de l'article:

L. Baratchart et J.-B. Pomet, "On local linearization of control systems"

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Abstract *We consider the problem of topological linearization of smooth ($\mathbf{C}^\infty$ or $\mathbf{C}^\omega$) control systems, i.e. of their local equivalence to a linear controllable system* via *point-wise transformations on the state and the control (static feedback transformations) that are topological but not necessarily differentiable. We prove that local topological linearization implies local smooth linearization, at generic points. At arbitrary points, it implies local conjugation to a linear system via a homeomorphism that induces a smooth diffeomorphism on the state variables, and, except at "strongly" singular points, this homeomorphism can be chosen to be a smooth mapping (the inverse map needs not be smooth). Deciding whether the same is true at "strongly" singular points is tantamount to solve an intriguing open question in differential topology.*

3.1 Introduction

Throughout the paper, *smooth* means of class $\mathbf{C}^\infty$.

In the early works [57, 50, 103], nice necessary and sufficient conditions were obtained for a smooth control system $\dot{x} = f(x,u)$, with state $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}^m$, to be locally smoothly linearizable, *i.e.* locally equivalent to a controllable linear system by means of a diffeomorphic change of variables on the state and the control. The afore-mentioned conditions require certain distributions of vector fields to be integrable, hence locally smoothly linearizable control systems are highly non generic among smooth control systems. Similar results hold for real analytic control systems with respect to real analytic linearizability.

Consider now the *topological* linearizability of a smooth control system, namely the property that it is locally equivalent to a controllable linear system via a homeomorphism on the state and the control which may *not*, this time, be differentiable. Obviously, smooth linearizability implies topological linearizability ; the extend to which the converse holds will be the main concern of the present paper. We address the real analytic case in the same stroke.

In brief, our goal is to *describe the class of smooth control systems that are locally topologically linearizable, yet not smoothly locally linearizable.* This class in nonempty : the smooth (even real-analytic) scalar system

$$\dot{x} = u^3 \qquad u \in \mathbb{R},\ x \in \mathbb{R}, \tag{3.1}$$

gets linearized locally around $(0,0)$ by the homeomorphism $(x,u) \mapsto (x,u^3)$, whereas the conditions for smooth linearizability fail at this point. However, we observe on this example that the conjugating homeomorphism has much more regularity than prescribed a priori :
1. it is a smooth (even real-analytic) local diffeomorphism around all points (x,u) such that $u \neq 0$,
2. it is triangular and induces a smooth (even real-analytic) diffeomorphism on the state variable (*i.e.* the identity map $x \mapsto x$),
3. it is a smooth (even real-analytic) map that fails to be a diffeomorphism only because its inverse is not smooth.

Theorem 3.5.2 of the present paper states that this example essentially depicts the general situation. More precisely, if a smooth control system is locally topologically linearizable at some point $(\bar{x},\bar{u})$ in the state-control space, then
$1'$. in a neighborhood of $(\bar{x},\bar{u})$, the system is locally smoothly linearizable around each point outside a closed subset of empty interior (an analytic variety of positive co-dimension in the analytic case),
$2'$. around $(\bar{x},\bar{u})$, there is a triangular linearizing homeomorphism that induces a smooth diffeomorphism on the state variable,
$3'$. the above-mentioned homeomorphism is smooth (although its inverse may not), at least if $\partial f/\partial u$ has constant rank around $(\bar{x},\bar{u})$ or if $\sup_{x,u} \mathrm{Rank} \partial f/\partial u(x,u) = m$ on every neighborhood of $(\bar{x},\bar{u})$.

Similar results hold for real-analytic linearization of a real-analytic system.

A homeomorphism satisfying $2'$ will be called *quasi-smooth* (see Definitions 3.3.9, 3.5.1), hence our main result is that local topological linearizability implies local quasi-smooth linearizability. A point $(\bar{x},\bar{u})$ where the first rank condition in $3'$ is satisfied is called *regular*, and at such points local smooth linearizability is equivalent to local topological linearizability (cf. Theorem 3.5.4). A point $(\bar{x},\bar{u})$ where none of the rank conditions in $3'$ are satisfied is called *strongly singular*. Whether the conclusion of $3'$ continues to hold at strongly singular points raises an intriguing question in differential topology, namely can one redefine the last components of a local homeomorphism whose first few components are smooth so as to obtain a new homeomorphism which is smooth ? The answer seems not to be known, see the discussion in section 3.5.1.

Motivations. They include the following.
1. For systems without controls, *i.e.* ordinary differential equations, local linearization around an equilibrium has generated a sizable literature, see Section 3.2 for a small sample. It tells us that, even for a real analytic o.d.e., linearizability much depends on the admissible class of transformations (formal, real analytic, C^k or topological). For instance, although analytic linearization requires subtle conditions relying upon a refined analysis of resonances and small divisors, the Grobman-Hartman theorem says nevertheless that topological linearization is always possible at a hyperbolic equilibrium. As one might suspect (this is indeed shown in section 3.5.4), no naive analog to the Grobman-Hartman theorem can hold for control systems because they feature a family of vector fields rather than a single one. However, it might still be expected that relaxing the smoothness of the allowable transformations increases the class of linearizable control systems. It is in fact hardly so : we knew already from [57, 50, 103] that C^1 linearizability of a smooth control system implies smooth linearizability, and we prove here that for C^0 linearizability this class does not get much bigger. In particular, there are no subtle questions about resonances and one may say that the most prominent feature of a control system is to be, or not to be linearizable, regardless of smoothness.
2. Linearizable control systems are systems with linear dynamics, whose nonlinear character

lies in their input-to-state and state-to-output maps only. Such models are advocated in [58, 95] for identification (in the discrete-time case), as their reduced complexity makes them more amenable to standard techniques. It is therefore natural to investigate this class, and topological equivalence is about the weakest possible from the point of view of identification.

3. From a control engineering point of view, it is common practice to design locally stabilizing feedback laws for a given system based on its linear approximation when the latter is controllable... and to a certain extent one believes that the latter and the former *locally* "look alike". It is therefore legitimate to ask about the relationship between them. Since no discriminating topological invariants are known, topological conjugacy might appear as a good candidate. The present paper shows that the relationship is almost never that strong : topological conjugacy to the linear approximation is almost as rare as differential conjugacy.

Incidentally, a system whose linear approximation is not controllable may still happen to be locally topologically linearizable, *i.e.* equivalent to a linear controllable system (which is *not* its linear approximation). This phenomenon is clarified in section 3.5.3.

Techniques. The conditions for smooth linearizability derived in [57, 50, 103] come up naturally in some sense. Indeed, to any control system, one may associate a sequence of distributions defined via a construction using Lie brackets of vector fields attached to the system ; it turns out that the instance of this sequence of distributions for linear systems yields "constant" –hence integrable– distributions that span the entire state space in a finite number of steps if the system is controllable. Since Lie brackets and integrability of distributions are preserved under local *diffeomorphisms*, this translates at once into necessary conditions for smooth linearizability, shown in [57, 50, 103] to be sufficient. In contrast, homeomorphisms do not allow to pull back Lie brackets or tangent vector fields ; hence the same conditions need not be necessary for topological linearization, and the proofs in the present paper are more intricate. Specifically, we have to rely upon the notion of orbits of families of smooth vector fields rather than integral manifolds. The proof of Theorem 3.5.2 uses classical results concerning such orbits, first established in [101], that we recall and slightly expand in Section 3.8. Incidentally, the lack of a theory dealing with orbits of $\mathbf{C}^k$ vector fields ($k \in \mathbb{N}$) is the main reason why the results of the present paper restrict to $\mathbf{C}^\infty$ or $\mathbf{C}^\omega$ (*i.e.* real analytic) control systems.

Hopefully our method can be useful to study local topological equivalence to other classes of systems than linear ones ; this is not investigated here.

Organization of the paper. Section 3.2 recalls classical facts on local linearization of ordinary differential equations. Section 3.3 introduces conjugation for *control systems* (under a homeomorphism, a diffeomorphism, etc.) and establishes basic properties of conjugating maps. Section 3.4 reviews (topological, smooth, linear) conjugacy between *linear* control systems after [17, 108]. Section 3.5 states the main result of the paper (Theorem 3.5.2), namely that local topological linearizability implies local quasi-smooth linearizability for smooth control systems (smooth meaning either $\mathbf{C}^\infty$ or $\mathbf{C}^\omega$), and discusses the gap between smooth and quasi-smooth linearizability, including geometric characterizations thereof. Section 3.6 contains the proofs of these results ; the proof of Theorem 3.5.2 relies upon section 3.3, results from [101] stated in Section 3.8, and technical lemmas from Section 3.7.

3.2 Local linearization for ordinary differential equations

Consider the differential equation

$$\dot{x}(t) = f(x(t)), \tag{3.2}$$

where $f \in \mathbf{C}^k(U, \mathbb{R}^n)$ with U an open subset of $\mathbb{R}^n$ and $k \in \mathbb{N} \cup \{\infty, \omega\}$, $k \geq 1$.

It is well known (the "flow box theorem", see *e.g.* [6]) that, around each $x_0 \in U$ such that $f(x_0) \neq 0$, there is a change of coordinates of class $\mathbf{C}^k$ that conjugates (3.2) to the equation $\dot{x}_1 = 1$, $\dot{x}_2 = 0$, ..., $\dot{x}_n = 0$. Hence all differentiable vector fields are equivalent to each other, at points where they do not vanish, *via* a diffeomorphism having the same degree of smoothness (including real analyticity).

At a point $x_0 \in U$ such that $f(x_0) = 0$, *i.e.* at an equilibrium of the dynamical system (3.2), its linear approximation is the system

$$\dot{x}(t) = Ax(t) - Ax_0 \tag{3.3}$$

where $A = Df(x_0)$ is the derivative of f at x_0. The equilibrium x_0 is said to be *hyperbolic* if the matrix A has no purely imaginary eigenvalue.

The problem of locally linearizing (3.2) is that of finding a local homeomorphism $h : V \to W$ around x_0 mapping the trajectories of (3.2) in V onto trajectories of (3.3) in W in a time-preserving manner. In other words, if ϕ_t denotes the flow of (3.2), we should have for each $x \in V$ that

$$h \circ \phi_t(x) = e^{At}\big(h(x) - h(x_0)\big) + h(x_0)$$

provided that $\phi_\rho(x) \in V$ for $0 \leq \rho \leq t$. When this is the case we say that h conjugates (3.2) and (3.3), and we speak of topological, $\mathbf{C}^k$, smooth, or analytic linearization depending on the regularity of h and h^{-1}.

Local linearization at an equilibrium is a very old issue. At the beginning of the twentieth century, H. Poincaré already identified the obstructions to the existence of a *formal* change of variables h that removes all the nonlinear terms when f is analytic. These are the so-called resonances, see e.g. [47, 6]. In fact, resonant monomials of order ℓ are obstructions to linearizing the Taylor expansion of f at order ℓ and consequently also obstructions to $\mathbf{C}^\ell$ linearization. However, although there exists a formal power series expansion for h when there are no resonant terms, the existence of a *convergent* power series for h (analytic linearization) is a delicate issue. When the eigenvalues of the Jacobian belong to the so-called Poincaré domain, the absence of resonances indeed implies analytic linearizability (the Poincaré theorem). If it is not the case, a famous theorem by Siegel gives additional Diophantine conditions on these eigenvalues to the same conclusion. These conditions are generically satisfied in the measure-theoretic sense [6]. If no eigenvalue of the Jacobian is purely imaginary, it turns out [89] that the absence of resonances is also sufficient for smooth (h, h^{-1} of class $\mathbf{C}^\infty$) but in general not real analytic linearization. This is still valid when f is merely of class $\mathbf{C}^\infty$.

In contrast, if one allows conjugation via a topological but not necessarily differentiable homeomorphism, the Grobman-Hartman theorem asserts that every ordinary differential equation with no purely imaginary eigenvalue of the Jacobian (hyperbolicity) can be locally linearized around an equilibrium, that is, resonances are no longer an obstruction. A proof of this classical result can be found in [47] :

Theorem 3.2.1 (Grobman-Hartman)**.** *Under the assumption that x_0 is a hyperbolic equilibrium point, system* (3.2) *is topologically conjugate to system* (3.3) *at x_0.*

In fact, it is proved in [105] that the conjugating homeomorphism h (together with its inverse h^{-1}) can be chosen Hölder-continuous, and even differentiable at x_0 (but not in a neighborhood). This brings additional rigidity to the mapping h.

The above theorem entails that the only invariant under local topological conjugacy, around a hyperbolic equilibrium, is the number of eigenvalues with positive real part in the Jacobian matrix, counting multiplicity. Indeed, as is well-known (cf. [5]), the linear system $\dot{x} = Ax$ where

A has no pure imaginary eigenvalue is topologically conjugate to $\dot{x} = DX$, where D is diagonal with diagonal entries ± 1, the number of $+1$ being the number of eigenvalues of A with positive real part.

3.3 Preliminaries on topological equivalence for control systems

3.3.1 Control systems and their solutions

Consider two *control systems* where n, m, n', m' are natural integers :

$$\dot{x} = f(x,u)\ ,\quad x \in \mathbb{R}^n\ ,\quad u \in \mathbb{R}^m\ , \tag{3.4}$$

$$\dot{z} = g(z,v)\ ,\quad z \in \mathbb{R}^{n'}\ ,\quad v \in \mathbb{R}^{m'}\ , \tag{3.5}$$

or expanded in coordinates :

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,\dots,x_n,u_1,\dots,u_m) & \dot{z}_1 &= g_1(z_1,\dots,z_{n'},v_1,\dots,v_{m'}) \\ &\ \vdots & &\ \vdots \\ \dot{x}_n &= f_n(x_1,\dots,x_n,u_1,\dots,u_m) & \dot{z}_{n'} &= g_{n'}(z_1,\dots,z_{n'},v_1,\dots,v_{m'}) \end{aligned}$$

where x or z is called the *state* and u or v the *control*.

Although our main results are stated (in section 3.5) for infinitely differentiable —or real analytic— control systems, their proofs deal with non-smooth objects because the transformations we consider are only assumed to be continuous. This leads us to keep smoothness assumptions to a minimum in the present section. Accordingly, the maps $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g_i : \mathbb{R}^{n'} \times \mathbb{R}^{m'} \to \mathbb{R}$ are assumed to be at least continuous ; *any additional regularity assumption will be stated explicitly.* We do not restrict their domains of definition ; this is no real loss of generality because they could anyway be extended using partitions of unity (real analyticity plays no role in the present section), and whenever a result is stated, the domain where it holds true is precisely stated and the value of f and g outside this domain does not matter.

If m is zero or f does not depend on u, equation (3.4) reduces to the ordinary differential equation (3.2). Of course "genuine" control systems are those whose right hand side does depend on the control.

Definition 3.3.1. *By* a solution of (3.4) that remains in an open set $\Omega \subset \mathbb{R}^{n+m}$, *we mean a mapping γ defined on a real interval I, say*

$$\begin{aligned} \gamma\ :\ I &\to \Omega \\ t &\mapsto \gamma(t) = (\,\gamma_{\mathrm{I}}(t)\,,\,\gamma_{\mathrm{II}}(t)\,) \end{aligned} \tag{3.6}$$

with $\gamma_{\mathrm{I}}(t) \in \mathbb{R}^n$ and $\gamma_{\mathrm{II}}(t) \in \mathbb{R}^m$, such that :
- *γ is measurable, locally bounded, and γ_{I} is absolutely continuous,*
- *whenever $[T_1, T_2] \subset I$, we have :*

$$\gamma_{\mathrm{I}}(T_2) - \gamma_{\mathrm{I}}(T_1) = \int_{T_1}^{T_2} f(\,\gamma_{\mathrm{I}}(t)\,,\,\gamma_{\mathrm{II}}(t)\,)\,\mathrm{d}t\ . \tag{3.7}$$

Solutions of (3.5) *that remain in $\Omega' \subset \mathbb{R}^{n'+m'}$ are likewise defined to be mappings*

$$\begin{aligned} \gamma'\ :\ I &\to \Omega' \\ t &\mapsto \gamma'(t) = (\,\gamma'_{\mathrm{I}}(t)\,,\,\gamma'_{\mathrm{II}}(t)\,) \end{aligned} \tag{3.8}$$

having the corresponding properties with respect to g.

If $(\bar{x}, \bar{u})$ is a point in Ω, $\mathcal{U}$ a neighborhood of $\bar{u}$ such that $\{\bar{x}\} \times \mathcal{U} \subset \Omega$, J a real interval, and $\gamma_{\mathrm{II}} : J \to \mathcal{U}$ a measurable and locally bounded map, then, by [24, Ch. 2, Theorem 1.1] and the continuity of f, there exists, on a possibly smaller interval $I \subset J$, a solution γ of (3.4) that remains in Ω subject to the initial condition $\gamma_{\mathrm{I}}(0) = \bar{x}$. This solution may not be unique without further assumptions on f, for instance that it is continuously differentiable, or merely locally Lipschitz in the first argument.

Remark 3.3.2. *Observe that Definition 3.3.1 assigns a definite value to $\gamma_{\mathrm{II}}(t)$ for* each $t \in I$*. Of course, since γ_{I} remains a solution to* (3.7) *when the control γ_{II} gets redefined over a set of measure 0, one could identify two control functions whose values agree a.e. on I, as is customary in integration theory. However, these values are in any case subject to the constraint that $\gamma(t) \in \Omega$ for* every *$t \in I$, and altogether we find it more convenient to adopt Definition 3.3.1.*

3.3.2 Feedbacks

In the terminology of control, a solution in the sense of Definition 3.3.1 would be termed *open loop* to emphasize that the value of the control at time t is a function of time only, namely that $\gamma_{\mathrm{II}}(t)$ bears no relation to the state x whatsoever. A central concept in control theory, though, is that of *closed loop* or *feedback* control, where the value of the control at time t is computed from the corresponding value of the state, namely is of the form $\alpha(x(t))$. To make a formal definition of a feedback defined on an arbitrary open set, we need one more piece of notation : if $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ is open, we let $\pi_n : \Omega \to \Omega_{\mathbb{R}^n}$ the natural projection that selects the first n components, where $\Omega_{\mathbb{R}^n} = \pi_n(\Omega) \subset \mathbb{R}^n$.

Definition 3.3.3. *Given an open set $\Omega \subset \mathbb{R}^{n+m}$, a* feedback *on Ω is a continuous mapping $\alpha : \Omega_{\mathbb{R}^n} \to \mathbb{R}^m$ such that $(x, \alpha(x)) \in \Omega$ for all $x \in \Omega_{\mathbb{R}^n}$. A $\mathbf{C}^\infty$ (resp. $\mathbf{C}^\omega$) feedback on Ω is one of class $\mathbf{C}^\infty$ (resp. $\mathbf{C}^\omega$).*

A feedback is nothing but a mapping α such that $x \mapsto (x, \alpha(x))$ is a continuous section of the natural fibration $\pi_n : \Omega \to \Omega_{\mathbb{R}^n}$. Of course, there are sets Ω whose topology prevents the existence of any feedback. However, if there is one there are plenty, among which $\mathbf{C}^\infty$feedbacks are uniformly dense. This is the content of the next proposition, that will be used in the proof of Theorem 3.5.2. To fix notations, let us agree throughout that the symbol $\| \;\|$ designates the Euclidean norm on $\mathbb{R}^\ell$ irrespectively of the positive integer ℓ, while $B(x, r)$ stands for the open ball centered at x of radius r and $\overline{B}(x, r)$ for the corresponding closed ball.

Proposition 3.3.4. *Let Ω be open in $\mathbb{R}^{n+m}$, and $\alpha : \Omega_{\mathbb{R}^n} \to \mathbb{R}^m$ be a feedback on Ω. To each $\varepsilon > 0$, there is a $\mathbf{C}^\infty$feedback $\beta : \Omega_{\mathbb{R}^n} \to \mathbb{R}^m$ such that $\|\alpha(x) - \beta(x)\| < \varepsilon$ for $x \in \Omega_{\mathbb{R}^n}$.*

Proof. Let $\varnothing = \mathcal{K}_0 \subset \mathcal{K}_1 \cdots \subset \mathcal{K}_k \subset \mathcal{K}_{k+1} \cdots$ be an increasing sequence of compact subsets of $\Omega_{\mathbb{R}^n}$, each of which contains the previous one in its interior, and whose union is all of $\Omega_{\mathbb{R}^n}$. For each $x \in \Omega_{\mathbb{R}^n}$, define an integer

$$k(x) \;\stackrel{\Delta}{=}\; \min\{k \in \mathbb{N};\; x \in \mathcal{K}_k\}\ . \tag{3.9}$$

To each k, by the continuity of α and the compactness of $\mathcal{K}_k$, there is $\mu_k > 0$ such that

$$x \in \mathcal{K}_k \;\Rightarrow \left\{ \begin{array}{l} \bullet\; B(x, \mu_k) \times \mathrm{Conv}\left\{\, \alpha\big(B(x, \mu_k)\big) \,\right\} \;\subset\; \Omega\,, \\ \bullet\; \forall u_1, u_2 \in \mathrm{Conv}\left\{\, \alpha\big(B(x, \mu_k)\big) \,\right\},\; \|u_1 - u_2\| < \varepsilon\,, \end{array} \right. \tag{3.10}$$

where the symbol Conv designates the convex hull. In addition, we may assume that the sequence (μ_k) is non increasing.

Denote by $\overset{\circ}{\mathcal{K}}_k$ the interior of $\mathcal{K}_k$, set $\mathcal{D}_k = \mathcal{K}_k \setminus \overset{\circ}{\mathcal{K}}_{k-1}$ for $k \geq 1$, and cover the compact set $\mathcal{D}_k$ with a finite collection $\mathcal{B}_k$ of open balls having the following properties :

– each of these balls is centered at a point of $\mathcal{D}_k$ and is contained in the open set $\overset{\circ}{\mathcal{K}}_{k+1} \setminus \mathcal{K}_{k-2}$ (with the convention that $\mathcal{K}_{-1} = \varnothing$),
– each of these balls has radius at most $\frac{\mu_{k+1}}{2}$.

The union $\mathcal{B} = \bigcup_{k\geq 1} \mathcal{B}_k$ is a countable locally finite collection of open balls that covers $\Omega_{\mathbb{R}^n}$, and it has the property that *every ball in* $\mathcal{B}$ *is included in* $B(x, \mu_{k(x)})$ *as soon as it contains* x. Let B_j, for $j \in \mathbb{N}$, enumerate $\mathcal{B}$, and h_j be a smooth partition of unity where h_j has support $\mathrm{supp} h_j \subset B_j$. If we pick $x_j \in B_j$ for each j, the map $\beta : \Omega_{\mathbb{R}^n} \to \mathbb{R}^m$ defined by

$$\beta(x) = \sum_{j\in\mathbb{N}} h_j(x)\alpha(x_j) \tag{3.11}$$

is certainly smooth. In addition, since by construction x_j belongs to $B(x, \mu_{k(x)})$ whenever $h_j(x) \neq 0$, we get that $\beta(x)$ lies in the convex hull of $\alpha\big(B(x,r)\big)$ for some $r < \mu_{k(x)}$, and therefore, from (3.10) and (3.9), that $(x, \beta(x)) \in \Omega$ and $\|\alpha(x) - \beta(x)\| < \varepsilon$. Hence β is a smooth feedback on Ω such that $\|\alpha(x) - \beta(x)\| < \varepsilon$ for all $x \in \Omega_{\mathbb{R}^n}$. □

3.3.3 Conjugacy

We turn to the notion of conjugacy for control systems, which is the central topic of the paper.

Definition 3.3.5. *Let*

$$\begin{array}{rcccl} \chi: & \Omega & \to & \Omega' & \\ & (x,u) & \mapsto & \chi(x,u) & = \ (\,\chi_{\mathrm{I}}(x,u)\,,\,\chi_{\mathrm{II}}(x,u)\,) \end{array} \tag{3.12}$$

be a bijective mapping between two open subsets of $\mathbb{R}^{n+m}$ *and* $\mathbb{R}^{n'+m'}$ *respectively. We say that* χ conjugates *systems* (3.4) *and* (3.5) *if, for any real interval* I*, a map* $\gamma : I \to \Omega$ *is a solution of* (3.4) *that remains in* Ω *if, and only if,* $\chi \circ \gamma$ *is a solution of* (3.5) *that remains in* Ω'.

Although this definition makes sense without any regularity assumption, we only consider the case when χ and χ^{-1} are at least continuous. Then Brouwer's invariance of the domain (see *e.g.* [77]) implies that $n' + m' = n + m$ if (3.4) and (3.5) are conjugate via such a χ. Proposition 3.3.6 below asserts that more in fact is true.

Proposition 3.3.6. *If the map* χ *in* (3.12) *is a homeomorphism that conjugates* (3.4) *to* (3.5)*, then* $n = n'$*,* $m = m'$*, and* χ_{I} *depends only on* x *:*

$$\chi(x,u) \;=\; (\,\chi_{\mathrm{I}}(x)\,,\,\chi_{\mathrm{II}}(x,u)\,)\ . \tag{3.13}$$

Moreover, $\chi_{\mathrm{I}} : \Omega_{\mathbb{R}^n} \to \Omega'_{\mathbb{R}^n}$ *is a homeomorphism. Here, one should recall the notation* $\Omega_{\mathbb{R}^n}$ *that was introduced before Definition 3.3.3.*

Proof. Let $\bar{x}$, $\bar{u}$, $\bar{u}'$ be such that $(\bar{x}, \bar{u})$ and $(\bar{x}, \bar{u}')$ belong to Ω. Let further $x(t)$ be a solution[1] to (3.4) with $x(0) = \bar{x}$ and

$$\begin{array}{ll} u(t) = \bar{u} & \text{if } \ t \leq 0, \\ u(t) = \bar{u}' & \text{if } \ t > 0\ . \end{array}$$

By conjugacy, $z(t) = \chi_{\mathrm{I}}(x(t), u(t))$ is a solution to (3.5) with v given by $v(t) = \chi_{\mathrm{II}}(x(t), u(t))$, for $t \in (-\epsilon, \epsilon)$ and some $\epsilon > 0$. In particular $\chi_{\mathrm{I}}(x(t), u(t))$ is continuous in t so its values at 0^+ and 0^- are equal. Hence $\chi_{\mathrm{I}}(\bar{x}, \bar{u}) = \chi_{\mathrm{I}}(\bar{x}, \bar{u}')$ so that $\chi_{\mathrm{I}} : \Omega_{\mathbb{R}^n} \to \Omega'_{\mathbb{R}^{n'}}$ is well defined and continuous. Similarly, $\left(\chi^{-1}\right)_{\mathrm{I}}$ induces a continuous inverse $\Omega'_{\mathbb{R}^{n'}} \to \Omega_{\mathbb{R}^n}$. By invariance of the domain $n = n'$. □

[1] This solution is not necessarily unique since here f and g are merely assumed to be continuous.

In view of this proposition, we will only consider conjugacy between systems having the same number of states and inputs. Hence the distinction between (n,m) and (n',m') from now on disappears.

Remark 3.3.7. *In the literature, there seems to be no general agreement on what should be called a solution of a control system, nor on the concept of equivalence. We discuss and compare some notions in use in section 3.3.5.*

Remark 3.3.8. *Taking into account the triangular structure of χ in Proposition 3.3.6, one may describe conjugacy as resulting from a change of coordinates in the state-space (upon setting $z=\chi_{\mathrm{I}}(x)$) and then feeding the system with a function both of the state and of a new control variable v (upon setting $u=(\chi^{-1})_{\mathrm{II}}(z,v)$), in such a way that the correspondence $(x,u)\mapsto(z,v)$ is invertible. In the language of control, this is known as a* static feedback transformation, *and two systems conjugate in the sense of Definition 3.3.10 would be termed* equivalent under static feedback.

This notion has received considerable attention (see for instance [54]), albeit only in the differentiable *case (i.e. when χ is a diffeomorphism). Differentiability has the following advantage : when χ_{I} and $(\chi_{\mathrm{I}})^{-1}$ are differentiable, χ conjugates systems* (3.4) *and* (3.5) *on some domain if, and only if*

$$g(\chi_{\mathrm{I}}(x),\chi_{\mathrm{II}}(x,u)) \quad = \quad \frac{\partial\chi_{\mathrm{I}}}{\partial x}(x)\,f(x,u) \tag{3.14}$$

holds true on this domain. Hence one may replace Definition 3.3.10, which is based on solutions *to* (3.4) *and* (3.5)*, by the equality above expressing the way in which χ transforms the* equations. *Note that the differentiability of χ_{II} is not required.*

Various degrees of regularity for χ give rise to corresponding notions of conjugacy in Definition 3.3.10 below.

Definition 3.3.9. *For $k\in\mathbb{N}\cup\{\infty,\omega\}$, $k\geq1$, a map χ as in (3.13) is called a* quasi-$\mathbf{C}^k$ diffeomorphism *if and only of it is $\mathbf{C}^0$ homeomorphism and χ_{I} is a $\mathbf{C}^k$ diffeomorphism $\Omega_{\mathbb{R}^n}\to\Omega'_{\mathbb{R}^n}$, i.e. χ_{I} and ${\chi_{\mathrm{I}}}^{-1}$ are of class $\mathbf{C}^k$.*

Definition 3.3.10. *Let $k\in\mathbb{N}\cup\{\infty,\omega\}$, $k\geq1$.*

Systems (3.4) *and* (3.5) *are* topologically *(resp.* $\mathbf{C}^k$*, resp.* quasi-$\mathbf{C}^k$*)* conjugate over the pair Ω,Ω' *if there exists a homeomorphism (resp. $\mathbf{C}^k$ diffeomorphism, resp. quasi-$\mathbf{C}^k$ diffeomorphism) $\chi:\Omega\to\Omega'$ that conjugates the two systems.*

System (3.4) *is* locally *topologically ($\mathbf{C}^k$, quasi-$\mathbf{C}^k$) conjugate to system* (3.5) *at $(\bar{x},\bar{u})\in\mathbb{R}^{n+m}$ if*[2] *the two systems are topologically ($\mathbf{C}^k$, quasi-$\mathbf{C}^k$) conjugate over a pair Ω, Ω', where Ω is a neighborhood of $(\bar{x},\bar{u})$.*

Remark 3.3.11. *All definitions are invariant under linear time re-parameterization, namely :* if $\chi:\Omega\to\Omega'$ conjugates systems (3.4) and (3.5), then for any $\lambda\in\mathbb{R}$ (if $\lambda<0$, this reverses time) the map χ also conjugates the systems

$$\dot{x}=\lambda f(x,u) \quad \text{and} \quad \dot{z}=\lambda g(z,v)\ .$$

Indeed, this is trivial for $\lambda=0$, otherwise, if $t\mapsto(x(t),u(t))$ is a solution of $\dot{x}=\lambda f(x,u)$ on a time-interval $[t_1,t_2]$, and $\tilde{x}(t)$ and $\tilde{u}(t)$ denote respectively $x(t/\lambda)$ and $u(t/\lambda)$, then $t\mapsto$

[2]It would be more natural to say that system (3.4) at $(\bar{x},\bar{u})\in\mathbb{R}^{n+m}$ is locally conjugate to system (3.5) at $(\bar{x}',\bar{u}')\in\mathbb{R}^{n+m}$ if the two systems are conjugate over a pair Ω, Ω', where Ω is a neighborhood of $(\bar{x},\bar{u})$ and Ω' is a neighborhood of $(\bar{x}',\bar{u}')$. However, prescribing $(\bar{x}',\bar{u}')$ would increase notational burden and add no relevant information.

$(\tilde{x}(t), \tilde{u}(t))$ is a solution of (3.4) *on* $[\lambda t_1, \lambda t_2]$*, hence* χ *sends* $(\tilde{x}(t), \tilde{u}(t))$ *to* $(\tilde{z}(t), \tilde{v}(t))$ *satisfying* $\dot{\tilde{z}}(t) = g(\tilde{z}(t), \tilde{v}(t))$*. Consequently,* χ *maps* $(x(t), u(t))$ *to* $(z(t), v(t)) = ((\tilde{z}(\lambda t), \tilde{v}(\lambda t)))$*, which is a solution of* $\dot{z} = \lambda g(z, v)$*.*

In case there is no control (i. e. $m = m' = 0$) so that neither u nor χ_{II} appear in (3.12), Definition 3.3.10 coincides with the usual notion of local conjugacy for ordinary differential equations.

3.3.4 Properties of conjugating maps

Below we derive some technical facts about conjugacy and feedback that are fundamental to the proof of Theorem 3.5.2, although they are not needed to understand the result itself.

In the proof of Proposition 3.3.6, we only used conjugacy on a very small class of solutions, namely those corresponding to piecewise constant controls with a single discontinuity. This raises the question whether smaller classes of solutions than prescribed in Definition 3.3.1 are still sufficiently rich to check for conjugacy. Under mild conditions on f and g, as we will see in the forthcoming proposition, conjugacy essentially holds if it is granted for a class of inputs that locally uniformly approximates piecewise continuous functions, and this fact will be of technical use in the proof of Lemma 3.6.3. To fix terminology, we agree that a function $I \to \mathbb{R}^m$, where I is a real interval, is called *piecewise continuous* if it is continuous except possibly at *finitely many* interior points of I where it has limits from both sides and is either right or left continuous. If in addition the function is constant (resp. affine, resp. $\mathbf{C}^\infty$) on every open interval not containing a discontinuity point, we say that it is *piecewise constant* (resp. *piecewise affine*, resp. *piecewise* $\mathbf{C}^\infty$).

Proposition 3.3.12 (Conjugacy from restricted classes of inputs)**.** *Assume that f and g are continuous $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and locally Lipschitz-continuous with respect to their first argument*[3]*. Let $\chi : \Omega \to \Omega'$ be a homeomorphism between two open subsets of $\mathbb{R}^{n+m}$, and denote by Ω_{II} and Ω'_{II} respectively the open subsets of $\mathbb{R}^m$ obtained by projecting Ω and Ω' onto the second factor. Let further $\mathcal{C}$ and $\mathcal{C}'$ be collections of locally bounded measurable functions $\mathbb{R} \to \mathbb{R}^m$ whose restrictions $\mathcal{C}|_J$ and $\mathcal{C}'|_J$ to any compact interval J contain in their respective closures, for the topology of uniform convergence, the set of all piecewise continuous functions $J \to \Omega_{\mathrm{II}}$ and $J \to \Omega'_{\mathrm{II}}$ respectively. If χ maps every solution* (3.6) *of* (3.4) *such that $\gamma_{\mathrm{II}}(t) \in \mathcal{C}|_I$ to a solution of* (3.5) *while, conversely, χ^{-1} maps every solution* (3.8) *of* (3.5) *such that $\gamma'_{\mathrm{II}}(t) \in \mathcal{C}'|_I$ to a solution of* (3.4)*, then the restriction of χ to any relatively compact open subset $O \subset \Omega$ conjugates systems* (3.4) *and* (3.5) *over the pair O, $\chi(O)$.*

Proof. Let us first show that

$$\left.\begin{array}{l}\text{for any solution } \gamma : I \to \Omega \text{ of (3.4) } \textit{such that } \gamma_{\mathrm{II}} \textit{ is} \\ \textit{piecewise continuous}, \chi \circ \gamma \text{ is a solution to (3.5).}\end{array}\right\} \tag{3.15}$$

Since the property of being a solution is local with respect to time, we may suppose that I is a compact interval. Then, there is an open set $\mathcal{O}$ and a compact set $\mathcal{K}$ such that $\gamma(I) \subset \mathcal{O} \subset \mathcal{K} \subset \Omega$. By the hypothesis on $\mathcal{C}$, there exists a sequence of functions $\gamma_{\mathrm{II},k} : I \to \mathbb{R}^m$ converging uniformly to γ_{II} such that $\gamma_{\mathrm{II},k} \in \mathcal{C}|_I$. Define for each $k \in \mathbb{N}$ a time-varying vector field X^k by $X^k(t, x) = f(x, \gamma_{\mathrm{II},k}(t))$. By the continuity of f, this sequence converges uniformly on compact subsets of $I \times \mathbb{R}^n$ to $X(t, x) = f(x, \gamma_{\mathrm{II}}(t))$; moreover, since γ_{II} is bounded (being piecewise continuous) $\gamma_{\mathrm{II},k}$ is also bounded, thus the local Lipschitz character of $f(x, u)$ with respect to x

[3]This means that each $(\bar{x}, \bar{u}) \in \Omega$ has a neighborhood $\mathcal{N}$ such that $\|f(x', u) - f(x, u)\| \le c\,\|x' - x\|$ for some constant c whenever (x, u) and (x', u) lie in $\mathcal{N}$.

implies by compactness that $X(t,x)$ and $X^k(t,x)$ are themselves locally Lipschitz with respect to x on $I \times \mathcal{O}_{\mathbb{R}^n}$. Pick $t_0 \in I$ and apply Lemma 3.7.3 with $I = [t_1, t_2]$, $x_0 = \gamma_{\mathrm{I}}(t_0)$, and $\mathcal{U} = \mathcal{O}_{\mathbb{R}^n}$. This yields, say for $k > K$, that the solution $\gamma_{\mathrm{I},k}$ to the Cauchy problem

$$\dot{\gamma}_{\mathrm{I},k}(t) = X^k(t, \gamma_{\mathrm{I},k}(t)), \qquad \gamma_{\mathrm{I},k}(t_0) = \gamma_{\mathrm{I}}(t_0),$$

maps I into $\mathcal{O}_{\mathbb{R}^n}$ and that the sequence $(\gamma_{\mathrm{I},k})_{k>K}$ converges uniformly on I to γ_{I}. Hence, if we let

$$\gamma_k(t) = (\gamma_{\mathrm{I},k}(t), \gamma_{\mathrm{II},k}(t)),$$

the sequence $(\gamma_k)_{k>K}$ converges to γ, uniformly on I. In particular $\gamma_k(I) \subset \mathcal{K} \subset \Omega$ for k large enough.

Now, since $\gamma_k : I \to \Omega$ is a solution to (3.4) with $\gamma_{\mathrm{II},k} \in \mathcal{C}|_I$, it follows from the hypothesis that $\chi \circ \gamma_k$ is a solution to (3.5) that remains in Ω', *i.e.* with the notations of (3.12) we have, for k large enough,

$$\chi_{\mathrm{I}} \circ \gamma_k(t) \;-\; \chi_{\mathrm{I}} \circ \gamma_k(t_0) \;=\; \int_{t_0}^{t} g(\chi \circ \gamma_k(s))\,\mathrm{d}s, \qquad t \in I. \tag{3.16}$$

By the continuity of χ, the convergence of $\gamma_k(t)$ to $\gamma(t)$, and the fact that g remains bounded on the compact set $\chi(\mathcal{K})$, we can apply the dominated convergence theorem to the right hand-side of (3.16) to obtain in the limit, as $k \to \infty$, that

$$\chi_{\mathrm{I}} \circ \gamma(t) \;-\; \chi_{\mathrm{I}} \circ \gamma(t_0) \;=\; \int_{t_0}^{t} g(\chi \circ \gamma(s))\,\mathrm{d}s, \qquad t \in I.$$

Thus $\chi \circ \gamma : I \to \mathbb{R}^{n+m}$ is a solution to (3.5) that remains in Ω', thereby proving (3.15).

The next step is to observe from (3.15) that, since piecewise constant controls are in particular piecewise continuous, the proof of Proposition 3.3.6 applies to show that $\chi : \Omega \to \Omega'$ has a triangular structure of the form (3.13).

With (3.15) and (3.13) at our disposal, let us now prove the proposition in its generality. Choose an arbitrary open subset $\mathcal{O}$ with compact closure $\overline{\mathcal{O}}$ in Ω, and fix two compact subsets $\mathcal{K}$ and $\mathcal{K}_1$ of Ω such that

$$\mathcal{O} \;\subset\; \overline{\mathcal{O}} \;\subset \overset{\circ}{\mathcal{K}} \subset\; \mathcal{K} \;\subset \overset{\circ}{\mathcal{K}}_1 \subset\; \mathcal{K}_1 \;\subset\; \Omega.$$

where $\overset{\circ}{\mathcal{K}}$ stands for the *interior* of $\mathcal{K}$.

Let $\gamma : I \to \mathcal{O}$ be a solution of (3.4). We need to prove that $\chi \circ \gamma$ is a solution to (3.5) and again, since the property of being a solution is local with respect to time, we may suppose that I is compact. Notations being as in (3.6), it follows by definition of a solution that γ_{II} is a bounded measurable function $I \to \mathbb{R}^m$. We shall proceed as before in that we again approximate γ by a sequence γ_k of trajectories of (3.4) that are mapped by χ to trajectories of (3.5). This time, however, the approximation process is slightly more delicate, because it is no longer granted by the hypothesis on $\mathcal{C}$ but it will rather depend on general point-wise approximation properties to measurable functions by continuous ones.

By the compactness of $\mathcal{K}$, there is $\varepsilon_{\mathcal{K}} > 0$ such that

$$(x,u) \in \mathcal{K} \;\Rightarrow\; B\big((x,u)\,,\,\varepsilon_{\mathcal{K}}\big) \subset \overset{\circ}{\mathcal{K}}_1\,. \tag{3.17}$$

Let $u_{\gamma_{\mathrm{I}}} : I \to \mathbb{R}^m$ be an auxiliary function with the following properties :

(i) $u_{\gamma_{\mathrm{I}}}$ is piecewise constant on I,

(ii) $(\xi(t), u_{\gamma_{\mathrm{I}}}(t)) \in \mathring{\mathcal{K}}_1$ for all $t \in I$ and every map $\xi : I \to \mathbb{R}^n$ that satisfies

$$\sup_{t\in I} \|\xi(t) - \gamma_{\mathrm{I}}(t)\| < \varepsilon_{\mathcal{K}}/2. \tag{3.18}$$

Such a function $u_{\gamma_{\mathrm{I}}}$ certainly exists. Indeed, by definition of a solution, γ_{I} is absolutely continuous thus a fortiori continuous $I \to \mathbb{R}^n$, and therefore we know for each $t \in I$ that the set

$$\gamma_{\mathrm{I}}^{-1}\big(B(\gamma_{\mathrm{I}}(t), \varepsilon_{\mathcal{K}}/2)\big)$$

is an open neighborhood of t in I, hence a disjoint union of open intervals in I one of which contains t; call this particular interval U_t. By the compactness of I, we may cover the latter with finitely many intervals U_{t_j} for $1 \le j \le \nu$. Let now $j(t)$ denote, for each $t \in I$, the smallest index $j \in \{1, \ldots, \nu\}$ such that $t \in U_{t_j}$. Then, the map

$$u_{\gamma_{\mathrm{I}}}(t) = \gamma_{\mathrm{II}}(t_{j(t)})$$

clearly satisfies (i), and since $(\gamma_{\mathrm{I}}(t_{j(t)}), \gamma_{\mathrm{II}}(t_{j(t)})) \in \mathcal{O} \subset \mathcal{K}$, it follows from (3.17) and the fact that $\|\gamma_{\mathrm{I}}(t) - \gamma_{\mathrm{I}}(t_{j(t)})\| < \varepsilon_{\mathcal{K}}/2$ by definition of $j(t)$ that $u_{\gamma_{\mathrm{I}}}$ also satisfies (ii).

Next, recall that γ_{II} is a bounded measurable function $I \to \mathbb{R}^m$ so, by Lusin's theorem [90, Theorem 2.23] applied component-wise, there is, for every integer $k \ge 1$, a continuous function $h_k : I \to \mathbb{R}^m$ that coincides with γ_{II} outside some set $\mathcal{T}_k \subset I$ of Lebesgue measure strictly less than $1/k^2$, and in addition such that

$$\sup_{t\in I} \|h_k(t)\| \le \sqrt{m} \sup_{t\in I} \|\gamma_{\mathrm{II}}(t)\|. \tag{3.19}$$

Put $E_k = \{t \in I;\ (\gamma_{\mathrm{I}}(t), h_k(t)) \notin \mathring{\mathcal{K}}\}$. Since h_k is continuous E_k is compact, and since $\gamma(I) \subset \mathcal{O} \subset \mathring{\mathcal{K}}$ it is clear that $E_k \subset \mathcal{T}_k$ hence E_k has Lebesgue measure strictly less than $1/k^2$. Consequently, by the outer regularity of Lebesgue measure, E_k can be covered by finitely many open real intervals $I_{k,1}, \ldots, I_{k,N_k}$ whose lengths add up to no more than $1/k^2$.

We now define the sequence of functions $\gamma_{\mathrm{II},k}$ on I by setting, for $k \ge 1$,

$$\begin{aligned} \gamma_{\mathrm{II},k}(t) &= h_k(t) \text{ if } t \in I \setminus \textstyle\bigcup_{j=1}^{N_k} I_{k,j}, \\ \gamma_{\mathrm{II},k}(t) &= u_{\gamma_{\mathrm{I}}}(t) \text{ if } t \in \textstyle\bigcup_{j=1}^{N_k} I_{k,j}. \end{aligned} \tag{3.20}$$

By construction $\gamma_{\mathrm{II},k}$ is piecewise continuous, and uniformly bounded independently of k in view of (3.19) and the fact that $u_{\gamma_{\mathrm{I}}}$, being piecewise constant, is bounded. Moreover, as $\sum_{k\ge 1} 1/k^2 < \infty$, the measure of the set $\cup_{j=1}^{N_k} I_{k,j}$ is the general term, indexed by k, of a convergent series, hence almost every $t \in I$ belongs at most to finitely many of these sets so that $\gamma_{\mathrm{II},k}$ converges point-wise a.e. to γ_{II} on I as $k \to \infty$.

Redefine now $X^k(t,x) = f(x, \gamma_{\mathrm{II},k}(t))$, $X(t,x) = f(x, \gamma_{\mathrm{II}}(t))$, and observe from what we just said and the continuity of f that $X^k(t,x)$ converges to $X(t,x)$ when $k \to \infty$, locally uniformly with respect to $x \in \mathcal{O}_{\mathbb{R}^n}$, as soon as $t \notin E$ where $E \subset I$ is a set of zero measure which is independent of k. Moreover, again from the boundedness of $\gamma_{\mathrm{II},k}$, γ_{II} and the local Lipschitz character of f, we have that $X^k(t,x)$, $X(t,x)$ are locally Lipschitz with respect to x. Pick $t_0 \in I$ and apply Lemma 3.7.3 with $\mathcal{U} = \mathcal{O}_{\mathbb{R}^n}$, $I = [t_1, t_2]$, and $x_0 = \gamma_{\mathrm{I}}(t_0)$. We get, say for $k > K$, that the solution $\gamma_{\mathrm{I},k}$ to the Cauchy problem

$$\dot{\gamma}_{\mathrm{I},k}(t) = X^k(t, \gamma_{\mathrm{I},k}(t)), \qquad \gamma_{\mathrm{I},k}(t_0) = \gamma_{\mathrm{I}}(t_0),$$

is defined over I, maps the latter into $\mathcal{O}_{\mathbb{R}^n}$, and that the sequence $(\gamma_{\mathrm{I},k})_{k>K}$ converges uniformly on $[t_1, t_2]$ to γ_{I}.

We claim that $\gamma_k(t) = (\gamma_{\mathrm{I},k}(t), \gamma_{\mathrm{II},k}(t))$ *lies in* $\overset{\circ}{\mathcal{K}}_1$ *for all* $t \in I$ *when* k *is so large that*

$$\sup_{t\in I} \|\gamma_{\mathrm{I},k}(t) - \gamma_{\mathrm{I}}(t)\| < \varepsilon_{\mathcal{K}}/2. \tag{3.21}$$

Indeed, if $t \in \cup_j I_{k,j}$, this follows automatically from definition (3.20) by property (ii) of $u_{\gamma_{\mathrm{I}}}$; if $t \notin \cup_j I_{k,j}$, then $(\gamma_{\mathrm{I}}(t), h_k(t)) \in \overset{\circ}{\mathcal{K}}$ by the very definition of $\cup_j I_{k,j}$, and since $\gamma_k(t) = (\gamma_{\mathrm{I},k}(t), h_k(t))$ in this case, we deduce from (3.17) and (3.21) that $\gamma_k(t) \in \overset{\circ}{\mathcal{K}}_1$. *This proves the claim.*

Altogether, we have shown that $\gamma_k : I \to \overset{\circ}{\mathcal{K}}_1$ is a solution of (3.4) as soon as k is large enough, with $\gamma_{\mathrm{II},k}$ a piecewise continuous function on I by construction. By (3.15), we now deduce that, for k large enough, $\gamma'_k = \chi \circ \gamma_k$ is a solution of (3.5) that stays in Ω'. Let us block-decompose γ'_k into

$$\gamma'_{\mathrm{I},k}(t) \;=\; \chi_{\mathrm{I}}(\,\gamma_{\mathrm{I},k}(t)\,)\;, \quad \gamma'_{\mathrm{II},k}(t) \;=\; \chi_{\mathrm{II}}(\,\gamma_{\mathrm{I},k}(t)\,,\,\gamma_{\mathrm{II},k}(t)\,)\;,$$

where we have taken into account the triangular structure of χ. That $\gamma'_k : I \to \Omega'$ is a solution of (3.5) means exactly that

$$\gamma'_{\mathrm{I},k}(t) \;-\; \gamma'_{\mathrm{I},k}(t_0) \;=\; \int_{t_0}^{t} g(\,\gamma'_{\mathrm{I},k}(s)\,,\,\gamma'_{\mathrm{II},k}(s)\,)\mathrm{d}s, \qquad t \in I. \tag{3.22}$$

Due to the continuity of χ, the functions $\gamma'_{\mathrm{I},k}$ and $\gamma'_{\mathrm{II},k}$ respectively converge uniformly and pointwise almost everywhere to $\gamma'_{\mathrm{I}} = \chi_{\mathrm{I}} \circ \gamma_{\mathrm{I}}$ and $\gamma'_{\mathrm{II}} = \chi_{\mathrm{II}} \circ \gamma$ on I. Since g is bounded on the compact set $\chi(\mathcal{K}_1)$ that contains $\gamma_k(I)$ for k large enough, we get on the one hand, by dominated convergence, that the right-hand side of (3.22) converges, as $k \to \infty$, to $\int_{t_0}^{t} g(\,\gamma'_{\mathrm{I}}(s)\,,\,\gamma'_{\mathrm{II}}(s)\,)\mathrm{d}s$, and on the other hand that the left-hand side converges to $\gamma'_{\mathrm{I}}(t) - \gamma'_{\mathrm{I}}(t_0)$. Therefore $(\gamma'_{\mathrm{I}}, \gamma'_{\mathrm{II}}) = \chi \circ \gamma : I \to \Omega'$ is a solution of (3.5).

This way we have shown that χ maps any solution of (3.4) that stays in a relatively compact open subset $\mathcal{O}$ of Ω to a solution of (3.5) that stays in Ω'. This achieves the proof, for the converse is obtained symmetrically upon swapping f and g, $\mathcal{C}$ and $\mathcal{C}'$, and replacing χ by χ^{-1}. □

The triangular structure of conjugating homeomorphisms asserted by Proposition 3.3.6 is to the effect that any such homeomorphism $\chi : \Omega \to \Omega'$ is a fiber preserving map from the bundle $\Omega \to \Omega_{\mathbb{R}^n}$ to the bundle $\Omega' \to \Omega'_{\mathbb{R}^n}$. Since feedbacks are naturally associated to sections of these bundles by Definition 3.3.3, χ gives rise to a natural transformation from feedbacks on Ω to feedbacks on Ω'. This transformation will prove important enough to deserve a notation : to any feedback α on Ω, we associate a feedback $\chi \blacksquare \alpha$ on Ω' by the formula

$$\chi \blacksquare \alpha(z) \;\stackrel{\Delta}{=}\; \chi_{\mathrm{II}}(\chi_{\mathrm{I}}^{-1}(z), \alpha(\chi_{\mathrm{I}}^{-1}(z)))\,. \tag{3.23}$$

We leave it to the reader to check that the properties of an action are satisfied, and in particular that

$$\chi^{-1} \blacksquare (\chi \blacksquare \alpha) \;=\; \alpha\,. \tag{3.24}$$

Naturally associated to a control system (3.4) and a feedback α is the following continuous vector field f_α on $\Omega_{\mathbb{R}^n}$:

$$f_\alpha(x) \;=\; f(x, \alpha(x))\,. \tag{3.25}$$

If the homeomorphism χ in (3.13) conjugates system (3.4) to system (3.5), then it is clear that χ_{I} maps the solutions of the ordinary differential equation $\dot{x} = f_\alpha(x)$ to the solutions of the ordinary differential equation $\dot{z} = g_{\chi \blacksquare \alpha}(z)$. Indeed if $x(t)$ is a solution of the former, then $(x(t), \alpha(x(t)))$ is a solution of the control system (3.4) in the sense of Definition 3.3.1 so the conjugacy assumption implies that $(\chi_{\mathrm{I}}(x(t)), \chi_{\mathrm{II}}(x(t), \alpha(x(t))))$ is a solution of (3.5), and setting $z(t) = \chi_{\mathrm{I}}(x(t))$ one clearly has $\chi_{\mathrm{II}}(x(t),\alpha(x(t)))) = \chi\blacksquare\alpha(z(t))$; hence $z(t)$ is a solution to $\dot{z} = g_{\chi\blacksquare\alpha}(z)$ because $(z(t),\chi\blacksquare\alpha(z(t)))$ is a solution of (3.5).

Now, if α_1 and α_2 are two feedbacks on Ω, and the two vector fields f_{α_1} and f_{α_2} are defined on $\Omega_{\mathbb{R}^n}$ by (3.25), we denote their difference by $\delta f_{\alpha_1,\alpha_2}$:

$$\delta f_{\alpha_1,\alpha_2} \;=\; f_{\alpha_1} \;-\; f_{\alpha_2}\ . \tag{3.26}$$

Such vector fields are similar to the difference vector fields used in [60], except that we consider arbitrary feedbacks instead of constant ones. To us, these vector fields will play an essential role. The next proposition states that a homeomorphism that conjugates two control systems also conjugates the integral curves of such difference vector fields.

Proposition 3.3.13 (preservation of difference vector fields)**.** *Suppose that f and g in* (3.4) *and* (3.5) *are continuous and locally Lipschitz continuous with respect to their first argument. Assume they are locally topologically conjugate at* $(0,0)$ *over the pair* Ω,Ω'. *Then, notations for* χ_{I} *and* χ_{II} *being as in Proposition 3.3.6, we have for every pair of feedbacks* α_1,α_2 *on* Ω *that* χ_{I} *conjugates any solution of*

$$\dot{x} \;=\; \delta f_{\alpha_1,\alpha_2}(x) \tag{3.27}$$

that remains in $\Omega_{\mathbb{R}^n}$ *to a solution of*

$$\dot{z} \;=\; \delta g_{\chi\blacksquare\alpha_1,\chi\blacksquare\alpha_2}(z) \tag{3.28}$$

that remains in $\Omega'_{\mathbb{R}^n}$.

It is perhaps worth emphasizing that the solutions of (3.27) and (3.28) need not be unique since α is merely assumed to be continuous.

Proof. Let $\eta : [t_1,t_2] \to \Omega_{\mathbb{R}^n}$ be an integral curve of $\delta f_{\alpha_1,\alpha_2}$, and set

$$u_1(t) \;=\; \alpha_1(\eta(t)) \quad , \quad u_2(t) \;=\; \alpha_2(\eta(t))\ . \tag{3.29}$$

Let further $\widehat{f} : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be bounded, continuous and Lipschitz continuous with respect to its first argument, and coincide with f on some compact neighborhood of

$$\eta([t_1,t_2]) \times \Big(\alpha_1(\eta([t_1,t_2])) \bigcup \alpha_2(\eta([t_1,t_2]))\Big)\ .$$

Such a $\widehat{f}$ is easily obtained upon multiplying f by a function of class $\mathbf{C}^\infty$with compact support. For $\ell \in \mathbb{N}$, let η^ℓ be the solution to the Cauchy problem

$$\eta^\ell(t) \;=\; \eta(t_1) \;+\; \int_{t_1}^{t} G_\ell(\tau,\eta^\ell(\tau))\mathrm{d}\tau\ , \tag{3.30}$$

with

$$\begin{aligned} G_\ell(t,x) &= 2\,\widehat{f}(x,u_1(t)) \\ &\qquad \text{if } t\in[t_1+\tfrac{j}{\ell}(t_2-t_1),t_1+(\tfrac{j}{\ell}+\tfrac{1}{2\ell})(t_2-t_1)), \\ G_\ell(t,x) &= -2\,\widehat{f}(x,u_2(t)) \\ &\qquad \text{if } t\in[t_1+(\tfrac{j}{\ell}+\tfrac{1}{2\ell})(t_2-t_1),t_1+\tfrac{j+1}{\ell}(t_2-t_1)), \\ G_\ell(t_2,x) &= -2\,\widehat{f}(x,u_2(t_2)), \qquad 0\le j\le \ell-1. \end{aligned} \tag{3.31}$$

The definition of η^ℓ is valid because, since $G_\ell(t,x)$ is bounded and locally Lipschitz with respect to the variable x, the solution to (3.30) uniquely exists.

From Lemma 3.7.4 applied to the case where $X^{1,\ell}(t,x) = \widehat{f}(x,u_1(t))$ and $X^{2,\ell}(t,x) = \widehat{f}(x,u_2(t))$ are in fact independent of ℓ, any accumulation point of the sequence (η^ℓ), say η^∞, is a solution to

$$\dot{\eta}^\infty(t) \;=\; \widehat{f}(\eta^\infty(t),u_1(t)) \;-\; \widehat{f}(\eta^\infty(t),u_2(t)) \;\;,\quad \eta^\infty(t_1) \;=\; \eta(t_1)\;.$$

Since $\widehat{f}$ is locally Lipschitz continuous with respect to its first argument, the solution to this Cauchy problem is unique and, since f and $\widehat{f}$ coincide at all points $(\eta(t),u_1(t))$ and $(\eta(t),u_2(t))$, this entails $\eta^\infty=\eta$. Thus (η^ℓ) converges uniformly to η on $[t_1,t_2]$ and, for ℓ large enough, η^ℓ remains a solution of (3.30) if $\widehat{f}$ is replaced by f in (3.31). Moreover, $\eta^\ell([t_1,t_2])\subset\Omega_{\mathbb{R}^n}$ for ℓ large since the same is true of η. Since χ conjugates the two systems, hence also by Remark 3.3.11 the systems where f and g are multiplied by 2 or -2, the map $\chi_{\mathrm{I}}\circ\eta^\ell:[t_1,t_2]\to\Omega'_{\mathbb{R}^n}$ is, for ℓ large enough, a solution to

$$\chi_{\mathrm{I}}\circ\eta^\ell(t) \;=\; \chi_{\mathrm{I}}\circ\eta(t_1) \;+\; \int_{t_1}^{t}\widetilde{G}_\ell(\tau,\chi_{\mathrm{I}}\circ\eta^\ell(\tau))\mathrm{d}\tau \tag{3.32}$$

with

$$\begin{aligned}\widetilde{G}_\ell(t,z) &= 2\,g(z,\chi_{\mathrm{II}}(\chi_{\mathrm{I}}^{-1}(z),u_1(t)))\\ &\qquad \text{if } t\in[t_1+\tfrac{j}{\ell}(t_2-t_1),t_1+(\tfrac{j}{\ell}+\tfrac{1}{2\ell})(t_2-t_1)),\\ \widetilde{G}_\ell(t,z) &= -2\,g(z,\chi_{\mathrm{II}}(\chi_{\mathrm{I}}^{-1}(z),u_2(t)))\\ &\qquad \text{if } t\in[t_1+(\tfrac{j}{\ell}+\tfrac{1}{2\ell})(t_2-t_1),t_1+\tfrac{j+1}{\ell}(t_2-t_1)),\\ \widetilde{G}_\ell(t_2,z) &= -2\,g(z,\chi_{\mathrm{II}}(\chi_{\mathrm{I}}^{-1}(z),u_2(t_2))).\end{aligned} \tag{3.33}$$

Since $(\chi_{\mathrm{I}}\circ\eta^\ell)$ converges uniformly to $\chi_{\mathrm{I}}\circ\eta$ by the continuity of χ, replacing g by a bounded and continuous $\widehat{g}:\mathbb{R}^{n+m}\to\mathbb{R}^n$ that coincides with g on a compact neighborhood of

$$\chi_{\mathrm{I}}\circ\eta([t_1,t_2])\times\Big(\chi_{\mathrm{II}}(\eta([t_1,t_2]),\alpha_1(\eta([t_1,t_2])))\bigcup\chi_{\mathrm{II}}(\eta([t_1,t_2]),\alpha_2(\eta([t_1,t_2])))\Big)$$

does not affect the validity of (3.32)-(3.33) for ℓ large enough. Lemma 3.7.4 now implies that all accumulation points of the sequence $(\chi_{\mathrm{I}}\circ\eta^\ell)$ in the uniform topology on $[t_1,t_2]$ are solutions of

$$\dot{z}=g(z,\chi_{\mathrm{II}}(\chi_{\mathrm{I}}^{-1}(z),u_1(t)))\;-\;g(z,\chi_{\mathrm{II}}(\chi_{\mathrm{I}}^{-1}(z),u_2(t))).$$

Because $\chi_{\mathrm{I}}\circ\eta$ is such an accumulation point, it is by (3.29) a solution to

$$\dot{z} \;=\; g(z,\chi_{\mathrm{II}}(\chi_{\mathrm{I}}^{-1}(z),\alpha_1(\chi_{\mathrm{I}}^{-1}(z))))\;-\;g(z,\chi_{\mathrm{II}}(\chi_{\mathrm{I}}^{-1}(z),\alpha_2(\chi_{\mathrm{I}}^{-1}(z))))\;,$$

which is nothing but (3.28). □

3.3.5 Alternative notions of conjugacy and equivalence

3.3.5.1 Transformations in functional spaces

Following [25], one may view the control system (3.4) as a flow on the product space $\mathbb{R}^n\times\mathcal{U}$, where $\mathcal{U}$ is a functional space of admissible controls whose dynamics is induced by the time-shift. Transformations on $\mathbb{R}^n\times\mathcal{U}$ then naturally arise ; they involve the *future and the past of the control*, unlike the mere homeomorphisms on finite dimensional spaces that we consider here. The corresponding notion of equivalence is obviously rather weak. In Chapter 2, a "Grobman-Hartman theorem" theorem is proved in this setting, i.e. generic control systems (3.4) are locally conjugate to a linear system via this kind of transformation. With the much stronger notion of equivalence that we use here, we shall see (section 3.5.4) that "almost" no system is conjugate to a linear system.

Let us also mention [21], where control systems are maps $(x(0),u(.))\mapsto x(.)$ that satisfy certain axioms, without reference to differential equations, and where the notion of topological equivalence involves transformations on the product $\mathbb{R}^n\times\mathcal{U}$.

3.3.5.2 x-conjugacy

Let us call *x-solution* of system (3.4) any map $t \mapsto \gamma_{\mathrm{I}}(t)$ *such that there exists* a map γ_{II} for which $\gamma = (\gamma_{\mathrm{I}}, \gamma_{\mathrm{II}})$ is a solution in the sense of Definition 3.3.1 ; the set of x-solutions is the projection on the x factor of the set of solutions. Let then x-conjugacy be defined in the same way as Definition 3.3.10 defines conjugacy, except that we replace solutions by x-solutions and the homeomorphism χ that acts on state and control with a homeomorphism $x \mapsto z = h(x)$ on the state only.

In the literature, both notions are used (without the prefix "x-"). For instance [108], devoted to the topological classification of *linear* control systems (see section 3.4.2) relies on x-conjugacy. We favor Definitions 3.3.5 and 3.3.10 of conjugacy and solutions because results have to be stated locally with respect both to x and u for nonlinear control systems.

Conjugacy implies x-conjugacy : use Proposition 3.3.6, take $h = \chi_{\mathrm{I}}$ and ignore χ_{II}. The converse is not true in general, as the reader may check easily.

3.4 The case of linear control systems

3.4.1 Kronecker indices

A *linear* control systems is a special instance of (3.4), of the form

$$\dot{x} \;=\; Ax \;+\; Bu \tag{3.34}$$

where A and B are constant $n \times n$ and $n \times m$ matrices respectively. When dealing with linear systems, it is natural to consider an equivalence relation similar to that of Definition 3.3.10, but where χ is restricted to be a linear isomorphism :

Definition 3.4.1. *Two linear systems*

$$\dot{x} \;=\; Ax \;+\; Bu \qquad \textit{and} \qquad \dot{z} \;=\; \widetilde{A}z \;+\; \widetilde{B}v$$

are linearly conjugate *if and only if any of the following two equivalent properties is satisfied :*

1. *There is a nonempty open set $\Omega \subset \mathbb{R}^{n+m}$, and a linear isomorphism χ of $\mathbb{R}^{n+m}$ whose restriction $\Omega \to \chi(\Omega)$ conjugates the two systems in the sense of Definition 3.3.10.*
2. *There exist matrices $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times m}$ and $K \in \mathbb{R}^{n\times m}$, with P and Q invertible, such that*

$$\begin{aligned} \widetilde{A} &= P(A - BK)P^{-1} \ , \\ \widetilde{B} &= PBQ^{-1} \ . \end{aligned} \tag{3.35}$$

Since, by Proposition 3.3.6, a linear conjugating homeomorphism is necessarily of the form $(x, u) \mapsto (Px, Kx + Qu)$, the equivalence between properties (1) and (2) follows at once from differentiating the solutions. Provided it exists, Ω plays absolutely no role in this context since (3.35) implies that the two systems are in fact linearly conjugate on all of $\mathbb{R}^{n+m}$.

Linear conjugacy actually defines an equivalence relation on linear control systems or equivalently on pairs (A, B), for which (3.35) can be read as "(A, B) is equivalent to $(\widetilde{A}, \widetilde{B})$". The classification of linear systems under this equivalence relation is well-known [17], and goes as follows. Each equivalence class contains a pair (A_c, B_c) of the form (block matrices) :

$$A_c \;=\; \begin{pmatrix} A_0^c & 0 & \cdots & 0 \\ 0 & A_1^c & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_m^c \end{pmatrix} \;, \quad B_c \;=\; \begin{pmatrix} 0 & \cdots & 0 \\ b_1^c & \ddots & \vdots \\ 0 & \ddots & 0 \\ \vdots & 0 & b_m^c \end{pmatrix} \tag{3.36}$$

where

$$A_i^c = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & 0 \\ & & & \ddots & 1 \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}_{(\kappa_i \times \kappa_i)}, \qquad b_i^c = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{(\kappa_i \times 1)}, \quad 1 \leq i \leq m. \tag{3.37}$$

The integers $(\kappa_1, \ldots, \kappa_m)$ are called the controllability indices of the control system, also known as the Kronecker indices of the matrix pencil (A, B), while A_0^c is a square matrix of dimension $n - (\kappa_1 + \cdots + \kappa_m)$ that may be assumed in Jordan canonical form. Note that $\kappa_1 + \cdots + \kappa_m \leq n$, and if $\kappa_1 + \cdots + \kappa_m = n$ there is no A_0^c; also, it may well happen that $\kappa_i = 0$, in which case A_i^c and b_i^c are empty and do not occur in (3.36) to the effect that there are less than m blocks beyond A_0^c. Normalizing so that

$$\kappa_1 \geq \cdots \geq \kappa_m \geq 0,$$

and ordering the Jordan blocks arbitrarily, there is one and only one such normal form per equivalence class. A complete set of invariants is then the list of Kronecker indices and the spectral invariants of the matrix A_0^c.

With the natural partition $z = (Z_0, Z_1, \ldots, Z_m)$ corresponding to the block decomposition (3.36), the control system associated to the pair (A_c, B_c) reads

$$\dot{Z}_0 = A_0 Z_0 \, , \quad \dot{Z}_1 = A_1 Z_1 + u_1 b_1^c \, , \quad \cdots \quad , \quad \dot{Z}_m = A_m Z_m + u_m b_m^c \, ,$$

where Z_0 is missing if $\kappa_1 + \cdots + \kappa_m = n$ and Z_i is missing if $\kappa_i = 0$. Because it is not influenced at all by the controls, Z_0 is sometimes called the non-controllable part of the state. In this paper, we are only interested in controllable linear systems, namely :

Definition 3.4.2. *A linear control system* (3.34) *is said to be* controllable *if, and only if, the following two equivalent properties are satisfied :*

1. *There is no bloc A_0^c in the associated normal form* (3.36)*.*
2. *Kalman's criterion for controllability :*

$$\mathrm{Rank}(B, AB, \ldots, A^{n-1}B) = n.$$

To see the equivalence of the two properties, observe that the $n - \kappa_1 - \cdots - \kappa_m$ first rows of the matrix P that puts (A, B) into canonical form (i.e. $z = Px$) form a basis of the smallest dual subspace that annihilates the columns of B and at the same time is invariant under right multiplication by A, i.e. they are a basis of the left kernel of $(B, AB, \ldots, A^{n-1}B)$. For controllable linear systems, the only invariant under linear conjugacy is thus the ordered list of Kronecker indices. These can be computed from $(B, AB, \ldots, A^{n-1}B)$ as follows : if we put

$$\begin{aligned} r_j &= \mathrm{Rank}(B, AB, \ldots, A^{j-1}B) \, , \; j \geq 1 \, , \quad r_0 = 0, \; r_{-1} = -m \, , \\ s_j &= r_j - r_{j-1} \, , \; j \geq 1 \, , \quad s_0 = m \, , \end{aligned} \tag{3.38}$$

then s_j does not increase with j and a moment's thinking will convince the reader that the number of Kronecker indices that are equal to i is $s_i - s_{i+1}$, or equivalently that s_k is the number of κ_j's that are no smaller than k.

To us, it will be more convenient to use as normal form the following permutation of the previous one. Let ρ be the smallest integer such that $s_\rho = 0$, so that

$$0 = s_\rho < s_{\rho-1} \leq s_{\rho-2} \leq \cdots \leq s_1 \leq s_0 = m \, ,$$

with $\sum_{j\geq 1} s_j = n$. From these we define, for $0 \leq i \leq \rho$:

$$\sigma_i \;=\; \sum_{j\geq i} s_j \;=\; n - r_{i-1}\,, \tag{3.39}$$

so that in particular $\sigma_\rho = 0$, $\sigma_{\rho-1} = s_{\rho-1} > 0$, $\sigma_1 = n$ and $\sigma_0 = n + m$. Note that, from (3.38), $\sigma_i = n - r_{i-1}$ for $i \geq 1$. We shall write our controllable canonical form as $\dot z = A_c z + B_c v$ with

$$A_c = \begin{pmatrix} 0 & J^{s_{\rho-1}}_{s_{\rho-2}} & & & & \\ & 0 & J^{s_{\rho-2}}_{s_{\rho-3}} & & & \\ & & 0 & J^{s_{\rho-3}}_{s_{\rho-4}} & & \\ & & & \ddots & \ddots & \\ & & & & 0 & J^{s_2}_{s_1} \\ & & & & & 0 \end{pmatrix}, \quad B_c = \begin{pmatrix} 0 \\ \hline J^{s_1}_{s_0} \end{pmatrix} \tag{3.40}$$

where for any integers r and s with $s \leq r$, J^s_r is the $s \times r$ matrix

$$J^s_r \;=\; \left(\begin{array}{c|c} I_s & 0 \end{array} \right) \tag{3.41}$$

where I_s is the $s \times s$ identity matrix.

3.4.2 Topological classification of linear control systems

In [108], which is devoted to the topological classification of *linear* control systems and uses the notion of x-conjugacy rather than conjugacy (cf. Section 3.3.5), the following result is proved :

Theorem 3.4.3 (Willems [108]). *If two linear control systems $\dot x = Ax + Bu$ and $\dot z = \widetilde{A}z + \widetilde{B}v$ are topologically x-conjugate, then they have the same list of Kronecker indices, and the non-controllable blocks A^c_0 and $\widetilde{A}^c_0$ in their respective canonical forms* (3.36) *are such that the two linear differential equations $\dot X_0 = A^c_0 X_0$ and $\dot Z_0 = \widetilde{A}^c_0 Z_0$ are topologically equivalent.*

As pointed out in Section 3.3.5, topological conjugacy implies topological x-conjugacy but not conversely. However, for linear control systems having the same number m of inputs, Theorem 3.4.3 implies that these notions are equivalent. Indeed, if two systems are respectively brought into their canonical form (3.36) by a linear change of variable on $\mathbb{R}^{n+m}$, and if in addition they are x-conjugate, then their non-controllable parts are topologically equivalent while the remaining blocks are identical by equality of the Kronecker indices. Hence, both in the above theorem and in the corollary below, one may use indifferently "x-conjugate" or "conjugate"

Corollary 3.4.4. *If two linear systems* $\dot{x} = Ax + Bu$ *and* $\dot{z} = \widetilde{A}z + \widetilde{B}v$ *are topologically conjugate and one of them is controllable, then the other one is controllable too and they are linearly conjugate.*

Proof. Controllability is preserved, since Kronecker indices are by the theorem. Linear conjugacy follows, as we saw that the list of Kronecker indices is a complete invariant for controllable systems under linear conjugacy. □

In some sense, the results of section 3.5 can be viewed as a generalization of Corollary 3.4.4 to a local setting where only one of the two systems is linear.

3.5 Local linearization for control systems

In this section, we consistently assume that the map f defining system (3.4) is either smooth or real-analytic.

Definition 3.5.1. *Let* $k \in \{\infty, \omega\}$*. The system* (3.4) *is said to be* locally topologically *(resp.* $\mathbf{C}^k$*, resp.* quasi-$\mathbf{C}^k$*)* linearizable at $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$ *if it is locally topologically (resp.* $\mathbf{C}^k$*, resp.* quasi-$\mathbf{C}^k$*) conjugate, in the sense of Definition 3.3.10, to a linear controllable system* $\dot{z} = Az + Bv$ *(cf. Definition 3.4.2).*

This definition of smooth linearizability coincides with linearizability by smooth static feedback as described in the textbooks [52, 79]. In subsection 3.5.2, we recall classical necessary and sufficient geometric conditions for a system to be smoothly (resp. analytically) linearizable, and we complement them with a characterization of quasi-smooth (resp. quasi-analytic) linearizability.

3.5.1 Main result

If a smooth control system is locally topologically linearizable, then the conjugating homeomorphism has a lot more regularity than required *a priori*. This is in contrast with the Grobman-Hartman theorem for ODE's and constitutes the central result of the paper :

Theorem 3.5.2. *Let* $k \in \{\infty, \omega\}$ *and assume that* f *is of class* $\mathbf{C}^k$ *on an open set* $\Omega \subset \mathbb{R}^{n+m}$*. Then system* (3.4) *is locally topologically linearizable at* $(\bar{x}, \bar{u}) \in \Omega$ *if, and only if, it is locally quasi-*$\mathbf{C}^k$ *linearizable at* $(\bar{x}, \bar{u})$*.*

Proof. See at the end of the paper, page 81. □

Observe from (3.14), that a quasi-$\mathbf{C}^k$ diffeomorphism χ is a linearizing homeomorphism if and only if it satisfies

$$\frac{\partial \chi_{\mathrm{I}}}{\partial x}(x)\, f(x,u) \;=\; A\,\chi_{\mathrm{I}}(x) \;+\; B\,\chi_{\mathrm{II}}(x,u)\;. \tag{3.42}$$

Hence quasi-smooth linearizability is much easier to handle than topological linearizability, that relies on conjugating *solutions* rather than equations.

System (3.1) of the introduction is topologically, quasi-$\mathbf{C}^\omega$and quasi-$\mathbf{C}^\infty$linearizable at $(0,0)$ but fails to be even $\mathbf{C}^1$ linearizable ; hence quasi-$\mathbf{C}^k$ cannot be replaced with $\mathbf{C}^k$ in Theorem 3.5.2. To study the gap between $\mathbf{C}^k$ and quasi-$\mathbf{C}^k$ linearizability, note that (3.42) imposes additional regularity on a linearizing quasi-$\mathbf{C}^k$ diffeomorphism :

Proposition 3.5.3. *Let $k \in \{\infty, \omega\}$ and f in (3.4) be $\mathbf{C}^k$. If $\chi : \Omega \to \Omega'$ is a quasi-$\mathbf{C}^k$ diffeomorphism that conjugates (3.4) to the linear system $\dot{z} = Az + Bv$, then :*

1. *the map $B\chi_{\mathrm{II}} : \Omega \to \mathbb{R}^m$ is of class $\mathbf{C}^k$,*
2. *for any $(x,u) \in \Omega$ in the neighborhood of which the rank of $\partial f/\partial u$ is constant, one has* $\mathrm{Rank}\, \dfrac{\partial f}{\partial u}(x,u) = \mathrm{Rank}\, B$.
3. *for any open subset O of Ω, one has* $\sup\limits_{(x',u')\in O} \mathrm{Rank}\, \dfrac{\partial f}{\partial u}(x',u') = \mathrm{Rank}\, B$.

Proof. Point (1) is direct consequence of (3.42) and the smoothness of χ_{I} and f. To establish (2) and (3), differentiate (3.42) with respect to u to obtain

$$\frac{\partial \chi_{\mathrm{I}}}{\partial x}(x)\frac{\partial f}{\partial u}(x,u) \quad = \quad \frac{\partial (B\chi_{\mathrm{II}})}{\partial u}(x,u) \,. \tag{3.43}$$

Let $\mathcal{V} \subset \Omega$ be open and such that $\mathrm{Rank}\, \partial f/\partial u(x,u) = \rho$ some integer ρ and all $(x,u) \in \mathcal{V}$. Define $\phi : \mathcal{V} \to \mathbb{R}^{n+m}$ by $\phi(x,u) = (\chi_{\mathrm{I}}(x), B\chi_{\mathrm{II}}(x,u))$. On the one hand, since χ_{I} is a diffeomorphism, (3.43) implies that the rank of the Jacobian of ϕ is $n+\rho$, hence, by the constant rank theorem, $\phi(\mathcal{V})$ is a $(n+\rho)$ dimensional immersed sub-manifold of $\mathbb{R}^{n+m}$; on the other hand, since χ is open, $\phi(\mathcal{V})$ is an open subset of the $(n + \mathrm{Rank}\, B)$-dimensional linear range of $I_n \times B$; hence $\rho = \mathrm{Rank}\, B$. This proves point 2, and at the same time point 3 because any $O \subset \Omega$ contains an open subset on which the rank of $\partial f/\partial u$ is constant while (3.43) clearly implies that, for all $(x,u) \in \Omega$, the rank of $\partial f/\partial u(x,u)$ is no larger than $\mathrm{Rank}\, B$. □

Based on Proposition 3.5.3, let us divide the points of Ω into three classes.

– A point $(\bar{x},\bar{u}) \in \Omega$ is called *regular* if it has a neighborhood on which $\partial f/\partial u$ has constant rank. It is easy to see that regular points form an open dense subset of Ω.
– If $(\bar{x},\bar{u})$ is not regular, it is termed *weakly singular* if each neighborhood $O \subset \Omega$ of this point satisfies

$$\sup_{(x,u)\in O} \mathrm{Rank}\, \frac{\partial f}{\partial u}(x,u) = m. \tag{3.44}$$

– A point $(\bar{x},\bar{u}) \in \Omega$ which is neither regular nor weakly singular is said to be *strongly singular*. This means it has a neighborhood $O \subset \Omega$, such that

$$\sup_{(x,u)\in O} \mathrm{Rank}\, \frac{\partial f}{\partial u}(x,u) = m' < m\,, \quad \mathrm{Rank}\, \frac{\partial f}{\partial u}(\bar{x},\bar{u}) < m'\,. \tag{3.45}$$

The distinction between topological and smooth linearizability may now be approached *via* the following theorem that complements Theorem 3.5.2.

Theorem 3.5.4. *Let $k \in \{\infty, \omega\}$ and f be of class $\mathbf{C}^k$ on an open set $\Omega \subset \mathbb{R}^{n+m}$.*

– *System (3.4) is locally $\mathbf{C}^k$ linearizable at $(\bar{x},\bar{u}) \in \Omega$ if, and only if it is locally topologically linearizable at $(\bar{x},\bar{u})$* and *the latter is a regular point.*
– *If (3.4) is locally topologically linearizable at $(\bar{x},\bar{u})$ and the latter is a weakly singular point, then a linearizing homeomorphism around $(\bar{x},\bar{u})$ may be chosen to be a map of class $\mathbf{C}^k$, although not necessarily a $\mathbf{C}^k$ diffeomorphism (its inverse may fail to be $\mathbf{C}^k$).*

Proof. The first assertion is a consequence of Theorem 3.5.2 together with Theorems 3.5.7 and 3.5.8 to come, observing that condition (2′) in the latter will automatically hold at a regular point by the constant rank theorem. Next, assume that $\chi : \Omega \to \Omega'$ is a quasi-$\mathbf{C}^k$ diffeomorphism that conjugates the $\mathbf{C}^k$ system (3.4) to the linear controllable system $\dot{z} = Az + Bv$ at some weakly singular point $(\bar{x},\bar{u})$. By (3) of Proposition 3.5.3, the rank of B is m hence it is left invertible; by (1) of the same proposition, χ_{II} is indeed $\mathbf{C}^k$. □

Whether Theorem 3.5.4 remains true if "weakly singular" gets replaced by "strongly singular" is unknown to the authors. This turns out to be equivalent to the following question in differential topology which is of interest in its own right and seems to have no answer so far.

Open question 3.5.5. *Let O be a neighborhood of the origin in $\mathbb{R}^{p+q}$ and $F : O \to \mathbb{R}^p$ a smooth (resp. real-analytic) map. Suppose $G : O \to \mathbb{R}^q$ is a* continuous *map such that $F \times G : O \to \mathbb{R}^p \times \mathbb{R}^q$ is a local homeomorphism at 0.*

Does there exist another neighborhood $O' \subset O$ of the origin and a smooth *(resp.* real-analytic*) map $H : O' \to \mathbb{R}^q$ such that $F \times H : O' \to \mathbb{R}^p \times \mathbb{R}^q$ is still a local homeomorphism at 0 ?*

If the answer to the open question was yes, then Definitions 3.5.1 and 3.3.9 of quasi-smooth (resp. quasi-analytic) linearizability might equivalently require χ to be smooth (resp. analytic) because, assuming the linear system is in normal form (3.40)-(3.41), one could set $F = \pi_{n+s_1} \circ \chi$ and smoothly (resp. analytically) redefine the last $m - s_1$ components of χ.

If the answer to the open question was no, then Definition 3.5.1 would really be more general than the one obtained by restricting χ to be smooth (resp. analytic). Indeed, if F provides a counterexample to the open question, say, in the $\mathbf{C}^\infty$ case, we may consider on $\mathbb{R}^p \times O$ the control system

$$\dot{x} \;=\; F(u) \;, \qquad x \in \mathbb{R}^p,\; u \in \mathbb{R}^{p+q} \tag{3.46}$$

which is locally quasi-smoothly linearizable at the origin because the local homeomorphism

$$(x, u) \;\mapsto\; (z, v) \;=\; (x, F(u), G(u))$$

conjugates (3.46) to

$$\dot{z} \;=\; B\,v, \qquad \text{with } B = (\,I_p\,|0). \tag{3.47}$$

However, no *smooth* homeomorphism

$$\chi \,:\, (x, u) \;\mapsto\; (z, v) \;=\; (\chi_{\mathrm{I}}(x), \chi_{\mathrm{II}}(x, u))$$

exists that quasi-smoothly linearizes (3.46) at 0 : if this was the case, by Corollary 3.4.4 we may assume up to a linear change of variables that χ conjugates (3.46) to (3.47). Then conjugacy would imply

$$\frac{\partial \chi_{\mathrm{I}}}{\partial x}(x) F(u) \;=\; B\,\chi_{\mathrm{II}}(x, u)$$

whence in particular

$$F(u) \;=\; \left(\frac{\partial \chi_{\mathrm{I}}}{\partial x}(0)\right)^{-1} B\,\chi_{\mathrm{II}}(0, u),$$

and the last q components of $\chi_{\mathrm{II}}(0, u)$ would yield a smooth H such that $F \times H$ is a local homeomorphism at 0 in R^{p+q}, contrary to the assumption.

3.5.2 Geometric characterization of quasi smooth linearization

Let $\mathcal{X}$ and $\mathcal{U}$ be two open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and assume that f is defined on $\mathcal{X} \times \mathcal{U}$. For each $u \in \mathbb{R}^m$, let f_u be the vector field on $\mathcal{X}$ defined by :

$$f_u(x) \;=\; f(x, u). \tag{3.48}$$

Also, for each $(x, u) \in \mathcal{X} \times \mathcal{U}$, we define below a subspace $D(x, u)$, that coincides with the range of the linear mapping $\partial f / \partial u\,(x, u)$ when its dimension is locally constant. First, we consider the subset $\mathcal{L}_{x,u} \subset \mathbb{R}^n$ (not a vector subspace) given by :

$$y \in \mathcal{L}_{x,u} \quad \Leftrightarrow \quad \exists (w_n) \in \mathcal{U}^{\mathbb{N}}, \quad \lim_{n\to\infty} w_n = u \qquad \text{and} \quad \lim_{n\to\infty} \frac{f(x,w_n)-f(x,u)}{\|f(x,w_n)-f(x,u)\|} = y; \tag{3.49}$$

subsequently we put

$$D(x,u) \quad = \quad \mathrm{Span}_{\mathbb{R}}\, \mathcal{L}_{x,u} \ . \tag{3.50}$$

In words, $D(x,u)$ is the vector space spanned by all limit directions of straight lines through $f(x,u)$ and $f(x,u')$ as u' approaches u in $\mathbb{R}^m$; it is of common use in stratified geometry to generalize the notion of tangent space. Note that the set $\mathcal{L}_{x,u}$ depends on the norm used in (3.49), but the subspace $D(x,u)$ does not.

Proposition 3.5.6. *If f is of class $\mathbf{C}^\infty$ and if we denote by $Ran\, L$ the range of a linear map L, we have that*

$$D(x,u) \quad \supset \quad Ran\, \frac{\partial f}{\partial u}(x,u) \tag{3.51}$$

and equality holds at every (x,u) where the rank of $\partial f/\partial u\,(x,u)$ is locally constant with respect to u.

Proof. The inclusion (3.51) holds because any nonzero element of $\mathrm{Ran}\,\partial f/\partial u(x,u)$ can be written $\partial f/\partial u(x,u).h$ for some h in $\mathbb{R}^m$, and one has

$$\frac{\partial f/\partial u\,(x,u).h}{\|\partial f/\partial u\,(x,u).h\|} = \lim_{t\to 0^+} \frac{f(x,u+th)-f(x,u)}{\|f(x,u+th)-f(x,u)\|} \ .$$

Now fix (x,u) and assume that the rank of $\partial f/\partial u$ is locally constant around (x,u), equal to $r \le m$ and use the constant rank-theorem. Up to a permutation of coordinates,

$$(h_1,\ldots,h_m) \overset{\lambda}{\mapsto} (f_1(x,u+h)-f_1(x,u),\ldots,f_r(x,u+h)-f_r(x,u),h_{r+1},\ldots,h_m)$$

is a local diffeomorphism around zero in $\mathbb{R}^m$ and, setting $\rho = \lambda^{-1}\circ z \circ \lambda$ with z given by $z(w_1,\ldots,w_m) = (w_1,\ldots,w_r,0,\ldots,0)$, there is a constant c such that

$$\|\rho(h)\| \le c\,\|f(x,u+h)-f(x,u)\| \ \text{ and } \ f(x,u+\rho(h)) = f(x,u+h) \tag{3.52}$$

for all h. Take $y \in \mathcal{L}_{x,u}$; by definition, there is a sequence (h_n) converging to zero and satisfying (3.49) with $w_n = u + h_n$; from (3.52), we may re-write it as

$$y = \lim_{n\to\infty} \frac{f(x,u+\rho(h_n))-f(x,u)}{\|\rho(h_n)\|} \, \frac{\|\rho(h_n)\|}{\|f(x,u+h_n)-f(x,u)\|} \tag{3.53}$$

where both ratios are bounded ; extracting a sequence such that both converge, the limit of the first ratio is, by definition of the derivative, $\frac{\partial f}{\partial u}(x,u).h$ with h a limit point of $\rho(h_n)/\|\rho(h_n)\|$; hence $y \in \mathrm{Ran}\,\partial f/\partial u\,(x,u)$. We have proved that $\mathcal{L}_{x,u} \subset \mathrm{Ran}\,\partial f/\partial u\,(x,u)$. From (3.50), this implies the reverse inclusion of (3.51) because the left-hand side is a linear subspace. □

We can now characterize smooth (resp. analytic) and quasi-smooth (resp. quasi-analytic) linearizability in parallel. The proofs are given pages 78 through 81.

Theorem 3.5.7 (smooth or analytic linearizability). *Let $k \in \{\infty, \omega\}$ and f be of class $\mathbf{C}^k$ on an open set $\Omega \subset \mathbb{R}^{n+m}$. The control system (3.4) is locally $\mathbf{C}^k$ linearizable at $(\bar{x}, \bar{u}) \in \Omega$ if, and only if there are open neighborhoods $\mathcal{X}$ and $\mathcal{U}$ of $\bar{x}$ and $\bar{u}$ in $\mathbb{R}^n$ and $\mathbb{R}^m$, with $\mathcal{X} \times \mathcal{U} \subset \Omega$, such that the following conditions are satisfied.*

1. *$D(x, u)$ does not depend on u for $(x, u) \in \mathcal{X} \times \mathcal{U}$.*
2. *The rank of $\dfrac{\partial f}{\partial u}(x, u)$ is constant in $\mathcal{X} \times \mathcal{U}$.*
3. *Defining on $\mathcal{X}$ the distribution Δ_0 by $\Delta_0(x) = D(x, u)$ — this is possible if point (1) holds true — and inductively the flag of distributions (Δ_k) by :*

$$\Delta_{k+1} \;=\; \Delta_k \;+\; [\, f_{\bar{u}} \,,\, \Delta_k \,] \tag{3.54}$$

where $[\;,\;]$ denotes the Lie bracket, then each Δ_k for $0 \leq k \leq n-1$ is integrable (i.e. has constant dimension over $\mathbb{R}$ and is closed under Lie bracket) and the rank of Δ_{n-1} is n.

Theorem 3.5.8 (quasi-smooth or quasi-analytic linearizability). *Let $k \in \{\infty, \omega\}$ and f be of class $\mathbf{C}^k$ on an open set $\Omega \subset \mathbb{R}^{n+m}$. The control system (3.4) is locally quasi-$\mathbf{C}^k$ linearizable at $(\bar{x}, \bar{u}) \in \Omega$ if, and only if there are open neighborhoods $\mathcal{X}$ and $\mathcal{U}$ of $\bar{x}$ and $\bar{u}$ in $\mathbb{R}^n$ and $\mathbb{R}^m$, with $\mathcal{X} \times \mathcal{U} \subset \Omega$, such that conditions (1) and (3) of Theorem 3.5.7 are met and, instead of condition (2), it holds that*

$2'$. *Denoting by $r_1 \leq m$ the constant rank of Δ_0, the mapping*

$$\begin{array}{rcccc} F & : & \mathcal{X} \times \mathcal{U} & \to & \mathcal{X} \times \mathbb{R}^n \\ & & (x, u) & \mapsto & (\, x \,,\, f(x, u) \,) \end{array} \tag{3.55}$$

restricts to a $\mathbf{C}^0$ fibration[4] $\mathcal{W} \to F(\mathcal{W})$ with fiber $\mathbb{R}^{m-r_1}$ on some neighborhood $\mathcal{W}$ of $(\bar{x}, \bar{u})$ in $\mathcal{X} \times \mathcal{U}$.

Theorem 3.5.7 is of course equivalent to the results in [57, 50, 103], but the conditions are stated here in a slightly different form to parallel Theorem 3.5.8.

Corollary 3.5.9. *Assume that f is real analytic on some open set $\Omega \subset \mathbb{R}^{n+m}$. If the control system (3.4) is locally $\mathbf{C}^\infty$ (resp. quasi-$\mathbf{C}^\infty$) linearizable at $(\bar{x}, \bar{u}) \in \Omega$, then it is also $\mathbf{C}^\omega$ (resp. quasi-$\mathbf{C}^\omega$) linearizable there.*

Proof. analyticity does not appear in the conditions of the theorems, except for the regularity of f itself. □

3.5.3 Linearization versus equivalence to the linear approximation

For a control system, *smooth* linearizability at an equilibrium implies conjugacy *to its linear approximation* :

Proposition 3.5.10. *Let $(\bar{x}, \bar{u})$ be an equilibrium point of (3.4), i.e. $f(\bar{x}, \bar{u}) = 0$, and let $\overline{A} = \partial f / \partial x\, (\bar{x}, \bar{u})$, $\overline{B} = \partial f / \partial u\, (\bar{x}, \bar{u})$ so that :*

$$f(x, u) \;=\; \overline{A}\,(x - \bar{x}) \;+\; \overline{B}\,(u - \bar{u}) \;+\; \varepsilon(x - \bar{x}, u - \bar{u})\,, \tag{3.56}$$

where ε is little $o(\|x - \bar{x}\| + \|u - \bar{u}\|)$.

[4] A $\mathbf{C}^0$ fibration with fiber $\mathcal{F}$ over $\mathcal{B}$ is a continuous map $g : \mathcal{E} \to \mathcal{B}$ for which every $\xi \in \mathcal{B}$ has a neighborhood $\mathcal{O}$ in $\mathcal{B}$ such that $g^{-1}(\mathcal{O}) \subset \mathcal{E}$ is homeomorphic to $\mathcal{O} \times \mathcal{F}$, the so-called *trivializing* homeomorphism $\psi : g^{-1}(\mathcal{O}) \to \mathcal{O} \times \mathcal{F}$ being such that $\pi \circ \psi = g$ where $\pi : \mathcal{O} \times \mathcal{F} \to \mathcal{O}$ is the natural projection onto the first factor.

If system (3.4) is locally smoothly linearizable at $(\bar{x}, \bar{u})$, then :
1. its linear approximation $(\overline{A}, \overline{B})$ is controllable (cf. Definition 3.4.2),
2. the system is smoothly conjugate to $(\overline{A}, \overline{B})$ at $(\bar{x}, \bar{u})$.

Proof. Let χ be a local diffeomorphism conjugating system (3.4) to $\dot{z} = Az + Bv$ at $(\bar{x}, \bar{u})$, and observe from (3.14) in Remark 3.3.8 that smooth linearizability translates into (3.42). If we write f as in (3.56), and if we set $\overline{P} = \frac{\partial \chi_{\mathrm{I}}}{\partial x}(\bar{x})$, $\overline{K} = \frac{\partial \chi_{\mathrm{II}}}{\partial x}(\bar{x}, \bar{u})$, $\overline{Q} = \frac{\partial \chi_{\mathrm{II}}}{\partial u}(\bar{x}, \bar{u})$, we get by differentiating (3.42) with respect to x and u at $(\bar{x}, \bar{u})$, using the relation $f(\bar{x}, \bar{u}) = 0$, that

$$\overline{P}\,\overline{A} \;=\; A\overline{P} \,+\, B\overline{K} \;\;, \quad \overline{P}\,\overline{B} \;=\; B\overline{Q}\,.$$

Since $\overline{P}$ and $\overline{Q}$ are square invertible matrices by the triangular structure of χ displayed in (3.13), this implies that the linear systems (A, B) and $(\overline{A}, \overline{B})$ are linearly conjugate, see (3.35). Since (A, B) is controllable by definition so is $(\overline{A}, \overline{B})$, thereby achieving the proof. □

Proposition 3.5.10 has no analog if the control system is only topologically linearizable (hence *quasi-smoothly* linearizable according to Theorem 3.5.2). For example, the system (3.1) in the introduction is quasi-$\mathbf{C}^{\omega}$linearizable at $(0, 0)$, but its linear approximation $\dot{x} = 0$ is not controllable and it is not topologically equivalent to $\dot{x} = 0$. Apart from such degenerate cases, there also exist systems that are quasi-analytically linearizable at some point with controllable linear approximation there, and still they are not conjugate to this linear approximation. An example when $m = n = 2$ is given by :

$$\dot{x}_1 \;=\; u_1 \;\;, \quad \dot{x}_2 \;=\; x_1 \,+\, u_2^3 \;,$$

This system is quasi-analytically conjugate at $(0, 0)$ to

$$\dot{z}_1 \;=\; v_1 \;\;, \quad \dot{z}_2 \;=\; v_2 \;, \tag{3.57}$$

via $z = x$, $v_1 = u_1$, $v_2 = u_2^3 \,+\, x_1$. However, its linear approximation at the origin is $\dot{x}_1 = u_1$, $\dot{x}_2 \;=\; x_1$, which is controllable yet not conjugate to (3.57) (cf. Theorem 3.4.3).

3.5.4 Non-genericity of linearizability

Except when $m \geq n$ or $(n, m) = (2, 1)$, the conditions of Theorem 3.5.7 require a certain number of *equalities* (involving f and its partial derivatives) to hold *everywhere.* For example, the integrability of a distribution entails that all Lie brackets be linearly dependent on the original vector fields, *i.e.* certain determinants must be identically zero. This makes smooth (resp. analytic) linearizability of a smooth (resp. analytic) control system highly non-generic in any reasonable sense, because when written in proper jet spaces it is contained in a set of infinite co-dimension. Moreover, small perturbations of a system that does not satisfy these condition will not satisfy them either, while most perturbations of a system which satisfies them will fail to do so. Compare for instance [102] where it is shown that the equivalence class of any system affine in the control has infinite co-dimension in some Whitney topology.

From Theorem 3.5.4, quasi-smooth or quasi-analytic linearizability, hence also topological linearizability by Theorem 3.5.2, require the same equalities to hold on an open dense set, although this time some singularities are allowed. This is no more "generic" than smooth linearizability, as opposed to ODE's for which the Grobman-Hartman theorem allows one to linearize around an equilibrium as soon as it is hyperbolic.

3.6 Proofs

Proof of Theorems 3.5.7 and 3.5.8

We begin with a lemma whose cumbersome index arrangement will be rewarded later when constructing the Kronecker indices of the linearized system.

Lemma 3.6.1. *Let $k \in \{\infty, \omega\}$. Let Δ_0 and $f_{\bar{u}}$ be respectively a distribution and a vector field, both of class $\mathbf{C}^k$ on a connected open neighborhood of $x \in \mathbb{R}^n$. Let further Δ_i, $i \geq 0$, be the distributions defined according to (3.54), and set for convenience $\Delta_{-1} = \{0\}$. Assume they satisfy point (3) of Theorem 3.5.7 or 3.5.8. Put*

$$r_i \;=\; \text{Rank}\Delta_{i-1}\,,\;\; i > 0, \qquad r_0 = 0\,,\;\; r_{-1} = -m\,, \tag{3.58}$$

so that $r_i = n$ for some $i \leq n-1$; let $\rho \in \{3, \ldots, n+1\}$ be the smallest integer such that $r_{\rho-1} = n$. Define also

$$s_i = r_i - r_{i-1}\,, \quad \sigma_i = \sum_{j=i}^{\rho} s_j = n - r_{i-1}\,, \quad 0 \leq i \leq \rho \tag{3.59}$$

(note that $s_\rho = \sigma_\rho = 0$).

Then, there exists coordinates $\chi_1, \ldots, \chi_n$ of class $\mathbf{C}^k$ on a neighborhood $\mathcal{X}$ of x such that

– $\chi_1, \ldots, \chi_{\sigma_i}$ are independent first integrals of Δ_{i-2} for $i \in \{1, \ldots, \rho-1\}$,

– $\chi_{\sigma_i + j} = f_{\bar{u}} \chi_{\sigma_{i+1}+j}$ for all integers i, j, $2 \leq i \leq \rho - 1$, $1 \leq j \leq s_i$

($f_{\bar{u}} \chi_{\sigma_{i+1}+j}$ is the Lie derivative of the function $\chi_{\sigma_{i+1}+j}$ along the vector field $f_{\bar{u}}$).

Proof. Note that when $i = 1$, the first point above means that $\chi_1, \ldots, \chi_n$ are indeed local coordinates. Now, the Frobenius theorem provides us with $n - r$ independent $\mathbf{C}^k$ first integrals for a $\mathbf{C}^k$ integrable distribution of rank r. This accounts for the regularity of the coordinates if we construct them as follows.

First pick $n - r_{\rho-2} = \sigma_{\rho-1}$ independent first integrals of $\Delta_{\rho-3}$ and call them $\chi_1, \ldots, \chi_{\sigma_{\rho-1}}$; define further $\chi_{1+\sigma_{\rho-1}}, \ldots, \chi_{2\sigma_{\rho-1}}$ by $\chi_{\sigma_{\rho-1}+j} = f_{\bar{u}} \chi_j$ for $1 \leq j \leq \sigma_{\rho-1} = s_{\rho-1}$. Clearly, $\chi_1, \ldots, \chi_{\sigma_{\rho-1}+s_{\rho-1}}$ satisfy the conditions for $i = \rho - 1$. Then proceed inductively : assume that, for some $i_0 \in \{2, \ldots, \rho-1\}$, the functions $\chi_1, \ldots, \chi_{\sigma_{i_0}+s_{i_0}}$ have been constructed and satisfy the conditions for $i \geq i_0$. *We claim* that the differentials $\mathrm{d}\chi_\ell$ are linearly independent at each point of $\mathcal{X}$. Indeed, assume that there is $\bar{x} \in \mathcal{X}$ and real coefficients μ_j and λ_k such that

$$\sum_{j=1}^{\sigma_{i_0}} \mu_j \mathrm{d}\chi_j(\bar{x}) + \sum_{k=1+\sigma_{i_0+1}}^{\sigma_{i_0}} \lambda_k \mathrm{d}\left(f_{\bar{u}} \chi_k\right)(\bar{x}) = 0. \tag{3.60}$$

Put $\omega_1 = \Sigma \mu_j \mathrm{d}\chi_j$ and $\omega_2 = \Sigma \lambda_k \mathrm{d}\chi_k$. Since d commutes with the Lie derivative, we may rewrite (3.60) as $\omega_1(\bar{x}) + f_{\bar{u}} \omega_2(\bar{x}) = 0$. In particular, for any $\mathbf{C}^k$-vector field X in Δ_{i_0-2}, we get as $\omega_1(X) \equiv 0$ that $f_{\bar{u}} \omega_2(X)(\bar{x}) = 0$. Now, by virtue of the formula

$$f_{\bar{u}}\left(\omega_2(X)\right) = f_{\bar{u}} \omega_2(X) + \omega_2([f_{\bar{u}}, X]), \tag{3.61}$$

we obtain since $\omega_2(X) \equiv 0$ that $\omega_2([f_{\bar{u}}, X])(\bar{x}) = 0$, that is, ω_2 annihilates Δ_{i_0-1} at $\bar{x}$. But $\mathrm{d}\chi_1(\bar{x}), \ldots, \mathrm{d}\chi_{\sigma_{i_0+1}}(\bar{x})$ are a basis of the orthogonal space to $\Delta_{i_0-1}(\bar{x})$ by the induction hypothesis, whereas $\omega_2(\bar{x})$ is a linear combination of the $\mathrm{d}\chi_k(\bar{x})$ for $\sigma_{i_0+1} < k \leq \sigma_{i_0}$. Therefore, since we know by the induction hypothesis that the $\mathrm{d}\chi_\ell$ are point-wise independent for $1 \leq \ell \leq \sigma_{i_0}$, we get that the λ_k are zero and then the μ_j are also zero by (3.60). *This proves* *the claim.* Next, recall that $\chi_1,\dots,\chi_{\sigma_{i_0}}$ are first integrals of Δ_{i_0-2}, thus *a fortiori* of Δ_{i_0-3}. For X a $\mathbf{C}^k$-vector field in the latter we deduce from (3.61), where ω_2 is replaced by $d\chi_\ell$ with $1+\sigma_{i_0+1}\leq\ell\leq\sigma_{i_0}$, that $\chi_{1+\sigma_{i_0}},\dots,\chi_{\sigma_{i_0}+s_{i_0}}$ are also first integrals of Δ_{i_0-3}. In case $\sigma_{i_0}+s_{i_0}<\sigma_{i_0-1}$, pick $\chi_{\sigma_{i_0}+s_{i_0}+1},\dots,\chi_{\sigma_{i_0-1}}$ so that χ_ℓ for $1\leq\ell\leq\sigma_{i_0-1}$ is a complete set of independent integrals of Δ_{i_0-3}. If $i_0=2$ we are done, otherwise define $\chi_{\sigma_{i_0-1}+j}=f_{\bar u}\chi_{\sigma_{i_0}+j}$ for $1\leq j\leq s_{i_0-1}$ in order to complete the induction step. □

Proof of Theorems 3.5.7 and 3.5.8. The two proofs run parallel to each other.

We first show necessity, assuming that $k=\infty$ for analyticity does not appear in the conclusions. Assume local (quasi) smooth linearizability, *cf.* Definitions 3.5.1 and 3.3.9. Without loss of generality, we assume that $\Omega=\mathcal{X}\times\mathcal{U}$ where $\mathcal{X}$ and $\mathcal{U}$ are open neighborhoods of $\bar x$ and $\bar u$ in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let $\chi:\mathcal{X}\times\mathcal{U}\to\Omega'\subset\mathbb{R}^{n+m}$ be as in (3.13) ; recall that χ_{I} is a smooth diffeomorphism $\mathcal{X}\to\chi_{\mathrm{I}}(\mathcal{X})$. We may also assume, after composing χ with a linear invertible map, that the pair (A,B) is in canonical form (3.40)-(3.41), but we still write A,B rather than A_c,B_c. Denote by $B_0,\dots,B_m$ the columns of B and define the vector fields $b_0,\dots,b_m$ on $\mathbb{R}^n$ by

$$b_i(z)=B_i\,,\ 1\leq i\leq m\,,\qquad b_0(z)=Az+B\bar u \tag{3.62}$$

and the distributions Λ_i by

$$\Lambda_0(z)=\mathrm{Span}_{\mathbb{R}}\{b_1(z),\dots,b_m(z)\}=\mathrm{Ran}B\qquad\Lambda_{i+1}=\Lambda_i+[b_0,\Lambda_i]\,,\ 1\leq i\leq m\,. \tag{3.63}$$

From (3.42), we have

$$\frac{\partial\chi_{\mathrm{I}}}{\partial x}(x)\,f(x,u)\ =\ b_0(\chi_{\mathrm{I}}(x))\ +\ B\left(\chi_{\mathrm{II}}(x,u)-\chi_{\mathrm{II}}(x,\bar u)\right). \tag{3.64}$$

Since χ is a triangular homeomorphism, $\chi_{\mathrm{II}}(x,w)-\chi_{\mathrm{II}}(x,u)$ covers an open neighborhood of 0 in $\mathbb{R}^m$ when w ranges around u in $\mathbb{R}^m$. Thus, in view of (3.64), $\mathcal{L}_{x,u}$ defined by (3.49) contains an open set in $\left(\frac{\partial\chi_{\mathrm{I}}}{\partial x}(x)\right)^{-1}\mathrm{Ran}B$, and by double inclusion

$$D(x,u)\ =\ \left(\frac{\partial\chi_{\mathrm{I}}}{\partial x}(x)\right)^{-1}\mathrm{Ran}B.$$

This proves point 1, and also proves that the distribution Δ_0 in point 3 is the pullback of Λ_0 by the diffeomorphism χ_{I}, *i.e.* $(\chi_{\mathrm{I}})_*\,\Delta_0=\Lambda_0$. Since (3.64) also implies $(\chi_{\mathrm{I}})_*\,f_{\bar u}=b_0$, we have $(\chi_{\mathrm{I}})_*\,\Delta_i=\Lambda_i$ for all i. This gives point 3 because it is obviously true with Λ_i instead of Δ_i, and integrability and ranks are preserved by conjugation with the smooth diffeomorphism χ_{I}. In the case of smooth linearizability, point 2 is easily obtained by differentiating (3.64) with respect to u and using invertibility of $\partial\chi_{\mathrm{II}}/\partial u(x,u)$.

To conclude the proof of necessity, let us prove point $2'$ in the case of quasi-smooth linearizability. Let

$$\mathcal{M}=\{\ \ (x,y)\in\mathcal{X}\times\mathbb{R}^n;\ \ \frac{\partial\chi_{\mathrm{I}}}{\partial x}(x)\,y\ -\ A\chi_{\mathrm{I}}(x)\ \ \in\ \ \mathrm{Ran}\,B\ \ \}\,.$$

This is a smooth embedded sub-manifold of $\mathcal{X}\times\mathbb{R}^n$ of dimension $n+r_1$, where $r_1=\mathrm{Rank}\,B\leq m$. If we define F as in (3.55), it is clear from (3.42) that

$$F(\mathcal{X}\times\mathcal{U})\ \subset\ \mathcal{M}\,.$$

Now, take some $(m-r_1)\times m$ matrix C whose rows complement r_1 independent rows of B into a basis of $\mathbb{R}^m$. Pick matrices E_1 and E_2 of appropriate sizes such that

$$E_1\,B\ +\ E_2\,C\ \ =\ \ I_m\,.$$

By (3.42) we get

$$E_1 \left[\frac{\partial \chi_{\mathrm{I}}}{\partial x}(x)\, f(x,u) \;-\; A\, \chi_{\mathrm{I}}(x) \right] \;+\; E_2 C \chi_{\mathrm{II}}(x,u) \;=\; \chi_{\mathrm{II}}(x,u)\;. \tag{3.65}$$

Define

$$\psi : \mathcal{X} \times \mathcal{U} \to \mathcal{M} \times \mathbb{R}^{m-r_1}$$

by the formula :

$$\psi(x,u) = (x, f(x,u), C\chi_{\mathrm{II}}(x,u)).$$

From (3.65), this mapping has an inverse given by

$$\begin{array}{rccl} \psi^{-1}: & \psi(\mathcal{X}\times\mathcal{U}) & \rightarrow & \mathcal{X}\times\mathcal{U} \\ & (x,y,z) & \mapsto & \chi^{-1}\left(\chi_{\mathrm{I}}(x)\;,\; E_1 \left[\dfrac{\partial \chi_{\mathrm{I}}}{\partial x}(x)\, y \;-\; A\,\chi_{\mathrm{I}}(x)\right] \;+\; E_2\, z \right) \end{array}$$

so that ψ defines a homeomorphism from $\mathcal{X} \times \mathcal{U}$ onto its image which is open in $\mathcal{M} \times \mathbb{R}^{m-r_1}$ by invariance of the domain. Let $\mathcal{O}$ be a neighborhood of $(\bar{x}, f(\bar{x},\bar{u}))$ in $\mathcal{M}$ and $\mathcal{S}$ an open ball centered at $C\chi_{\mathrm{II}}(\bar{x},\bar{u})$ in $\mathbb{R}^{m-r_1}$ such that $\mathcal{O} \times \mathcal{S} \subset \psi(\mathcal{X} \times \mathcal{U})$, and take $\mathcal{W} = \psi^{-1}(\mathcal{O} \times \mathcal{S})$. Then $F : \mathcal{W} \to F(\mathcal{W}) = \mathcal{O}$ is a $\mathbf{C}^0$ fibration with fiber $\mathcal{S}$ and trivializing homeomorphism $\psi : \mathcal{W} \to \mathcal{O} \times \mathcal{S}$. Since $\mathcal{S}$ is homeomorphic to $\mathbb{R}^{m-r_1}$, condition $2'$ follows.

We turn to sufficiency. Points 1, 3, and either 2 or $2'$ imply, for all $x \in \mathcal{X}$,

$$\Delta_0(x) \;=\; \mathrm{Span}_{\mathbb{R}}\{f(x,w) - f(x,u), (u,w) \in \mathcal{U} \times \mathcal{U}\}\;. \tag{3.66}$$

Indeed the right-hand side always contains $D(x,u)$ because it contains all the differences $f(x,w_n) - f(x,u)$ in (3.49), and point 1 implies the reverse inclusion because $f(x,w) - f(x,u)$ can be computed as the integral on the segment $[u,w] \subset \mathcal{U}$ of a function that, thanks to Proposition 3.5.6, belongs constantly to $\Delta_0(x)$.

From (3.66), the distribution Δ_0 is of class $\mathbf{C}^k$. Considering point 3, we may apply Lemma 3.6.1. We thus obtain some, with r_i, s_i and σ_i the integers defined by (3.58) and (3.59), some $\mathbf{C}^k$ coordinates $\chi_1, \ldots, \chi_n$ on a neighborhood of $\bar{x}$ possibly smaller than $\mathcal{X}$ (but that we continue to denote by $\mathcal{X}$), i.e. a diffeomorphism $\chi_{\mathrm{I}} : \mathcal{X} \to \chi_{\mathrm{I}}(\mathcal{X})$, with $\chi_{\mathrm{I}} = (\chi_1, \ldots, \chi_n)$, meeting the conclusions of Lemma 3.6.1. In particular, $\chi_1, \ldots, \chi_{n-r_1}$ are first integrals of the distribution Δ_0, and from (3.66), this implies that $\partial\chi_i/\partial x(x)\, f(x,u)$ does not depend on u, and is there fore equal to its value for $u = \bar{u}$:

$$\frac{\partial \chi_i}{\partial x}(x) f(x,u) = f_{\bar{u}}\chi_i\,(x)\,, \quad 1 \le i \le n - r_1\;. \tag{3.67}$$

For larger i, the left-hand side depends on x and u : define $\lambda : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^{m_1}$ by

$$\lambda(x,u) \;=\; (\;\frac{\partial \chi_{n-r_1+1}}{\partial x}(x) f(x,u)\;, \;\ldots\;,\; \frac{\partial \chi_n}{\partial x}(x) f(x,u)\;)\;. \tag{3.68}$$

Then, defining coordinates $z_1, \ldots, z_n$ by $z = \chi_{\mathrm{I}}(x)$. The equations of system (3.4) are as follows (the first line gives the derivatives of the $n - r_1$ first coordinates and the second line the last r_1 ones) :

$$\begin{array}{rcll} \dot{z}_{\sigma_{i+1}+j} & = & z_{\sigma_i+j}\,, & 2 \le i \le \rho - 1\,,\; 1 \le j \le s_i\,, \\ \dot{z}_\ell & = & \lambda_{n-\ell}(\,\chi_{\mathrm{I}}^{-1}(z)\,,\,u\,)\,, & n - m_1 + 1 \le \ell \le n\;. \end{array} \tag{3.69}$$

If point 2 is satisfied, the rank of the map $(x,u) \mapsto (\chi_{\mathrm{I}}(x), \frac{\partial \chi_{\mathrm{I}}}{\partial (x)} f(x,u))$ is constant and thus, according to (3.66), it is equal to $n + r_1$, r_1 being the rank of Δ_0. From (3.67), the map

$(x,u) \mapsto (\chi_{\mathrm{I}}(x), \lambda(x,u))$ has the same constant rank $n+r_1$. Hence there exists $\phi\ : \mathcal{X}\times\mathcal{U} \to \mathbb{R}^{m-r_1}$ such that

$$(x,u) \mapsto (\, \chi_{\mathrm{I}}(x)\, ,\, \lambda(x,u)\, ,\, \phi(x,u)\,) \tag{3.70}$$

is a diffeomorphism of class $\mathbf{C}^k$. Obviously, defining χ_{II} by $\chi_{\mathrm{II}}(x,u) = (\lambda(x,u), \phi(x,u))$ yields a $\mathbf{C}^k$ diffeomorphism χ that conjugates (3.4) to a linear controllable system $\dot{z} = Az + Bu$. This proves sufficiency in Theorem 3.5.7.

If point $2'$ is satisfied instead, let $\psi : \mathcal{W} \to F(\mathcal{W}) \times \mathbb{R}^{n-r_1}$ be the "trivializing" homeomorphism. Recall that, with $\pi : F(\mathcal{W}) \times \mathbb{R}^{n-r_1} \to F(\mathcal{W})$ the natural projection, one has $\pi \circ \psi = F$; call $\phi : \mathcal{W} \to \mathbb{R}^{n-r_1}$ the map such that $\psi = F \times \phi$ Composing F with $(x,\xi) \mapsto (\chi_{\mathrm{I}}(x), \frac{\partial \chi_{\mathrm{I}}}{\partial(x)}\xi)$, one gets that $(x,u) \mapsto \chi(x,u) = (\chi_{\mathrm{I}}(x), \lambda(x,u), \phi(x,u))$ is a homeomorphism. It clearly conjugates (3.4) to a linear controllable system $\dot{z} = Az + Bu$. This proves sufficiency in Theorem 3.5.8. □

Proof of Theorem 3.5.2

This theorem for $k = \omega$ is consequence of this theorem for $k = \infty$ and of Corollary 3.5.9. Hence we only have to prove it for $k = \infty$, *i.e.* we assume that f is infinitely differentiable and we prove that topological linearizability implies quasi-$\mathbf{C}^\infty$linearizability.

Without loss of generality, we suppose that $(\bar{x}, \bar{u}) = (0,0)$. Assume there exists a homeomorphism χ from a neighborhood of the origin in $\mathbb{R}^{n+m}$ to an open subset of $\mathbb{R}^{n+m}$ that conjugates system (3.4) to the linear controllable system

$$\dot{z} \;=\; Az \;+\; Bv \tag{3.71}$$

with $z \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$. Composing χ with a linear invertible map allows us to suppose that the pair (A,B) is in canonical form (3.40)-(3.41), i.e. that (3.71) can be read

$$\dot{z}_{\sigma_i+k} \;=\; z_{\sigma_{i-1}+k}\;, \quad 2 \le i \le \rho,\ 1 \le k \le s_{i-1}, \tag{3.72}$$

where the integers s_i and σ_i were defined in (3.38) and (3.39) and where, for notational compactness, we have set :

$$z_{n+k} \;\triangleq\; v_k\;; \tag{3.73}$$

recall here that $s_0 = m$, and notice that $s_1 < m$ may well occur as it simply means that $\operatorname{Rank} B < m$, in which case some of the controls do not appear in the canonical form. With the aggregate notation :

$$Z_j \;\triangleq\; \begin{pmatrix} z_{\sigma_{j+1}+1} \\ \vdots \\ z_{\sigma_j} \end{pmatrix},\ 1 \le j \le \rho-1\;, \qquad Z_0 \;\triangleq\; \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}\;, \tag{3.74}$$

and the matrices J^s_r defined in (3.41), system (3.72) can be rewritten as

$$\begin{array}{rcl} \dot{Z}_{\rho-1} &=& J^{s_{\rho-1}}_{s_{\rho-2}} Z_{\rho-2} \\ \dot{Z}_{\rho-2} &=& J^{s_{\rho-2}}_{s_{\rho-3}} Z_{\rho-3} \\ &\vdots& \\ \dot{Z}_2 &=& J^{s_2}_{s_1} Z_1 \\ \dot{Z}_1 &=& J^{s_1}_{s_0} Z_0 \end{array} \tag{3.75}$$

and is viewed as a control system with state $(Z_{\rho-1}, \ldots, Z_1)$ and control Z_0. We also make the convention, similar to (3.73), that

$$x_{n+k} \;\triangleq\; u_k\;, \tag{3.76}$$

and we use for the controls the aggregate notation :

$$X_0 \triangleq \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} . \tag{3.77}$$

Let us now prove that property $\mathcal{P}_\ell$ below is true for $0 \le \ell \le \rho - 1$.

Property $\mathcal{P}_\ell$ ***:*** *there exists a smooth local change of coordinates around* 0 *in* $\mathbb{R}^n$*, say*

$$(x_1, \ldots, x_n) \ \mapsto \ (\widehat{X}, X_\ell, \ldots, X_2, X_1),$$

with $\widehat{X} \in \mathbb{R}^{\sigma_{\ell+1}}$ *and* $X_i \in \mathbb{R}^{s_i}$ *for* $0 \le i \le \ell$ *(if* $\ell = 0$ *there are no* X_i*'s beyond* X_0 *whereas if* $\ell = \rho - 1$ *there is no* $\widehat{X}$*), after which system* (3.4) *reads :*

$$\begin{array}{rcl} \dot{\widehat{X}} &=& \widehat{F}(\widehat{X}, X_\ell) \\ \dot{X}_\ell &=& F_\ell(\widehat{X}, X_\ell, X_{\ell-1}) \\ \vdots && \\ \dot{X}_2 &=& F_2(\widehat{X}, X_\ell, \ldots, X_1) \\ \dot{X}_1 &=& F_1(\widehat{X}, X_\ell, \ldots, X_1, X_0) \end{array} , \tag{3.78}$$

and such that (3.78)*, viewed as a control system with state* $(\widehat{X}, X_\ell, \ldots, X_1)$ *and control* X_0*, is locally topologically conjugate at* $(0,0)$ *to system* (3.75) *via a local homeomorphism*

$$(\widehat{X}, X_\ell, \ldots, X_1, X_0) \mapsto (Z_{\rho-1}, \ldots, Z_0)$$

which is, together with its inverse, of the block triangular form :

$$\begin{array}{lcl} (Z_{\rho-1}, \ldots, Z_{\ell+1}) &=& \widehat{\Phi}(\widehat{X}) \\ Z_\ell &=& \Phi_\ell(\widehat{X}, X_\ell) \\ \vdots && \\ Z_1 &=& \Phi_1(\widehat{X}, X_\ell, \ldots, X_1) \\ Z_0 &=& \Phi_0(\widehat{X}, X_\ell, \ldots, X_1, X_0) \end{array} \qquad \begin{array}{lcl} \widehat{X} &=& \widehat{\Psi}(Z_{\rho-1}, \ldots, Z_{\ell+1}) \\ X_\ell &=& \Psi_\ell(Z_{\rho-1}, \ldots, Z_\ell) \\ \vdots && \\ X_1 &=& \Psi_1(Z_{\rho-1}, \ldots, Z_1) \\ X_0 &=& \Psi_0(Z_{\rho-1}, \ldots, Z_1, Z_0) \end{array}$$

where Φ_i *and* Ψ_i *are, for* $1 \le i \le \ell$*, continuously differentiable with respect to* X_i *and* Z_i *respectively, have an invertible derivative, and satisfy for* $1 \le i \le \ell$ *the relation :*

$$\begin{array}{rcl} F_i(\widehat{X}, X_\ell, \ldots, X_i, X_{i-1}) &=& F_i(\widehat{X}, X_\ell, \ldots, X_i, 0) \\ + \ \left(\frac{\partial \Phi_i}{\partial X_i}(\widehat{X}, X_\ell, \ldots, X_i)\right)^{-1} J^{s_i}_{s_{i-1}} \Big(&& \\ && \Phi_{i-1}(\widehat{X}, X_\ell, \ldots, X_i, X_{i-1}) - \Phi_{i-1}(\widehat{X}, X_\ell, \ldots, X_i, 0)\Big) \ ; \end{array} \tag{3.79}$$

furthermore, the partial homeomorphism

$$(\widehat{X}, X_\ell) \ \mapsto \ (Z_{\rho-1}, \ldots, Z_\ell) \tag{3.80}$$

locally topologically conjugates, at $(0,0) \in \mathbb{R}^{\sigma_{\ell+1}+s_\ell}$*, the reduced control system*

$$\dot{\widehat{X}} \ = \ \widehat{F}(\widehat{X}, X_\ell), \tag{3.81}$$

with state $\widehat{X}$ *and input* X_ℓ*, to the reduced linear control system*

$$\begin{array}{rcl} \dot{Z}_{\rho-1} & = & J^{s_{\rho-1}}_{s_{\rho-2}} Z_{\rho-2} \,, \\ & \vdots & \\ \dot{Z}_{\ell+1} & = & J^{s_{\ell+1}}_{s_\ell} Z_\ell \end{array} \tag{3.82}$$

wit state $(Z_{\rho-1}, \ldots, Z_{\ell+1})$ *and input* Z_ℓ*.*

Indeed, $\mathcal{P}_0$ is merely the original assumption on local topological conjugacy of systems (3.4) and (3.75), where the triangular structure (3.13) of the conjugating homeomorphism was taken into account ; note that, in $\mathcal{P}_0$, (3.79) is empty and that the reduced system (3.81) is the original system. Next, supposing that $\mathcal{P}_\ell$ holds for some $\ell \geq 0$, we apply Lemmas 3.6.2 and 3.6.3 (see below) to the reduced systems (3.81), (3.82), and to the partial homeomorphism (3.80), with

$$d = \sigma_{\ell+1}, \; r = s_\ell, \; s = s_{\ell+1}, \; U = X_\ell, \; (x_1, \ldots, x_d) = \widehat{X},$$
$$Z^1 = (Z_{\rho-1}, \ldots, Z_{\ell+2}), \; Z^2 = Z_{\ell+1}, \text{ and } V = Z_\ell,$$

and then, upon renaming $\widetilde{X}^2$ as $X_{\ell+1}$, $\widetilde{f}^2$ as $F_{\ell+1}$, and choosing $\widetilde{X}^1$ to be the new $\widehat{X}$, we get $\mathcal{P}_{\ell+1}$.

Now, $\mathcal{P}_{\rho-1}$, where we specialize (3.79) to $i = 1$, provides us with a *smooth* change of variables around 0 in $\mathbb{R}^n$:

$$(x_1, \ldots, x_n) \;\mapsto\; (X_{\rho-1}, \ldots, X_2, X_1)$$

with $X_i \in \mathbb{R}^{s_i}$ such that, in the new coordinates, system (3.4) reads

$$\begin{array}{rcl} \dot{X}_{\rho-1} & = & F_{\rho-1}(X_{\rho-1}, X_{\rho-2}) \\ \dot{X}_{\rho-2} & = & F_{\rho-2}(X_{\rho-1}, X_{\rho-2}, X_{\rho-3}) \\ & \vdots & \\ \dot{X}_2 & = & F_2(X_{\rho-1}, \ldots, X_1) \\ \dot{X}_1 & = & F_1(X_{\rho-1}, \ldots, X_1, X_0), \end{array} \tag{3.83}$$

and also such that the local homeomorphism Φ that topologically conjugates system (3.83) to system (3.75) at $(0,0)$ is, together with its inverse Ψ, of the triangular form :

$$\begin{array}{rclcrcl} Z_{\rho-1} & = & \Phi_{\rho-1}(X_{\rho-1}) & \quad & X_{\rho-1} & = & \Psi_{\rho-1}(Z_{\rho-1}) \\ Z_{\rho-2} & = & \Phi_{\rho-2}(X_{\rho-1}, X_{\rho-2}) & & X_{\rho-2} & = & \Psi_{\rho-2}(Z_{\rho-1}, Z_{\rho-2}) \\ \vdots & & & & \vdots & & \\ Z_1 & = & \Phi_1(X_{\rho-1}, \ldots, X_1) & & X_1 & = & \Psi_1(Z_{\rho-1}, \ldots, Z_1) \\ Z_0 & = & \Phi_0(X_{\rho-1}, \ldots, X_1, X_0) & & X_0 & = & \Psi_0(Z_{\rho-1}, \ldots, Z_1, Z_0), \end{array} \tag{3.84}$$

where the following three properties hold :

1. Each Φ_k and Ψ_k for $k \geq 1$ is continuously differentiable with respect to X_k and Z_k respectively ; in particular, $\partial\Phi_k/\partial X_k$ is invertible throughout the considered neighborhood.
2. For $k \geq 2$, $\text{Rank}\dfrac{\partial F_k}{\partial X_{k-1}}(0, \ldots, 0) \;=\; s_k$, i.e. this rank is maximum, equal to the number of rows.
3. F_1 satisfies

$$\begin{array}{rcl} F_1(X_{\rho-1}, \ldots, X_1, X_0) & = & F_1(X_{\rho-1}, \ldots, X_1, 0) \\ & & + \left(\dfrac{\partial\Phi_1}{\partial X_1}(X_{\rho-1}, \ldots, X_1) \right)^{-1} J^{s_1}_m \Bigg(\\ & & \Phi_0(X_{\rho-1}, \ldots, X_1, X_0) - \Phi_0(X_{\rho-1}, \ldots, X_1, 0) \Bigg) \,. \end{array} \tag{3.85}$$

From the maximum rank assumption on $\partial F_{\rho-1}/\partial X_{\rho-2}$, it is possible to define $Y_{\rho-2}$ whose first $s_{\rho-1}$ entries are those of $F_{\rho-1}(X_{\rho-1}, X_{\rho-2})$ and whose remaining $s_{\rho-2} - s_{\rho-1}$ entries are suitable components of $X_{\rho-2}$, in such a way that

$$(X_{\rho-1}, \ldots, X_1) \;\; \mapsto \;\; (X_{\rho-1}, Y_{\rho-2}, X_{\rho-3} \ldots, X_1)$$

is a local *smooth* change of coordinates around 0 in $\mathbb{R}^n$. After performing this change of coordinates and setting $Y_{\rho-1} = X_{\rho-1}$ for notational homogeneity, system (3.83) reads

$$\begin{array}{rcl} \dot{Y}_{\rho-1} & = & J^{s_{\rho-1}}_{s_{\rho-2}} Y_{\rho-2} \\ \dot{Y}_{\rho-2} & = & \widetilde{F}_{\rho-2}(Y_{\rho-1}, Y_{\rho-2}, X_{\rho-3}) \\ & \vdots & \\ \dot{X}_2 & = & \widetilde{F}_2(Y_{\rho-1}, Y_{\rho-2}, X_{\rho-3}, \ldots, X_1) \\ \dot{X}_1 & = & \widetilde{F}_1(Y_{\rho-1}, Y_{\rho-2}, X_{\rho-3}, \ldots, X_1, X_0) \end{array}$$

where the $\widetilde{F}$'s enjoy the same properties than the F's, in particular the maximality of $\operatorname{Rank} \partial \widetilde{F}_k / \partial X_{k-1}(0, \ldots, 0)$ for $\rho-2 \geq k \geq 2$. One may iterate this procedure, limited only by the fact that the maximum rank property mentioned above only holds for $k \geq 2$ but not necessarily for $k = 1$. Altogether, this yields a *smooth* local change of coordinates around 0 in $\mathbb{R}^n$:

$$(X_{\rho-1}, \ldots, X_1) \;\; \mapsto \;\; (Y_{\rho-1}, \ldots, Y_1),$$

after which system (3.83) is of the form

$$\begin{array}{rcl} \dot{Y}_{\rho-1} & = & J^{s_{\rho-1}}_{s_{\rho-2}} Y_{\rho-2} \\ & \vdots & \\ \dot{Y}_2 & = & J^{s_2}_{s_1} Y_1 \\ \dot{Y}_1 & = & F_1(Y_{\rho-1}, , \ldots, Y_1, X_0) \;, \end{array} \tag{3.86}$$

where we abuse the notation F_1 for simplicity because, although it needs not be the same as in (3.83), this new F_1 enjoys the same property (3.85) for some suitably redefined Φ_1 and Φ_0. Now, we may rewrite (3.85) as

$$F_1(Y_{\rho-1}, , \ldots, Y_1, X_0) \;\; = \;\; J^{s_1}_m \, H(Y_{\rho-1}, , \ldots, Y_1, X_0) \tag{3.87}$$

where H, in the aggregate notation $Y = (Y_{\rho-1}, , \ldots, Y_1)$, is defined by

$$H(Y, X_0) \;=\; \begin{pmatrix} F_1(Y, 0) \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial \Phi_1}{\partial Y_1}(Y)^{-1} & 0 \\ 0 & I_{m-s_1} \end{pmatrix} \big(\Phi_0(Y, X_0) - \Phi_0(Y, 0) \big).$$

Since Φ has the triangular structure displayed in (3.84), the map $X_0 \mapsto \Phi_0(Y, X_0)$ is injective for fixed $Y = (Y_{\rho-1}, \ldots, Y_1)$ in the neighborhood of 0 where it is defined in $\mathbb{R}^m$. Consequently, $(Y, X_0) \mapsto (Y, H(Y, X_0))$ is also injective in the neighborhood of 0 where it is defined in $\mathbb{R}^{n+m}$; since it is continuous, it is a local homeomorphism of $\mathbb{R}^{n+m}$ at $(0, 0)$ by invariance of the domain, and then (3.86), (3.87) make it clear that system (3.83) is locally quasi-smoothly linearizable at this point.

Since (3.83) is smoothly conjugate to the original system (3.4), this proves local quasi-smooth linearizability of the latter hence the theorem.

Two lemmas. The following two lemmas are applied recursively in the above proof of Theorem 3.5.2 to obtain the forms (3.83), (3.75), and (3.84). Although these lemmas team up into a single result in the above-mentioned proof, they have been stated here separately for the sake of clarity.

We will consider two control systems with state in $\mathbb{R}^d$ and control in $\mathbb{R}^r$. Expanded in coordinates, the first system reads

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1,\ldots,x_d,x_{d+1},\ldots,x_{d+r}) \\ & \vdots & \\ \dot{x}_d & = & f_d(x_1,\ldots,x_d,x_{d+1},\ldots,x_{d+r})\ , \end{array} \tag{3.88}$$

with state variable $(x_1,\ldots,x_d)$ and control variable $(x_{d+1},\ldots,x_{d+r}) \in \mathbb{R}^r$, the functions $f_1,\cdots,f_d$ being smooth $\mathbb{R}^{d+r} \to \mathbb{R}$. The second system has state variable $(z_1,\ldots,z_d)$ and control variable $(z_{d+1},\ldots,z_{d+r}) \in \mathbb{R}^r$, and it assumes the special form :

$$\begin{array}{rcl} \dot{z}_1 & = & g_1(z_1,\ldots,z_d) \\ & \vdots & \\ \dot{z}_{d-s} & = & g_{d-s}(z_1,\ldots,z_d) \\ \dot{z}_{d-s+1} & = & z_{d+1} \\ & \vdots & \\ \dot{z}_d & = & z_{d+s}\ , \end{array} \tag{3.89}$$

where $0 < s \leq d$ and $s \leq r$ while $g_1,\cdots,g_{d-s}$ are again smooth $\mathbb{R}^d \to \mathbb{R}$. Nothing prevents us here from having $s < r$, in which case some of the controls do not enter the equation. It will be convenient to use the aggregate notations

$$\begin{array}{ll} X \triangleq (x_1,\ldots,x_d)\,, & U \triangleq (x_{d+1},\ldots,x_{d+r})\,, \\ Z \triangleq (z_1,\ldots,z_d)\,, & V \triangleq (z_{d+1},\ldots,z_{d+r})\,, \end{array}$$

and to further split Z into (Z^1,Z^2) with

$$Z^1 \triangleq (z_1,\ldots,z_{d-s})\,,\quad Z^2 \triangleq (z_{d-s+1},\ldots,z_d)\,, \tag{3.90}$$

so as to write (3.88) in the form

$$\dot{X} \;=\; f(X,U) \tag{3.91}$$

and (3.89) as

$$\begin{array}{rcl} \dot{Z}^1 & = & g^1(Z^1,Z^2) \\ \dot{Z}^2 & = & J_r^s\,V\ , \end{array} \tag{3.92}$$

with J_r^s the $s \times r$ matrix, defined in (3.41), that selects the first s entries of a vector.

Lemma 3.6.2. *Let d, r and s be strictly positive integers with $s \leq d$ and $s \leq r$. Suppose, for some $\varepsilon > 0$, that*

$$\varphi : (-\varepsilon,\varepsilon)^{d+r} \to \mathbb{R}^{d+r}$$

is a homeomorphism onto its image, with inverse ψ, that conjugates system (3.91) *to system* (3.92)*. Then, there exists $0 < \varepsilon' < \varepsilon$ and a* smooth *local change of coordinates around $0 \in \mathbb{R}^d$:*

$$\theta :\ (-\varepsilon',\varepsilon')^d \ \to\ \theta\big((-\varepsilon',\varepsilon')^d\big) \subset (-\varepsilon,\varepsilon)^d$$

that fixes the origin and is such that, in the new coordinates $\widetilde{X} = \theta^{-1}(X)$, both the system (3.91) *and the conjugating homeomorphism $\widetilde{\varphi} = \varphi \circ (\theta \times \mathrm{id})$ assume a block triangular structure with respect to the partition $\widetilde{X} = (\widetilde{X}^1,\widetilde{X}^2)$, where $\widetilde{X}^1 \triangleq (\widetilde{x}_1,\ldots,\widetilde{x}_{d-s})$ and $\widetilde{X}^2 \triangleq (\widetilde{x}_{d-s+1},\ldots,\widetilde{x}_d)$; that is to say, on $(-\varepsilon',\varepsilon')^{d+r}$, we have that*

– *system* (3.88) *reads :*

$$\begin{array}{rcl}\dot{\widetilde{X}}^1 & = & \widetilde{f}^1(\widetilde{X}^1,\widetilde{X}^2)\\ \dot{\widetilde{X}}^2 & = & \widetilde{f}^2(\widetilde{X}^1,\widetilde{X}^2,U),\end{array} \tag{3.93}$$

– *On their respective domains of definition, the homeomorphism* $\widetilde{\varphi}$ *and its inverse* $\widetilde{\psi} = (\theta^{-1}\times id)\circ\psi$ *read :*

$$\begin{array}{rclcrcl} Z^1 & = & \widetilde{\varphi}^1(\widetilde{X}^1) & & \widetilde{X}^1 & = & \widetilde{\psi}_1(Z^1)\\ Z^2 & = & \widetilde{\varphi}^2(\widetilde{X}^1,\widetilde{X}^2) & & \widetilde{X}^2 & = & \widetilde{\psi}_2(Z^1,Z^2)\\ V & = & \widetilde{\varphi}^3(\widetilde{X}^1,\widetilde{X}^2,U) & & U & = & \widetilde{\psi}_3(Z^1,Z^2,V)\,.\end{array} \tag{3.94}$$

Lemma 3.6.3. *Let*

$$\widetilde{\varphi} : (-\varepsilon',\varepsilon')^{d+r} \to \mathbb{R}^{d+r}$$

be a homeomorphism onto its image, having the block triangular structure displayed in (3.94)*, and assume that it conjugates the smooth system* (3.93) *to the smooth system* (3.92)*. Necessarily then,* $\widetilde{\varphi}$ *has the following properties :*

1. *The map* $\widetilde{\varphi}^2$ *is continuously differentiable with respect to its second argument* $\widetilde{X}^2$*, and* $\dfrac{\partial\widetilde{\varphi}^2}{\partial\widetilde{X}^2}(0,0)$ *is invertible.*
2. *On some neighborhood of* $0\in\mathbb{R}^{d+r}$ *included in* $(-\varepsilon',\varepsilon')^{d+r}$*, one has :*

$$\begin{array}{l}\widetilde{f}^2(\widetilde{X}^1,\widetilde{X}^2,U) \;=\\ \widetilde{f}^2(\widetilde{X}^1,\widetilde{X}^2,0) \;+\; \left(\dfrac{\partial\widetilde{\varphi}^2}{\partial\widetilde{X}^2}(\widetilde{X}^1,\widetilde{X}^2)\right)^{-1} J^s_r\left(\widetilde{\varphi}^3(\widetilde{X}^1,\widetilde{X}^2,U)-\widetilde{\varphi}^3(\widetilde{X}^1,\widetilde{X}^2,0)\right)\end{array} \tag{3.95}$$

3. *On some neighborhood of* $0\in\mathbb{R}^d$ *included in* $(-\varepsilon',\varepsilon')^d$*, the partial homeomorphism*

$$(\,\widetilde{X}^1\,,\,\widetilde{X}^2\,) \;\mapsto\; (\,\widetilde{\varphi}^1(\widetilde{X}^1)\,,\,\widetilde{\varphi}^2(\widetilde{X}^1,\widetilde{X}^2)\,) \tag{3.96}$$

conjugates the control system

$$\dot{\widetilde{X}}^1 = \widetilde{f}^1(\widetilde{X}^1,\widetilde{X}^2), \tag{3.97}$$

with state $\widetilde{X}^1$ *and control* $\widetilde{X}^2$*, to the control system*

$$\dot{Z}^1 = g^1(Z^1,Z^2) \tag{3.98}$$

with state Z^1 *and input* Z^2*.*

Note that (3.97) and (3.98) are reduced systems from (3.93) and (3.92).

Proof of Lemma 3.6.2. Since the homeomorphism φ conjugates (3.91) to (3.92), we know, by Proposition 3.3.6, that φ and ψ split component-wise into :

$$\begin{array}{rclcrcl} Z & = & \varphi_{\mathrm{I}}(X) & & X & = & \psi_{\mathrm{I}}(Z)\\ V & = & \varphi_{\mathrm{II}}(X,U) & & U & = & \psi_{\mathrm{II}}(Z,V)\;.\end{array} \tag{3.99}$$

Consider the map $f : (-\varepsilon,\varepsilon)^{d+r}\to\mathbb{R}^d$ given in (3.91), and let us define $g : \varphi((-\varepsilon,\varepsilon)^{d+r})\to\mathbb{R}^d$ analogously from (3.92), namely g is the concatenated map whose first $d-s$ components are given by $g^1(Z)$ and whose last s components are given by J^s_rV. Define two families of continuous vector fields $\mathcal{F}'$ and $\mathcal{G}'$, on $(-\varepsilon,\varepsilon)^d$ and $\varphi_{\mathrm{I}}((-\varepsilon,\varepsilon)^d)$ respectively, by the following formulas (compare (3.142)) :

$$\mathcal{F}' = \{\, \delta f_{\alpha_1,\alpha_2}\,;\alpha_1,\alpha_2 \text{ feedbacks on } (-\varepsilon,\varepsilon)^{d+r}\,\}\ , \tag{3.100}$$

$$\mathcal{G}' = \{\, \delta g_{\beta_1,\beta_2}\,;\beta_1,\beta_2 \text{ feedbacks on } \varphi\big((-\varepsilon,\varepsilon)^{d+r}\big)\,\}\ . \tag{3.101}$$

Applying Proposition 3.3.13 twice, first to $\chi=\varphi$ and then to $\chi=\psi$, we see that each integral curve of a vector field in $\mathcal{F}'$ is mapped by φ_{I} to some integral curve of a vector field in $\mathcal{G}'$ and vice-versa upon replacing φ_{I} by ψ_{I}. This shows in particular that uniqueness of solutions to the Cauchy problem associated to vector fields is preserved, i.e. if we define the families of vector fields (compare (3.143)) :

$$\mathcal{F}'' = \{\, Y\in\mathcal{F}'\,,\ Y \text{ has a flow}\,\}\ , \tag{3.102}$$

$$\mathcal{G}'' = \{\, Y\in\mathcal{G}'\,,\ Y \text{ has a flow}\,\}\ , \tag{3.103}$$

we also have that each integral curve of a vector field in $\mathcal{F}''$ is mapped by φ_{I} to an integral curve of a vector field in $\mathcal{G}''$ and vice-versa upon replacing φ_{I} by ψ_{I}. By concatenation, using Proposition 3.8.5, it follows that

$$\left.\begin{array}{l}\textit{for any } X\in(-\varepsilon,\varepsilon)^d,\ \varphi_{\mathrm{I}} \textit{ defines a homeomorphism,}\\ \textit{for the orbit topologies, from the orbit of } \mathcal{F}'' \textit{ through } \mathrm{X}\\ \textit{onto the orbit of } \mathcal{G}'' \textit{ through } \varphi_{\mathrm{I}}(X),\end{array}\right\} \tag{3.104}$$

where the orbit topology as described in Proposition 3.8.5 (by definition the restriction of φ_{I} is bi-continuous for the topologies induced by the ambient space; bi-continuity for the orbit topologies requires the description of these topologies as given in Proposition 3.8.5).

Now, the vector fields $\delta g_{\beta_1,\beta_2}$ appearing in (3.101) inherit from the structure of g, displayed in (3.92), the following particular form :

$$\delta g_{\beta_1,\beta_2}(Z) = \begin{pmatrix} 0\\ \vdots\\ 0\\ \beta_{1,1}(Z)-\beta_{2,1}(Z)\\ \vdots\\ \beta_{1,s}(Z)-\beta_{2,s}(Z)\end{pmatrix}, \tag{3.105}$$

where $\beta_{i,1},\ldots,\beta_{i,s}$ designate, for $i=1,2$, the first s component of the feedback β_i. This will allow for us to describe explicitly the orbits of $\mathcal{G}''$, namely :

$$\left.\begin{array}{l}\textit{the orbit of } \mathcal{G}'' \textit{ through } Z_0=(c_1,\ldots,c_d)\\ \textit{is the connected component containing } Z_0 \textit{ of the set}\\ \{Z\in\varphi_{\mathrm{I}}\left((-\varepsilon,\varepsilon)^d\right)\,,\ z_1=c_1,\ldots,z_{d-s}=c_{d-s}\}.\end{array}\right\} \tag{3.106}$$

Indeed, the orbit in question is contained in this set, because it is connected, and because all the vector fields in $\mathcal{G}''$ have their first $d-s$ components equal to zero by (3.105).

To prove the reverse inclusion, it is enough to show that the orbit of $\mathcal{G}''$ through Z_0, denoted hereafter by $\mathcal{O}_{\mathcal{G}'',Z_0}$, contains all the points sufficiently close to Z_0 having the same first $d-s$ coordinates as Z_0. Indeed, since Z_0 was arbitrary, this will imply that the connected component defined by (3.106) splits into a disjoint union of open orbits hence consists of a single one by connectedness. That is to say, putting $Z_0=(Z_0^1,Z_0^2)$ according to (3.90), 3.106 will follow from the existence of a $\rho>0$ such that

$$\{Z_0^1\}\times B(Z_0^2,\rho)=B(Z_0,\rho)\cap\mathcal{O}_{\mathcal{G}'',Z_0}. \tag{3.107}$$

Now, it follows from Remark 3.8.3 that, for sufficiently small ρ, each connected component of $B(Z_0,\rho)\cap\mathcal{O}_{\mathcal{G}'',Z_0}$ is an embedded sub-manifold of $B(Z_0,\rho)$. Then, the connected component of $B(Z_0,\rho)\cap\mathcal{O}_{\mathcal{G}'',Z_0}$ containing Z_0 is, by inclusion, an embedded sub-manifold of the linear manifold $\{Z_0^1\}\times B(Z_0^2,\rho)$. In particular, since no strict sub-manifold can be densely embedded in a given manifold, we see that (3.107) will hold is only we can prove that

$$\text{\textit{The connected component containing } } Z_0 \text{ \textit{of} } B(Z_0,\rho)\cap\mathcal{O}_{\mathcal{G}'',Z_0} \text{ \textit{is dense in} } \{Z_0^1\}\times B(Z_0^2,\rho) \text{ \textit{for the Euclidean topology.}} \tag{3.108}$$

To prove (3.108), pick V_0 such that $(Z_0,V_0)\in\varphi\big((-\varepsilon,\varepsilon)^{d+r}\big)$ and observe, since the latter is an open set, that shrinking ρ further, if necessary, allows us to assume $\overline{B}(Z_0,\rho)\times\overline{B}(V_0,\rho)\subset\varphi\big((-\varepsilon,\varepsilon)^{d+r}\big)$. *We claim that any continuous map* $\overline{B}(Z_0,\rho)\to\overline{B}(V_0,\rho)$ *extends to a feedback on* $\varphi\big((-\varepsilon,\varepsilon)^{d+r}\big)$. Indeed, in view of the one-to-one correspondence $\beta\to\psi\blacksquare\beta$ between feedbacks on $\varphi\big((-\varepsilon,\varepsilon)^{d+r}\big)$ and feedbacks on $(-\varepsilon,\varepsilon)^{d+r}$ (cf. the discussion leading to (3.23)-(3.24)), it is enough to prove that every continuous map $\psi_{\mathrm{I}}\big(\overline{B}(Z_0,\rho)\big)\to(-\varepsilon,\varepsilon)^r$ extends to a continuous map $(-\varepsilon,\varepsilon)^d\to(-\varepsilon,\varepsilon)^r$, and this in turn follows from the Tietze extension theorem since $\psi_{\mathrm{I}}\big(\overline{B}(Z_0,\rho)\big)$ is closed in $(-\varepsilon,\varepsilon)^d$ and since $(-\varepsilon,\varepsilon)^r$ is a poly-interval. *This proves the claim.*

From the claim, it follows that the restriction to $\overline{B}(Z_0,\rho)$ of the $\mathbb{R}^s$-valued vector field $J_r^s(\beta_1(Z)-\beta_2(Z))$, accounting for the lower half of the right-hand side in (3.105), can be assigned *arbitrarily*, by choosing adequately the feedbacks β_1 and β_2, among *continuous* vector fields $\overline{B}(Z_0,\rho)\to\overline{B}(0,\rho)$ (take β_2 to extend the constant map V_0 on $\overline{B}(Z_0,\rho)$). Of course, the corresponding vector field $\delta g_{\beta_1,\beta_2}$ in (3.105) belongs to $\mathcal{G}'$ but not necessarily to $\mathcal{G}''$ since continuous vector fields need not have a flow. However, since $\delta g_{\beta_1,\beta_2}$ has a flow at least when β_1 and β_2 are smooth, we deduce from Proposition 3.3.4 that the restriction to $\overline{B}(Z_0,\rho)$ of the vector fields in $\mathcal{G}''$ are of the form $\{0\}\times Y$, where Y ranges over a uniformly dense subset Υ of all $\mathbb{R}^s$-valued continuous maps $\overline{B}(Z_0,\rho)\to\overline{B}(0,\rho)$. Now, every point in $B(Z_0^2,\rho)$ can be attained from Z_0^2 upon integrating, *within* $B(Z_0^2,\rho)$, a constant vector field of arbitrary small norm. By Lemma 3.7.2 applied with $\mathcal{U}=B(Z_0^2,\rho)$ and $K=\{Z_0^2\}$, the corresponding trajectory can be approximated uniformly by integral curves *that remain in* $B(Z_0^2,\rho)$ of vector fields in Υ. Therefore, every point in $\{z_0^1\}\times B(Z_0^2,\rho)$ is the limit of endpoints of integral curves of $\mathcal{G}''$ *that remain in* $\{z_0^1\}\times B(Z_0^2,\rho)$, which proves (3.108) and thus (3.106). In particular, the orbits of $\mathcal{G}''$ are *embedded* sub-manifolds in $\varphi_{\mathrm{I}}\big((-\varepsilon,\varepsilon)^d\big)$.

Next, we turn to the orbits of $\mathcal{F}''$, and we designate by $\mathcal{O}_{\mathcal{F}'',p}$ the orbit of $\mathcal{F}''$ in $]-\varepsilon,\varepsilon[^d$ through the point p. On the one hand, Proposition 3.8.5 and Theorem 3.8.2 show that $\mathcal{O}_{\mathcal{F}'',p}$ is a smooth immersed sub-manifold of $]-\varepsilon,\varepsilon[^d$. On the other hand, by (3.104), this immersed sub-manifold is sent homeomorphically by φ_{I}, both for the orbit topology and the ambient topology, onto $\mathcal{O}_{\mathcal{G}'',\varphi_{\mathrm{I}}(p)}$ which is a smooth *embedded* s-dimensional sub-manifold of $\varphi_{\mathrm{I}}\big((-\varepsilon,\varepsilon)^d\big)$, as we saw from (3.106). This entails that all orbits of $\mathcal{F}''$ in $]-\varepsilon,\varepsilon[^d$ are *embedded* sub-manifolds of dimension s. Consequently, still from Proposition 3.8.5 and Theorem 3.8.2, there are coordinates $(\xi_1,\ldots,\xi_d)$ defined on an open neighborhood W_0 of the origin in $]-\varepsilon,\varepsilon[^d$ —this neighborhood may be assumed to be of the form $\{(\xi_1,\ldots,\xi_d),\ |\xi_i|<\varepsilon'\}$ — such that, in these coordinates,

$$W_0\cap\mathcal{O}_{\mathcal{F}'',0} \;=\; \{\,(\xi_1,\ldots,\xi_d),\text{ with }(\xi_{s+1},\ldots,\xi_d)\in T\,\}\ ,$$

with T a subset of $]-\varepsilon',\varepsilon'[^{d-s}$ containing $(0,\ldots,0)$, the tangent space to $W_0\cap\mathcal{O}_{\mathcal{F}'',0}$ at each of its points being *spanned* by $\partial/\partial\xi_1,\ldots,\partial/\partial\xi_s$, while at any point $p\in W_0$ the vector fields $\partial/\partial\xi_1,\ldots,\partial/\partial\xi_s$ *belong* to the tangent space of $\mathcal{O}_{\mathcal{F}'',p}$. But since we saw that *all* orbits are smooth sub-manifolds of dimension s, these vector fields actually *span* the tangent space to the orbit at every point. Hence all the vector fields $\delta f_{\alpha_1,\alpha_2}$ in $\mathcal{F}''$ have their last $d-s$ components equal to zero on W_0 in the ξ coordinates, and this holds in particular when α_1, α_2 range over all constant feedbacks $(-\varepsilon,\varepsilon)^d \to (-\varepsilon,\varepsilon)^r$. This implies, by the very definition of $\delta f_{\alpha_1,\alpha_2}$, that $(\dot\xi_{s+1},\ldots,\dot\xi_d)$ — as computed from (3.91) upon performing the change of variable $X \mapsto (\xi_1,\ldots,\xi_d)$ — does not depend on the control variable U. Choose for $\widetilde X$ the ξ coordinates arranged in reverse order, and let $\widetilde f$ be the analog of f in the new coordinates $(\widetilde X, U)$. Then the first $d-s$ components of $\widetilde f$ do not depend on U so that (3.93) holds. Moreover, if $\widetilde\varphi$ denotes the new homeomorphism that conjugates (3.93) to (3.92) over $(-\varepsilon,\varepsilon)^{d+r}$, $\widetilde\varphi((-\varepsilon,\varepsilon)^{d+r})$, and if $\widetilde\psi$ denotes its inverse, it follows from (3.104) and the above characterization of the orbits that $\widetilde\varphi_{\mathrm I}$ maps the sets where $\tilde x_1,\ldots,\tilde x_{d-s}$ are constant to those where $z_1,\ldots,z_{d-s}$ are constant, thus the functions $\widetilde\varphi_1,\ldots,\widetilde\varphi_{d-s}$ and $\widetilde\psi_1,\ldots,\widetilde\psi_{d-s}$ depend only on their $d-s$ first arguments whence (3.94) follows. □

Proof of Lemma 3.6.3. We use again the concatenated notation $\widetilde\varphi_{\mathrm I} = (\widetilde\varphi^1,\widetilde\varphi^2)$, $\widetilde\psi_{\mathrm I} = (\widetilde\psi^1,\widetilde\psi^2)$, these partial homeomorphisms being inverse of each other. Let $(Z_0,V_0)\in\widetilde\varphi((-\varepsilon',\varepsilon')^{d+r})$ and ε'' be so small that the product neighborhood $(Z_0,V_0)+(-\varepsilon'',\varepsilon'')^{d+r}$ lies entirely within $\widetilde\varphi((-\varepsilon',\varepsilon')^{d+r})$. The restriction to $(Z_0,V_0)+(-\varepsilon'',\varepsilon'')^{d+r}$ of $\widetilde\psi$ conjugates (3.92) to (3.93). Consequently, for any $\overline V\in(-\varepsilon'',\varepsilon'')^r$, we may apply Proposition 3.3.13 to this restriction and to the constant feedbacks $\alpha_1(Z)=V_0+\overline V$ and $\alpha_2(Z)=V_0$; this yields that $\widetilde\psi_{\mathrm I}$, given by

$$(Z^1,Z^2) \;\mapsto\; (\widetilde X^1,\widetilde X^2) \;=\; (\widetilde\psi^1(Z^1),\widetilde\psi^2(Z^1,Z^2)),$$

maps every solution of

$$\dot Z^1 \;=\; 0 \;\;,\qquad \dot Z^2 \;=\; J_r^s\overline V \tag{3.109}$$

that remains in $Z_0+(-\varepsilon'',\varepsilon'')^d$ to a solution of

$$\begin{aligned}\dot{\widetilde X}^1 = 0\;,\quad \dot{\widetilde X}^2 &= \widetilde f^2(\widetilde X^1,\widetilde X^2,\widetilde\psi^3(\widetilde\varphi^1(\widetilde X^1),\widetilde\varphi^2(\widetilde X^1,\widetilde X^2),V_0+\overline V))\\ &\quad - \widetilde f^2(\widetilde X^1,\widetilde X^2,\widetilde\psi^3(\widetilde\varphi^1(\widetilde X^1),\widetilde\varphi^2(\widetilde X^1,\widetilde X^2),V_0))\end{aligned}\tag{3.110}$$

that remains in $\widetilde\psi_{\mathrm I}(Z_0+(-\varepsilon'',\varepsilon'')^d)$, and vice versa upon applying Proposition 3.3.13 in the other direction.

Integrating (3.109) explicitly with initial condition $Z(0)=Z_0$, we get that

$$t \;\mapsto\; \begin{pmatrix}\widetilde\psi^1(Z_0^1)\\ \widetilde\psi^2(Z_0^1,Z_0^2\,+\,tJ_r^s\overline V)\end{pmatrix}$$

solves (3.110) for sufficiently small t, hence $\widetilde\psi^2(Z^1,Z^2)$ is differentiable at Z_0 with respect to its second argument in the direction $J_r^s\overline V$, with directional derivative

$$\begin{aligned}\frac{\partial\widetilde\psi^2}{\partial Z^2}(Z_0^1,Z_0^2)\,J_r^s\overline V \;&=\; \widetilde f^2(\widetilde\psi^1(Z_0^1),\widetilde\psi^2(Z_0^1,Z_0^2),\widetilde\psi^3(Z_0^1,Z_0^2,V_0+\overline V))\\ &\qquad - \widetilde f^2(\widetilde\psi^1(Z_0^1),\widetilde\psi^2(Z_0^1,Z_0^2),\widetilde\psi^3(Z_0^1,Z_0^2,V_0))\;.\end{aligned}\tag{3.111}$$

In particular, since Z_0 can be any member of $\widetilde\varphi_{\mathrm I}((-\varepsilon',\varepsilon')^d)$ while $J_r^s\overline V$ can be assigned arbitrarily in $(-\varepsilon'',\varepsilon'')^s$, we conclude that $\partial\widetilde\psi^2/\partial Z^2(Z^1,Z^2)$ exists and is continuous since this holds for the partial derivatives. Next we prove that $\partial\widetilde\psi^2/\partial Z^2$ is invertible at every point by showing that its kernel reduces to zero. In fact, if the left-hand side of (3.111) vanishes, so does the right-hand side which is also the value of the right-hand side of (3.110) for $\widetilde X=\widetilde\psi_{\mathrm I}(Z_0)$. Therefore the constant map $t\mapsto\widetilde\psi_{\mathrm I}(Z_0)$ is a solution to (3.110) over a suitable time interval, and by conjugation the constant map $t\mapsto Z_0$ is a solution to (3.109) over that time interval which clearly entails

$J_r^s\overline{V} = 0$, as desired. Now, since $\partial\widetilde{\psi}^2/\partial Z^2$ is invertible at every $(Z^1, Z^2) \in \widetilde{\varphi}_{\mathrm{I}}((-\varepsilon', \varepsilon')^d)$, the triangular structure of (3.94) and the inverse function theorem together imply that

$$\frac{\partial\widetilde{\varphi}^2}{\partial\widetilde{X}^2}(\widetilde{X}^1, \widetilde{X}^2) = \left(\frac{\partial\widetilde{\psi}^2}{\partial Z^2}(\widetilde{\varphi}^1(\widetilde{X}^1), \widetilde{\varphi}^2(\widetilde{X}^1, \widetilde{X}^2))\right)^{-1} \tag{3.112}$$

continuously exists and is invertible for $(\widetilde{X}^1, \widetilde{X}^2) \in (-\varepsilon', \varepsilon')^d$. This proves point 1.

Let us turn to point 2. Select an open neighborhood $\mathcal{W}$ of 0 having compact closure in $(-\varepsilon', \varepsilon')^d$, so there is $\eta > 0$ such that $\widetilde{\varphi}(\widetilde{X}, 0) + (-\eta, \eta)^{d+r}$ is included in $\widetilde{\varphi}((-\varepsilon', \varepsilon')^{d+r})$ whenever $\widetilde{X} \in \mathcal{W}$. If $\overline{V} \in (-\eta, \eta)^r$, we can apply (3.111) to $(Z_0, V_0) = \widetilde{\varphi}(\widetilde{X}, 0)$ with $\widetilde{X} \in \mathcal{W}$, and we obtain in view of (3.112) :

$$\begin{aligned}\left(\frac{\partial\widetilde{\varphi}^2}{\partial\widetilde{X}^2}(\widetilde{X}^1, \widetilde{X}^2)\right)^{-1} J_r^s\overline{V} &= -\widetilde{f}^2(\widetilde{X}^1, \widetilde{X}^2, 0) \\ &+ \widetilde{f}^2\left(\widetilde{X}^1, \widetilde{X}^2, \widetilde{\psi}^3(\widetilde{\varphi}^1(\widetilde{X}^1), \widetilde{\varphi}^2(\widetilde{X}^1, \widetilde{X}^2), \widetilde{\varphi}^3(\widetilde{X}^1, \widetilde{X}^2, 0) + \overline{V})\right).\end{aligned} \tag{3.113}$$

Set

$$U = \widetilde{\psi}^3(\widetilde{\varphi}^1(\widetilde{X}^1), \widetilde{\varphi}^2(\widetilde{X}^1, \widetilde{X}^2), \widetilde{\varphi}^3(\widetilde{X}^1, \widetilde{X}^2, 0) + \overline{V}) \tag{3.114}$$

and observe that $(\widetilde{X}, \overline{V}) \mapsto (\widetilde{X}, U) = \widetilde{\psi}(\widetilde{\varphi}(\widetilde{X}, 0) + (0, \overline{V}))$ defines a continuous map $h : \mathcal{W} \times (-\eta, \eta)^r \to (-\varepsilon', \varepsilon')^{d+r}$, such that $h(0) = 0$, which is injective. By invariance of the domain, h is a homeomorphism onto some open neighborhood of 0, say $\mathcal{N} \subset (-\varepsilon', \varepsilon')^{d+r}$. For $(\widetilde{X}, U) \in \mathcal{N}$, (3.114) can be inverted as

$$\overline{V} = \widetilde{\varphi}^3(\widetilde{X}, U) - \widetilde{\varphi}^3(\widetilde{X}, 0), \tag{3.115}$$

and substituting (3.114) and (3.115) in (3.113) yields (3.95).

Finally we prove point 3, keeping in mind the previous definitions and properties of h, $\mathcal{W}$, η and $\mathcal{N}$. For $\widetilde{X} = (\widetilde{X}^1, \widetilde{X}^2) \in (-\varepsilon', \varepsilon')^d$, define $\overline{V}(\widetilde{X}) \in \mathbb{R}^s \times \{0\} \subset \mathbb{R}^r$ by the formula :

$$J_r^s\overline{V}(\widetilde{X}) = \frac{\partial\widetilde{\varphi}^2}{\partial\widetilde{X}^2}(\widetilde{X}^1, \widetilde{X}^2)\big(\widetilde{f}^2(0,0,0) - \widetilde{f}^2(\widetilde{X}^1, \widetilde{X}^2, 0)\big). \tag{3.116}$$

Clearly $\overline{V} : (-\varepsilon', \varepsilon')^d \to \mathbb{R}^r$ is continuous and $\overline{V}(0) = 0$, so there exists an open neighborhood $\mathcal{V} \subset \mathcal{W}$ of 0 in $\mathbb{R}^d$ such that $\overline{V}(\widetilde{X}) \in (-\eta, \eta)^r$ as soon as $\widetilde{X} \in \mathcal{V}$; then, if we set $h(\widetilde{X}, \overline{V}(\widetilde{X})) = (\widetilde{X}, U(\widetilde{X})) \in \mathcal{N}$, it follows from (3.116), (3.115), and (3.95) that

$$\widetilde{f}^2(\widetilde{X}^1, \widetilde{X}^2, U(\widetilde{X})) = \widetilde{f}^2(0,0,0), \qquad \widetilde{X} \in \mathcal{V}. \tag{3.117}$$

We will show, using Proposition 3.3.12, that the restriction of $\widetilde{\varphi}_{\mathrm{I}}$ to any relatively compact open subset $\mathcal{X}$ of $\mathcal{V}$ conjugates (3.97) *and* (3.98) *over $\mathcal{X}$, $\widetilde{\varphi}(\mathcal{X})$, and this will achieve the proof.* To this effect, let $\mathcal{C}$ to be the collection of all piecewise affine maps $\mathbb{R} \to \mathbb{R}^s$ with constant slope $\widetilde{f}^2(0,0,0)$ (cf. the discussion before Proposition 3.3.12) and note that, for any open set $\mathcal{O} \subset \mathbb{R}^s$ and any compact interval $J \subset \mathbb{R}$, the restriction of $\mathcal{C}$ to J contains, in its uniform closure, the set all piecewise continuous maps $J \to \mathcal{O}$. Now, consider a solution $\gamma : I \to \mathcal{V}$ of the control system :

$$\dot{\widetilde{X}}^1 = \widetilde{f}^1(\widetilde{X}^1, \Upsilon) \tag{3.118}$$

with state $\widetilde{X}^1$ and control Υ ; hereafter, $\mathcal{V}_{\mathrm{I}} \subset \mathbb{R}^{d-s}$ and $\mathcal{V}_{\mathrm{II}} \subset \mathbb{R}^s$ will indicate the projections of $\mathcal{V}$ onto the first $d-s$ and the last s components respectively, and similarly for any other open set in $\mathbb{R}^d$. Assume that the control function $\gamma_{\mathrm{II}} : I \to \mathcal{V}_{\mathrm{II}}$ is the restriction to I of some member of $\mathcal{C}$. By definition, if a, b are the endpoints of I (that may belong to I or not), there are time instants $a = t_0 < t_1 < \cdots < t_N = b$, and vectors $\bar{\xi}_1, \ldots, \bar{\xi}_N \in \mathbb{R}^s$ such that, for $1 \leq j < N$, one has

$$t_{j-1} < t < t_j \;\Rightarrow\; \gamma_{\mathrm{II}}(t) \;=\; \bar{\xi}_j + t\widetilde{f}^2(0,0,0), \tag{3.119}$$

while at the points t_j themselves γ_{II} is either right or left continuous when $1 < j < N$. *We claim* that $\widetilde{\varphi}_{\mathrm{I}}(\gamma(t))$ is a solution that remains in $\widetilde{\varphi}_{\mathrm{I}}(\mathcal{V})$ of the control system :

$$\dot{Z}^1 \;=\; g^1(Z^1, \Gamma) \tag{3.120}$$

with state Z^1 and control Γ. In fact, since γ_{I} is continuous by definition of a solution, so is $\widetilde{\varphi}^1(\gamma_{\mathrm{I}})$ and therefore, as $\widetilde{\varphi}_{\mathrm{I}}(\gamma(t))$ lies in $\widetilde{\varphi}_{\mathrm{I}}(\mathcal{V})$ for all $t \in I$ by construction, it is enough to check that

$$\widetilde{\varphi}^1(\gamma_{\mathrm{I}}(T_2)) - \widetilde{\varphi}^1(\gamma_{\mathrm{I}}(T_1)) = \int_{T_1}^{T_2} g^1\big(\widetilde{\varphi}^1(\gamma_{\mathrm{I}}(t)), \widetilde{\varphi}^2(\gamma_{\mathrm{I}}(t), \gamma_{\mathrm{II}}(t))\big)\,\mathrm{d}t \tag{3.121}$$

whenever $t_{j-1} < T_1 < T_2 < t_j$ for some $j > 1$. However, the restriction of $\gamma(t)$ to (t_{j-1}, t_j) is a solution that remains in $\mathcal{V}$ of the differential equation :

$$\begin{aligned} \dot{\gamma}_{\mathrm{I}} &= \widetilde{f}^1(\gamma_{\mathrm{I}}, \gamma_{\mathrm{II}}) \\ \dot{\gamma}_{\mathrm{II}} &= \widetilde{f}^2(0,0,0), \end{aligned}$$

hence $(\gamma(t), U(\gamma(t))$ is, by (3.117), a solution of (3.93) that remains in $\mathcal{N}$, and therefore (3.121) follows from the triangular structure (3.94) of $\widetilde{\varphi}$ and the fact that it conjugates system (3.93) to system (3.92). *This proves the claim.*

In the other direction, we observe since it is included in $\mathcal{W}$ that $\mathcal{V}$ has compact closure in $(-\varepsilon', \varepsilon')^d$, and therefore that $\widetilde{\varphi}_{\mathrm{I}}(\mathcal{V})$ in turn has compact closure in $\widetilde{\varphi}_{\mathrm{I}}\big((-\varepsilon', \varepsilon')^d\big)$. Pick $\eta' > 0$ such that $\widetilde{\varphi}_{\mathrm{I}}(\mathcal{V}) \times (-\eta', \eta')^r \subset \widetilde{\varphi}((-\varepsilon', \varepsilon')^{d+r})$, and let $\mathcal{C}'$ denote the collection of all piecewise smooth maps $\mathbb{R} \to \mathbb{R}^s$ whose derivative is strictly bounded by η' component-wise. The restriction of $\mathcal{C}'$ to any compact real interval J is uniformly dense in the set all piecewise continuous maps $J \to \mathcal{O}$, for any open set $\mathcal{O} \subset \mathbb{R}^s$. Clearly, any solution $\gamma' : I \to \widetilde{\varphi}_{\mathrm{I}}(\mathcal{V})$ of system (3.120), whose control function $\gamma'_{\mathrm{II}} : I \to \big(\widetilde{\varphi}_{\mathrm{I}}(\mathcal{V})\big)_{\mathrm{II}}$ is the restriction to I of some member of $\mathcal{C}'$, satisfies the differential equation

$$\begin{aligned} \dot{\gamma}'_{\mathrm{I}} &= g^1(\gamma'_{\mathrm{I}}, \gamma'_{\mathrm{II}}) \\ \dot{\gamma}'_{\mathrm{II}} &= J^s_r\,(d\gamma'_{\mathrm{II}}/dt\,,\,0) \end{aligned}$$

on every interval where it is smooth. By the very definition of η' and $\mathcal{C}'$, it follows that $\big(\gamma'(t), (d\gamma'_{\mathrm{II}}(t)/dt, 0)\big)$ is, on such intervals, a solution to (3.92) that remains in $\widetilde{\varphi}((-\varepsilon', \varepsilon'))^{d+r}$ and, since $\widetilde{\psi}$ conjugates system (3.92) to system (3.93), we argue as before to the effect that $\widetilde{\psi}_{\mathrm{I}}(\gamma')$ is a solution to system (3.118) that remains in $\mathcal{V}$. Appealing to Proposition 3.3.12, we conclude that $\widetilde{\varphi}_{\mathrm{I}}$ conjugates system (3.118) to system (3.120) on relatively compact open subsets of $\mathcal{V}$, as desired. □

3.7 Appendix : Four lemmas on ODEs

Throughout this section, we let $\mathcal{U}$ be an open subset of $\mathbb{R}^d$. We say that a continuous vector field $X : \mathcal{U} \to \mathbb{R}^d$ *has a flow* if the Cauchy problem $\dot{x}(t) = X(x(t))$ with initial condition $x(0) = x_0$ has a unique solution, defined for $t \in (-\varepsilon, \varepsilon)$ with $\varepsilon = \varepsilon(x_0) > 0$. The flow of X at time t is denoted by X_t, in other words we have with the preceding notations that $X_t(x_0) = x(t)$. It is easy to see that the domain of definition of $(t, x) \mapsto X(t, x)$ is open in $\mathbb{R} \times \mathcal{U}$.

Lemma 3.7.1. *If $X : \mathcal{U} \to \mathbb{R}^d$ is a continuous vector field that has a flow, the map $(t, x) \mapsto X_t(x)$ is continuous on the open subset of $\mathbb{R} \times \mathcal{U}$ where it is defined.*

Proof. This is an easy consequence of the Ascoli-Arzela theorem, and actually a special case of [47, chap. V, Theorem 2.1]. □

Lemma 3.7.2. *Assume that the sequence of continuous vector fields $X^k : \mathcal{U} \to \mathbb{R}^d$ converges to X, uniformly on compact subsets of $\mathcal{U}$, and that all the X^k as well as X itself have a flow. Suppose that $X_t(x)$ is defined for all $(t, x) \in [0, T] \times K$ with $T > 0$ and $K \subset \mathcal{U}$ compact. Then $X^k_t(x)$ is also defined on $[0, T] \times K$ for k large enough, and the sequence of mappings $(t, x) \mapsto X^k_t(x)$ converges to $(t, x) \mapsto X_t(x)$, uniformly on $[0, T] \times K$.*

Proof. By assumption,

$$K_1 = \{X_t(x);\ (t, x) \in [0, T] \times K\}$$

is a well-defined subset of $\mathcal{U}$ that contains K, and it is compact by Lemma 3.7.1. Let K_0 be another compact subset of $\mathcal{U}$ whose interior contains K_1, and put $d(K_1, \mathcal{U} \setminus K_0) = \eta > 0$ where $d(E_1, E_2)$ indicates the distance between two sets E_1, E_2. From the hypothesis there is $M > 0$ such that $\|X^k\| \leq M$ on K_0 for all k, hence the maximal solution to $\dot{x}(t) = X^k(x(t))$ with initial condition $x(0) = x_0 \in K$ remains in K_0 as long as $t \leq \eta/2M$. Consequently the flow $(t, x) \mapsto X^k_t(x)$ is defined on $[0, \eta/2M] \times K$ for all k, with values in K_0. *We claim* that it is a bounded equicontinuous sequence of functions there. Boundedness is clear since these functions are K_0-valued, so we must show that, to every $(t, x) \in [0, \eta/2M] \times K$ and every $\varepsilon > 0$, there is $\alpha > 0$ such that $\|X^k(t', x') - X^k(t, x)\| < \varepsilon$ for all k as soon as $|t - t'| + \|x - x'\| < \alpha$. By the mean-value theorem and the uniform majorization $\|X^k(X^k_t(x))\| \leq M$, it is sufficient to prove this when $t = t'$. Arguing by contradiction, assume for some subsequence k_l and some sequence x_l converging to x in K that

$$\|X^{k_l}_t(x) - X^{k_l}_t(x_l)\| \geq \varepsilon \quad \text{for all } l \in \mathbb{N}. \tag{3.122}$$

Then, by Lemma 3.7.1, the index k_l tends to infinity with l. Next consider the sequence of maps $F_l : [0, \eta/2M] \to K_0$ defined by $F_l(t) = X^{k_l}_t(x_l)$. Again, by the mean value theorem, it is a bounded equicontinuous family of functions and, by the Ascoli-Arzela theorem, it is relatively compact in the topology of uniform convergence (compare [47, chap. II, Theorem 3.2]). But if $\Phi : [0, \eta/2M] \to K_0$ is the uniform limit of some subsequence F_{l_j}, and since $X^{k_{l_j}}$ converges uniformly to X on K_0 as $j \to \infty$, taking limits in the relation

$$X^{k_{l_j}}_t(x_{l_j}) = x_{l_j} + \int_0^t X^{k_{l_j}}(X^{k_{l_j}}_s(x_{l_j}))\, ds$$

gives us

$$\Phi(t) = x + \int_0^t X(\Phi(s))\, ds$$

so that $\Phi(t) = X_t(x)$ since X has a flow. Altogether $F_l(t)$ converges uniformly to $X_t(x)$ on $[0, \eta/2M]$ because this is the only accumulation point, and then (3.122) becomes absurd. *This proves the claim.* From the claim it follows, using the Ascoli-Arzela theorem again, that the family of functions $(t, x) \mapsto X^k_t(x)$ is relatively compact for the topology of uniform convergence $[0, \eta/2M] \times K \to K_0$, and in fact it converges to $(t, x) \mapsto X_t(x)$ because, by the same limiting argument as was used to prove the claim, every accumulation point $\Phi(t, x)$ must be a solution to

$$\Phi(t, x) = x + \int_0^t X(s, \Phi(s, x))\, ds$$

hence for fixed x is an integral curve of X with initial condition x. In particular, by definition of K_1, we shall have that $d(X_t^k(x), K_1) < \eta/2$ for all $(t,x) \in [0,\eta/2M] \times K$ as soon as k is large enough. For such k the flow $(t,x) \mapsto X_t^k(x)$ will be defined on $[0,\eta/M]\times K$ with values in K_0, and we can repeat the whole argument again to the effect that $X_t^k(x)$ converges uniformly to $X_t(x)$ there. Proceeding inductively, we obtain after $[2TM/\eta]+1$ steps at most that $(t,x) \mapsto X_t^k(x)$ is defined on $[0,T] \times K$ with values in K_0 for k large enough, and converges uniformly to $(t,x) \mapsto X_t(x)$ there, as was to be shown. □

The next lemma stands analogous to Lemma 3.7.2 for time-dependent vector fields, assuming that the convergence holds boundedly almost everywhere in time. The assumption that the vector fields have a flow is replaced here by a local Lipschitz condition that we now comment upon.

By definition, a time-dependent vector field $X : [t_1,t_2] \times \mathcal{U} \to \mathbb{R}^d$ is locally Lipschitz with respect to the second variable if every $(t_0,x_0) \in [t_1,t_2] \times \mathcal{U}$ has a neighborhood there such that $\|X(t,x') - X(t,x)\| < c\|x' - x\|$, for some constant c, whenever (t,x) and (t,x') belong to that neighborhood. This of course entails that X is bounded on compact subsets of $[t_1,t_2] \times \mathcal{U}$. Next, by the compactness of $[t_1,t_2]$, the local Lipschitz character of X strengthens to the effect that each $x_0 \in \mathcal{U}$ has a neighborhood $\mathcal{N}_{x_0}$ such that $\|X(t,x') - X(t,x)\| < c_{x_0}\|x' - x\|$, for some constant c_{x_0}, whenever $x,x' \in \mathcal{N}_{x_0}$ and $t \in [t_1,t_2]$. If now $\mathcal{K} \subset \mathcal{U}$ is compact, we can cover it by finitely many $\mathcal{N}_{x_{0,k}}$ as above and find $\varepsilon > 0$ such that $x,x' \in \mathcal{K}$ and $\|x - x'\| < \varepsilon$ is impossible unless x,x' lie in some common $\mathcal{N}_{x_0}$. Consequently there is $c_{\mathcal{K}} > 0$ such that $\|X(t,x') - X(t,x)\| < c_{\mathcal{K}}\|x' - x\|$ whenever $x,x' \in \mathcal{K}$ and $t \in [t_1,t_2]$, because if $\|x - x'\| < \varepsilon$ we can take $c_{\mathcal{K}} \geq \max_k c_{x_{0,k}}$, whereas if $\|x - x'\| \geq \varepsilon$ it is enough to take $c_{\mathcal{K}} > 2M/\varepsilon$ where M is a bound for $\|X\|$ on $[t_1,t_2] \times \mathcal{K}$. Finally, if $X(t,x)$ happens to vanish identically for x outside some compact $\mathcal{K}' \subset \mathcal{U}$, we can choose $\mathcal{K}$ such that

$$\mathcal{K}' \subset \mathring{\mathcal{K}} \subset \mathcal{K} \subset \mathcal{U}$$

and construct $c_{\mathcal{K}}$ as before except that we also pick $\varepsilon > 0$ so small that $\|x - x'\| < \varepsilon$ is impossible for $x \in \mathcal{K}'$ and $x' \notin \mathcal{K}$. Then it holds that $\|X(t,x') - X(t,x)\| < c_{\mathcal{K}}\|x' - x\|$ for all $x,x' \in \mathcal{U}$ and all $t \in [t_1,t_2]$, that is to say $X(t,x)$ becomes globally Lipschitz with respect to x. These remarks will be used in the proof to come.

Lemma 3.7.3. *Let $t_1 < t_2$ be two real numbers and $X^k : [t_1,t_2] \times \mathcal{U} \to \mathbb{R}^d$ a sequence of time-dependent vector fields, measurable with respect to t, locally Lipschitz continuous with respect to $x \in \mathcal{U}$, and bounded on compact subsets of $[t_1,t_2]\times\mathcal{U}$ independently of k. Let $X : [t_1,t_2]\times\mathcal{U} \to \mathbb{R}^d$ be another time-dependent vector field, measurable with respect to t, locally Lipschitz continuous with respect to $x \in \mathcal{U}$, and assume that, to each compact $\mathcal{K} \subset \mathcal{U}$, there is $E_{\mathcal{K}} \subset [t_1,t_2]$ of zero measure such that, whenever $t \notin E_{\mathcal{K}}$, the sequence $X^k(t,x)$ converges to $X(t,x)$ as $k \to \infty$, uniformly with respect to $x \in \mathcal{K}$. Suppose finally that $\gamma : [t_1,t_2] \to \mathcal{U}$ is, for some $(t_0,x_0) \in [t_1,t_2] \times \mathcal{U}$, a solution to the Cauchy problem*

$$\dot{\gamma}(t) = X(t,\gamma(t)), \qquad \gamma(t_0) = x_0. \tag{3.123}$$

Then, for k large enough, there is a unique solution $\gamma_k : [t_1,t_2] \to \mathcal{U}$ to the Cauchy problem

$$\dot{\gamma}_k(t) = X^k(t,\gamma_k(t)), \qquad \gamma_k(t_0) = x_0, \tag{3.124}$$

and the sequence (γ_k) converges to γ, uniformly on $[t_1,t_2]$.

Proof. Upon multiplying $X^k(t,x)$ and $X(t,x)$ by a smooth function $\varphi(x)$ which is compactly supported $\mathcal{U} \to \mathbb{R}$ and identically 1 on a neighborhood of $\gamma([t_1,t_2])$, we may assume in view of the discussion preceding the lemma that $X(t,x)$ and $X^k(t,x)$ are defined and bounded $[t_1,t_2]\times\mathbb{R}^d \to \mathbb{R}^d$ independently of k, measurable with respect to t, and (globally) Lipschitz continuous with respect to x.

Then, by classical results [98, Proposition C 3.8., Theorem 54], the solution to (3.124), say γ_k uniquely exists $[t_1,t_2] \to \mathbb{R}^d$ for each k :

$$\gamma_k(t) = x_0 + \int_{t_0}^{t} X^k(s,\gamma_k(s))\, ds, \qquad t \in [t_1,t_2]. \tag{3.125}$$

From the boundedness of X^k, it is clear that γ_k is an equicontinuous and bounded family of functions, hence it is relatively compact in the topology of uniform convergence on $[t_1,t_2]$. All we have to prove then is that every accumulation point of γ_k coincides with γ. Extracting a subsequence if necessary, let us assume that γ_k converges to some $\bar\gamma$, uniformly on $[t_1,t_2]$. Let $\mathcal{K} \subset \mathbb{R}^d$ be a compact set containing $\gamma_k([t_1,t_2])$ for all k ; such a set exists by the boundedness of γ_k. If we let $E_{\mathcal{K}} \subset [t_1,t_2]$ be the set of zero measure granted by the hypothesis, there exists to each $s \in [t_1,t_2]\backslash E_{\mathcal{K}}$ and each $\varepsilon > 0$ an integer $k_{s,\varepsilon}$ such that $\|X^k(s,x) - X(s,x)\| < \varepsilon$ as soon as $x \in \mathcal{K}$ and $k > k_{s,\varepsilon}$. In another connection, the Lipschitz character of X with respect to the second argument and the uniform convergence of γ_k to $\bar\gamma$ shows that that $\|X(s,\gamma_k(s)) - X(s,\bar\gamma(s))\| < \varepsilon$ for k large enough. Altogether, by a 2-ε majorization , we find that

$$\lim_{k\to\infty} \|X^k(s,\gamma_k(s)) - X(s,\bar\gamma(s))\| = 0,$$

that is to say the integrand in the right-hand side of (3.125) converges point-wise almost everywhere to $X(s,\bar\gamma(s))$. Since X^k is bounded we can apply the dominated convergence theorem and, taking limits on both sides of (3.125) as $k \to \infty$, we find that $\bar\gamma$ is a solution to (3.123) whereas the latter is unique. Hence $\bar\gamma - \gamma$ as desired. □

The following averaging lemma for continuous vector fields is less classical than in the locally Lipschitz case, where the Cauchy problem has a unique solution.

Lemma 3.7.4. *Let $t_1 < t_2$ be real numbers and $(X^{1,\ell})_{\ell\in\mathbb{N}}$, $(X^{2,\ell})_{\ell\in\mathbb{N}}$, be two sequences of continuous time-dependent vector fields $[t_1,t_2]\times\mathbb{R}^d \to \mathbb{R}^d$, uniformly bounded with respect to ℓ, that converge uniformly on compact subsets of $[t_1,t_2]\times\mathbb{R}^d$ to some vector fields X^1 and X^2 respectively. Denoting by $L = t_2 - t_1$ the length of the time interval, define, for each $\ell \in \mathbb{N}$, the "average" vector field $G_\ell : [t_1,t_2]\times\mathbb{R}^d \to \mathbb{R}^d$ by :*

$$\begin{array}{lcl} t \in [t_1 + \frac{j}{\ell}L\,,\, t_1 + \frac{2j+1}{2\ell}L) \;\Rightarrow\; G_\ell(t,x) &=& X^{1,\ell}(t,x)\ , \\ t \in [t_1 + \frac{2j+1}{2\ell}L\,,\, t_1 + \frac{j+1}{\ell}L) \;\Rightarrow\; G_\ell(t,x) &=& X^{2,\ell}(t,x)\ \ , \end{array} \tag{3.126}$$

for $j \in \{0,\ldots,\ell-1\}$ and, say, $G_\ell(t_2,x) = X^{2,\ell}(t_2,x)$ for definiteness.

Let $\gamma_\ell : [t_1,t_2] \to \mathbb{R}^d$ be a solution to

$$\gamma_\ell(t) \;-\; \bar{x} \;\;=\;\; \int_{t_1}^{t} G_\ell(\tau,\gamma_\ell(\tau))\mathrm{d}\tau\ . \tag{3.127}$$

Then the sequence (γ_ℓ) is compact in $\mathbf{C}^0([t_1,t_2],\mathbb{R}^d)$, and every accumulation point γ_∞ is a solution to

$$\gamma_\infty(t) \;-\; \bar{x} \;\;=\;\; \frac{1}{2}\int_{t_1}^{t} \left(X^1(\tau,\gamma_\infty(\tau)) + X^2(\tau,\gamma_\infty(\tau))\right)\mathrm{d}\tau\ . \tag{3.128}$$

Proof. Let

$$M \;=\; \sup_{t,x,i,\ell} \|X^{i,\ell}(t,x)\| \;. \tag{3.129}$$

From (3.126)-(3.127), it is clear that M is a Lipschitz constant for γ_ℓ, regardless of ℓ. In particular $\gamma_\ell(t)$ stays in a fixed compact ball B of radius ML, and the family (γ_ℓ) is equicontinuous. From Ascoli-Arzela's theorem this implies compactness of the sequence (γ_ℓ) in the uniform topology on $[t_1, t_2]$.

Rewrite (3.127) as

$$\begin{aligned}\gamma_\ell(t) \,-\, \bar{x} \;=\; & \int_{t_1}^{t} \left(G_\ell(\tau,\gamma_\ell(\tau)) \;-\; \frac{X^{1,\ell}(\tau,\gamma_\ell(\tau)) + X^{2,\ell}(\tau,\gamma_\ell(\tau))}{2} \right) \mathrm{d}\tau \\ + & \int_{t_1}^{t} \left(\frac{X^{1,\ell}(\tau,\gamma_\ell(\tau)) + X^{2,\ell}(\tau,\gamma_\ell(\tau))}{2} \;-\; \frac{X^{1}(\tau,\gamma_\ell(\tau)) + X^{2}(\tau,\gamma_\ell(\tau))}{2} \right) \mathrm{d}\tau \\ + & \int_{t_1}^{t} \frac{X^{1}(\tau,\gamma_\ell(\tau)) + X^{2}(\tau,\gamma_\ell(\tau))}{2} \,\mathrm{d}\tau \;. \end{aligned} \tag{3.130}$$

By the uniform convergence of $X^{i,\ell}$ to X^i, it will clearly follow that any accumulation point γ_∞ of (γ_ℓ) satisfies (3.128) if only we can show that the first integral in the right-hand side of (3.130) converges to zero as $\ell \to \infty$.

To prove this, we compute, from the definition of G_ℓ :

$$\begin{aligned} & \int_{t_1+\frac{j}{\ell}L}^{t_1+\frac{j+1}{\ell}L} \left(G_\ell(\tau,\gamma_\ell(\tau)) \;-\; \frac{X^{1,\ell}(\tau,\gamma_\ell(\tau)) + X^{2,\ell}(\tau,\gamma_\ell(\tau))}{2} \right) \mathrm{d}\tau \\ & = \int_{t_1+\frac{j}{\ell}L}^{t_1+\frac{2j+1}{2\ell}L} \frac{X^{1,\ell}(\tau,\gamma_\ell(\tau)) - X^{2,\ell}(\tau,\gamma_\ell(\tau))}{2}\,\mathrm{d}\tau \\ & \qquad - \int_{t_1+\frac{2j+1}{2\ell}L}^{t_1+\frac{j+1}{\ell}L} \frac{X^{1,\ell}(\tau,\gamma_\ell(\tau)) - X^{2,\ell}(\tau,\gamma_\ell(\tau))}{2}\,\mathrm{d}\tau \\ & = \int_{t_1+\frac{j}{\ell}L}^{t_1+\frac{2j+1}{2\ell}L} \left(\Delta_\ell(\tau,\gamma_\ell(\tau)) - \Delta_\ell(\tau+\tfrac{L}{2\ell},\gamma_\ell(\tau+\tfrac{L}{2\ell})) \right) \mathrm{d}\tau \end{aligned} \tag{3.131}$$

with $\Delta_\ell = \frac{1}{2}(X^{1,\ell} - X^{2,\ell})$. On the compact set $[t_1,t_2]\times B$, the vector field Δ_ℓ is uniformly continuous with a modulus of continuity that does not depend on ℓ ; consequently, by the uniform Lipschitz property of γ_ℓ, we see for arbitrary $\varepsilon > 0$ that the norm of the last integral is less that $\varepsilon/2\ell$ as soon as ℓ is large enough, independently of j.

Now, the first integral in (3.130) can be decomposed into a sum of at most ℓ integrals like these we just studied plus an integral over an interval of length smaller that $1/\ell$. Since the norm of the integrand is bounded by $2M$, the norm of the last term is less than $2M/\ell$. Summing over j, the above estimates tell us that, for $t \in [t_1,t_2]$ and for ℓ is large enough,

$$\int_{t_1}^{t} \left(G_\ell(\tau,\gamma_\ell(\tau)) \;-\; \frac{X^{1}(\tau,\gamma_\ell(\tau)) + X^{2}(\tau,\gamma_\ell(\tau))}{2} \right) \mathrm{d}\tau \;\le\; \frac{\varepsilon}{2} \,+\, \frac{2M}{\ell}\,.$$

This achieves the proof since $\varepsilon > 0$ was arbitrary. □

3.8 Appendix : Orbits of families of vector fields

In the proof of lemma 3.6.2 we need results from [101] on orbits[5] of families of smooth vector fields, that were recently exposed in the textbook [59, chapter II]. We recall them below, in a slightly expanded form.

Let $\mathcal{F}$ be a family of smooth vector fields defined on an open subset U of $\mathbb{R}^d$. For any positive integer N and vector fields $X^1, \ldots, X^N$ belonging to $\mathcal{F}$, given $m \in U$, consider the map F given by

$$(t_1, \ldots, t_N) \quad \mapsto \quad X^1_{t_1}(X^2_{t_2}(\cdots(X^N_{t_N}(m))\cdots)) \tag{3.132}$$

where the standard notation $X_t(x)$ indicates the flow of X from x at time t ; of course, F depends on the choice of the vector fields X^j and of the point m. This map is defined on some open connected neighborhood of the origin, hereafter denoted by $\mathrm{dom}(F)$, and takes values in U. In fact, $(t_1, \ldots, t_N) \in \mathrm{dom}(F)$ if, and only if, for every $j \in \{1, \ldots, N\}$, the solution $x(\tau)$ to $\dot{x} = X^j(x)$, with initial condition $x(0) = X^{j-1}_{t_{j-1}}(\cdots(X^1_{t_1}(m))\cdots)$, exists in U for all $\tau \in [0, t_j]$ (or $[t_j, 0]$ if $t_j < 0$).

The *orbit* of the family $\mathcal{F}$ through a point $m \in U$ is the set of all points that lie in the image of F for at least one choice of the vector fields $X^1, \ldots, X^N$. In words, the orbit of the family $\mathcal{F}$ through m is the set of points that may be linked to m in U upon concatenating finitely many integral curves of vector fields in the family. We shall denote by $\mathcal{O}_{\mathcal{F},p}$ the orbit of $\mathcal{F}$ through m.

Note that the definition depends on U in a slightly subtle manner : if $\mathcal{F}$ defines by restriction a family of vector fields $\mathcal{F}|_V$ on a smaller open set $V \subset U$ and if $m \in V$, then

$$V \cap \mathcal{O}_{\mathcal{F},m} \quad \supset \quad \mathcal{O}_{\mathcal{F}|_V,m}, \tag{3.133}$$

but the inclusion is generally strict because of the requirement that the integral curves used to construct $\mathcal{O}_{\mathcal{F}|_V,m}$ should lie entirely in V.

We turn to topological considerations. The topology of U is the usual Euclidean topology. The topology of $\mathcal{O}_{\mathcal{F},m}$ *as an orbit* is the finest that makes all the maps F, arising from (3.132), continuous on their respective domains of definition, the latter being endowed with the Euclidean topology. The classical smoothness of the flow implies that each F is continuous $\mathrm{dom}(F) \to \mathbb{R}^d$, hence the topology of $\mathcal{O}_{\mathcal{F},m}$ as an orbit is finer than the Euclidean topology induced by the ambient space U. *It can be strictly finer*, and this is why we speak of the *orbit topology*, as opposed to the *induced topology.*

Starting from $\mathcal{F}$, one defines a larger family of vector fields $P_{\mathcal{F}}$, consisting of all the push-forwards[6] of vector fields in $\mathcal{F}$ through all local diffeomorphisms of the form $X^1_{t_1} \circ X^2_{t_2} \circ \cdots \circ X^N_{t_N}$ where $X^1, \ldots, X^N$ belong to $\mathcal{F}$. That is to say, vector fields in $P_{\mathcal{F}}$ are of the form

$$\left(X^1_{t_1} \circ \cdots \circ X^N_{t_N}\right)_\star X^0 \tag{3.134}$$

where $X^0, X^1, \ldots, X^N$ belong to $\mathcal{F}$.

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[5] One of the motivations in [101] was to generalize the notion of integral manifolds to vector fields that are smooth but not real analytic. Note that the orbits of a family of *real analytic* vector fields actually coincide with the maximal integral manifolds of the closure of this family under Lie brackets [101, 67, 78]. However, even if we assume the control system (3.4) to be real analytic, integral manifolds are of no help to us because topological conjugacy does not preserve tangency nor Lie brackets. Using orbits of families of vector fields instead is much more efficient, because topological conjugacy does preserve integral curves.

[6] Recall that the push-forward of a vector field $X : V \to \mathbb{R}^d$ through a diffeomorphism $\varphi : V \to \varphi(V)$ is the vector field $\varphi_\star X$ on $\varphi(V)$ whose flow at each time is the conjugate of the flow of X under the diffeomorphism φ ; it can be defined as $\varphi_\star X(\varphi(x)) = D\varphi(x)X(x)$, where $D\varphi(x)$ is the derivative of φ at $x \in V$.

Remark 3.8.1. *Note that a member of $P_{\mathcal{F}}$ is defined on an open set which is generally a strict subset of U, whereas members of $\mathcal{F}$ are defined over the whole of U, and it is understood that a curve $\gamma : I \to U$, where I is a real interval, will be called an integral curve of $Y \in P_{\mathcal{F}}$ only when $\gamma(I)$ is included in the domain of definition of Y.*

For $x \in U$, we denote by $P_{\mathcal{F}}(x)$ the subspace of $\mathbb{R}^d$ spanned by all the vectors $Y(x)$, where $Y \in P_{\mathcal{F}}(x)$ is defined in a neighborhood of x.

Theorem 3.8.2 below, which is the central result in this appendix, describes the topological nature of the orbits. To interpret the statement correctly, it is necessary to recall (see for instance [99]) that an *immersed* sub-manifold of a manifold is a subset of the latter which is a manifold in its own right, and is such that the inclusion map is an immersion. This allows one to naturally identify the tangent space to an immersed sub-manifold at a given point with a linear subspace of the tangent space to the ambient manifold at the same point. The topology of an immersed sub-manifold is in general finer than the one induced by the ambient manifold ; when these two topologies coincide, the sub-manifold is called *embedded*.

Theorem 3.8.2 (Orbit Theorem, Sussmann [101])**.** *Let $\mathcal{F}$ be a family of smooth vector fields defined on an open set $U \subset \mathbb{R}^d$, and m be a point in U. If $\mathcal{O}_{\mathcal{F},m}$ denotes the orbit of $\mathcal{F}$ through m, then :*

(i) Endowed with the orbit topology, $\mathcal{O}_{\mathcal{F},m}$ has a unique differential structure that makes it a smooth connected immersed sub-manifold of U, for which the maps (3.132) *are smooth.*

(ii) The tangent space to $\mathcal{O}_{\mathcal{F},m}$ at $x \in \mathcal{O}_{\mathcal{F},m}$ is $P_{\mathcal{F}}(x)$.

(iii) There exists an open neighborhood W of m <u>in U</u>, and smooth local coordinates $\xi : W \to (-\eta,\eta)^d \subset \mathbb{R}^d$, with $\xi(m) = 0$, such that

(a) in these coordinates, $W \cap \mathcal{O}_{\mathcal{F},m}$ is a product :

$$W \cap \mathcal{O}_{\mathcal{F},m} \quad = \quad (-\eta,\eta)^q \times T \tag{3.135}$$

where $\eta > 0$, q is the dimension of $\mathcal{O}_{\mathcal{F},m}$, and T is some subset of $(-\eta,\eta)^{d-q}$ containing the origin. The orbit topology of $\mathcal{O}_{\mathcal{F},m}$ induces on $W \cap \mathcal{O}_{\mathcal{F},m}$ the product topology where $(-\eta,\eta)^q$ is endowed with the usual Euclidean topology and T with the discrete topology.

(b) if $\gamma : [t_1,t_2] \to W \cap \mathcal{O}_{\mathcal{F},m}$ is an integral curve of a vector field $Y \in P_{\mathcal{F}}$ (see remark 3.8.1), then $t \mapsto \xi_i(\gamma(t))$, $q+1 \le i \le d$, are constant mappings,

(c) the tangent space to $\mathcal{O}_{\mathcal{F},m}$ at each point $p \in W \cap \mathcal{O}_{\mathcal{F},m}$ is spanned by the vector fields $\partial/\partial\xi_1, \ldots, \partial/\partial\xi_q$,

(d) at any point $p \in W$, the vector fields $\partial/\partial\xi_1, \ldots, \partial/\partial\xi_q$ belong to the tangent space to the orbit of $\mathcal{F}$ through p.

Remark 3.8.3. *Another description of the product topology in point $(iii)-(a)$ is as follows. The connected components of $W \cap \mathcal{O}_{\mathcal{F},m}$ are the sets*

$$S_{W,a} \quad = \quad (-\eta,\eta)^q \times \{a\} \tag{3.136}$$

for $a \in T$, and the topology on each of these connected components is the topology induced by the ambient Euclidean topology. In particular each $S_{W,a}$ is an embedded *sub-manifold of U.*

Proof of Theorem 3.8.2. Assertion (i) is the standard form of the orbit theorem (cf. e.g. [59, Chapter 2, Theorem 1]), while assertion (ii) is a rephrasing of [101, Theorem 4.1, point (b)]. Assertion (iii) apparently cannot be referenced exactly in this form, but we shall deduce it from the previous ones as follows.

By point (ii), the tangent space to $\mathcal{O}_{\mathcal{F},m}$ at $m \in S$ is the linear span over $\mathbb{R}$ of $Y^1(m), \ldots, Y^q(m)$, where $Y^1, \ldots, Y^q$ are q vector fields belonging to $P_{\mathcal{F}}$, defined on some neighborhood of m, and such that $Y^1(m),\ldots,Y^q(m)$ are linearly independent (recall that q is the dimension of $\mathcal{O}_{\mathcal{F},m}$). Let us write

$$Y^j \;=\; \left(X^{j,1}_{t_{j,1}}\circ\cdots\circ X^{j,N_j}_{t_{j,N_j}}\right)_{\star} X^{j,0}, \qquad 1\le j\le q,$$

where $X^{j,k}\in\mathcal{F}$ for $0\le k\le N_j$, and where the $t_{j,k}$'s are real numbers for which the concatenated flow exists, locally around m (compare (3.134)).

Since $Y^1(m),\ldots,Y^q(m)$ are linearly independent, one may complement them into a basis of $\mathbb{R}^d$ by adjunction of $d-q$ independent vectors that may, without loss of generality, be regarded as values at m of $d-q$ smooth vector fields in U, say $Y^{q+1},\ldots,Y^d$. Then, the smooth map

$$L(\xi_1,\ldots,\xi_d) \;=\; \left(Y^1_{\xi_1}\circ\cdots\circ Y^q_{\xi_q}\circ Y^{q+1}_{\xi_{q+1}}\circ\cdots\circ Y^d_{\xi_d}\right)(m) \tag{3.137}$$

defines a diffeomorphism from some poly-interval $\mathcal{I}_\eta=\{(\xi_1,\ldots,\xi_d),\ |\xi_i|<\eta\}$ onto an open neighborhood W of m in U, simply because the derivative of L is invertible at the origin as $Y^1(m),\ldots,Y^d(m)$ are linearly independent by construction. Let $\xi:W\to\mathcal{I}_\eta$ denote its inverse.

By the characteristic property of push-forwards, we locally have, for $1\le j\le q$, that

$$Y^j_{\xi_j} = X^{j,1}_{t_{j,1}}\circ\cdots\circ X^{j,N_j}_{t_{j,N_j}}\circ X^{j,0}_{\xi_j}\circ X^{j,N_j}_{-t_{j,N_j}}\circ\cdots\circ X^{j,1}_{-t_{j,1}}\,. \tag{3.138}$$

This implies that, in (3.137), the images under L of those d-tuples sharing a common value of $\xi_{q+1},\ldots,\xi_d$ all lie in the same orbit $\mathcal{O}_{\mathcal{F},L(0,\ldots,0,\xi_{q+1},\ldots,\xi_d)}$. In particular, the map

$$\tau_1,\ldots,\tau_q \;\mapsto\; \left(Y^1_{\tau_1+\xi_1}\circ\cdots\circ Y^q_{\tau_q+\xi_q}\circ Y^{q+1}_{\xi_{q+1}}\circ\cdots\circ Y^d_{\xi_d}\right)(m)$$

is defined $\Pi_{j=1}^q(-\eta-\xi_j,\eta-\xi_j)\to W\cap\mathcal{O}_{\mathcal{F},L(\xi_1,\ldots,\xi_d)}$, and this map is smooth from the Euclidean to the orbit topology by (3.138) and point (i). If we compose it with the immersive injection $J_W: W\cap\mathcal{O}_{\mathcal{F},L(\xi_1,\ldots,\xi_d)}\to W$ (keeping in mind that $W\cap\mathcal{O}_{\mathcal{F},L(\xi_1,\ldots,\xi_d)}$ is open in $\mathcal{O}_{\mathcal{F},L(\xi_1,\ldots,\xi_d)}$ since the orbit topology is finer than the Euclidean one), and if we subsequently apply ξ, we get the affine map

$$\tau_1,\ldots,\tau_q \;\mapsto\; (\tau_1+\xi_1,\cdots,\tau_q+\xi_q,\xi_{q+1},\cdots,\xi_d). \tag{3.139}$$

Thus the derivative of (3.139) factors through the derivative of $\xi\circ J_W$ at $L(\xi_1,\ldots,\xi_d)$, which implies (d) ; from this (c) follows, because q is the dimension of the orbit through m. If $Y\in P_{\mathcal{F}}$ is defined over an open subset of W, and if we write in the ξ coordinates $Y(\xi)=\sum_i a_i(\xi)\partial/\partial\xi_i$, then, since $Y(\xi)$ is tangent to $\mathcal{O}_{\mathcal{F},\xi}$ by (ii), we deduce from (c), that the functions $a_{q+1},\ldots,a_d$ vanish on $\mathcal{O}_{\mathcal{F},m}$, whence (b) holds.

We finally prove (a). Considering (3.137) and (3.138), a moment's thinking will convince the reader that $W\cap\mathcal{O}_{\mathcal{F},m}$ consists exactly, in the ξ coordinates, of those $(\xi_1,\ldots,\xi_d)$ such that

$$\left(Y^{q+1}_{\xi_{q+1}}\circ\cdots\circ Y^d_{\xi_d}\right)(m)\in\mathcal{O}_{\mathcal{F},m}, \tag{3.140}$$

which accounts for (3.135) where T is the set of $(d-q)$-tuples $(\xi_{q+1},\ldots,\xi_d)$ such that (3.140) holds. To prove that the orbit topology is the product topology on $(-\eta,\eta)^q\times T$ where T is discrete, consider a map F as in (3.132), and pick $\bar t=(\bar t_1,\ldots,\bar t_N)\in\mathrm{dom}(F)$ such that $F(\bar t)\in W$ (hence $F(\bar t)\in W\cap\mathcal{O}_{\mathcal{F},m}$) ; then F is continuous at $\bar t$ for the product topology because, for t close enough to $\bar t$, the values $\xi_{q+1}(F(t)),\ldots,\xi_d(F(t))$ do not depend on t by (b) (moving t_i means following the flow of a vector field in $P_{\mathcal{F}}$, namely the push-forward of X^i through $X^1_{t_1}\circ\cdots\circ X^{i-1}_{t_{i-1}}$) while $\xi_1(F(t)),\ldots,\xi_q(F(t))$ vary continuously with t according to the continuous dependence on time and initial conditions of solutions to differential equations. Since this is true for all maps F, the orbit topology on $W\cap\mathcal{O}_{\mathcal{F},m}$ is finer than the product topology. To show that it cannot be strictly finer, it is enough to prove that the orbit topology coincides with the Euclidean topology on each set $S_{W,a}$ defined in (3.136), a basis of which consists of the sets $O \times \{a\}$ where O is open in $(-\eta, \eta)^q$. Being open for the product topology, these sets are open the orbit topology as well by what precedes and, since $\mathcal{O}_{\mathcal{F},m}$ is a manifold by (i), each point $(y, a) \in O \times \{a\}$ has, in the orbit topology, a neighborhood $\mathcal{N}_y \subset O \times \{a\}$ which is homeomorphic to an open ball of $\mathbb{R}^q$ via some coordinate map. When viewed in these coordinates, the injection $\mathcal{N}_y \to O \times \{a\}$ from the orbit topology to the Euclidean topology is a continuous injective map from an open ball in $\mathbb{R}^q$ into $\mathbb{R}^q$, and therefore it is a homeomorphism onto its image by invariance of the domain. As (y, a) was arbitrary in $O \times \{a\}$, this shows the latter is a union of open sets for the orbit topology, as desired. □

Consider now the control system :

$$\dot{x} \;=\; f(x, u), \tag{3.141}$$

with state $x \in \mathbb{R}^d$ and control $u \in \mathbb{R}^r$, the function f being smooth on $\mathbb{R}^d \times \mathbb{R}^r$. Let Ω be an open subset of $\mathbb{R}^d \times \mathbb{R}^r$ and, following the notation introduced in section 3.3, put $\Omega_{\mathbb{R}^d}$ to denote its projection onto the first factor. In the proof of Theorem 3.5.2, we shall be concerned with the following family of vector fields on $\Omega_{\mathbb{R}^d}$:

$$\mathcal{F}' \;=\; \{\, \delta f_{\alpha_1,\alpha_2}\,,\ \alpha_1, \alpha_2 \text{ feedbacks on } \Omega \,\}\;, \tag{3.142}$$

where feedbacks on Ω were introduced in Definition 3.3.3 and the notation $\delta f_{\alpha_1,\alpha_2}$ was fixed in (3.25), (3.26).

Since feedbacks are only required to be *continuous*, $\mathcal{F}'$ is a family of continuous *but not necessarily differentiable* vector fields on $\Omega_{\mathbb{R}^d}$ and, though the existence of solutions to differential equations with continuous right-hand side makes it still possible to define the orbit as the collection of endpoints of all concatenated integrations like (3.132), Theorem 3.8.2 does not apply in this case.

To overcome this difficulty, we will consider instead of $\mathcal{F}'$ the smaller family :

$$\mathcal{F}'' \;=\; \{\, X \in \mathcal{F}'\,,\ X \text{ has a flow} \,\}\;, \tag{3.143}$$

where the sentence "X has a flow" means, as in appendix 3.7, that the Cauchy problem $\dot{x}(t) = X(x(t))$, $x(0) = x_0$, has a unique solution, defined for $|t| < \varepsilon_0$ where ε_0 may depend on x_0, whenever x_0 lies in the domain of definition of X. Let us consider the orbit $\mathcal{O}_{\mathcal{F}'',m}$ of $\mathcal{F}''$ through $m \in \Omega_{\mathbb{R}^d}$, which is still defined as the union of images of all maps (3.132) where $X^j \in \mathcal{F}''$, the domain of each such map F being again a connected open neighborhood $\mathrm{dom}(F)$ of the origin in $\mathbb{R}^N$ by repeated application of Lemma 3.7.1. As before, we define the orbit topology on $\mathcal{O}_{\mathcal{F}'',m}$ to be the finest that makes all the maps (3.132) continuous, and since uniqueness of solutions implies continuous dependence on initial conditions (see Lemma 3.7.1), the orbit topology is again finer than the Euclidean topology. A priori, we know very little about $\mathcal{O}_{\mathcal{F}'',m}$ and its orbit topology as Theorem 3.8.2 does not apply. However, Proposition 3.8.5 below will establish that these notions coincide with those arising from the family $\mathcal{F}$ of *smooth* vector fields obtained by setting :

$$\mathcal{F} \;=\; \{\, \delta f_{\alpha_1,\alpha_2}\,,\ \alpha_1, \alpha_2 \ \underline{\text{smooth}} \text{ feedbacks on } \Omega \,\}. \tag{3.144}$$

Note that, from the definitions (3.142), (3.143) and (3.144), we obviously have

$$\mathcal{F} \;\subset\; \mathcal{F}'' \;\subset\; \mathcal{F}'\;, \tag{3.145}$$

hence the orbits of these families through a given point obey the same inclusions.

Remark 3.8.4. *It may of course happen that the family $\mathcal{F}'$ is empty because Ω admits no feedback at all. However, if $\mathcal{F}'$ is not empty, then $\mathcal{F}$ is not empty either by Proposition 3.3.4.*

Proposition 3.8.5. *Suppose that $f : \mathbb{R}^d \times \mathbb{R}^r \to \mathbb{R}^d$ is smooth, and let Ω be an open subset of $\mathbb{R}^d \times \mathbb{R}^r$. Let $\mathcal{F}''$ be defined by* (3.142)-(3.143)*.*

For any $m \in \Omega_{\mathbb{R}^d}$, the orbit $\mathcal{O}_{\mathcal{F}'',m}$ of $\mathcal{F}''$ through m coincides with the orbit through m of the family $\mathcal{F}$ of smooth *vector fields defined by* (3.144)*, and the topology of $\mathcal{O}_{\mathcal{F}'',m}$, as an orbit of $\mathcal{F}$, coincides with its topology as an orbit of $\mathcal{F}''$. In particular, the conclusions of Theorem 3.8.2 hold if we replace $\mathcal{F}$ by $\mathcal{F}''$ and U by $\Omega_{\mathbb{R}^d}$.*

Remark 3.8.6. *With a limited amount of extra-work, it is possible to show that the orbits of $\mathcal{F}'$ also coincide with those of $\mathcal{F}$. Hence they turn out to be manifolds despite the possible non-uniqueness of solutions to the Cauchy problem. However,* (3.132) *is no longer convenient to define the orbit topology in this case because the maps F may be multiply-valued when $X^j \in \mathcal{F}'$, and it is simpler to work with the family $\mathcal{F}''$ anyway.*

The proof of the proposition is based on the following lemma.

Lemma 3.8.7. *For $m \in \Omega_{\mathbb{R}^d}$ and $X^1, \ldots, X^N \in \mathcal{F}''$, let $F : \mathrm{dom}(F) \to \Omega_{\mathbb{R}^d}$ be defined by* (3.132)*. Fix $\bar{t} = (\bar{t}_1, \ldots, \bar{t}_N) \in \mathrm{dom}(F)$ and set $\overline{m} = F(\bar{t})$.*

Then, there is a neighborhood $\mathcal{T}$ of $\bar{t}$ in dom(F)*, with $F(\mathcal{T}) \subset \mathcal{O}_{\mathcal{F},\overline{m}}$, such that $F : \mathcal{T} \to \mathcal{O}_{\mathcal{F},\overline{m}}$ is continuous from the Euclidean topology to the orbit topology.*

Assuming the lemma for a while, we first prove the proposition.

Proof of Proposition 3.8.5. We noticed already from (3.145) that the orbit of $\mathcal{F}''$ through m contains the orbit of $\mathcal{F}$ through m. To get the reverse inclusion, consider the map F defined by (3.132) for some vector fields $X^1, \ldots, X^N$ belonging to $\mathcal{F}''$. Then, observe from Lemma 3.8.7 that F takes values in a disjoint union of orbits of $\mathcal{F}$, and that it is continuous if each orbit in this union is endowed with the orbit topology. Since dom(F) is connected, F takes values in a single orbit, which can be none but $\mathcal{O}_{\mathcal{F},m}$. As F was arbitrary, we conclude that $\mathcal{O}_{\mathcal{F}'',m} \subset \mathcal{O}_{\mathcal{F},m}$ and therefore the two orbits agree as sets. Moreover, since each map F was continuous dom(F) $\to \mathcal{O}_{\mathcal{F},m}$, the orbit topology of $\mathcal{O}_{\mathcal{F}'',m}$ is by definition finer than the orbit topology of $\mathcal{O}_{\mathcal{F},m}$; but since it is also coarser, by definition of the orbit topology on $\mathcal{O}_{\mathcal{F},m}$, because $\mathcal{F} \subset \mathcal{F}''$, the two topologies in turn agree as desired. □

Proof of Lemma 3.8.7. Theorem 3.8.2 applied to the family $\mathcal{F}$, at the point $\overline{m} = F(\bar{t})$, yields an open neighborhood W of $\overline{m}$ in $\Omega_{\mathbb{R}^d}$ and smooth local coordinates $(\xi_1, \ldots, \xi_d) : W \to (-\eta, \eta)^d$ satisfying properties $(iii)-(a)$ to $(iii)-(d)$ of that theorem. For $\varepsilon > 0$ denote by $\mathcal{T}_\varepsilon$ the compact poly-interval :

$$\mathcal{T}_\varepsilon \quad = \quad \{t = (t_1, \ldots, t_N) \in \mathbb{R}^N,\ |t_i - \bar{t}_i| \leq \varepsilon\} \ .$$

By Lemma 3.7.1, F is continuous dom(F) $\to \Omega_{\mathbb{R}^d}$ and, since dom(F) is an open neighborhood of $\bar{t}$ in $\mathbb{R}^N$, we can pick $\varepsilon > 0$ such that

$$\mathcal{T}_\varepsilon \subset \mathrm{dom}(F) \quad \text{and} \quad F(\mathcal{T}_\varepsilon) \subset W \ .$$

As $X^1, \ldots, X^N$ belong to $\mathcal{F}'' \subset \mathcal{F}'$, we can write

$$X^\ell \quad = \quad \delta f_{\alpha_1^\ell, \alpha_2^\ell} \ , \quad 1 \leq \ell \leq N$$

for some collection of feedbacks α_1^ℓ, α_2^ℓ on Ω. From Proposition 3.3.4, there exists for each $(\ell, l) \in \{1, \ldots, N\} \times \{1, 2\}$ a sequence of *smooth* feedbacks on Ω, say $(\beta_l^{\ell,k})_{k\in\mathbb{N}}$, converging to

α_l^ℓ uniformly on $\Omega_{\mathbb{R}^d}$. Subsequently, we let $Y^{\ell,k}$ denote, for $1 \leq \ell \leq N$ and $k \in \mathbb{N}$, the smooth vector field on $\Omega_{\mathbb{R}^d}$

$$Y^{\ell,k} \;=\; \delta f_{\beta_1^{\ell,k},\beta_2^{\ell,k}} \;.$$

Clearly $Y^{\ell,k} \in \mathcal{F}$ and, for each ℓ, we have that $Y^{\ell,k}$ converges to X^ℓ as $k \to \infty$, uniformly on compact subsets of $\Omega_{\mathbb{R}^d}$.

Now, pick $j \in \{1, \ldots, N\}$ and consider a N-tuple $t^{(j)} \in \mathcal{T}_\varepsilon$ of the form :

$$t^{(j)} = (\bar{t}_1, \ldots, \bar{t}_{j-1}, t_j, \ldots, t_N)\,, \qquad |t_\ell - \bar{t}_\ell| \leq \varepsilon \;\text{ for } j \leq \ell \leq N.$$

Let also $\mathbf{1}_j$ designate, for simplicity, the N-tuple $(0, \ldots, 1, \ldots, 0)$ with zero entries except for the j-th one which is 1. Then, for $|\lambda| \leq \varepsilon$, we have that

$$t^{(j)} + \lambda \mathbf{1}_j = (\bar{t}_1, \ldots, \bar{t}_{j-1}, \bar{t}_j + \lambda, t_{j+1}, \ldots, t_N) \;\in \mathcal{T}_\varepsilon,$$

and a simple computation allows us to rewrite $F(t + \lambda \mathbf{1}_j)$ as :

$$F(t^{(j)} + \lambda \mathbf{1}_j) \;=\; X^1_{\bar{t}_1} \circ \cdots \circ X^{j-1}_{\bar{t}_{j-1}} \circ X^j_\lambda \circ X^{j-1}_{-\bar{t}_{j-1}} \circ \cdots \circ X^1_{-\bar{t}_1}(F(t)).$$

Let us set

$$A_k(\lambda) \;=\; Y^{1,k}_{\bar{t}_1} \circ \cdots \circ Y^{j-1,k}_{\bar{t}_{j-1}} \circ Y^{j,k}_\lambda \circ Y^{j-1,k}_{-\bar{t}_{j-1}} \circ \cdots \circ Y^{1,k}_{-\bar{t}_1}(F(t)).$$

Repeated applications of Lemmas 3.7.1 and 3.7.2 show that, for fixed j and $t^{(j)}$, the map $\lambda \mapsto A_k(\lambda)$ is well-defined $[-\varepsilon, \varepsilon] \to W$ as soon as the integer k is sufficiently large, and moreover that $A_k(\lambda)$ converges to $F(t^{(j)} + \lambda \mathbf{1}_j)$ as $k \to +\infty$, uniformly with respect to $\lambda \in [-\varepsilon, \varepsilon]$. Now, by the characteristic property push forwards, $\lambda \mapsto A_k(\lambda)$ is an integral curve of the smooth vector field

$$Z^k \;=\; \left(Y^{1,k}_{\bar{t}_1} \circ \cdots \circ Y^{j-1,k}_{\bar{t}_{j-1}}\right)_\star Y^{j,k} \;,$$

which is defined on a neighborhood of $\{F(t^{(j)} + \lambda \mathbf{1}_j);\ |\lambda| \leq \varepsilon\}$ in W. Since $Z^k \in P_{\mathcal{F}}$ (cf. equation (3.134)), it follows from point $(iii) - (b)$ of Theorem 3.8.2 that, for k large enough,

$$\xi_i \circ A_k(\lambda) \;=\; \xi_i \circ A_k(0)\;, \quad \forall \lambda \in [-\varepsilon, \varepsilon],\ i \in \{q+1, \ldots, d\}.$$

It is clear from the definition that $A_k(0) = F(t^{(j)})$; hence, using the continuity of ξ_i and taking, in the above equation, the limit as $k \to +\infty$, we get

$$\xi_i \circ F(t^{(j)} + \lambda \mathbf{1}_j) = \xi_i \circ F(t^{(j)}),\ \forall \lambda \in [-\varepsilon, \varepsilon],\ i \in \{q+1, \ldots, d\}. \tag{3.146}$$

Since $\xi_{q+1} \circ F(\bar{t}) = \cdots = \xi_d \circ F(\bar{t}) = 0$ by definition of W, successive applications of (3.146) for $j = N, \ldots, 1$ lead us to the conclusion that

$$\xi_{q+1} \circ F(t) = \cdots = \xi_d \circ F(t) = 0, \quad \forall t \in \mathcal{T}_\varepsilon \;. \tag{3.147}$$

Equation (3.147) means that, in the ξ-coordinates, $F(\mathcal{T}_\varepsilon) \subset (-\eta, \eta)^q \times \{0\}$. Hence, from the local description of the orbits in (3.135) (where m is to be replaced by $\overline{m}$), we deduce that $F(\mathcal{T}_\varepsilon) \subset \mathcal{O}_{\mathcal{F},\overline{m}}$. Actually, with the notations of (3.136), we even get the stronger conclusion that

$$F(\mathcal{T}_\varepsilon) \;\subset\; S_{W,0}$$

which achieves the proof of the lemma, with $\mathcal{T} = \mathcal{T}_\varepsilon$, because the orbit topology on $S_{W,0}$ is the Euclidean topology by Remark 3.8.3. □

Chapitre 4

Reproduction de l'article:

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Cet article est une discussion sur les deux précédents.

Abstract

This note deals with "Grobman-Hartman like" theorems for control systems (or in other words under-determined systems of ordinary differential equations). The main results (proved elsewhere) is that when a control system is topologically conjugate to a linear controllable one, then it is also "almost" differentiably conjugate. We focus on the meaning of this result, and on an open question resulting from it.

4.1 Introduction

In this note, we discuss the *local* behavior of a nonlinear control system

$$\dot{x} = f(x,u) \ , \quad x \in \mathbb{R}^n \ , \quad u \in \mathbb{R}^m \ , \tag{4.1}$$

say around $(0,0) \in \mathbb{R}^{n+m}$. For general control systems (as opposed e.g. to affine in the control), "local" has to be understood with respect to both state *and* control.

The first reaction when dealing with local properties is to compute the linear approximation of (4.1). When this linear control system happens to be controllable, all the *local* usual control objectives can be met using linear control, based on the linear approximation. For instance, a linear control that asymptotically stabilizes the linear approximation will also stabilize the nonlinear system, locally ; minimizing a quadratic cost can also be achieved up to first order based on the linear approximation only. Hence, the linear approximation is a good enough model for the purpose of designing controllers achieving a desired behavior for small states and controls. We believe that all control engineers or control theorists agree on this statement, arising from practice, although we would welcome some contradiction.

Rephrasing the above statement without reference to control objectives leads to an imprecise statement, grounded mostly on some necessarily subjective intuition, and that should rather be taken as an opening sentence to launch a debate than as a conjecture :

$$\begin{cases} \textit{nothing distinguishes qualitatively the behavior of a nonlinear control} \\ \textit{system from the one of its linear approximation if the latter is controllable.} \end{cases} \tag{4.2}$$

It is natural to try to formalize this statement, as a prerequisite to any proper theory of nonlinear modeling and identification of control systems, in a very preliminary manner since it only deals with *local* phenomena. A nice way to turn that belief into a sound, and correct, assertion would be to find some equivalence relation between control systems (or models) that preserves at least “qualitative” behavior, and for which these two systems (a nonlinear system and its controllable linear approximation) are in general equivalent.

We assume controllability of the linear approximation. When this fails none of the above is correct, at least in the most common case when the nonlinear system is itself controllable. Indeed, (non-)controllability is a qualitative phenomenon : for instance, feeding a linear non controllable system with “random” inputs, one observes that the state is confined in leafs of positive codimension, while for a controllable system the whole state space is explored.

To enlighten the discussion on local behavior of control systems, let us recall the situation for ordinary differential equations $\dot{x} = F(x)$ (particular case of (4.1) where the control u has dimension 0) :

- If $F(0) \neq 0$, the “flow-box theorem” (see e.g. [5, §7]), gives local coordinates, smooth if F is smooth, in which F is of the form $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.
- If $F(0) = 0$ and the square matrix $F'(0)$ has no pure imaginary eigenvalue (hyperbolic equilibrium), then Grobman-Hartman Theorem [47, Theorem IX-7.1] tells us that the flow of the differential equation is locally conjugate to the flow of its linear approximation via a homeomorphism that need not, in general, be smooth if F is smooth (and in fact smooth conjugation requires more assumption, resonances are obstructions to it) (see e.g. [6, §22]).
- If $F(0) = 0$ and the square matrix $F'(0)$ has some pure imaginary eigenvalue, then the situation is more intricate even locally, namely the phase portrait of the nonlinear dynamical system $\dot{x} = F(x)$ can be very different locally from the one of a linear system. This case is of high interest in the theory of dynamical systems, but can be considered as “degenerate”, in the same way as non controllability of the linear approximation for control systems.

Since conjugation of flows does preserve qualitative phenomena like the overall aspect of the phase portrait, one can indeed assert that, locally around all points except non hyperbolic equilibria, a differentiable dynamical system “behaves like” a linear one, and this is translated by conjugation via a homeomorphism, although conjugation via a smooth diffeomorphism preserves some more subtle local invariants (resonances, etc...).

Coming back to control systems, first of all, the equivalent of conjugation by a smooth change of coordinates is *(smooth) feedback equivalence*, whose study was initiated in [15], see a survey in [54]. In fact this is conjugation via a smooth diffeomorphism on the state and control, forced to have a triangular structure (see Proposition 4.2.5 below). The conditions under which a control system (4.1) is smoothly feedback equivalent to a linear controllable one are well known [57, 50] (and contrary to the case of ordinary differential equations, they are very simple), but they reveal that very few nonlinear systems are locally feedback equivalent to a linear one, even when the linear approximation is controllable. This remark and the review of the situation for ordinary differential equations naturally brings about the question whether for control systems, relaxing the regularity of the conjugating maps, i.e. considering conjugacy by homeomorphisms instead of smooth diffeomorphisms would make more systems equivalent to a linear one.

After recalling some basic facts in section 4.2, we give in section 4.3 an essentially negative answer to the question evoked above, based on quoting a result from Chapter 3, that topological conjugacy to a linear controllable system implies conjugacy by “almost” smooth feedback (but the gap is really small). Section 4.4 recalls, also from Chapter 3, a technical open question that would allow a nicer result and a nicer description of that "almost" smooth conjugacy, and finally section 4.5 extends the discussion of the results from section 4.3, their implications, and the questions they raise in nonlinear modeling.

4.2 Preliminaries on equivalence of control systems

4.2.1 Definitions

Consider two smooth control systems with state x (resp. z) and input u (resp. v) :

$$\begin{aligned} \dot{x} &= f(x,u)\ , \quad x \in \mathbb{R}^n\ , \quad u \in \mathbb{R}^m\ , && (4.3)\\ \dot{z} &= g(z,v)\ , \quad z \in \mathbb{R}^{n'}\ , \quad v \in \mathbb{R}^{m'}\ , && (4.4)\end{aligned}$$

or, expanded in coordinates,

$$\dot{x}_i \;=\; f_i(x_1,\ldots,x_n,u_1,\ldots,u_m), \qquad \dot{z}_j \;=\; g_j(z_1,\ldots,z_{n'},v_1,\ldots,v_{m'}),$$

$1 \le i \le n$, $1 \le j \le n'$, with the f_i's and g_i's some smooth (i.e. C^∞) maps.

We assume that f and g are defined respectively on the whole of $\mathbb{R}^n \times \mathbb{R}^m$ and $\mathbb{R}^{n'} \times \mathbb{R}^{m'}$ because it simplifies many of the statements below ; this is actually no loss of generality to us for all the results we prove are local with respect to x, u, z, v, so that f and g can be extended using partitions of unity outside some neighborhoods of the arguments under consideration without affecting the results.

Definition 4.2.1. *By* a solution of (4.3) that remains in an open set $\Omega \subset \mathbb{R}^{n+m}$, *we mean a mapping γ defined on a real interval :*

$$\begin{array}{rccl} \gamma : & I & \to & \Omega \\ & t & \mapsto & \gamma(t) \;=\; (\,\gamma_{\mathrm{I}}(t)\,,\,\gamma_{\mathrm{II}}(t)\,)\ , \end{array} \qquad (4.5)$$

with $\gamma_{\mathrm{I}}(t) \in \mathbb{R}^n$ and $\gamma_{\mathrm{II}}(t) \in \mathbb{R}^m$, such that γ is measurable, locally bounded, γ_{I} is absolutely continuous and, whenever $[T_1, T_2] \subset I$, we have :

$$\gamma_{\mathrm{I}}(T_2) - \gamma_{\mathrm{I}}(T_1) \;=\; \int_{T_1}^{T_2} f(\,\gamma_{\mathrm{I}}(t)\,,\,\gamma_{\mathrm{II}}(t)\,)\,\mathrm{d}t\ .$$

Solutions of (4.4) that remain in $\Omega' \subset \mathbb{R}^{n'+m'}$ are likewise defined to be mappings $\gamma' : I \to \Omega'$ having the corresponding properties with respect to g.

We now define the notion of conjugacy for control systems.

Definition 4.2.2. *Let*

$$\begin{array}{rccl} \chi : & \Omega & \to & \Omega' \\ & (x,u) & \mapsto & \chi(x,u) \;=\; (\,\chi_{\mathrm{I}}(x,u)\,,\,\chi_{\mathrm{II}}(x,u)\,) \end{array} \qquad (4.6)$$

be a bijective mapping between two open subsets of $\mathbb{R}^{n+m}$ and $\mathbb{R}^{n'+m'}$ respectively. We say that χ conjugates *$\gamma : I \to \Omega$ and $\gamma' : I \to \Omega'$ if and only if $\gamma' = \chi \circ \gamma$.*

We say that χ conjugates *systems (4.3) and (4.4) if, for any real interval I, a map $\gamma : I \to \Omega$ is a solution of (4.3) that remains in Ω if, and only if, $\chi \circ \gamma$ is a solution of (4.4) that remains in Ω'.*

We say that systems (4.3) and (4.4) are locally topologically conjugate *at $(0,0)$ if we can chose Ω and Ω' to be neighborhoods of the origin and χ a homeomorphism. We say that they are* locally smoothly conjugate *if, in addition, χ and χ^{-1} are smooth. Here the word smooth means C^∞.*

In case there is no control, so that $m = m' = 0$ and we omit u and χ_{II}, Definition 4.2.2 coincides with the classical notion of local topological conjugacy for non controlled differential equations, and may serve as a definition in this case too. Let us write more formally the classical local results on ordinary differential equations that we recalled in the introduction :

Theorem 4.2.3 (Flow-box theorem). *If $m = 0$ and $f(0) \neq 0$, system (4.3) is locally smoothly conjugate at 0 to the linear system $\dot{z}_1 = 1, \dot{z}_2 = \cdots = \dot{z}_n = 0$.*

Theorem 4.2.4 (The Grobman-Hartman theorem). *If $m = 0$, $f(0) = 0$, and $f'(0)$ has no pure imaginary eigenvalue, system (4.3) is locally topologically conjugate at 0 to the linear system $\dot{z} = f'(0)z$.*

From now on, we consider control system, i.e. we assume $m \geq 1$.

4.2.2 Some properties of conjugating maps

It turns out that conjugating homeomorphisms preserve the dimension of both the state and the control and must have a triangular structure :

Proposition 4.2.5. *With the notations of Definition 4.2.2, suppose that (4.3) and (4.4) are topologically conjugate via a homeomorphism $\chi : \Omega \to \Omega'$. Then $n = n'$, $m = m'$, and χ_{I} depends only on x :*

$$\chi(x, u) \quad = \quad (\,\chi_{\mathrm{I}}(x)\,,\,\chi_{\mathrm{II}}(x, u)\,)\ . \tag{4.7}$$

Moreover, $\chi_{\mathrm{I}} : \Omega_{\mathbb{R}^n} \to \Omega'_{\mathbb{R}^n}$ is a homeomorphism.

Proof. Let $\bar{x}$, $\bar{u}$, $\bar{u}'$ be such that $(\bar{x}, \bar{u})$ and $(\bar{x}, \bar{u}')$ belong to Ω. Let further $x(t)$ be the solution to (4.3) with $x(0) = \bar{x}$ and $u(t) = \bar{u}$ for $t \leq 0$ and $u(t) = \bar{u}'$ for $t > 0$. By conjugacy, $z(t) = \chi_{\mathrm{I}}(x(t), u(t))$ is a solution to (4.4) with v given by $v(t) = \chi_{\mathrm{II}}(x(t), u(t))$, for $t \in (-\epsilon, \epsilon)$ and some $\epsilon > 0$. In particular $\chi_{\mathrm{I}}(x(t), u(t))$ is continuous in t so its values at 0^+ and 0^- are equal. Hence $\chi_{\mathrm{I}}(\bar{x}, \bar{u}) = \chi_{\mathrm{I}}(\bar{x}, \bar{u}')$ so that $\chi_{\mathrm{I}} : \Omega_{\mathbb{R}^n} \to \Omega'_{\mathbb{R}^{n'}}$ is well defined and continuous. Similarly, $\left(\chi^{-1}\right)_{\mathrm{I}}$ induces a continuous inverse $\Omega'_{\mathbb{R}^{n'}} \to \Omega_{\mathbb{R}^n}$. □

In view of Proposition 4.2.5, we will only consider conjugacy between systems having the same number of states and inputs. Hence the distinction between (n, m) and (n', m') from now on disappears.

Taking into account the triangular structure of χ in Proposition 4.2.5, one may describe conjugation as the result of changing coordinates in the state-space (by setting $z = \chi_{\mathrm{I}}(x)$) and feeding the system with a function both of the state and of a new control variable v (by setting $u = (\chi^{-1})_{\mathrm{II}}(z, v)$), in such a way that the correspondence $(x, u) \mapsto (z, v)$ is invertible. In the language of control, this is known as a *static feedback transformation*, and two conjugate systems in the sense of Definition 4.2.2 would be termed *equivalent under static feedback*. This notion has received much attention, although only in the *differentiable* setting (i.e. when the triangular transformation χ is a diffeomorphism), see e.g. [15, 54].

4.2.3 Linearization

Recall that f is assumed to be smooth (of class C^∞). Let us make a formal definition of topological and smooth linearizability.

Definition 4.2.6. *The system (4.3) is said to be* locally topologically linearizable at $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$ *if it is locally topologically conjugate, in the sense of Definition 4.2.2, to a linear controllable system $\dot{z} = Az + Bv$.*

Definition 4.2.7. *The system (4.3) is said to be* locally smoothly linearizable at $(\bar{x},\bar{u}) \in \mathbb{R}^{n+m}$ *if it is locally smoothly conjugate, in the sense of Definition 4.2.2, to a linear controllable system* $\dot{z} = Az + Bv$.

Explicit necessary and sufficient conditions for a nonlinear system to be locally smoothly linearizable at a point were given in [57, 50], and also in [103] (the previous two references dealt with control affine systems only), and is recalled in many nonlinear control textbooks. Without mentioning these conditions, let us simply say that they require a certain number of distributions to be involutive, and that this is a very non-generic property.

4.3 Main result on topological linearization

Let us now give a —basically negative— answer to the natural question raised at the end of section 4.1 : for control systems, removing the differentiability requirement on the conjugacy does *not* allow many more control systems to be (topologically) conjugate to a linear controllable system, contrary to the situation of ordinary differential equations (without control), (see section 4.1 and Theorems 4.2.3 and 4.2.4). Recall that f is assumed to be smooth (of class C^∞).

Theorem 4.3.1 (from Chapter 3)**.** *System (4.3) is locally topologically linearizable at* $(0,0)$ *if, and only if there exists an open neighborhood* $\widetilde{\Omega}$ *of* $(0,0)$ *in* $\mathbb{R}^{n+m}$ *and a homeomorphism*

$$\begin{array}{rccl} \widetilde{\chi}: & \widetilde{\Omega} & \to & \widetilde{\Omega}' \\ & (x,u) & \mapsto & \widetilde{\chi}(x,u) \;=\; (\,\widetilde{\chi}_{\mathrm{I}}(x)\,,\,\widetilde{\chi}_{\mathrm{II}}(x,u)\,) \end{array} \tag{4.8}$$

(possibly different from the homeomorphism defining topological linearizability of the system) such that

1. $\widetilde{\chi}$ *conjugates system (4.3) to a linear controllable system* $\dot{z} = Az + Bv$*, in the sense of Definition 4.2.2,*
2. $\widetilde{\chi}_{\mathrm{I}} : \widetilde{\Omega}_n \to \widetilde{\Omega}'_n$ *defines a smooth* (C^∞) *diffeomorphism.*

This does not state that topological linearizability implies smooth linearizability for $\widetilde{\chi}$ need not be a diffeomorphism even though $\widetilde{\chi}_{\mathrm{I}}$ is. In Chapter 3, the conclusion of the theorem is called *quasi smooth linearizability.* A thorough discussion as well as the proof of Theorem 4.3.1 is given there. Let us recall here what is necessary to make this theorem clearer.

Proposition 4.3.2. *The conclusions of Theorem 4.3.1 imply that*

1. $B\widetilde{\chi}_{\mathrm{II}} : \widetilde{\Omega} \to \mathbb{R}^m$ *is smooth,*
2. *the rank of* B *is the maximum rank of* $\partial f/\partial u$ *in small neighborhoods of the origin.*

Proof. Computing $\dot{z}$ at the origin of a trajectory starting from $(x,u) \in \Omega$ implies, by the smoothness of $\widetilde{\chi}_{\mathrm{I}}$,

$$\frac{\partial \widetilde{\chi}_{\mathrm{I}}}{\partial x}(x)\,f(x,u) \;=\; A\,\widetilde{\chi}_{\mathrm{I}}(x) \;+\; B\,\widetilde{\chi}_{\mathrm{II}}(x,u)\;. \tag{4.9}$$

This gives an obviously smooth expression of $B\widetilde{\chi}_{\mathrm{II}}$. The second point is proved using Corollary 4.3.5[1] at points close to the origin where the rank of $\partial f/\partial u$ is maximum, and hence locally constant. □

If B is left invertible (i.e. has rank m), the first point implies that $\widetilde{\chi}_{\mathrm{II}}$ itself is smooth, and we have the following immediate corollary :

[1] The proof of Corollary 4.3.5 does not use Proposition 4.3.2.

Corollary 4.3.3. *If there are points arbitrarily close to* $(0,0)$ *where the rank of* $\partial f/\partial u$ *is* m *(i.e. where this linear map is injective), then* $\widetilde{\chi}$ *in Theorem 4.3.1 is a smooth mapping.*

Of course if B has rank strictly less that m, $\widetilde{\chi}_{\mathrm{II}}$ need not be smooth. This is discussed in section 4.4.

Note that the assumption of Corollary 4.3.3 is very "reasonable" : for instance for single input systems, the only case where it is not met is when f does not depend on u in a neighborhood of $(0,0)$, but then the system cannot be topologically conjugate to a *controllable* linear system :

Corollary 4.3.4. *If* $m=1$, *i.e. if (4.3) is a single input system, then* $\widetilde{\chi}$ *in Theorem 4.3.1 is a smooth mapping.*

This is however still not "smooth linearizability" because even though χ is smooth, its inverse might fail to be differentiable at the point of interest. The simplest example is the system

$$\dot{x} = u^3\ ,\ \ x \in \mathbb{R}\ ,\ \ u \in \mathbb{R}\ \ , \tag{4.10}$$

clearly conjugate by $(z,v)=\chi(x,u)=(x,u^3)$ to the linear controllable system $\dot{z}=v$. Obviously, χ is smooth, χ^{-1} is continuous, χ_{I} is the identity smooth diffeomorphism, but the inverse of χ itself fails to be differentiable at the origin. In fact, no smooth diffeomorphism can conjugate these two systems. This can easily be proved but is also a consequence of the necessity part of the following result that tells us exactly when smooth linearizability is implied by topological linearizability :

Corollary 4.3.5. *When* f *is of class* C^∞, *system (4.3) is locally smoothly linearizable at* $(0,0)$ *if and only if it is locally topologically linearizable at* $(0,0)$ *and the rank of* $\partial f/\partial u$ *is constant around* $(0,0)$.

Proof. Smooth linearizability is a particular case of topological linearizability, and it implies constant rank of $\partial f/\partial u$ because differentiability of the smooth diffeomorphism and its inverse allow one to get a formula for $\partial f/\partial u(x,u)$.

Let us prove the converse. Suppose that the rank of $\partial f/\partial u$ is $r \le m$ in a neighborhood of $(0,0)$ and that system (4.3) is locally topologically linearizable at $(0,0)$. From Theorem 4.3.1, this implies that there exists a triangular homeomorphism $(x,u)\mapsto(z,v)=\widetilde{\chi}(x,u)=(\widetilde{\chi}_{\mathrm{I}}(x),\widetilde{\chi}_{\mathrm{II}}(x,u))$ that conjugates system (4.3) to a linear controllable system $\dot{z}=Az+Bv$ with the additional property that $\widetilde{\chi}_{\mathrm{I}}$ defines a *smooth diffeomorphism* from a neighborhood of $0\in\mathbb{R}^n$ onto its image.

Let $r' \le m$ be the rank of the matrix B. There are invertible $n\times n$ and $m\times m$ matrices P and Q such that

$$B_c \;=\; PBQ \;=\; \left(\begin{array}{c|c} I_{r'} & 0 \\ \hline 0 & 0 \end{array}\right) \tag{4.11}$$

where $I_{r'}$ is the $r'\times r'$ identity matrix.

Computing $\dot{z}$ at the origin of a trajectory starting from $(x,u)\in\Omega$ implies (4.9) by the smoothness of $\widetilde{\chi}_{\mathrm{I}}$. Hence the map $B_cQ^{-1}\widetilde{\chi}_{\mathrm{II}}$ is smooth where it is defined, and differentiating (4.9) with respect to u yields :

$$P\frac{\partial\widetilde{\chi}_{\mathrm{I}}}{\partial x}(x)\,\frac{\partial f}{\partial u}(x,u) \;=\; \frac{\partial(B_cQ^{-1}\widetilde{\chi}_{\mathrm{II}})}{\partial u}(x,u)\ .$$

Since $P\frac{\partial \widetilde{\chi}_{\rm I}}{\partial x}(x)$ is invertible and the rank of $\partial f/\partial u$ is r, both sides have constant rank r. This implies $r \leq r'$. This also implies that the mapping $\mathbb{R}^{n+m} \to \mathbb{R}^{n+r'}$ defined by

$$(x,u) \quad \mapsto \quad (\, x \,,\, B_c Q^{-1} \widetilde{\chi}_{\rm II}(x,u)\,)\ ,$$

has a constant rank $n+r$ in a neighborhood of the origin, hence, by the constant rank theorem applied to this mapping, there is a $r \times m$ matrix K of rank r (selects r lines that are independent among the m lines of $\frac{\partial (B_c Q^{-1} \widetilde{\chi}_{\rm II})}{\partial u}(x,u)$), a neighborhood Ω of $(0,0)$ in $\mathbb{R}^{n+m}$, two smooth mappings $\alpha : \mathbb{R}^{n+r} \to \mathbb{R}^{n+m}$ and $\beta : \mathbb{R}^{n+m} \to \mathbb{R}^{r'-r}$ (in fact they only need to be defined in suitable neighborhoods of the origin) such that $\widetilde{\chi}$ is defined on Ω, and

$$B_c Q^{-1} \widetilde{\chi}_{\rm II}(x,u) \quad = \quad \begin{pmatrix} \alpha(\, x \,,\, K B_c Q^{-1} \widetilde{\chi}_{\rm II}(x,u)\,) \\ 0 \end{pmatrix} \tag{4.12}$$

for all $(x,u) \in \Omega$ and

$$(x,u) \quad \mapsto \quad (\, x \,,\, K B_c Q^{-1} \widetilde{\chi}_{\rm II}(x,u) \,,\, \beta(x,u)\,)\ , \tag{4.13}$$

defines a smooth diffeomorphism from Ω onto its image. This implies that $r = r'$ because from (4.12, $r < r'$ would prevent $\widetilde{\chi}$ from being one-to-one. Hence K can be taken the identity matrix. Define $\widetilde{\widetilde{\chi}} : \Omega \to \mathbb{R}^{n+m}$ by $\widetilde{\widetilde{\chi}} = L \circ \psi$ with

$$\psi(x,u) \quad = \quad (\, P\widetilde{\chi}_{\rm I}(x) \,,\, K B_c Q^{-1} \widetilde{\chi}_{\rm II}(x,u) \,,\, \beta(x,u)\,)$$

and $L(z,v) \ = \ (P^{-1}z, Q^{-1}v)$. ψ is a smooth diffeomorphism because (4.13) is one, and L is obviously a (linear) smooth diffeomorphism. Setting $(\tilde{z},\tilde{v}) = \widetilde{\widetilde{\chi}}(x,u)$ conjugates system (4.3) to $\dot{\tilde{z}} = A\tilde{z} + B\tilde{v}$. □

4.4 An open question

It is a reasonable question to ask whether the conclusion of Corollary 4.3.3 holds in general, namely whether Theorem 4.3.1 can be strengthened so as to state that $\widetilde{\chi}$ is, on top of its other properties, a smooth mapping (when the rank of $\partial f/\partial u$ is not locally constant, $\widetilde{\chi}$ would fail to be be differentiable, from the necessity part of Corollary 4.3.5).

Let us examine the case where the assumptions of Corollaries 4.3.3, 4.3.4 and 4.3.5 fail (these three corollaries already state the desired conclusion), namely the case of systems with $m \leq 2$ controls where the rank of $\partial f/\partial u$ is everywhere strictly smaller than m, studied locally around a point where this rank is not constant (i.e. the rank at the point is strictly less than the maximum rank in arbitrary small neighborhoods of this point, itself strictly smaller than m.

The smallest dimensions where this occurs is $n = 1$, $m = 2$, i.e. systems $\dot{x} = f(x,u_1,u_2)$ with x, u_1 and u_2 scalar. In order to state our open question in the smallest dimension possible, let us drop the dependence on the right-hand side on x and consider systems

$$\dot{x} = a(u_1,u_2)\,,\ x \in \mathbb{R}\,,\ u = (u_1,u_2) \in \mathbb{R}^2 \quad , \tag{4.14}$$

where $a : \mathbb{R}^2 \to \mathbb{R}$ is smooth. Let us assume that this system is locally topologically linearizable around $(x,u) = (0,0,0)$. The only canonical controllable linear system with one state $z \in \mathbb{R}$ and two controls $(v_1,v_2) \in \mathbb{R}^2$ is $\dot{z} = v_1$, hence local topological linearizability means existence of a homeomorphism

$$\chi \ : \ (x,u_1,u_2) \ \mapsto \ (z,v_1,v_2) \quad = \quad (\, \chi_1(x) \,,\, \chi_2(x,u_1,u_2) \,,\, \chi_3(x,u_1,u_2)\,) \tag{4.15}$$

(in the terms of Definition 4.2.6, χ_{I} is χ_1 and χ_{II} is (χ_2, χ_3)) that conjugates (4.14) to the linear system $\dot{z} = v_1$. From Theorem 4.3.1, this implies existence of another homeomorphism $\widetilde{\chi}$ of the same triangular form, that we denote by χ instead of $\widetilde{\chi}$, such that χ_1 is a smooth diffeomorphism (from a real interval containing zero onto an open interval) and χ_2 is a smooth mapping from an open neighborhood of the origin in $\mathbb{R}^3$ to $\mathbb{R}$, while our results do not grant that χ_3 has any more regularity than continuity. In fact the conjugation reads

$$\frac{\partial \chi_1}{\partial x}(x) a(u_1, u_2) \quad = \quad \chi_2(x, u_1, u_2). \tag{4.16}$$

This implies in particular that χ_2 does not depend on x, and then one can replace $v_2 = \chi_3(x, u_1, u_2)$ with $v_2 = \chi_3(0, u_1, u_2)$ without changing the conjugating property. Composing χ given by (4.15) with $(z, v_1, v_2) \mapsto (\chi_1^{-1}(z), \frac{\partial \chi_1}{\partial x}(\chi_1^{-1}(z))^{-1} v_1, v_2)$, one finally gets a conjugating homeomorphism of the form

$$(x, u_1, u_2) \;\mapsto\; (\, x \,,\, a(u_1, u_2) \,,\, \beta(u_1, u_2)\,) \tag{4.17}$$

where $\beta(u_1, u_2) = \chi_3(0, u_1, u_2)$. Hence local topological linearizability amounts to existence of a continuous mapping β from an open neighborhood of the origin in $\mathbb{R}^2$ to $\mathbb{R}$ such that $(u_1, u_2) \mapsto (a(u_1, u_2), \beta(u_1, u_2))$ defines a homeomorphism from a neighborhood of the origin in $\mathbb{R}^2$ onto its image (we just proved it is necessary, but conversely, it makes (4.17) a local homeomorphism, that obviously conjugates (4.14) to $\dot{z} = v_1$). Similarly, conjugacy via a homeomorphism that is a smooth map amounts to existence of a *smooth* mapping having the same property. Hence the question whether $\widetilde{\chi}$ can be taken a smooth mapping in Theorem 4.3.1 reduces to the following

Open question 4.4.1. *Let a and β be two mappings $]-\varepsilon, \varepsilon[^2 \to \mathbb{R}$, $\varepsilon > 0$, such that a is* smooth*, β is* continuous*, and $(u_1, u_2) \mapsto (a(u_1, u_2), \beta(u_1, u_2))$ defines a homeomorphism from $]-\varepsilon, \varepsilon[^2$ onto its image. Does there exist a* smooth *mapping $b :]-\varepsilon', \varepsilon'[^2 \to \mathbb{R}$, $0 < \varepsilon' < \varepsilon$, such that $(u_1, u_2) \mapsto (a(u_1, u_2), b(u_1, u_2))$ defines a homeomorphism from $]-\varepsilon', \varepsilon'[^2$ onto its image ?*

This question in differential topology can be posed in higher dimension of course, see below. It is of interest in its own right and seems to have no answer so far, even for $p = q = 1$.

Open question 4.4.2. *Let O be a neighborhood of the origin in $\mathbb{R}^{p+q}$ and $F : O \to \mathbb{R}^p$ a smooth map. Suppose there is a* continuous *map $G : O \to \mathbb{R}^q$ such that $F \times G : O \to \mathbb{R}^p \times \mathbb{R}^q$ is a local homeomorphism at 0.*

Does there exist another neighborhood of the origin $O' \subset O$ and a smooth *mapping $H : O' \to \mathbb{R}^q$ such that $F \times H : O' \to \mathbb{R}^p \times \mathbb{R}^q$ is again a local homeomorphism at 0 ?*

4.5 Implications in Control Theory

Let us come back to the discussion we started in the Introduction. Consider a control system (4.1), assume for simplicity that we work around an equilibrium, i.e. $f(0,0) = 0$, and let us write its linear approximation, i.e.

$$f(x, u) \quad = \quad Ax \;+\; Bu \;+\; F(x, u) \tag{4.18}$$

$$\text{with} \quad F(0,0) = \frac{\partial F}{\partial x}(0,0) = \frac{\partial F}{\partial u}(0,0) = 0 \;, \tag{4.19}$$

so that the nonlinear system (4.1) reads

$$\dot{x} \quad = \quad Ax \;+\; Bu \;+\; F(x, u) \;. \tag{4.20}$$

From the remarks made in the introduction, the relevant situation is the one where the linear system

$$\dot{z} \;=\; Az + Bv \tag{4.21}$$

is controllable. Let us assume slightly more to rule out the pathologies described in the previous section. The additional assumption is very mild and is, for instance, always true when the constant $n \times m$ matrix B has rank m; it is implied by linear controllability for single input systems.

Assumption 4.5.1. *The pair (A, B) is controllable and the rank of $\frac{\partial F}{\partial u}(x, u)$ is equal to the rank of B for small (x, u).*

The question raised in the introduction was the one of finding a reasonable equivalence relation that would make the two systems (4.20) and (4.21) locally equivalent. Comparing the situation of ordinary differential equations (without control), a candidate was local topological conjugacy as in Definitions 4.2.2 and 4.2.6, and if that candidate was successful, we would have a result making precise the vague statement (4.2).

Corollary 4.3.5 implies that, for A, B and F satisfying Assumption 4.5.1, systems (4.20) and (4.21) are locally topologically conjugate if and only if they are locally smoothly conjugate, and it is known from [57, 50] that this is false for a generic F, even satisfying (4.19). This discards topological conjugacy as a candidate for the above mentioned equivalence relation, but this does *not* contradict the basic belief behind statement (4.2).

A way to contradict that statement would be to find at least one example satisfying the assumption, but where the nonlinear system (4.20) displays some local "qualitative" phenomenon that do not occur for the linear system (4.21). In the qualitative theory of dynamical systems (without control), the phase portrait gives a picture of the behavior, on which phenomena like attractors, invariant set, (stable) closed orbits can just be "seen". A control system is more complex : it describes how the behavior of the state (at least in the state space representation) is linked to the control. It is not very clear what a qualitative phenomenon should be for a control system. The least to require is that it be invariant by topological conjugacy as defined here. In the introduction, we pointed out that (non-)controllability *is* a qualitative property, but it is of no help here since (4.2) only refers to controllable systems.

We do believe that clarifying the status of a statement like (4.2) is very relevant to control theory and modeling. Our negative results (section 4.3) say that topological conjugacy is not the right tool to answer this. A looser equivalence could be a way to state (4.2) properly. It could also be that the intuition behind (4.2) is totally wrong and that some nonlinearities F allow system (4.20) to display some qualitative phenomena locally that cannot occur on a linear system (4.21).

Chapitre 5

Reproduction de l'article:

J.-B. Pomet,
"A necessary condition for dynamic equivalence",

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Abstract. If two control systems on manifolds of the same dimension are dynamic equivalent, we prove that either they are static equivalent –*i.e.* equivalent via a classical diffeomorphism– or they are both ruled ; for systems of different dimensions, the one of higher dimension must be ruled. A ruled system is one whose equations define at each point in the state manifold, a ruled submanifold of the tangent space. Dynamic equivalence is also known as equivalence by endogenous dynamic feedback, or by a Lie-Bäcklund transformation when control systems are viewed as underdetermined systems of ordinary differential equations ; it is very close to absolute equivalence for Pfaffian systems. It was already known that a differentially flat system must be ruled ; this was a particular case of the present result, in which one of the systems was assumed to be "trivial" (or linear controllable).

5.1 Introduction

We consider time-invariant control systems, or underdetermined systems of ordinary differential equations (ODEs) where the independent variable is time. *Static equivalence* refers to equivalence via a diffeomorphism in the variables of the equation, or in the state and control variables, with a triangular structure that induces a diffeomorphism (preserving time) in the state variables too. It is also known as "feedback equivalence". *Dynamic equivalence* refers to equivalence via invertible transformations in jet spaces that do not induce any diffeomorphism in a finite number of variables, except when it coincides with static equivalence ; these transformations are also known as endogenous dynamic feedback [68, 37], or Lie-Bäcklund transformations ([1, 37] and Chapter 6), although this terminology is more common for systems of partial differential equations (PDEs) ; dynamic equivalence is also very close to absolute equivalence for Pfaffian systems [19, 93, 94].

The literature on classification and invariants for static equivalence is too large to be quoted here ; let us only recall that, as evidenced by all detailed studies and mentioned in [102], each equivalence class (within control systems on the same manifold, or germs of control systems) is very very thin, indeed it has infinite co-dimension except in trivial cases. Since dynamic equivalence is a priori more general, it is natural to ask how more general it is. Systems on manifolds of different dimension may be dynamic equivalent, but not static equivalent. Restricting our attention to systems on the same manifold and considering dynamic equivalence instead of static, how bigger are the equivalence classes ?

The literature on dynamic feedback linearization [53, 23], differential flatness [37, 68], or absolute equivalence [93] tends to describe the classes containing linear controllable systems or "trivial" systems. The authors of [37, 68, 93] made the link with deep differential geometric questions dating back to [44, 19, 48] ; see [7] for a recent overview. Despite these efforts, no characterization is available except for systems with one control, *i.e.* whose general solution depends on one function of one variable ; there are many systems that one suspects to be non-flat –*i.e.* dynamic equivalent to no trivial system– while no proof is available, see the remark on (5.23) in Section 5.4.1. There is however one powerful necessary condition [88, 96] : a flat system must be ruled, *i.e.* its equations must define a ruled submanifold in each tangent space. As pointed out in [88], this proves that the equivalence class of linear systems for dynamic equivalence, although bigger than for static equivalence, still has infinite co-dimension.

Deciding whether two general systems are dynamic equivalent is at least as difficult. There is no method to prove that two systems are not dynamic equivalent. The contribution of this paper is a necessary condition for two systems to be dynamic equivalent, that generalizes [88, 96] : if they live on manifolds of the same dimension, they must be either both ruled or static equivalent ; if not, the one of higher dimension must be ruled. Besides being useful to prove that some pairs of systems are not dynamic equivalent, it also implies that "generic" equivalence classes for dynamic equivalence are the same as for static equivalence.

Outline Notations on jet bundles and differential operators are recalled in Section 5.2 ; the notions of systems, ruled systems, dynamic and static equivalence are precisely defined in Section 5.3. Our main result is stated and commented in Section 5.4, and proved in Section 5.5.

5.2 Miscellaneous notations

Let M be an n-dimensional manifold, either $\mathbf{C}^\infty$ (infinitely differentiable) or $\mathbf{C}^\omega$ (real analytic).

5.2.1 Jet bundles

Using the notations and definitions of [43, Chapter II, §2], $J^k(\mathbb{R}, M)$ denotes the k^{th} jet bundle of maps $\mathbb{R} \to M$. It is a bundle both over $\mathbb{R}$ and over M. If $(x^1, \ldots, x^n)$ is a system of coordinates on an open subset of M, coordinates on the lift of this open subset are given by $t, x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n, \cdots, (x^1)^{(k)}, \ldots, (x^n)^{(k)}$ where t is the projection on $\mathbb{R}$.

As an additive group, $\mathbb{R}$ acts on $J^k(\mathbb{R}, M)$ by translation of the t-component ; the quotient by this action is well defined and we denote it by

$$J^k(M) \;=\; J^k(\mathbb{R}, M) \Big/ \mathbb{R}\ . \tag{5.1}$$

Since we only study time-invariant systems, we prefer to work with $J^k(M)$. Quotienting indeed drops the t information : local coordinates on $J^k(M)$ are given by $x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n, \cdots, (x^1)^{(k)}, \ldots, (x^n)^{(k)}$; for short, we write $x, \dot{x}, \ldots, x^{(k)}$. For $\ell < k$, there is a canonical projection

$$\pi_{k,\ell} :\; J^k(M) \to J^\ell(M) \tag{5.2}$$

that makes $J^k(M)$ a bundle over $J^\ell(M)$; in particular it is a bundle over $M = J^0(M)$ and over $\mathrm{T}M = J^1(M)$. In coordinates,

$$\pi_{k,\ell}(x, \dot{x}, \ldots, x^{(\ell)}, \ldots, x^{(k)}) = (x, \dot{x}, \ldots, x^{(\ell)})\ .$$

Notation To a subset $\Omega \subset J^k(M)$, we associate, for all ℓ, a subset $\Omega_\ell \subset J^\ell(M)$ in the following manner (obviously, $\Omega_k = \Omega$) :

$$\Omega_\ell = \begin{cases} \pi_{k,\ell}(\Omega) & \text{if } \ell \leq k, \\ {\pi_{\ell,k}}^{-1}(\Omega) & \text{if } \ell \geq k. \end{cases} \tag{5.3}$$

5.2.2 The k^{th} jet of a smooth ($\mathbf{C}^\infty$) map $x(.) : I \to M$

With $I \subset \mathbb{R}$ a time interval, it is a smooth map $j^k(\,x(.)\,) : I \to J^k(M)$ (see again [43]) ; in coordinates,

$$j^k(\,x(.)\,)\,(t) \;=\; (x(t), \dot{x}(t), \ddot{x}(t), \ldots, x^{(k)}(t)) \;.$$

By *a smooth map whose k^{th} jet remains in* Ω, for some $\Omega \subset J^k(M)$, we mean a smooth $x(.) : I \to M$ such that $j^k(\,x(.)\,)(t) \in \Omega$ for all t in I.

5.2.3 Differential operators

If Ω is an open subset of $J^k(M)$, and M' is a manifold of dimension n', a smooth ($\mathbf{C}^\infty$ or $\mathbf{C}^\omega$) map $\Phi : \Omega \to M'$ defines the smooth differential operator of order[1] k

$$\mathcal{D}^k_\Phi \;=\; \Phi \circ j^k \;. \tag{5.4}$$

Obviously, $\mathcal{D}^k_\Phi$ sends smooth maps $I \to M$ whose k^{th} jet remains in Ω to smooth maps $I \to M'$. In coordinates, the image of $t \mapsto x(t)$ is $t \mapsto \Phi(x(t), \dot{x}(t), \ddot{x}(t), \ldots, x^{(k)}(t))$. Note that we do not require that k be minimal, so Φ *might* not depend on $x^{(k)}$

We call $j^r \circ \mathcal{D}^k_\Phi$ the r^{th} *prolongation of* the differential operator $\mathcal{D}^k_\Phi$; it sends smooth maps $I \to M$ whose k^{th} jet remains in Ω to smooth maps $I \to J^r(M')$; it is indeed the differential operator $\mathcal{D}^{k+r}_{\Phi^{[r]}}$, of order $k+r$, with $\Phi^{[r]}$ the unique smooth map ${\pi_{k+r,k}}^{-1}(\Omega) \to J^r(M')$ such that

$$j^r \circ \Phi \circ j^k \;=\; \Phi^{[r]} \circ j^{k+r} \;. \tag{5.5}$$

We call $\Phi^{[r]}$ the r^{th} prolongation of Φ. One has $\pi_{r,0} \circ \Phi^{[r]} = \Phi \circ \pi_{k+r,k}$ and more generally, for $s < r$,

$$\pi_{r,s} \circ \Phi^{[r]} = \Phi^{[s]} \circ \pi_{k+r,k+s} \;. \tag{5.6}$$

5.3 Systems and equivalence

5.3.1 Systems

Definition 5.3.1. *A* $\mathbf{C}^\infty$ *or* $\mathbf{C}^\omega$ regular system *with m controls on a smooth manifold M is a* $\mathbf{C}^\infty$ *or* $\mathbf{C}^\omega$ *sub-bundle Σ of the tangent bundle* $\mathrm{T}M$

$$\begin{array}{ccc} \Sigma & \overset{i}{\hookrightarrow} & \mathrm{T}M \\ & \pi\searrow & \downarrow \\ & & M \end{array} \tag{5.7}$$

with fiber Υ, a $\mathbf{C}^\infty$ *or* $\mathbf{C}^\omega$ *manifold of dimension m (e.g. an open subset of $\mathbb{R}^m$). The* velocity set *at a point $x \in M$ is the fiber $\Sigma_x = \pi^{-1}(\{x\})$, a submanifold of T_xM diffeomorphic to Υ.*

[1] "Of order no larger than k" would be more accurate : if Φ does not depend on k^{th} derivatives, the order in the usual sense would be smaller than k. See for instance Ψ in example (5.22).

Definition 5.3.2 (Solutions of a system)**.** *A solution of system* Σ *on the real interval* I *is a* smooth *(*$\mathbf{C}^\infty$*)* $x(.) : I \to M$ *such that* $j^1(x(.))(t) \in \Sigma$ *for all* $t \in I$*.*

Although a general solution of a system need not be smooth, we only consider *smooth* solutions. They form a rich enough class in the sense that systems are fully characterized by their set of smooth solutions.

Locally, one may write “explicit” equations of Σ in the following form. Of course there are many choices of coordinates and the map f depends on this choice.

Proposition 5.3.3. *For each* $\xi \in \Sigma$*, with* $\Sigma \hookrightarrow \mathrm{T}M$ *a regular system (5.7), there is*
- *an open neighborhood* $\mathcal{U}$ *of* ξ *in* $\mathrm{T}M$*,* $\mathcal{U}_0$ *its projection on* M*,*
- *a system of local coordinates* $(x_{\mathrm{I}}, x_{\mathrm{II}})$ *on* $\mathcal{U}_0$*, with* x_{I} *a block of dimension* $n-m$ *and* x_{II} *of dimension* m*,*
- *an open subset* U *of* $\mathbb{R}^{n+m}$ *and a smooth (*$\mathbf{C}^\infty$ *or* $\mathbf{C}^\omega$*) map* $f : U \to \mathbb{R}^{n-m}$*,*

such that the equation of $\Sigma \cap \mathcal{U}$ *in these coordinates is*

$$\dot{x}_{\mathrm{I}} = f(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}) , \quad (x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}) \in U \ . \tag{5.8}$$

Proof. Consequence of the implicit function theorem. □

Control systems A more usual representation of a system with m controls is

$$\dot{x} = F(x,u) \ , \quad x \in M \, , \ u \in \mathcal{B} \, , \tag{5.9}$$

with $\mathcal{B}$ an open subset of $\mathbb{R}^m$ and $F : M \times \mathcal{B} \to \mathrm{T}M$ smooth enough. It can be brought locally, in block coordinates $(x_{\mathrm{I}}, x_{\mathrm{II}})$, to the form

$$\dot{x}_{\mathrm{I}} = f(x_{\mathrm{I}}, x_{\mathrm{II}}, u) \ , \quad \dot{x}_{\mathrm{II}} = u \tag{5.10}$$

modulo a static feedback on u, at least around nonsingular points (x,u) where

$$\mathrm{rank} \frac{\partial F}{\partial u}(x,u) = m \ . \tag{5.11}$$

Equation (5.8) can be obtained by eliminating the control u in (5.10).

If (5.11) holds, (5.9) defines a system in the sense of Definition 5.3.1. All results on systems in that sense may easily be translated to control systems (5.9).

Implicit systems of ODEs A smooth *system of* $n-m$ *ODEs* on M : $R(x,\dot{x}) = 0$ with $R : \mathrm{T}M \to \mathbb{R}^{n-m}$ also defines a system in the sense of Definition 5.3.1 if it is nonsingular, *i.e.* $\mathrm{rank} \frac{\partial R}{\partial \dot{x}}(x,\dot{x}) = n-m$.

Singularities With the above rank assumptions, or the one that Σ is a sub-bundle in Definition 5.3.1, we carefully avoid singular systems. This paper does not apply to singular control systems or singular implicit systems of ODEs.

Prolongations of Σ

For integers $k \geq 1$, we denote by Σ_k the prolongation of the system Σ to k^{th} order ; it is *the* subbundle $\Sigma_k \hookrightarrow J^k(M)$ with the following property : for any smooth map $x(.) : I \to M$, with $j^k(x(.))$ defined in section 5.2.2,

$$j^1(x(.))(t) \in \Sigma \, , \ t \in I \qquad \Leftrightarrow \qquad j^k(x(.))(t) \in \Sigma_k \, , \ t \in I \, . \tag{5.12}$$

The left-hand side means that $x(.)$ is a *solution* of Σ according to Definition 5.3.2. Obviously, $\Sigma_1 = \Sigma$. We may describe Σ_k in coordinates.

Proposition 5.3.4. *Let K be a positive integer. There is a unique sub-bundle $\Sigma_K \hookrightarrow J^K(M)$ such that :*

$$\begin{array}{l}\text{a smooth map } x(.) : I \to M \text{ is a solution of system } \Sigma \text{ on the real interval } I \\ \text{if and only if } j^K(x(.))(t) \in \Sigma_K \text{ for all } t \in I.\end{array} \tag{5.13}$$

For all $\xi \in \Sigma_K$, its projection $\xi_1 = \pi_{K,1}(\xi)$ is in Σ and, with $\mathcal{U}$ the neighborhood of ξ_1, $(x_{\rm I}, x_{\rm II})$ the coordinates on $\mathcal{U}_0$, U the open subset $\mathbb{R}^{n+m}$ and $f : U \to \mathbb{R}^m$ the map given by Proposition 5.3.3, the equations of $\mathcal{U}_K \cap \Sigma_K$ in $J^K(M)$ are, in the coordinates $(x_{\rm I}, x_{\rm II}, \dot{x}_{\rm I}, \dot{x}_{\rm II}, \ldots, x_{\rm I}^{(K)}, x_{\rm II}^{(K)})$ induced on $\mathcal{U}$ by $(x_{\rm I}, x_{\rm II})$,

$$\begin{array}{l} x_{\rm I}^{(i)} = f^{(i-1)}(x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}, \ldots, x_{\rm II}^{(i)}) , \quad 1 \le i \le K , \\ (x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}, \ldots, x_{\rm II}^{(K)}) \in U \times \mathbb{R}^{(K-1)m} , \end{array} \tag{5.14}$$

where, for a smooth map $f : U \to \mathbb{R}^{n-m}$, and $\ell \ge 0$, $f^{(\ell)}$ is the smooth map $U \times \mathbb{R}^{Km} \to \mathbb{R}^{n-m}$ defined by $f^{(0)} = f$ and, for $i \ge 1$,

$$f^{(i)}(x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}, \ldots, x_{\rm II}^{(i+1)}) = \frac{\partial f^{(i-1)}}{\partial x_{\rm I}} f(x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}) + \sum_{i=0}^{i} \frac{\partial f^{(i-1)}}{\partial x_{\rm II}^{(i)}} x_{\rm II}^{(i+1)} . \tag{5.15}$$

Proof. This is classical, and obvious in coordinates. □

Remark 5.3.5. Each Σ_{k+1} ($k \ge 1$) is an affine bundle over Σ_k, and may be viewed as an affine sub-bundle of $\mathrm{T}\Sigma_k$, *i.e.* it is a system in the sense of Section 5.3.1 on the manifold Σ_k instead of M.

In particular $\Sigma_2 \hookrightarrow \mathrm{T}\Sigma$ is the system obtained by “adding an integrator in each control” of the system $\Sigma \hookrightarrow \mathrm{T}M$. It is an affine system (i.e. affine sub-bundle) even when Σ is not.

5.3.2 Ruled systems

Recall that a smooth submanifold of an affine space is ruled if and only if it is a union of straight lines, i.e. if through each point of the submanifold passes a straight line contained in the submanifold. Such a manifold must be unbounded ; since we want to consider the intersection of a submanifold with an arbitrary open set and allow this patch to be “ruled”, we use the same slightly abusive notion as [64] : a submanifold N is ruled if and only if, through each point of it, passes a straight line which is contained in N “until it reaches the boundary of N”. Here, the boundary of the submanifold N is $\partial N = \overline{N} \setminus N$.

A system will be called ruled if and only if Σ_x is, for all x, a ruled submanifold of $\mathrm{T}_x M$. This is formalized below in a self-contained manner.

Definition 5.3.6. *Let $\mathcal{O}$ be an open subset of* $\mathrm{T}M$. *System Σ (see (5.7)) is* ruled *in $\mathcal{O}$ if and only if, for all $(x, \dot{x}) \in (\mathcal{O} \cap \Sigma)$, there is a nonzero vector $w \in \mathrm{T}_x M \setminus \{0\}$ and two possibly infinite numbers $\lambda^- \in [-\infty, 0)$ and $\lambda^+ \in (0, +\infty]$ such that*
$(x, \dot{x} + \lambda w) \in \mathcal{O} \cap \Sigma$ for all λ, $\lambda^- < \lambda < \lambda^+$ and

$$\begin{array}{l} \lambda^- > -\infty \Rightarrow (x, \dot{x} + \lambda^- w) \in \partial\,(\mathcal{O} \cap \Sigma) , \\ \lambda^+ < +\infty \Rightarrow (x, \dot{x} + \lambda^+ w) \in \partial\,(\mathcal{O} \cap \Sigma) . \end{array} \tag{5.16}$$

Recall that, by definition, $\partial\,(\mathcal{O} \cap \Sigma) = \overline{\mathcal{O} \cap \Sigma} \setminus (\mathcal{O} \cap \Sigma)$.

We shall need the following characterisation.

Proposition 5.3.7 ([64]). *Let $\mathcal{O}$ be an open subset of* $\mathrm{T}M$. *Σ is ruled in $\mathcal{O}$ if and only if, for all $\xi = (x, \dot{x})$ in $\Sigma \cap \mathcal{O}$, there is a straight line in $\mathrm{T}_x M$ passing through $\dot{x}$ that has contact of infinite order with Σ_x at $\dot{x}$.*

Proof. From [64, Theorem 1], a "patch of" submanifold of dimension m in a manifold of dimension n is ruled if and only if there is, through each point, a straight line that has contact of order $n+1$. This is of course implied by infinite order. □

5.3.3 Dynamic equivalence

The following notion is usually called dynamic equivalence, or equivalence by (endogenous) dynamic feedback transformations in control theory, see [68, 41, 56] and Chapter 6. It is in fact also the notion of Lie-Bäcklund transformation, limited to ordinary differential equation, as noted in [41] or Chapter 6.

Definition 5.3.8. *Let $\Sigma \hookrightarrow \mathrm{T}M$ and $\Sigma' \hookrightarrow \mathrm{T}M'$ be $\mathbf{C}^\infty$ (resp. $\mathbf{C}^\omega$) regular systems (see (5.7)) on two manifolds M and M' of dimension n and n', K, K' two integers, $\Omega \subset J^K(M)$ and $\Omega' \subset J^{K'}(M')$ two open subsets.*

Systems Σ and Σ' are dynamic equivalent over Ω and Ω' *if and only if there exists two mappings of class $\mathbf{C}^\infty$ (resp. $\mathbf{C}^\omega$) :*

$$\Phi : \Omega \to M', \qquad \Psi : \Omega' \to M \tag{5.17}$$

inducing differential operators $\mathcal{D}_\Phi^K$ and $\mathcal{D}_\Psi^{K'}$ –see (5.4)– such that, for any interval I,

– for any solution $x(.) : I \to M$ of Σ whose K^{th} jet remains inside Ω, $\mathcal{D}_\Phi^K(\,x(.)\,)$ is a solution of Σ' whose K'^{th} jet remains inside Ω' and $\mathcal{D}_\Psi^{K'}(\,\mathcal{D}_\Phi^K(\,x(.)\,)\,) \;=\; x(.)$,

– for any solution $z(.) : I \to M'$ of Σ' whose K'^{th} jet remains inside Ω', $\mathcal{D}_\Psi^{K'}(\,z(.)\,)$ is a solution of Σ whose K^{th} jet remains inside Ω and $\mathcal{D}_\Phi^K(\,\mathcal{D}_\Psi^{K'}(\,z(.)\,)\,) \;=\; z(.)$.

Remark 5.3.9. Since all properties are tested on *solutions*, only the restriction of Φ and Ψ to Σ_K and $\Sigma_{K'}$ (see Proposition 5.3.4) matter ; for instance, Φ can be arbitrarily modified away from Σ_K without changing any conclusions. Borrowing this language from the literature on Lie-Bäcklund transformations, Φ and Ψ above are "external" correspondences.

In [41] or in Chapter 6, the "internal" point of view prevails : for instance Φ and Ψ are replaced, in [41], by diffeomorphisms between diffieties. This is more intrinsic because maps are defined only where they are to be used. However the definitions are equivalent because these internal maps admit infinitely many "external" prolongations.

Here, this external point of view is adopted because it makes the statement of the main result less technical. Note however that, as a preliminary to the proofs, an "internal" translation is given in section 5.5.1.

Remark 5.3.10. In the theorems, we shall require that Ω and Ω' satisfy

$$\Omega_1 \cap \Sigma \;\subset\; (\Omega \cap \Sigma_K)_1 \quad \text{and} \quad \Omega'_1 \cap \Sigma' \;\subset\; \left(\Omega' \cap \Sigma'_{K'}\right)_1 \,, \tag{5.18}$$

i.e. any (jet of) solution whose first jet is in Ω_1 lifts to at least one (jet of) solution whose K^{th} jet is in Ω. Note the following facts about this requirement.

- These inclusions are equalities for the reverse inclusions always hold.

- Replacing the original Ω with $\Omega \setminus \left(\overline{(\Omega_1 \cap \Sigma) \setminus (\Omega \cap \Sigma_K)_1}\right)_K$ and Ω' accordingly forces (5.18) ; alternatively, keeping arbitrary open sets, Theorem 5.4.2 and Theorem 5.4.1 would hold with Ω_1 replaced with $\Omega_1 \setminus \overline{(\Omega_1 \cap \Sigma) \setminus (\Omega \cap \Sigma_K)_1}$.

- When $\Sigma' = \mathrm{T}M'$ is the trivial system (see section 5.3.5), any open Ω' satisfies (5.18).

5.3.4 Static equivalence

Definition 5.3.11. *Let $\mathcal{O} \subset \mathrm{T}M$ and $\mathcal{O}' \subset \mathrm{T}M'$ be open subsets. Systems Σ and Σ' are* static equivalent *over $\mathcal{O}$ and $\mathcal{O}'$ if and only if there is a smooth diffeomorphism $\Phi : \mathcal{O}_0 \to \mathcal{O}'_0$ such that the following holds :*

$$\left.\begin{array}{l}\text{a smooth map } t \mapsto x(t) \text{ is a solution of } \Sigma \text{ whose first jet remains in } \mathcal{O} \\ \text{if and only if } t \mapsto \Phi(x(t)) \text{ is a solution of } \Sigma' \text{ whose first jet remains in } \mathcal{O}'.\end{array}\right\} \tag{5.19}$$

Definition 5.3.12 (Local static equivalence)**.** *Let $\mathcal{O} \subset \mathrm{T}M$ and $\mathcal{O}' \subset \mathrm{T}M'$ be open subsets. Systems Σ and Σ' are* locally *static equivalent over $\mathcal{O}$ and $\mathcal{O}'$ if and only if there are coverings of $\mathcal{O} \cap \Sigma$ and $\mathcal{O}' \cap \Sigma'$:*

$$\Sigma \cap \mathcal{O} \ \subset \ \Sigma \cap \bigcup_{\alpha \in A} \mathcal{O}^\alpha , \quad \Sigma' \cap \mathcal{O}' \ \subset \ \Sigma' \cap \bigcup_{\alpha \in A} \mathcal{O}'^\alpha$$

where A is a set of indices, $\mathcal{O}^\alpha$ and $\mathcal{O}'^\alpha$ are open subsets of $\mathcal{O}$ and $\mathcal{O}'$, such that, for all α, systems Σ and Σ' are static equivalent over $\mathcal{O}^\alpha$ and $\mathcal{O}'^\alpha$.

This definition, stated in terms of solutions, is translated into point (a) below, that only relies on the geometry of Σ and Σ' as submanifolds. Point (b) is used for instance in [63, 107] where "centro-affine" geometry of each Σ_x is studied.

Proposition 5.3.13. (a) *Systems Σ and Σ' are* static equivalent *over $\mathcal{O} \subset \mathrm{T}M$ and $\mathcal{O}' \subset \mathrm{T}M'$ if and only there is a smooth diffeomorphism $\Phi : \mathcal{O}_0 \to \mathcal{O}'_0$ such that $\Phi_\star$ maps $\mathcal{O} \cap \Sigma$ to $\mathcal{O}' \cap \Sigma'$.*

(b) *If systems Σ and Σ' are static equivalent over $\mathcal{O} \subset \mathrm{T}M$ and $\mathcal{O}' \subset \mathrm{T}M'$, there is, for each $x \in \mathcal{O}_0$* a linear isomorphism $\mathrm{T}_x M \to \mathrm{T}_{\Phi(x)} M'$ *that maps Σ_x to $\Sigma'_{\Phi(x)}$.*

(c) *Static equivalence preserves ruled systems.*

Proof. (b) and (c) are easy consequences of (a), which in turn is clear by differentiating solutions in Definition 5.3.2. □

5.3.5 Examples

1 We call *trivial system* on a smooth manifold M the tangent bundle itself $\mathrm{T}M$. Any smooth $x(.) : I \to M$ is a solution of this system ; it corresponds to "no equation", or to the control system $\dot{x} = u$, or to the "affine diffieties" in [41]. Following [37, 41], a system $\Sigma \hookrightarrow \mathrm{T}M$ is called *differentially flat* (on $\Omega \subset J^K(M)$) if and only if it is dynamic equivalent (over Ω and Ω') to the trivial system $\mathrm{T}M'$ for some manifold M'.

2 Any system $\Sigma \hookrightarrow \mathrm{T}M$ is dynamic equivalent to the one obtained by "adding integrators". It was described in Remark 5.3.5 as an affine sub-bundle $\Sigma_2 \hookrightarrow \mathrm{T}\Sigma$; Σ and Σ_2 are equivalent in the sense of Definition 5.3.8 with $M' = \Sigma$, $K = 1$, $K' = 0$, Ω an open neighborhood of Σ in $J^1(M) = \mathrm{T}M$ such that there is a $\Phi : \Omega \to \Sigma$ that coincides with identity on Σ, $\Omega' = M' = \Sigma$ and $\Psi = \pi$ (see (5.7)).

This may be easier to follow in the coordinates of Proposition 5.3.3. The prolongation of (5.8) has state $(y_{\mathrm{I}}, y_{\mathrm{II}}) \in U$, with y_{I} a block of dimension n and y_{II} of dimension m, and equation $\dot{y}_{\mathrm{I}} = \big(f(y_{\mathrm{I}}, y_{\mathrm{II}})\,,\, y_{\mathrm{II}}\big)$. In coordinates, the transformations $\Phi : J^1(U_0) \to U$ and $\Psi : U \to U_0$ are given by $(y_{\mathrm{I}}, y_{\mathrm{II}}) = \Phi(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{I}}, \dot{x}_{\mathrm{II}}) = (x, \dot{x}_{\mathrm{II}})$ and $x = \Psi(y) = y_{\mathrm{I}}$.

Static equivalence between these systems of different dimension does not hold.

3 Let us now give, mostly to illustrate the role of the integers K, K' and the open sets Ω and Ω', two more specific examples of systems $\Sigma \hookrightarrow \mathrm{T}\mathbb{R}^3$ and $\Sigma' \hookrightarrow \mathrm{T}\mathbb{R}^3$ with the following equations in $\mathrm{T}\mathbb{R}^3$, with coordinates $(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3)$ or $(y_1, y_2, y_3, \dot{y}_1, \dot{y}_2, \dot{y}_3)$, clearly defining sub-bundles with fiber diffeomorphic to $\mathbb{R}^2$:

$$\Sigma : \ \dot{x}_1 = x_2 \ , \qquad \Sigma' : \ \dot{y}_1 = y_2 + (\dot{y}_2 - y_1\dot{y}_3)\,\dot{y}_3 \ . \tag{5.20}$$

These equations are even globally in the “explicit” form given by Proposition 5.3.3.

First of all, Σ is dynamic equivalent to the trivial system $\Sigma'' = \mathrm{T}\mathbb{R}^2$, with $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\Phi(x_1, x_2, x_3) = (x_1, x_3)$ and $\Psi : J^1(\mathbb{R}^2) \to \mathbb{R}^3$ given by $\Psi(z_1, z_2, \dot{z}_1, \dot{z}_2) = (z_1, \dot{z}_1, z_2)$. Here $K = 0, K' = 1, \Omega = \mathbb{R}^2, \Omega' = J^1(\mathbb{R}^2)$.

Also, with $K = 1$ and $K' = 2$, systems Σ *and* Σ' *are dynamic equivalent* over $\Omega \subset J^1(\mathbb{R}^3)$ and $\Omega' \subset J^2(\mathbb{R}^3)$ defined by

$$\Omega = \{(x_1, x_2, x_3,\, \dot{x}_1, \dot{x}_2, \dot{x}_3),\ 1 - \dot{x}_2 - {x_2}^3 \neq 0\} \, ,$$
$$\Omega' = \{(y_1, y_2, y_3,\, \dot{y}_1, \dot{y}_2, \dot{y}_3,\, \ddot{y}_1, \ddot{y}_2, \ddot{y}_3),\ 1 - \ddot{y}_3 - \dot{y}_3^3 \neq 0\} \, .$$

The maps $\Phi : \Omega \to \mathbb{R}^3$ and $\Psi : \Omega' \to \mathbb{R}^3$ are given by

$$\Phi(x_1, x_2, x_3,\, \dot{x}_1, \dot{x}_2, \dot{x}_3) = \Big(\frac{(1 - \dot{x}_2)x_3 + x_2\, \dot{x}_3}{1 - \dot{x}_2 - {x_2}^3} \, , \, \frac{{x_2}^2\, x_3 + \dot{x}_3}{1 - \dot{x}_2 - {x_2}^3} \, , \, x_1 \Big) \, , \tag{5.21}$$

$$\Psi(y_1, y_2, y_3,\, \dot{y}_1, \dot{y}_2, \dot{y}_3,\, \ddot{y}_1, \ddot{y}_2, \ddot{y}_3) = (\, y_3 \, , \, \dot{y}_3 \, , \, y_1 - \dot{y}_3\, y_2 \,) \, . \tag{5.22}$$

Remark 5.3.14. Since Ψ does not depend on second derivatives, $K' = 2$ is not the order of the differential operator $\mathcal{D}_\Psi^{K'}$ in the usual sense ; this illustrates the footnote after (5.4) ; it is however necessary to go to second jets to describe the domain Ω' where the restriction to solutions of Σ' of this first order operator can be inverted.

Finally, note that systems Σ *and* Σ' *are not static equivalent* because, from Proposition 5.3.13-(b), this would imply that each Σ_x is sent to some Σ'_y by a linear isomorphism $\mathrm{T}_xM \to \mathrm{T}_yM'$, which is not possible because each Σ_x is an affine subspace of T_xM and Σ'_y a non degenerate quadric of T_yM'.

4 Consider two more systems, $\Sigma \hookrightarrow \mathrm{T}\mathbb{R}^3$ and $\Sigma' \hookrightarrow \mathrm{T}\mathbb{R}^3$ described as in (5.20) :

$$\Sigma : \ \dot{x}_1 = x_2 + (\dot{x}_2 - x_1\dot{x}_3)^2\, \dot{x}_3^2 \ , \qquad \Sigma' : \ \dot{y}_1 = y_2 + (\dot{y}_2 - y_1\dot{y}_3)^2\, \dot{y}_3 \ . \tag{5.23}$$

System Σ is ruled –each Σ_y is the union of lines $\dot{y}_2 - y_1\dot{y}_3 = \lambda$, $\dot{y}_1 = y_2 + \lambda^2\, \dot{y}_3$ for λ in $\mathbb{R}$– while Σ' is not. Hence, from point (c) of Proposition 5.3.13, Σ *and* Σ' *are not static equivalent.* We shall come back to these two systems from the point of view of flatness and dynamic equivalence in sections 5.4.1 and 5.4.3.

5.4 Necessary conditions

5.4.1 The case of flatness

It has been known since [88, 96] that a system which is dynamic equivalent to a *trivial system* –see the beginning of section 5.3.5 ; such a system is called differentially flat– must be ruled ; of course, at least in the smooth case, this is true only on the domain where equivalence is assumed.

Theorem 5.4.1 ([88, 96])**.** *If Σ is dynamic equivalent to the trivial system $\Sigma' = \mathrm{T}M'$ over $\Omega \subset J^K(M)$ and $\Omega' \subset J^{K'}(M')$ satisfying (5.18), then Σ is ruled in Ω_1.*

Application Since Σ in (5.23) is not ruled, this theorem implies that it is not flat, i.e. not dynamic equivalent to the trivial system $\mathrm{T}\mathbb{R}^2$. On the contrary, Σ' in (5.23) is ruled, hence the result does not help deciding it being flat or not; in fact, one conjectures that this system is not flat, but no proof is available; see Chapter 9.

5.4.2 Main idea of the proofs

Our main result, stated in next section, studies what remains of Theorem 5.4.1 when Σ' is not the trivial system. Due to many technicalities concerning regularity conditions, the main ideas may be difficult to grasp in the proof given in section 5.5.2. In order to enlighten these ideas, and even the result itself, let us first sketch the proof of the above theorem, following the line of [88] (itself inspired from [48]), but *without assuming a priori that* Σ' *is trivial.*

Take two arbitrary systems Σ and Σ', and assume that they are dynamic equivalent. From Proposition 5.3.3, one may use locally the explicit forms

$$\Sigma: \ \dot{x}_{\mathrm{I}} = f(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}) \ , \quad \Sigma': \ \dot{z}_{\mathrm{I}} = g(z_{\mathrm{I}}, z_{\mathrm{II}}, \dot{z}_{\mathrm{II}}) \ .$$

Recall that n and n' denote the dimensions of x and z; assume $n \leq n'$. Since we work only on *solutions* (see Remark 5.3.9 and also Section 5.5.1) and the above equations allow one to express each time-derivative $x_{\mathrm{I}}^{(j)}$, $j \geq 1$, as a function of $x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(j)}$, we may work with the variables $x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ddot{x}_{\mathrm{II}}, x_{\mathrm{II}}^{(3)}, \ldots$ and $z_{\mathrm{I}}, z_{\mathrm{II}}, \dot{z}_{\mathrm{II}}, \ddot{z}_{\mathrm{II}}, z_{\mathrm{II}}^{(3)}, \ldots$ only. The map Φ of Definition 5.3.8 translates, in these coordinates, into a correspondence $z_{\mathrm{I}} = \phi_{\mathrm{I}}(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)})$, $z_{\mathrm{II}} = \phi_{\mathrm{II}}(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)})$; here the number K is chosen such that *the dependence of* ϕ *versus* $x_{\mathrm{II}}^{(K)}$ *is effective.*

If $K = 0$, this reads $z = \phi(x)$, and $n < n'$ is absurd because it would imply (around points where the rank of ϕ is constant) some nontrivial relations $R(z) = 0$. Hence $n = n'$, ϕ is a local diffeomorphism and static equivalence holds locally.

If $K \geq 1$, note that Φ mapping solutions of Σ to solution of Σ' implies (plug the expression of z given by ϕ into state equations of Σ') the following identity, valid for all $x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K+1)}$:

$$\begin{aligned} &\frac{\partial \phi_{\mathrm{I}}}{\partial x_{\mathrm{I}}} f(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}) + \frac{\partial \phi_{\mathrm{I}}}{\partial x_{\mathrm{II}}} \dot{x}_{\mathrm{II}} + \frac{\partial \phi_{\mathrm{I}}}{\partial \dot{x}_{\mathrm{II}}} \ddot{x}_{\mathrm{II}} + \cdots + \frac{\partial \phi_{\mathrm{I}}}{\partial x_{\mathrm{II}}^{(K)}} x_{\mathrm{II}}^{(K+1)} \\ &= \ g\left(\phi_{\mathrm{I}}, \phi_{\mathrm{II}}, \ \frac{\partial \phi_{\mathrm{II}}}{\partial x_{\mathrm{I}}} f(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}) + \frac{\partial \phi_{\mathrm{II}}}{\partial x_{\mathrm{II}}} \dot{x}_{\mathrm{II}} + \frac{\partial \phi_{\mathrm{II}}}{\partial x_{\mathrm{II}}} \ddot{x}_{\mathrm{II}} + \cdots + \frac{\partial \phi_{\mathrm{II}}}{\partial x_{\mathrm{II}}^{(K)}} x_{\mathrm{II}}^{(K+1)} \right) \end{aligned}$$

where ϕ_{I} and ϕ_{II} depend on $x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)}$ only and, at least at generic points,

$$\left(\frac{\partial \phi_{\mathrm{I}}}{\partial x_{\mathrm{II}}^{(K)}}, \frac{\partial \phi_{\mathrm{II}}}{\partial x_{\mathrm{II}}^{(K)}} \right) \neq (0,0) \, .$$

Fixing such $x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)}$ and consequently $z = \phi(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)})$, and examining Σ'_z as a submanifold of T_zM' with equation $\dot{z}_{\mathrm{I}} = g(z, \dot{z}_{\mathrm{II}})$, it is clear that moving $x_{\mathrm{II}}^{(K+1)}$ in a direction which is not in the kernel of $\frac{\partial \phi_{\mathrm{II}}}{\partial x_{\mathrm{II}}^{(K)}}(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)})$ provides a straight line of T_zM' contained in Σ'_z and, since this covers all points of Σ'_z, proves that the latter is a ruled submanifold of T_zM' and finally that system Σ' is ruled. We only examined regular points; see Section 5.5.2 for a proper proof.

Collecting the two cases, we have proved that, if $n \leq n'$, either Σ' is ruled or $n = n'$ and Σ' is static equivalent to Σ. This is stated formally in Theorem 5.4.2.

5.4.3 The result for general systems

The contribution of this paper is the following strong necessary condition for dynamic equivalence between two general systems. Ω_1 and Ω'_1 are defined by (5.3).

Theorem 5.4.2. *Let Σ and Σ' be systems on manifolds of dimension n and n', K, K' two integers and $\Omega \subset J^K(M)$, $\Omega' \subset J^{K'}(M')$ two open subsets satisfying (5.18).*

If Σ and Σ' are dynamic equivalent over Ω and Ω', then

if $n > n'$, *system Σ is ruled in Ω_1,*

if $n < n'$, *system Σ' is ruled in Ω'_1,*

if $n = n'$, *then* *(see Definition 5.3.12 for "locally static equivalent")*

- in the real analytic case, *and if $\Omega_1 \cap \Sigma$ and $\Omega'_1 \cap \Sigma'$ are connected,*
  either systems Σ and Σ' are ruled in Ω_1 and Ω'_1 respectively,
  or they are locally static equivalent over Ω_1 and Ω'_1,
- in the smooth ($\mathbf{C}^\infty$) case, *there are open subsets $\mathcal{R}, \mathcal{S}$ of Ω_1 and $\mathcal{R}', \mathcal{S}'$ of Ω'_1 such that Ω_1 and Ω'_1 are covered as*

$$\Omega_1 = \overline{\mathcal{R}} \cup \mathcal{S} = \mathcal{R} \cup \overline{\mathcal{S}}\,, \quad \Omega'_1 = \overline{\mathcal{R}'} \cup \mathcal{S}' = \mathcal{R}' \cup \overline{\mathcal{S}'} \tag{5.24}$$

  and the systems have the following properties on these sets :
  1. *Σ and Σ' are ruled in $\mathcal{R}$ and $\mathcal{R}'$ respectively,*
  2. *Σ and Σ' are* locally *static equivalent over $\mathcal{S}$ and $\mathcal{S}'$.*

Proof. See Section 5.5.2. □

A few remarks are in order :

1 Theorem 5.4.1 is a consequence. Indeed, $n'=m'$ because Σ' is trivial, dynamic equivalence implies $m'=m$ (this is common knowledge ; see Theorem 6.1 or [19]), and $n \geq m$ for any system ; hence $n \geq n'$ and Theorem 5.4.2 directly implies that Σ is ruled except if the systems are static equivalent, but this also implies that Σ is ruled from point (c) of Proposition 5.3.13 and the fact that the trivial system Σ' is ruled.

Static equivalence still appears explicitly in Theorem 5.4.2 because two general systems can be static equivalent without being ruled.

2 The part "$n > n'$ or $n < n'$" can be rephrased as follows : if a system is not ruled, it cannot be dynamic equivalent to any system of smaller dimension. No necessary condition is given on the system of lower dimension ; indeed *any* system is dynamic equivalent to at least its first prolongation, see Example 2 in Section 5.3.5.

3 The case $n = n'$ states that dynamic equivalence, except when it reduces to static equivalence, forces both systems to be ruled (in the real analytic case, the added rigidity prevents the two situations from occurring simultaneously).

In other words, if two systems are not static equivalent and at least one of them is not ruled, they are not dynamic equivalent. Since the two conditions can be checked rather systematically, this yields a new and powerful method for proving that two systems are *not* dynamic equivalent, a difficult task in general because very few invariants of dynamic equivalence are known.

For instance, to the best of our knowledge, the state of the art does not allow one to decide whether Σ and Σ' in (5.23) are dynamic equivalent or not. In section 5.3.5, it was noted that they are not static equivalent and Σ' is not ruled. This implies :

Corollary 5.4.3. *Σ and Σ' in (5.23) are not dynamic equivalent over any domains.*

4 Since being ruled is non-generic [88], we have the following general consequence (in terms of germs of systems because the conclusion in the theorem is only local).

Corollary 5.4.4. *Generic static equivalence classes for germs of systems of the same dimension at a point are also dynamic equivalence classes.*

Note that this is in the mathematical sense of "generic" : this does not prevent many interesting systems from being dynamic equivalent without being static equivalent... it might even be that "most interesting systems" fall in this case!

5.5 Proofs

Recall that subscripts always refer to the order of the jet space. The notation (5.3) is constantly used.

5.5.1 Preliminaries : a re-formulation of dynamic and static equivalence

The maps Φ and Ψ are always applied to jets of solutions, and, according to (5.12), the K^{th} jets of solutions of Σ remain in Σ_K ; hence the only information to retain about Φ and Ψ is their restriction to, respectively,

$$\widetilde{\Omega} = \Omega \cap \Sigma_K \quad \text{and} \quad \widetilde{\Omega}' = \Omega' \cap {\Sigma'}_{K'} . \tag{5.25}$$

We need one more piece of notation : according to Section 5.2.3, the ℓ^{th} prolongation of a smooth map $\widetilde{\Phi} : \widetilde{\Omega} \to M'$, is a map $\pi^{-1}_{K+\ell,\ell}(\widetilde{\Omega}) \to J^\ell M'$; again, only its restriction to $\widetilde{\Omega}_{K+\ell}$ will matter; for this reason, the notations $\widetilde{\Phi}^{[\ell]}$ and $\widetilde{\Psi}^{[\ell]}$ will not stand for the prolongations as defined earlier, but rather these restrictions :

$$\widetilde{\Phi}^{[\ell]} : \widetilde{\Omega}_{K+\ell} \to J^\ell(M') , \quad \widetilde{\Psi}^{[\ell]} : \widetilde{\Omega}'_{K'+\ell} \to J^\ell(M) , \tag{5.26}$$

$$\text{with} \quad \widetilde{\Omega}_{K+\ell} = \Omega_{K+\ell} \cap \Sigma_{K+\ell} , \quad \widetilde{\Omega}'_{K'+\ell} = {\Omega'}_{K'+\ell} \cap {\Sigma'}_{K'+\ell} . \tag{5.27}$$

We may now state the following proposition. Smooth ($\mathbf{C}^\infty$ or $\mathbf{C}^\omega$) maps on $\widetilde{\Omega}_{K+\ell}$ or $\widetilde{\Omega}'_{K'+\ell}$ can be defined in a standard way because, from Proposition 5.3.3, these are smooth embedded submanifolds.

Proposition 5.5.1 (Dynamic Equivalence)**.** *Let K, K' be integers, $\Omega \subset J^K(M)$ and $\Omega' \subset J^{K'}(M')$ two open subsets. Systems Σ and Σ' are* dynamic equivalent over Ω and Ω' *if and only if, with $\widetilde{\Omega}, \widetilde{\Omega}'$ defined in (5.25), there exist two smooth (real analytic, in the real analytic case) mappings*

$$\widetilde{\Phi} : \widetilde{\Omega} \to M' \quad \textit{and} \quad \widetilde{\Psi} : \widetilde{\Omega}' \to M ,$$

such that
$$\widetilde{\Phi}^{[1]}(\widetilde{\Omega}_{K+1}) \subset \Sigma' , \quad \widetilde{\Psi}^{[1]}(\widetilde{\Omega}'_{K'+1}) \subset \Sigma , \tag{5.28}$$

and, with $\widetilde{\Phi}^{[K]}$ and $\widetilde{\Psi}^{[K]}$ defined by (5.26),

$$\widetilde{\Phi}^{[K']}(\widetilde{\Omega}_{K+K'}) \subset \Omega' , \ \widetilde{\Psi}^{[K]}(\widetilde{\Omega}'_{K+K'}) \subset \Omega , \tag{5.29}$$

$$\widetilde{\Psi} \circ \widetilde{\Phi}^{[K']} = \pi_{K+K',0}\Big|_{\widetilde{\Omega}_{K+K'}} , \quad \widetilde{\Phi} \circ \widetilde{\Psi}^{[K]} = \pi_{K+K',0}\Big|_{\widetilde{\Omega}'_{K+K'}} . \tag{5.30}$$

Proof. If the above conditions on Φ and Ψ are satisfied, and $x(.) : I \to M$ is a solution of Σ whose K^{th} jet remains inside Ω, then the first part of (5.28) implies that $\mathcal{D}^K_\Phi(\, x(.)\,)$ is a solution of Σ', the first part of (5.29) implies that its K^{th} jet remains inside Ω', and the first part of

(5.30) implies that $\mathcal{D}_\Psi^{K'}(\,\mathcal{D}_\Phi^K(\,x(.)\,)\,) = x(.)$. This proves the first item of Definition 5.3.8 ; the second item follows in the same way from the second part of (5.28), (5.29) and (5.30).

Conversely, if Φ and Ψ satisfy the properties of Definition 5.3.8, their restrictions $\widetilde{\Phi}$ and $\widetilde{\Psi}$ to $\widetilde{\Omega}$ and $\widetilde{\Omega}'$ respectively satisfy the above relations because through each point in $\widetilde{\Omega}_{K+1}$, $\widetilde{\Omega}'_{K'+1}$, $\widetilde{\Omega}_{K+K'}$ or $\widetilde{\Omega}_{K+K'}$ passes a jet of order $K+1$, $K'+1$ or $K+K'$ of a solution of Σ or Σ' ; differentiating yields the required relations. □

Proposition 5.5.2 (Static Equivalence). *With $\Omega_1 \subset J^1(M) = \mathrm{T}M$ and $\Omega'_1 \subset J^1(M') = \mathrm{T}M$ two open subsets, systems Σ and Σ' are* static equivalent over Ω_1 and Ω'_1 *if and only if, with $\widetilde{\Omega}_1, \widetilde{\Omega}'_1$ defined in (5.25), there exist a smooth diffeomorphism $\Phi_0 : \widetilde{\Omega}_0 \to \widetilde{\Omega}'_0$, and its inverse Ψ_0 such that $\widetilde{\Phi}_0^{[1]}(\widetilde{\Omega}_1) = \widetilde{\Omega}'_1$ (and $\widetilde{\Psi}_0^{[1]}(\widetilde{\Omega}'_1) = \widetilde{\Omega}_1$).*

Proof. This is a re-phrasing of point (a) of Proposition 5.3.13. □

5.5.2 Proof of Theorem 5.4.2

Assume that Σ and Σ' are dynamic equivalent over the open sets $\Omega \subset J^K(M)$ and $\Omega' \subset J^{K'}(M')$; let $\widetilde{\Phi} : \widetilde{\Omega} \to M'$ and $\widetilde{\Psi} : \widetilde{\Omega}' \to M$ be the smooth maps given by Proposition 5.5.1 (recall that $\widetilde{\Omega}$ and $\widetilde{\Omega}'$ are open subsets of Σ_K and $\Sigma'_{K'}$). We define open subsets $\widetilde{\Omega}^S \subset \widetilde{\Omega}$ and $\widetilde{\Omega}'^S \subset \widetilde{\Omega}'$ and state four lemmas concerning these :

$$\xi \in \widetilde{\Omega}^S \;\Leftrightarrow\; \begin{array}{l}\text{There is a neighborhood } V \text{ of } \xi \text{ in } \widetilde{\Omega} \text{ and a smooth map}\\ \widetilde{\Phi}_0 : V_0 \to M' \text{ such that } \widetilde{\Phi}\Big|_V = \widetilde{\Phi}_0 \circ \pi_{K,0}\,,\end{array} \tag{5.31}$$

$$\xi' \in \widetilde{\Omega}'^S \;\Leftrightarrow\; \begin{array}{l}\text{There is a neighborhood } V' \text{ of } \xi' \text{ in } \widetilde{\Omega}' \text{ and a smooth map}\\ \widetilde{\Psi}_0 : V'_0 \to M \text{ such that } \widetilde{\Psi}\Big|_{V'} = \widetilde{\Psi}_0 \circ \pi_{K,0}\,.\end{array} \tag{5.32}$$

Lemma 5.5.3. *In the analytic case, and if $\widetilde{\Omega} = \Omega \cap \Sigma$ and $\widetilde{\Omega}' = \Omega' \cap \Sigma'$ are connected, one has either $\widetilde{\Omega}^S = \widetilde{\Omega}$ or $\widetilde{\Omega}^S = \varnothing$, and either $\widetilde{\Omega}'^S = \widetilde{\Omega}'$ or $\widetilde{\Omega}'^S = \varnothing$.*

Lemma 5.5.4. *One has the following identities, where the two first ones hold for any subsets $S \subset \widetilde{\Omega}$, $S' \subset \widetilde{\Omega}'$ and any integer ℓ, $0 \le \ell \le K + K'$,*

$$\pi_{K+K',\ell}\left(\widetilde{\Phi}^{[K']^{-1}}(S')\right) = \widetilde{\Psi}^{[\ell]}\left(S'_{K'+\ell}\right), \quad \pi_{K+K',\ell}\left(\widetilde{\Psi}^{[K]^{-1}}(S)\right) = \widetilde{\Psi}^{[\ell]}\left(S_{K+\ell}\right), \tag{5.33}$$

$$\widetilde{\Phi}^{[1]}(\widetilde{\Omega}_{K+1}) = \widetilde{\Omega}'_1\,, \qquad \widetilde{\Psi}^{[1]}(\widetilde{\Omega}'_{K'+1}) = \widetilde{\Omega}_1\,. \tag{5.34}$$

Lemma 5.5.5. ***If $n < n'$,*** *then $\widetilde{\Omega}^S = \varnothing$.* ***If $n > n'$,*** *then $\widetilde{\Omega}'^S = \varnothing$.*

If $n = n'$, *there is, for all $\xi_K \in \widetilde{\Omega}^S$, a neighborhood $\mathcal{V}_1$ of $\xi_1 = \pi_{K,1}(\xi_K)$ in Ω_1 and an open subset $\mathcal{V}'_1$ of Ω'_1 such that systems Σ and Σ' are static equivalent over $\mathcal{V}_1$ and $\mathcal{V}'_1$. There is also, for all $\xi'_{K'} \in \widetilde{\Omega}'^S$, a neighborhood $\mathcal{W}'$ of $\xi'_1 = \pi_{K',1}(\xi'_{K'})$ in Ω'_1 and an open subset $\mathcal{W}_1$ of Ω_1 such that systems Σ and Σ' are static equivalent over $\mathcal{W}_1$ and $\mathcal{W}'_1$. Finally,*

$$\pi_{K+K',K'}\left(\widetilde{\Psi}^{[K]^{-1}}\left(\widetilde{\Omega}^S\right)\right) = \widetilde{\Phi}^{[K']}\left(\widetilde{\Omega}^S_{K+K'}\right) \;=\; \widetilde{\Omega}'^S\,, \tag{5.35}$$

$$\pi_{K+K',K}\left(\widetilde{\Phi}^{[K']^{-1}}\left(\widetilde{\Omega}'^S\right)\right) = \widetilde{\Psi}^{[K]}\left(\widetilde{\Omega}'^S_{K'+K}\right) \;=\; \widetilde{\Omega}^S\,. \tag{5.36}$$

Lemma 5.5.6. *For all $\xi_{K+1} \in \widetilde{\Omega}_{K+1}$ such that $\xi_K = \pi_{K+1,K}(\xi_{K+1}) \in \widetilde{\Omega} \setminus \widetilde{\Omega}^S$, there is a straight line in $\mathrm{T}_{\widetilde{\Phi}(\xi_K)}M'$ that has contact of infinite order with Σ' at $\widetilde{\Phi}^{[1]}(\xi_{K+1})$.*

These lemmas will be proved later. Let us finish the proof of the Theorem.

If $n < n'$, (5.34) implies existence, for each $\xi' \in \widetilde{\Omega}'_1 = \Omega_1 \cap \Sigma'$, of some $\xi_{K+1} \in \widetilde{\Omega}_{K+1}$ such that $\widetilde{\Phi}^{[1]}(\xi_{K+1}) = \xi'$ and finally, since $\widetilde{\Omega}^S$ is empty according to Lemma 5.5.5, Lemma 5.5.6 yields a straight line in $\mathrm{T}_{\xi'_0}M'$ that has contact of infinite order with Σ' at ξ'; from Proposition 5.3.7, this implies that system Σ' is ruled over Ω_1. If $n > n'$, one concludes in the same way.

Now assume $\boldsymbol{n = n'}$. For all ξ' in $\widetilde{\Phi}^{[1]}\left((\widetilde{\Omega} \setminus \widetilde{\Omega}^S)_{K+1}\right)$, there is, according to Lemma 5.5.6, a straight line in $\mathrm{T}_{\xi'_0}M'$ that has contact of infinite order with Σ' at ξ'. By continuity, this is also true for all ξ' in the topological closure

$$\widetilde{R}' \;=\; \overline{\widetilde{\Phi}^{[1]}\left((\widetilde{\Omega} \setminus \widetilde{\Omega}^S)_{K+1}\right)} = \overline{\pi_{K+K',1}\left({\widetilde{\Psi}^{[K]}}^{-1}\left(\widetilde{\Omega} \setminus \widetilde{\Omega}^S\right)\right)}\,, \tag{5.37}$$

where the second equality come from (5.33). Let $i(\widetilde{R}')$ be the interior of $\widetilde{R}'$ for the induced topology on Σ'; since $\widetilde{R}' = \overline{i(\widetilde{R}')}$, there is an open subset $\mathcal{R}'$ of $\Omega'_1 \subset \mathrm{T}M'$, enjoying the property that it is the interior of its topological closure, and such that $\mathcal{R}' \cap \Sigma' = i(\widetilde{R}')$ and $\overline{\mathcal{R}'} \cap \Sigma' = \widetilde{R}'$. From Proposition 5.3.7, Σ' is ruled over $\mathcal{R}'$. Setting $\mathcal{S}' = \Omega'_1 \setminus \overline{\mathcal{R}'}$, one has $\Omega'_1 = \overline{\mathcal{R}'} \cup \mathcal{S}' = \mathcal{R}' \cup \overline{\mathcal{S}'}$. Along the same lines, Σ is ruled over $\mathcal{R}$, open subset of $\Omega_1 \subset \mathrm{T}M$ such that $\mathcal{R} \cap \Sigma$ is the relative interior of

$$\widetilde{R} \;=\; \overline{\widetilde{\Psi}^{[1]}\left((\widetilde{\Omega}' \setminus \widetilde{\Omega}'^S)_{K'+1}\right)} = \overline{\pi_{K+K',1}\left({\widetilde{\Phi}^{[K']}}^{-1}\left(\widetilde{\Omega}' \setminus \widetilde{\Omega}'^S\right)\right)}\,, \tag{5.38}$$

and such that $\Omega_1 = \overline{\mathcal{R}} \cup \mathcal{S} = \mathcal{R} \cup \overline{\mathcal{S}}$ with $\mathcal{S} = \Omega_1 \setminus \mathcal{R}$.

We have proved (5.24) and point 1; let us prove point 2. Obviously,

$$\mathcal{S} \cap \Sigma \subset \pi_{K+K',1}\left({\widetilde{\Phi}^{[K']}}^{-1}(\widetilde{\Omega}'^S)\right) \quad \text{and} \quad \mathcal{S}' \cap \Sigma' \subset \pi_{K+K',1}\left({\widetilde{\Psi}^{[K]}}^{-1}(\widetilde{\Omega}^S)\right)\ .$$

Using identities (5.35) and (5.36), this implies

$$\mathcal{S} \cap \Sigma \subset \pi_{K,1}\big(\widetilde{\Omega}^S\big) \;\text{ and }\; \mathcal{S}' \cap \Sigma' \subset \pi_{K',1}\big(\widetilde{\Omega}'^S\big)\ . \tag{5.39}$$

For all ξ in $\mathcal{S} \cap \Sigma$, there is one $\xi_K \in \widetilde{\Omega}^S$ such that $\xi = \pi_{K,1}(\xi_K)$ and, from Lemma 5.5.5, a neighborhood $\mathcal{V}_1^\xi$ of ξ in Ω_1 and an open subset $\mathcal{V}'^\xi_1$ of Ω'_1 such that systems Σ and Σ' are static equivalent over $\mathcal{V}_1^\xi$ and $\mathcal{V}'^\xi_1$. For all ξ' in $\mathcal{S}' \cap \Sigma'$, there is one $\xi'_{K'} \in \widetilde{\Omega}'^S$ such that $\xi' = \pi_{K',1}(\xi'_{K'})$ and, from Lemma 5.5.5, a neighborhood $\mathcal{W}'^{\xi'}$ of $\xi'_1 = \pi_{K',1}(\xi'_{K'})$ in Ω'_1 and an open subset $\mathcal{W}_1^{\xi'}$ of Ω_1 such that systems Σ and Σ' are static equivalent over $\mathcal{W}_1^{\xi'}$ and $\mathcal{W}'^{\xi}_1$.

Now, $(\mathcal{V}_1^\xi)_{\xi \in \mathcal{S} \cap \Sigma}$ is an open covering of $\mathcal{S} \cap \Sigma$ and $(\mathcal{W}'^{\xi'}_1)_{\xi' \in \mathcal{S}' \cap \Sigma'}$ is an open covering of $\mathcal{S}' \cap \Sigma'$. Take for $(\widetilde{\mathcal{S}}^\alpha)_{\alpha \in A}$ the union of $(\mathcal{V}_1^\xi)_{\xi \in \mathcal{S} \cap \Sigma}$ and $(\mathcal{W}_1^{\xi'})_{\xi' \in \mathcal{S}' \cap \Sigma'}$; take for $(\widetilde{\mathcal{S}}'^\alpha)_{\alpha \in A}$ the union of $(\mathcal{V}'^{\,\xi}_1)_{\xi \in \mathcal{S} \cap \Sigma}$ and $(\mathcal{W}'^{\xi'}_1)_{\xi' \in \mathcal{S}' \cap \Sigma'}$.

This proves the smooth case, and obviously implies the real analytic one from Lemma 5.5.3.

Let us now prove the four lemmas used in the above proof.

Proof of Lemma 5.5.3. If $\widetilde{\Omega}^S \neq \varnothing$, then there is at least an open set in $\widetilde{\Omega}$ derivatives of $\widetilde{\Phi}$ along any vertical vector field (preserving fibers of $\Sigma_K \to M$) are identically zero; since these are real analytic they must be zero all over $\widetilde{\Omega}$, assumed connected, hence $\widetilde{\Omega}^S = \widetilde{\Omega}$. The proof is similar in $\widetilde{\Omega}'$.

Proof of Lemma 5.5.4. The first relation in (5.33) is a consequence of the two identities

$$\pi_{K'+\ell,K'} \circ \widetilde{\Phi}^{[K'+\ell]} = \widetilde{\Phi}^{[K']} \circ \pi_{K+K'+\ell,K+K'} \;\text{ and }\; \widetilde{\Psi}^{[\ell]} \circ \widetilde{\Phi}^{[K'+\ell]} = \pi_{K+K'+\ell,\ell}\,, \tag{5.40}$$

respectively (5.6) with $(r,s) = (K'+\ell, K')$ and the ℓ^{th} prolongation of (5.30). The second relation follows from interchanging K, Φ, S with K', Ψ, S'.

From equations (5.28) and (5.29), one has, for any positive integer ℓ,

$$\widetilde{\Phi}^{[\ell]}(\widetilde{\Omega}_{K+\ell}) \subset \widetilde{\Omega}'_\ell \ \text{ and } \ \widetilde{\Psi}^{[\ell]}(\widetilde{\Omega}'_{K'+\ell}) \subset \widetilde{\Omega}_\ell \tag{5.41}$$

(for instance, (5.28) implies $\widetilde{\Phi}^{[\ell]}(\widetilde{\Omega}_{K+\ell}) \subset \Sigma'_\ell$, (5.29) implies $\widetilde{\Phi}^{[\ell]}(\widetilde{\Omega}_{K+\ell}) \subset \Omega'_\ell$, hence the first relation above because $\widetilde{\Omega}'_\ell = \Omega'_\ell \cap \Sigma'_\ell$). We only need to prove the reverse inclusions for $\ell = 1$. Let us do it for the second one. The second relation in (5.40) for $\ell = 1$ implies $\widetilde{\Omega}_1 = \widetilde{\Psi}^{[1]}\left(\widetilde{\Phi}^{[K'+1]}(\widetilde{\Omega}_{K+K'+1})\right)$, and finally $\widetilde{\Omega}_1 \subset \widetilde{\Psi}^{[1]}(\widetilde{\Omega}'_{K'+1})$ from the first relation in (5.40) with $\ell = K'+1$.

Proof of Lemma 5.5.5. Assume for instance that $\widetilde{\Omega}^S$ is non-empty ; then it contains an open subset V and there is a smooth $\widetilde{\Phi}_0 : V_0 \to M'$ such that, in restriction to V, $\widetilde{\Phi} = \widetilde{\Phi}_0 \circ \pi_{K,0}$. Hence (5.30) implies, on the open subset $V' = \left(\widetilde{\Psi}^{[K]}\right)^{-1}(V)$ of ${\Sigma'}_{K+K'}$,

$$\widetilde{\Phi}_0 \circ \pi_{K,0} \circ \widetilde{\Psi}^{[K]} = \pi_{K+K',0}\big|_{V'} . \tag{5.42}$$

The rank of the map on the left-hand side is n' while the rank of the right-hand side is no larger than n (rank of $\pi_{K,0}$), hence $\widetilde{\Omega}^S \neq \varnothing$ implies $n' \leq n$. By interchanging the two systems, this proves the fist sentence of the Lemma.

Let us now turn to the case where $n = n'$. Consider ξ_K in $\widetilde{\Omega}^S$. By definition of $\widetilde{\Omega}^S$, there is a neighborhood V and a smooth (real analytic in the real analytic case) map $\widetilde{\Phi}_0 : V_0 \to M'$ such that $\widetilde{\Phi} = \widetilde{\Phi}_0 \circ \pi_{K,0}$ on V. Let V' be defined from V as

$$V' = \pi_{K+K',K'}\left({\widetilde{\Psi}^{[K]}}^{-1}(V)\right) = \widetilde{\Phi}^{[K']}(V_{K+K'}) , \tag{5.43}$$

where the second equality comes from (5.33). Applying $\widetilde{\Psi}$ and $\widetilde{\Psi}^{[1]}$ to both sides of the first equality in (5.6) and using (5.43) with $(r,s) = (K,0)$ and $(r,s) = (K,1)$ yields

$$\widetilde{\Psi}(V') = V_0 , \quad \widetilde{\Psi}^{[1]}(V'_{K'+1}) = V_1 . \tag{5.44}$$

Substituting $\widetilde{\Phi} = \widetilde{\Phi}_0 \circ \pi_{K,0}$ in (5.30), one has $\widetilde{\Phi}_0 \circ \widetilde{\Psi} \circ \pi_{K+K',K'} = \pi_{K+K',0}$ on ${\widetilde{\Psi}^{[K]}}^{-1}(V)$, and finally

$$\widetilde{\Phi}_0 \circ \widetilde{\Psi} = \pi_{K',0} \text{ on } V' ; \tag{5.45}$$

in a similar way, substituting $\widetilde{\Phi}^{[1]} = \widetilde{\Phi}_0^{[1]} \circ \pi_{K+1,1}$ in the first prolongation of (5.30),

$$\widetilde{\Phi}_0^{[1]} \circ \widetilde{\Psi}^{[1]} = \pi_{K'+1,1} \text{ on } V'_{K'+1} . \tag{5.46}$$

Applying $\widetilde{\Phi}_0$ to both sides of the first relation and $\widetilde{\Phi}_0^{[1]}$ to both sides of the second relation in (5.44), one has, using (5.45) and (5.46),

$$\widetilde{\Phi}_0(V_0) = V'_0 , \quad \widetilde{\Phi}_0^{[1]}(V_1) = V'_1 . \tag{5.47}$$

Since the rank of $\pi_{K',0}$ in the right-hand side of (5.45) is $n' = n$ at all points of V', $\widetilde{\Phi}_0$ must be a local diffeomorphism at all point of $\widetilde{\Psi}(V') = V_0$ and in particular at ξ_0 : by the inverse function theorem, there is a neighborhood O of $\xi_0 = \pi_{K,0}(\xi)$ in V_0 and a neighborhood O' of $\Phi_0(\xi_0)$ in M' such that Φ_0 defines a diffeomorphism $O \to O'$.

Let us now replace V with $V \cap {\pi_{K,0}}^{-1}(O)$, a smaller neighborhood of ξ_K ; V' is still defined by (5.43) from this smaller V, one has $V_0 = O$, the former $\widetilde{\Phi}_0$ is replaced by its restriction to this smaller V_0, and the above relations still hold. In particular, $O' = \widetilde{\Phi}_0(O)$ must be all V'_0 according to (5.47), i.e. $\widetilde{\Phi}_0$ defines a diffeomorphism $V_0 \to V'_0$; let $\widetilde{\Psi}_0$ be its inverse. Composing each side of (5.45) with $\widetilde{\Psi}_0$, one gets $\widetilde{\Psi} = \widetilde{\Psi}_0 \circ \pi_{K',0}$ on V' ; hence, by (5.32), one has $V' \subset \widetilde{\Omega}'^S$ and, since this is true for all ξ_K in $\widetilde{\Omega}^S$, one has

$$\pi_{K+K',K'}\left({\widetilde{\Psi}^{[K]}}^{-1}(\widetilde{\Omega}^S)\right) = \widetilde{\Phi}^{[K']}(\widetilde{\Omega}^S_{K+K'}) \subset \widetilde{\Omega}'^S . \tag{5.48}$$

Let $\mathcal{V}_1$ and $\mathcal{V}'_1$ and be open subsets of Ω_1 and Ω'_1 such that

$$V_1 = \Sigma \cap \mathcal{V}_1 , \quad V'_1 = \Sigma \cap \mathcal{V}'_1 . \tag{5.49}$$

From Proposition 5.5.2, the second relation in (5.47) implies that systems Σ and Σ' are static equivalent over $\mathcal{V}_1$ and $\mathcal{V}'_1$. Interchanging the two systems, one proves that

$$\pi_{K+K',K}\left({\widetilde{\Phi}^{[K']}}^{-1}(\widetilde{\Omega}'^S)\right) = \widetilde{\Psi}^{[K]}(\widetilde{\Omega}'^S_{K+K'}) \subset \widetilde{\Omega}^S . \tag{5.50}$$

and that, for all $\xi'_{K'} \in \widetilde{\Omega}'^S$, there are a neighborhood $\mathcal{W}'$ of $\xi'_1 = \pi_{K',1}(\xi'_{K'})$ in Ω'_1 and an open subset $\mathcal{W}_1$ of Ω_1 such that systems Σ and Σ' are static equivalent over $\mathcal{W}_1$ and $\mathcal{W}'_1$.

Now, $\widetilde{\Phi}^{[K']}(\widetilde{\Omega}^S_{K+K'}) \subset \widetilde{\Omega}'^S$ in (5.48) implies $\widetilde{\Omega}^S_{K+K'} \subset {\widetilde{\Phi}^{[K']}}^{-1}(\widetilde{\Omega}'^S)$, and hence $\widetilde{\Omega}^S \subset \pi_{K+K',K}\left({\widetilde{\Phi}^{[K']}}^{-1}(\widetilde{\Omega}'^S)\right)$. Hence (5.48) implies the converse inclusion in (5.50); in a similar way (5.50) implies the converse inclusion in (5.48). This proves (5.35) and (5.36), and ends the proof of Lemma 5.5.5.

Proof of Lemma 5.5.6. Denote by $\bar{\xi}_{K+1}$ the point ξ_{K+1} in the lemma statement and set $\bar{\xi}_K = \pi_{K+1,K}(\bar{\xi}_{K+1}) \in \widetilde{\Omega}\backslash\widetilde{\Omega}^S$, $\bar{\xi}_0 = \pi_{K,0}(\bar{\xi}_{K+1})$, $\bar{\xi}_1 = \pi_{K,1}(\bar{\xi}_{K+1})$. From Proposition 5.3.4, and after possibly shrinking $\mathcal{U}_K$ so that it is contained in Ω, there exist a neighborhood $\mathcal{U}_K \subset \Omega$ of $\bar{\xi}_K$ in $J^K(M)$, coordinates $(x_{\mathrm{I}}, x_{\mathrm{II}})$ on $\mathcal{U}_0 = \pi_{K,0}(\mathcal{U}_K)$ inducing coordinates $(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{I}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{I}}^{(K)}, x_{\mathrm{II}}^{(K)})$ on $\mathcal{U}_K$, and an open subset $U_K \subset \mathbb{R}^{n+Km}$ such that the equations of $\widetilde{\mathcal{U}}_K = \mathcal{U}_K \cap \Sigma_K$ in $J^K(M)$ in these coordinates are

$$\begin{aligned} &x_{\mathrm{I}}^{(i)} = f^{(i-1)}(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(i)}) , \quad 1 \le i \le K , \\ &(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)}) \in U_K . \end{aligned} \tag{5.51}$$

By substitution, there is a unique smooth map $\phi_K : U_K \to M'$ such that
$\widetilde{\Phi}(\xi) = \phi_K(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)})$ for all ξ in $\widetilde{\mathcal{U}}_K$ with coordinate vector $(x_{\mathrm{I}}, x_{\mathrm{II}}, \ldots, x_{\mathrm{I}}^{(K)}, x_{\mathrm{II}}^{(K)})$.

Let $\overline{X}_i = (\overline{x_{\mathrm{I}}}, \overline{x_{\mathrm{II}}}, \overline{\dot{x}_{\mathrm{I}}}, \overline{\dot{x}_{\mathrm{II}}}, \ldots, \overline{x_{\mathrm{I}}^{(i)}}, \overline{x_{\mathrm{II}}^{(i)}})$ be the coordinate vector of $\bar{\xi}_i$ for $i \le K+1$ and $\bar{\rho}$ the smallest integer such that ϕ_K does not depend on $x_{\mathrm{II}}^{(\bar{\rho}+1)}, \ldots, x_{\mathrm{II}}^{(K)}$ on at least one neighborhood of $\overline{X}_K$. Shrinking U_K to this neighborhood, and $\widetilde{\mathcal{U}}_K$ accordingly, we may define $\phi : U_{\bar{\rho}} \to M'$, with $U_{\bar{\rho}}$ the projection of U_K on $\mathbb{R}^{n+\bar{\rho}m}$, such that $\widetilde{\Phi}(\xi) = \phi_K(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)}) = \phi(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(\bar{\rho})})$. If $\bar{\rho}$ was zero, one would have $\widetilde{\Phi}(\xi) = \phi(x_{\mathrm{I}}, x_{\mathrm{II}})$, hence the right-hand side of (5.31) would be satisfied for $\xi = \bar{\xi}_K$ with $V = \widetilde{\mathcal{U}}_K$; this is impossible because we assumed $\bar{\xi}_K \in \widetilde{\Omega} \setminus \widetilde{\Omega}^S$. Hence $\bar{\rho} \ge 1$.

For all ξ_{K+1} in $\widetilde{\mathcal{U}}_{K+1}$ with coordinate vector $(x_{\mathrm{I}}, x_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K+1)})$, one has

$$\widetilde{\Phi}^{[1]}(\xi_{K+1}) = \chi(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(\bar{\rho})}, x_{\mathrm{II}}^{(\bar{\rho}+1)}) \tag{5.52}$$

with $\chi : U_{\bar\rho+1} \to \mathrm{T}M'$ the map defined by

$$\chi(x_{\mathrm{I}} \dots x_{\mathrm{I\!I}}^{(\bar\rho+1)}) = \Big(\ \phi(x_{\mathrm{I}} \dots x_{\mathrm{I\!I}}^{(\bar\rho)})\ ,\ a(x_{\mathrm{I}} \dots x_{\mathrm{I\!I}}^{(\bar\rho)}) + \frac{\partial\phi}{\partial x_{\mathrm{I\!I}}^{(\bar\rho)}}(x_{\mathrm{I}} \dots x_{\mathrm{I\!I}}^{(\bar\rho)})\ x_{\mathrm{I\!I}}^{(\bar\rho+1)}\ \Big) \tag{5.53}$$

with $a = \frac{\partial\phi}{\partial x_{\mathrm{I}}} f + \sum_{i=0}^{\bar\rho-1} \frac{\partial\phi}{\partial x_{\mathrm{I\!I}}^{(i)}}\, x_{\mathrm{I\!I}}^{(i+1)}$. According to (5.29), (5.51) and (5.52), Σ' contains $\chi(U_{\bar\rho+1})$. Now, for any $(x_{\mathrm{I}}, \dots, x_{\mathrm{I\!I}}^{(\bar\rho+1)}) \in U_{\bar\rho+1}$ such that the linear map

$$\frac{\partial\phi}{\partial x_{\mathrm{I\!I}}^{(\rho)}}(x_{\mathrm{I}}, \dots, x_{\mathrm{I\!I}}^{(\bar\rho)}) : \quad \mathbb{R}^m \ \to\ \mathrm{T}_{\phi(x_{\mathrm{I}},\dots,x_{\mathrm{I\!I}}^{(\bar\rho)})}M'$$

is nonzero, picking $\underline{w} \neq 0$ in its range, (5.53) implies that the straight line Δ in $\mathrm{T}_{\phi(x_{\mathrm{I}},\dots,x_{\mathrm{I\!I}}^{(\bar\rho)})}M'$ passing through $\chi(x_{\mathrm{I}} \dots x_{\mathrm{I\!I}}^{(\bar\rho+1)})$ with direction $\underline{w}$ has a segment around $\chi(x_{\mathrm{I}} \dots x_{\mathrm{I\!I}}^{(\bar\rho+1)})$ contained in Σ', hence in particular Δ has contact of infinite order with Σ' at point $\chi(x_{\mathrm{I}}, \dots, x_{\mathrm{I\!I}}^{(\bar\rho+1)})$. To sum up, we have proved so far that, for all ξ_{K+1} in $\widetilde{\mathcal{U}}_{K+1}$ with coordinate vector $(x_{\mathrm{I}}, x_{\mathrm{I\!I}}, \dots, x_{\mathrm{I\!I}}^{(K+1)})$ such that $\frac{\partial\phi}{\partial x_{\mathrm{I\!I}}^{(\rho)}}(x_{\mathrm{I}} \dots x_{\mathrm{I\!I}}^{(\bar\rho)})$ is nonzero, there is a straight line $\Delta_{\xi_{K+1}}$ in $\mathrm{T}_{\widetilde\Phi(\xi_K)}M'$ passing through $\widetilde\Phi^{[1]}(\xi_{K+1})$ that has contact of infinite order with Σ' at $\widetilde\Phi^{[1]}(\xi_{K+1})$. The set of such points ξ_{K+1} may not contain $\bar\xi_{K+1}$ but its topological closure does, by minimality of $\bar\rho$; taking a sequence of points ξ_{K+1} that converges to $\bar\xi_{K+1}$, any accumulation point of the compact sequence $\big(\Delta_{\xi_{K+1}}\big)$ is a straight line in $\mathrm{T}_{\widetilde\Phi(\bar\xi_K)}M'$ passing through $\widetilde\Phi^{[1]}(\bar\xi_{K+1})$ that has contact of infinite order with Σ' at $\widetilde\Phi^{[1]}(\bar\xi_{K+1})$.

Chapitre 6

Reproduction de l'article:

J.-B. Pomet,

"A differential geometric setting for dynamic equivalence and dynamic linearization",

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Abstract. This paper presents an (infinite dimensional) geometric framework for control system, based on infinite jet bundles, where a system is represented by a single vector field and dynamic equivalence (to be precise : equivalence by endogenous dynamic feedback) is conjugation by diffeomorphisms. These diffeomorphisms are very much related to Lie-Bäcklund transformations. It is proved in this framework that dynamic equivalence of single-input systems is the same as static equivalence.

6.1 Introduction

For a control system

$$\dot{x} \;=\; f(x,u) \tag{6.1}$$

where $x \in \mathbb{R}^n$ is the state, and $u \in \mathbb{R}^m$ is the input, what one usually means by a dynamic feedback is a system with a certain state z, input (x, v) and output u :

$$\dot{z} = g(x,z,v)\ , \qquad u = \gamma(x,z,v) \quad . \tag{6.2}$$

When applying this dynamic feedback to system (6.1), one gets a system with state (x, z) and input v : $\dot{x} = f(x, \gamma(x, z, v))$, $\dot{z} = g(x, z, v)$. This system may be transformed with a change of coordinates $X = \phi(x, z)$ in the extended variables to a system $\dot{X} = h(X, v)$. The problem of dynamic feedback linearization is stated in [23] by B. Charlet, J. Lévine and R. Marino as the one of finding g, γ and ϕ such that $\dot{X} = h(X, v)$ be a linear controllable system. When z is not present, γ and ϕ define a static feedback transformation in the usual sense. This transformation is said to be invertible if ϕ is a diffeomorphism and γ is invertible with respect to v ; these transformations form a group of transformations. On the contrary, when z is present, the simple fact that the general "dynamic feedback transformation" (6.2), defined by g, γ and ϕ increases the size of the state prevents dynamic feedbacks in this sense from being "invertible".

In [36, 37], M. Fliess, J. Lévine, P. Martin and P. Rouchon introduced a notion of equivalence in a differential algebraic framework where two systems are *equivalent by endogenous dynamic* *feedback* if the two corresponding differential fields are algebraic over one another. This is translated in a state-space representation by some (implicit algebraic) relations between the "new" and the "old" state, output and many derivatives of outputs transforming one system into the other and vice-versa. It is proved that equivalence to controllable linear system is equivalent to *differential flatness*, which is defined as existence of m elements in the field which have the property to be a "linearizing output" or "flat output". In [68, "Point de vue analytique"], P. Martin introduced the notion of *endogenous dynamic feedback* as a dynamic feedback (6.2) where, roughly speaking, z is a function of $x, u, \dot{u}, \ddot{u} \ldots$. He proved that a system may be obtained from another one by nonsingular endogenous feedback if and only if there exists a transformation of the same kind as in [36, 37] but explicit and analytic which transforms one system into the other. This is called equivalence by endogenous dynamic feedback as in the algebraic case. These transformations may either increase or decrease the dimension of the state.

B. Jakubczyk gives in [55, 56] a notion of dynamic equivalence in terms of transformations on "trajectories" of the system; different types of transformations are defined there in terms of infinite jets of trajectories. One of them is proved there to be exactly the one studied here. See after Definition 6.1 for further comparisons.

In [93], W.F. Shadwick makes (prior to [36, 37, 55, 56]) a link between dynamic feedback linearization and the notion of absolute equivalence defined by E. Cartan for Pfaffian systems. It is not quite clear that this notion of equivalence coincides with equivalence in the sense of [36, 37] or [55, 56], the formulation is very different.

The contribution of the present paper –besides Theorem 6.3 which states that dynamic equivalent single input systems with the same number of states are static equivalent– is to give a geometric meaning to transformations which are exactly these introduced by P. Martin in [68] (endogenous dynamic feedback transformations). Our system is represented by a single vector field on a certain "infinite-dimensional manifold", and our transformations are diffeomorphisms on this manifold. Then the action of these transformations on systems is translated by the usual transformation diffeomorphisms induce on vector fields. There are of course many technical difficulties in defining vector fields, diffeomorphisms or smooth functions in these "infinite-dimensional manifolds". The original motivation was to "geometrize" the constructions made in [4, 83]; it grew up into the present framework which, we believe, has some interest in itself, the geometric exposition of [4, 83] is contained in the paper reprinted in Chapter 7.

Note finally that the described transformations are very closely related to *infinite order contact transformations* or *Lie-Bäcklund transformations* or $\mathcal{C}$*-transformations*, see [45, 1] and that the geometric context we present here is the one of infinite jet spaces used in [31, 62, 106, 92] for example to describe and study Lie-Bäcklund transformations. These presentations however are far from being unified, for instance smooth functions do not have to depend only on a finite number of variables in [92], and are not explicitly defined in [1]. They also had to be adapted for many reasons in order to get a technically workable framework; for instance, we prove an inverse function theorem which characterizes local diffeomorphisms without having to refer to an inverse mapping which is of the same type. The language of jet spaces and differential systems has been used already in control theory by M. Fliess [32] and by J.-F. Pommaret [84], with a somewhat different purpose.

Some recent work by M. Fliess [33] (see also a complete exposition on this topic in E. Delaleau's [30]) points out that a more natural state-space representation than (6.1) for a nonlinear system involves not only x and u, but also an arbitrary number of time-derivatives of u; this is referred to as "generalized-state" representation, and we keep this name for the infinite dimensional state-manifold, see section 6.3. In [33, 30], the "natural" state-space representation is $F(x, \dot{x}, u, \dot{u}, \ddot{u}, \ldots, u^{(J)}) = 0$ rather than (6.1). Here not only do we suppose that $\dot{x}$ is an explicit function of the other variables ("explicit representation" according to [33, 30]) but also that $J = 0$ ("classical representation"). Almost everything in this paper may be adapted to the "non-classical" case, i.e. to the case where some time-derivatives of the input would appear in the right-hand side of (6.1) ; we chose the classical representation for simplicity and because, as far as dynamic equivalence is concerned, a non-classical system is equivalent to a classical one by simply "adding some integrators" ; on the contrary, the implicit case is completely out of the scope of this paper, see the end of section 6.2.

Very recently the authors of [36, 37] have independently proposed a "differential geometric" approach for dynamic equivalence, see [38, 41], which is similar in spirit to the present approach, although the technical results do differ. This was brought to the attention of the author too late for a precise comparison between the two approaches.

The paper is organized as follows : section 6.2 presents briefly the point of view of jet spaces and contact structure for system (6.1) considered as a differential relation $\dot{x} - f(x, u) = 0$ (no theoretical material from this section is used elsewhere in the paper). Section 6.3 presents in details the differential structure of the "generalized state-space manifold" where coordinates are $x, u, \dot{u}, \ldots$, where we decide to represent a system by a single vector field. Section 6.4 defines in this context dynamic equivalence and relates it to notions already introduced in the literature. Section 6.5 deals with static equivalence. Section 6.6 is devoted to the single-input case, and states the result that dynamic equivalence and static equivalence are then the same. Finally section 6.7 is devoted to dynamic linearization, it introduces in a geometric way the "linearizing outputs" defined for for dynamic linearization in [36, 37, 68].

6.2 Control systems as differential relations

This section is only meant to relate the approach described subsequently to some better known theories. It does not contain rigorous arguments.

In the spirit of the work of J. Willems [109], or also of M. Fliess [33], one may consider that the control system (6.1) is simply a differential relation on the functions of time $x(t), u(t)$ and that the object of importance is the set of solutions, i.e. of functions $t \mapsto (x(t), u(t))$ such that $\frac{dx}{dt}(t)$ is identically equal to $f(x(t), u(t))$. Of course this description does not need precisely a state-space description like (6.1).

The geometric way of describing the solution of this first order relation in the "independent variable" t (time) and the "dependent variables" x and u is to consider, as in [1, 84, 62, 106, 31], the fibration

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R}^{n+m} & \stackrel{\pi}{\rightarrow} & \mathbb{R} \\ (t, x, u) & \mapsto & t \end{array} \tag{6.3}$$

and its first jet manifold $J^1(\pi)$, which is simply $T\left(\mathbb{R}^n \times \mathbb{R}^m\right) \times \mathbb{R}$. A canonical set of coordinates on $J^1(\pi)$ is $(t, x, u, \dot{x}, \dot{u})$. The relation $R(t, x, u, \dot{x}, \dot{u}) = \dot{x} - f(x, u) = 0$ defines a sub-manifold $\mathcal{R}$ of the fiber bundle (6.3), which is obviously a sub-bundle. The contact module on $J^1(\pi)$ is the module of 1-forms (or the codistribution) generated by the 1-forms $\mathrm{d}x_i - \dot{x}_i \mathrm{d}t$ and $\mathrm{d}u_j - \dot{u}_j \mathrm{d}t$, $1 \leq i \leq n$, $1 \leq j \leq m$. A "solution" of the differential system is a section $t \mapsto (t, x(t), u(t), \dot{x}(t), \dot{u}(t))$ of the sub-bundle $\mathcal{R}$, which annihilates the contact forms (this simply means that $\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}$ and $\frac{\mathrm{d}u}{\mathrm{d}t} = \dot{u}$, i.e. that this section is the jet of a section of (6.3)).

Since we wish to consider some transformations involving an arbitrary number of derivatives, we need the infinite jet space $J^\infty(\pi)$ of the fibration (6.3). For short, it is the projective limit of the finite jet spaces $J^k(\pi)$, and some natural coordinates on this "infinite-dimensional manifold" are $(t, x, u, \dot{x}, \dot{u}, \ddot{x}, \ddot{u}, x^{(3)}, u^{(3)}, \ldots\ldots)$. The contact forms are

$$\mathrm{d}x_i^{(j)} - x_i^{(j+1)}\mathrm{d}t\,, \quad \mathrm{d}u_k^{(j)} - u_k^{(j+1)}\mathrm{d}t\,, \qquad 1 \leq i \leq n\,,\ 1 \leq k \leq m\,,\ j \geq 0\,. \tag{6.4}$$

This infinite dimensional “manifold” is described in [62] for example, and we will recall in next section what we really need. The “Cartan distribution” is the one annihilated by all these forms, it is spanned by the single vector field

$$\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{u}\frac{\partial}{\partial u} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \ldots\ldots \tag{6.5}$$

where $\dot{x}\frac{\partial}{\partial x}$ stands for $\sum_i \dot{x}_i\frac{\partial}{\partial x_i}$, $\dot{u}\frac{\partial}{\partial u}$ for $\sum_i \dot{u}_i\frac{\partial}{\partial u_i}$... The relation R has to be replaced by its infinite prolongation, i.e. R itself plus all its “Lie derivatives” along (6.5) :

$$\begin{array}{rcl} R(t,x,u,\dot{x},\dot{u}) & = & \dot{x} - f(x,u) = 0 \\ R_1(t,x,u,\dot{x},\dot{u},\ddot{x},\ddot{u}) & = & \ddot{x} - \frac{\partial f}{\partial x}\dot{x} - \frac{\partial f}{\partial u}\dot{u} = 0 \\ R_2(t,x,u,\dot{x},\dot{u},\ddot{x},\ddot{u},x^{(3)},u^{(3)}) & = & x^{(3)} - \ldots = 0 \\ & \vdots & \end{array} \tag{6.6}$$

This defines a sub-bundle $\mathcal{R}_\infty$ of $J^\infty(\pi)$. A “solution” of the differential system is a section $t \mapsto (t, x(t), u(t), \dot{x}(t), \dot{u}(t), \ddot{x}(t), \ddot{u}(t), \ldots)$ of the sub-bundle $\mathcal{R}_\infty$, which annihilates the contact forms; it is obviously defined uniquely by $x(t)$ and $u(t)$ such that $\frac{dx}{dt}(t) = f(x(t), u(t))$ with the functions $u^{(j)}$ and $x^{(j)}$ obtained by differentiating $x(t)$ and $u(t)$.

$\mathcal{R}_\infty$ is a sub-bundle of $J^\infty(\pi)$ which has a particular form : since the relations allow one to explicitly express all the time-derivatives $\dot{x}, \ddot{x}, x^{(3)}, \ldots$ of x as functions of $x, u, \dot{u}, \ddot{u}, u^{(3)}, \ldots$, a natural set of coordinates on this sub-manifold is $(t, x, u, \dot{u}, \ddot{u}, \ldots)$; note that if, instead of the explicit form (6.1), we had an implicit system $f(x, u, \dot{u}) = 0$, this would not be true. The vector field (6.5), which spans the Cartan distribution is tangent to $\mathcal{R}_\infty$, and its expression in the coordinates $(t, x, u, \dot{u}, \ddot{u}, \ldots)$ *considered as coordinates on* $\mathcal{R}_\infty$ is

$$\frac{\partial}{\partial t} + f(x,u)\frac{\partial}{\partial x} + \dot{u}\frac{\partial}{\partial u} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \ldots + u^{(k+1)}\frac{\partial}{\partial u^{(k)}} + \ldots\ldots \tag{6.7}$$

and the restriction of the contact forms are $\mathrm{d}x - f\mathrm{d}t$, $\mathrm{d}u^{(j)} - u^{j+1}\mathrm{d}t$, $j \geq 0$. The sub-bundles $\mathcal{R}_\infty$ obtained for different systems are therefore all diffeomorphic to a certain “canonical object” independent of the system, and where coordinates are $(t, x, u, \dot{u}, \ddot{u}, \ldots)$, let this object be $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ where $\mathcal{M}_\infty^{m,n}$ is described in more details in next section and the first factor $\mathbb{R}$ is time, with an embedding ψ of $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ into $J_\infty(\pi)$ which defines a diffeomorphism between $\mathcal{R}_\infty$ and $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$; this embedding depends on the system and completely determines it ; it pulls back the contact module on $J^\infty(\pi)$ to a certain module of forms on $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ and the Cartan vector field (6.5) into (6.7). The points in $J^\infty(\pi)$ which are outside $\mathcal{R}_\infty$ are not really of interest to the system, so that we only need to retain $\mathcal{R}_\infty$, and it turns out that all the information is contained in $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ and the vector field (6.7) which translates the way the contact module is pulled back by the embedding of $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ into $J_\infty(\pi)$ whose image is $\mathcal{R}_\infty$. This is the point of view defended in [106] for example where such a manifold endowed with what it inherits from the contact structure on $J^\infty(\pi)$ is called a “diffiety”. It is only in the special case of explicit systems like (6.1) that all diffieties can be parameterized by $x, u, \dot{u}, \ldots$ and therefore can all be represented by the single object $\mathcal{M}_\infty^{m,n}$, endowed with a contact structure, or a Cartan vector field, which of course depends on the system.

Finally, since everything is time-invariant, one may “drop” the variable t (or quotient by time-translations, or project on the sub-manifold $\{t = 0\}$ which is possible because all objects are invariant along the fibers) and work with the coordinates $(x, u, \dot{u}, \ddot{u}, \ldots)$ only, with $f\frac{\partial}{\partial x} + \dot{u}\frac{\partial}{\partial u} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \ldots$ instead of (6.7) ; solutions are curves which are tangent to this vector field. This is the point of view we adopt here, and this is described in details in next section.

6.3 The generalized state-space manifold

The phrase "generalized state" denotes the use of many derivatives of the input as in [33, 30]. The "infinite-dimensional manifold" $\mathcal{M}_\infty^{m,n}$ we are going to consider is parameterized by $x, u, \dot{u}, \ddot{u}, \ldots$; in order to keep things simple, we define it in coordinates, i.e. a point of $\mathcal{M}_\infty^{m,n}$ is simply a sequence of numbers, as in [80] for example. It may be extended to x and u living in arbitrary manifolds via local coordinates, but, since dynamic equivalence is local in nature, the present description is suitable.

6.3.1 The manifold, functions and mappings

For $k \geq -1$, let $\mathcal{M}_k^{m,n}$ be $\mathbb{R}^n \times (\mathbb{R}^m)^{k+1}$ ($\mathcal{M}_{-1}^{m,n}$ is $\mathbb{R}^n$), and let us denote the coordinates in $\mathcal{M}_k^{m,n}$ by

$$(x \, , \, u \, , \, \dot{u} \, , \, \ddot{u} \, , \ldots \, , \, u^{(k)} \,)$$

where x is in $\mathbb{R}^n$ and $u, \dot{u}, \ldots$ are in $\mathbb{R}^m$. $\mathcal{M}_\infty^{m,n}$ is the space of infinite sequences

$$(x \, , \, u \, , \, \dot{u} \, , \, \ddot{u} \, , \ldots \, , \, u^{(j)} \, , \, u^{(j+1)} \, , \, \ldots) \; .$$

For simplicity, we shall use the following notation :

$$\mathcal{U} \;=\; (u, \dot{u}, \ddot{u}, u^{(3)}, \ldots) \; , \quad \mathcal{X} \;=\; (x, \mathcal{U}) \;=\; (x, u, \dot{u}, \ddot{u}, u^{(3)}, \ldots) \; . \tag{6.8}$$

Let, for $k \geq -1$, the projection π_k, from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_k^{m,n}$ be defined by :

$$\pi_{-1}(\mathcal{X}) = x \; , \quad \text{and } \pi_k(\mathcal{X}) = (x, u, \dot{u}, \ldots, u^{(k)}) \; , \; k \geq 0 \; . \tag{6.9}$$

$\mathcal{M}_\infty^{m,n}$ may be constructed as the projective limit of $\mathcal{M}_k^{m,n}$, and this naturally endows it with the weakest such that all these projections are continuous (product topology) ; a basis of the topology are the sets

$$\pi_k^{-1}(O) \; , \quad O \text{ open subset of } \mathcal{M}_k^{m,n} \; .$$

This topology makes $\mathcal{M}_\infty^{m,n}$ a topological vector space, which is actually a Fréchet space (see for instance [14]). It is easy to see that continuous linear forms are these which depend only on a finite number of coordinates. This leads one to the (false) idea that there is a natural way of defining differentiability so that differentiable functions depend only on a finite number of variables, which is exactly the class of smooth functions we wish to consider (as in most of the literature on differential system and jet spaces [1, 62, 80, 31, 106]), since they translate into realistic dynamic feedbacks from the system theoretic point of view. It is actually possible to define a very natural notion of differentiability in Fréchet spaces (see for instance the very complete [46]) but there is nothing wrong in this framework with smooth functions depending on infinitely many variables. For instance the function mapping $(u, \dot{u}, \ddot{u}, u^{(3)}, \ldots)$ to $\sum_{j=0}^{\infty} \frac{1}{2^j} \rho(\frac{u^{(j)}}{j})$, with ρ a smooth function with compact support containing 0 vanishing at 0 as well as its derivatives of all orders depends on *all* the variables at zero, but it is smooth in this framework. It is hard to imagine a local definition of differentiability which would classify this function non-smooth.

Here, we do not wish to consider smooth functions or smooth maps depending on infinitely many variables ; we therefore define another differentiable structure, which agrees with the one usually used for differential systems [31, 1, 80, 62, 106] :

- A function h from an open subset V of $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$ (or to any finite-dimensional manifold) is a **smooth function** at $\mathcal{X} \in V$ if and only if, locally at each point, it depends only on a finite number of derivatives of u and, as a function of a finite number of variables, it is smooth (of class $\mathbf{C}^\infty$) ; more technically : if and only if there exists an open neighborhood U of $\mathcal{X}$ in V, an integer ρ, and a smooth function h_ρ from an open subset of $\mathcal{M}_\rho^{m,n}$ to $\mathbb{R}$ (or to the finite-dimensional manifold under consideration) such that $h(\mathcal{Y}) = h_\rho \circ \pi_\rho(\mathcal{Y})$ for all $\mathcal{Y}$ in U. It is a smooth function on V if it is a smooth function at all $\mathcal{X}$ in V. The highest ρ such that h actually depends on the ρth derivative of u on any neighborhood of $\mathcal{X}$ (-1 if it depends on x only on a certain neighborhood of $\mathcal{X}$) we will call the **order** of h at $\mathcal{X}$, and we denote it by $\boldsymbol{\delta(h)(\mathcal{X})}$. It is also the largest integer such that $\frac{\partial h}{\partial u^{(\rho)}}$ (this may be defined in coordinates and is obviously a smooth function) is not identically zero on any neighborhood of $\mathcal{X}$. Note that $\delta(h)$ may be unbounded on $\mathcal{M}_\infty^{m,n}$. We denote by $\mathbf{C}^\infty(V)$ the algebra of smooth functions from V to $\mathbb{R}$, $\mathbf{C}^\infty(\mathcal{M}_\infty^{m,n})$ if $V = \mathcal{M}_\infty^{m,n}$.

– A **smooth mapping** from an open subset V of $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ is a map φ from V to $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ such that, for any ψ in $\mathbf{C}^\infty(\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}})$, $\psi \circ \varphi$ is in $\mathbf{C}^\infty(V)$. It is a smooth mapping at $\mathcal{X}$ if it is a smooth mapping from a certain neighborhood of $\mathcal{X}$ to $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$. Of course, in coordinates, it is enough that this be true for ψ any coordinate function. For such a map and for all k, there exists locally an integer ρ_k and a (unique) smooth map φ_k from $\pi_{\rho_k}(V) \subset \mathcal{M}_{\rho_k}^{m,n}$ to $\mathcal{M}_k^{\widetilde{m},\widetilde{n}}$ such that

$$\pi_k \circ \varphi \quad = \quad \varphi_k \circ \pi_{\rho_k} \ . \tag{6.10}$$

The smallest possible ρ_k at a point $\mathcal{X}$ is $\delta(\pi_k \circ \varphi)(\mathcal{X})$.

– A **diffeomorphism** from an open subset V of $\mathcal{M}_\infty^{m,n}$ to an open subset $\widetilde{V}$ of $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ is a smooth mapping φ from V to $\widetilde{V}$ which is invertible and is such that φ^{-1} is a smooth mapping from $\widetilde{V}$ to V.

– A **static diffeomorphism** φ from an open subset V of $\mathcal{M}_\infty^{m,n}$ to an open subset $\widetilde{V}$ of $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ is a diffeomorphism from V to $\widetilde{V}$ such that for all k, $\delta(\pi_k \circ \varphi)(\mathcal{X})$ is constant equal to k.

– A **(local) system of coordinates** on $\mathcal{M}_\infty^{m,n}$ (at a certain point) is a sequence $(h_\alpha)_{\alpha \geq 0}$ of smooth functions (defined on a neighborhood of the point under consideration) such that the smooth mapping $\mathcal{X} \mapsto (h_\alpha(\mathcal{X}))_{\alpha \geq 0}$ is a local diffeomorphism onto an open subset of $\mathbb{R}^{\mathbb{N}}$, considered as $\mathcal{M}_\infty^{1,0}$.

Note that the functions $x_1, \ldots, x_n, u_1, \ldots, u_m, \dot{u}_1, \ldots, \dot{u}_m, \ldots$ are coordinates in this sense. Actually, this makes all the "manifolds" $\mathcal{M}_\infty^{m,n}$ globally diffeomorphic to $\mathcal{M}_\infty^{1,0}$, so that they are all diffeomorphic to one another (this can be viewed as renumbering the natural coordinates). The following proposition shows that static diffeomorphisms are much more restrictive : they preserve n and m.

Proposition 6.1. *Let φ be a **static** diffeomorphism from an open set U of $\mathcal{M}_\infty^{m,n}$ to an open set V of $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$. Its inverse φ^{-1} is also a static diffeomorphism and φ induces, for all $k \geq 0$, a diffeomorphism φ_k from $\mathcal{M}_k^{m,n}$ to $\mathcal{M}_k^{\widetilde{m},\widetilde{n}}$ (from $\mathbb{R}^n$ to $\mathbb{R}^{\widetilde{n}}$ for $k = -1$). Its existence therefore implies $\widetilde{n} = n$ and $\widetilde{m} = m$.*

Proof : For all $k \geq -1$, since $\delta(\varphi \circ \pi_k) = k$, there exists a mapping φ_k from $\pi_k(U)$ to $\pi_k(V)$ satisfying (6.10) with $\rho_k = k$. All these mappings are onto because if one of them was not onto, (6.10) would imply that φ is onto either. Now let us consider φ^{-1} ; it is a diffeomorphism from V to U and there exists therefore, for all k, an integer σ_k and a smooth map $\left(\varphi^{-1}\right)_k$ from $\pi_{\sigma_k}(V) \subset \mathcal{M}_{\sigma_k}^{\widetilde{m},\widetilde{n}}$ to $\mathcal{M}_k^{m,n}$ such that

$$\pi_k \circ \varphi^{-1} \quad = \quad \left(\varphi^{-1}\right)_k \circ \pi_{\sigma_k} \ . \tag{6.11}$$

Applying φ on the right to both sides and using the fact that $\pi_{\sigma_k} \circ \varphi = \varphi_{\sigma_k} \circ \pi_{\sigma_k}$, we get

$$\pi_k \quad = \quad \left(\varphi^{-1}\right)_k \circ \varphi_{\sigma_k} \circ \pi_{\sigma_k} \ . \tag{6.12}$$

Applied to $(x, u, \dot{u}, \ldots)$, this means

$$\begin{aligned}(x, u, \dot{u}, \ldots, u^{(k)}) &= \left(\varphi^{-1}\right)_k (y, v, \dot{v}, \ldots, v^{(k)}, \ldots, v^{(\sigma_k)}) \\ \text{with } (y, v, \dot{v}, \ldots, v^{(k)}, \ldots, v^{(\sigma_k)}) &= \varphi_{\sigma_k} (x, u, \dot{u}, \ldots, u^{(k)}, \ldots, u^{(\sigma_k)})\end{aligned} \tag{6.13}$$

Since φ_{σ_k} is onto and each $v^{(j)}$ depends only on $x, u, \ldots, u^{(j)}$, (6.13) implies that $\left(\varphi^{-1}\right)_k$ depends only on $y, v, \dot{v}, \ldots, v^{(k)}$. Therefore σ_k might have been taken to be k, and then one has (6.12) with $\sigma_k = k$ and therefore

$$\left(\varphi^{-1}\right)_k \circ \varphi_k = \mathrm{Id}_{\mathcal{M}_k^{\tilde{m},\tilde{m}}} \tag{6.14}$$

which proves that each φ_k is a diffeomorphism and ends the proof. ∎

Let us define, as examples of diffeomorphisms, the (non static!) diffeomorphisms $\Upsilon_{n,(p_1,\ldots,p_m)}$ from $\mathcal{M}_\infty^{m,n}$ $\mathcal{M}_\infty^{m,n+p_1+\ldots+p_m}$ which "adds p_k integrators on the kth input" :

$$\Upsilon_{n,(p_1,\ldots,p_m)}(x,\mathcal{U}) = (z,\mathcal{V}) \text{ with } \begin{array}{l} z = (x, u_1, \dot{u}_1 \ldots u_1^{(p_1-1)}, \ldots, u_m, \dot{u}_m, \ldots u_m^{(p_m-1)}) \\ v_k^{(j)} = u_k^{(j+p_k)} . \end{array} \tag{6.15}$$

It is invertible : one may define $\Upsilon_{N,(-p_1,\ldots,-p_m)}$ from $\mathcal{M}_\infty^{m,N}$ to $\mathcal{M}_\infty^{m,N-p_1-\ldots-p_m}$ for $N \geq p_1 + \ldots + p_m$ by $\Upsilon_{n,(-p_1,\ldots,-p_m)}(z,\mathcal{V}) = (x,\mathcal{U})$ where x is the $N - p_1 - \ldots - p_m$ first coordinates of z, and $u_k^{(j)}$ is $v_k^{(j-p_k)}$ if $j \geq p_k$ and one of the remaining components of z if $0 \leq j \leq p_k - 1$, so that $\Upsilon_{n,(p_1,\ldots,p_m)} \circ \Upsilon_{n,(-p_1,\ldots,-p_m)} = \mathrm{Id}$.

6.3.2 Vector fields and differential forms

The "tangent bundle" to the infinite dimensional manifold $\mathcal{M}_\infty^{m,n}$ is, since $\mathcal{M}_\infty^{m,n}$ is a vector space, $\mathcal{M}_\infty^{m,n} \times \mathcal{M}_\infty^{m,n}$, which is a (trivial) vector bundle over $\mathcal{M}_\infty^{m,n}$. A **smooth vector field** is a smooth (as a mapping from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{m,n} \times \mathcal{M}_\infty^{m,n}$, considered as $\mathcal{M}_\infty^{2m,2n}$) section of this bundle. It is of the form

$$F = f \frac{\partial}{\partial x} + \sum_0^\infty \alpha_j \frac{\partial}{\partial u^{(j)}} \tag{6.16}$$

where f is a smooth function from $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}^n$ and the α_j's are smooth functions from $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}^m$, where $f \frac{\partial}{\partial x}$ stands for $\sum_i f_i \frac{\partial}{\partial x_i}$ and $\alpha_j \frac{\partial}{\partial u^{(j)}}$ for $\sum_i \alpha_{j,i} \frac{\partial}{\partial u_i^{(j)}}$, and the $\frac{\partial}{\partial x_i}$'s and $\frac{\partial}{\partial u_i^{(j)}}$'s are the canonical sections corresponding to the "coordinate vector fields" associated with the canonical coordinates. Vector fields obviously define smooth differential operators on smooth functions : in coordinates, $L_F h$ is an infinite sum with finitely many nonzero terms.

Smooth differential forms are smooth sections of the cotangent bundle, which is simply $\mathcal{M}_\infty^{m,n} \times (\mathcal{M}_\infty^{m,n})^*$ where $(\mathcal{M}_\infty^{m,n})^*$ is the topological dual of$\mathcal{M}_\infty^{m,n}$, i.e. the space of infinite sequences with only a finite number of nonzero entries; they can be written :

$$\omega = g \mathrm{d}x + \sum_{\text{finite}} \beta_j \mathrm{d}u^{(j)} . \tag{6.17}$$

This defines the $\mathbf{C}^\infty(\mathcal{M}_\infty^{m,n})$ module $\Lambda^1(\mathcal{M}_\infty^{m,n})$ of smooth differential forms on $\mathcal{M}_\infty^{m,n}$. One may also define differential forms of all degree.

Of course, one may apply a differential form to a vector field according to $\langle \omega, F \rangle = fg + \sum \alpha_j \beta_j$ (compare (6.16)-(6.17)), where the sum is finite because finitely many β_j's are nonzero. One may also define the **Lie derivative** of a smooth function h, of a differential form ω,... along a vector field F, which we denote by $L_F h$ or $L_F \omega$. The **Lie bracket** of two vector fields may also be defined. All this may be defined exactly as in the finite-dimensional case because, on a computational point of view, all the sums to be computed are finite.

Finally, note that a diffeomorphism carries differential forms, vector fields, functions from a manifold to another, exactly as in the finite dimensional case; for example, if φ is a diffeomorphism from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$, F is given by (6.16) and $z, v, \dot{v}, \ddot{v}, \ldots$ are the canonical coordinates on $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$, the vector field $\varphi_* F$ on $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ is given by $\sum_i \widetilde{f}_i \frac{\partial}{\partial x_i} + \sum_{j,k} \widetilde{\alpha}_{j,k} \frac{\partial}{\partial u_k^{(j)}}$ with $\widetilde{f}_i = (L_F(z_i \circ \varphi)) \circ \varphi^{-1}$ and $\widetilde{\alpha}_{j,k} = (L_F(v_k^{(j)} \circ \varphi)) \circ \varphi^{-1}$.

6.3.3 Systems

A **system** is a vector field F on $\mathcal{M}_\infty^{m,n}$ –with $n \geq 0$ and $m \geq 1$ some integers– of the form

$$F(\mathcal{X}) \;=\; f(x,u)\frac{\partial}{\partial x} \;+\; \sum_{j=0}^{+\infty} u^{(j+1)} \frac{\partial}{\partial u^{(j)}}, \tag{6.18}$$

i.e. the x-component of F is a function of x and u only, and its $u^{(j)}$-component is $u^{(j+1)}$. This may be rewritten, in a more condensed form,

$$F \;=\; f \;+\; C \tag{6.19}$$

where C is the **canonical vector field** on $\mathcal{M}_\infty^{m,n}$, given by

$$C \;=\; \sum_0^{\infty} u^{(j+1)} \frac{\partial}{\partial u^{(j)}}, \tag{6.20}$$

and the vector field f is such that

$$\begin{array}{rcll} \langle \mathrm{d}u_i^{(j)}\,,\, f\rangle &=& 0 & i = 1, \ldots, m\,,\; j \geq 0 \\ [\frac{\partial}{\partial u_i^{(j)}}\,,\, f] &=& 0 & i = 1, \ldots, m\,,\; j \geq 1\quad. \end{array} \tag{6.21}$$

m will be called the **number of inputs** of the system, and n its **state dimension**. Note that in the (explicit) non-classical case [33, 30] (i.e. the case when some derivatives of u would appear in the right-hand side of (6.1), there would be no restriction on f, besides being smooth, i.e. the second relation in (6.21) would no longer be there (note however that any smooth vector field has zero Lie Bracket with $\frac{\partial}{\partial u^{(j)}}$ for j large enough, or in other words f depending on infinitely many time-derivatives of u in (6.1) is ruled out).

In the special case where $n = 0$, there is only one system (with "no state") on $\mathcal{M}_\infty^{m,0}$. We call this system the **canonical linear system** with m inputs; it is simply represented by the canonical vector field C given by (6.20).

In section 6.2, a system was an embedding of $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ as a sub-bundle of J^∞; this defines canonically the vector field F on $\mathcal{M}_\infty^{m,n}$ as, more or less, the pull back of the Cartan vector field (annihilating the contact forms) in $J^\infty(\pi)$.

F is the vector field defining the "total derivation along the system", i.e. the derivative of a smooth function (depending on $x, u, \dot{u}, \ldots, u^{(j)}$) knowing that $\dot{x} = f(x,u)$ is exactly its Lie derivative along this vector field. In [55], B. Jakubczyk attaches a differential algebra to the smooth system (6.1) which is exactly $\mathbf{C}^\infty(\mathcal{M}_\infty^{m,n})$ endowed with the Lie derivative along the vector field F. Of course, this is very much related to the differential algebraic approach introduced in control theory by M. Fliess [33], based on differential Galois theory, and where a system is represented by a certain differential field. In the analytic case, as explained in [28], this differential field may be realized as the field of fractions of the integral domain $\mathbf{C}^\omega(\mathcal{M}_\infty^{m,n})$. The present framework is more or less dual to these differential algebra representations since it describes the set of "points" on which the objects manipulated in differential algebra are "functions".

The following proposition gives an intrinsic definition of the number of inputs, which will be useful to prove that it is invariant under dynamic equivalence :

Proposition 6.2. *The number of inputs m is the largest integer q such that there exists q smooth functions $h_1, \ldots, h_q$ from $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$ such that all the functions*

$$L_F^j h_k \quad 1 \leq k \leq q\,,\ j \geq 0$$

are independent (the Jacobian of a finite collection of them has maximum rank).

Proof : On one hand, $h_k(x,\mathcal{U}) = u_k$ provides m functions enjoying this property. On the other hand, consider $m+1$ smooth functions $h_1, \ldots, h_{m+1}$, let $\rho \geq 0$ be such that they are functions only of $x, u, \dot{u}, \ldots, u^{(\rho)}$, and consider the $(m+1)(n+m\rho+1)$ functions functions

$$L_F^j h_k \quad 1 \leq k \leq m+1\,,\ 0 \leq j \leq n+m\rho\ ;$$

from the form of F (see (6.19) and (6.20)), they depend only on $x, u, \dot{u}, \ldots, u^{(\rho+n+m\rho)}$, i.e. on $n+m(\rho+n+m\rho+1)$ coordinates ; since this integer is strictly smaller than $(m+1)(n+m\rho+1)$, the considered functions cannot be independent. ∎

6.3.4 Differential calculus ; an inverse function theorem

All the identities from differential calculus involving functions, vector fields, differential forms apply on the "infinite-dimensional manifold" $\mathcal{M}_\infty^{m,n}$ exactly as if it were finite-dimensional : if it is an equality between functions or forms, it involves only a finite number of variables (i.e. both sides are constant along the vector fields $\frac{\partial}{\partial u_k^{(j)}}$ for j larger than a certain $J > 0$) so that all the vector fields appearing in the formula may be truncated (replaced by a vector fields with a zero component on $\frac{\partial}{\partial u_k^{(j)}}$ for $j > J$), and everything may then be projected by a certain π_K (K possibly larger than J), yielding an equivalent formula on the finite-dimensional manifold $\mathcal{M}_K^{m,n}$; if it is an equality between vector fields, it may be checked component by component, yielding equalities between functions, and the preceding remark applies.

Of course, theorems from differential calculus yielding existence of an object do not follow so easily, and often do not hold in infinite dimension. For instance, locally around a point where it is nonzero, a vector field on a manifold of dimension n has $n-1$ independent first integrals (functions whose Lie derivative along this vector field is zero) whereas this is false on $\mathcal{M}_\infty^{m,n}$ in general : for the vector field C on $\mathcal{M}_\infty^{m,0}$ given by (6.20), any function h such that $L_C h = 0$ is a constant function.

One fundamental theorem in differential calculus is the inverse function theorem stating that a smooth function from a manifold to another one whose tangent map at a certain point is an isomorphism admits locally a smooth inverse. In infinite dimension, the situation is to more intricate, see for instance [46] for a very complete discussion of this subject and general inverse function theorems on Fréchet spaces, which are not exactly the kind of theorem we will need since more general smooth functions are considered there. Here, for a mapping φ from $\mathcal{M}_\infty^{m,n}$ (coordinates : $x, u, \dot{u}, \ldots$) to $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ (coordinates : $z, v, \dot{v}, \ldots$), the function assigning to each point the tangent map to F at this point may be represented by the collection of differential forms $\mathrm{d}(z_i \circ \varphi)$, $\mathrm{d}(v_k^{(j)} \circ \varphi)$, and a way of saying that, at all point, the linear mapping is invertible with a continuous inverse, and that it depends smoothly on the point, is to say that these forms are a basis of the module $\Lambda^1(\mathcal{M}_\infty^{m,n})$; equivalently, this tangent map might be represented by an infinite matrix whose lines are finite (each line represents one of the above differential forms), and which is invertible for matrix multiplication with an inverse having also finite lines. It is clear that for a diffeomorphism this linear invertibility holds ; the additional assumption we add to get a converse is that the mapping under consideration carries a control system (as defined by (6.19)) on $\mathcal{M}_\infty^{m,n}$ to a control system on $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$; note also that we require that the tangent map be invertible *in a neighborhood* of the point under consideration whereas the finite-dimensional theorem just asks for invertibility *at* the point.

Besides its intrinsic interest, the following result will be required to prove theorem 6.5 which characterizes "linearizing outputs" in terms of their differentials.

Proposition 6.3 (local inverse function Theorem). *Let $m, n, \widetilde{m}, \widetilde{n}$ be nonnegative integers with m and $\widetilde{m}$ nonzero. Let $z_1, \ldots, z_{\widetilde{n}}, v_1, \ldots, v_{\widetilde{m}}, \dot{v}_1, \ldots, \dot{v}_{\widetilde{m}}, \ldots\ldots$ be the canonical coordinates on $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$, and $\overline{\mathcal{X}} = (\bar{x}, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \ldots)$ be a point in $\mathcal{M}_\infty^{m,n}$. Let φ be a smooth mapping from a neighborhood of $\overline{\mathcal{X}}$ in $\mathcal{M}_\infty^{m,n}$ to a neighborhood of $\varphi(\overline{\mathcal{X}})$ in $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ such that*

1. *on a neighborhood of $\overline{\mathcal{X}}$, the following set of 1-forms on $\mathcal{M}_\infty^{m,n}$:*

$$\{\,\mathrm{d}(z_i \circ \varphi)\,\}_{1\le i\le \widetilde{n}} \;\cup\; \{\,\mathrm{d}(v_k^{(j)} \circ \varphi)\,\}_{1\le k\le \widetilde{m},\, j\ge 0} \quad , \tag{6.22}$$

   form a basis of the $\mathbf{C}^\infty(\mathcal{M}_\infty^{m,n})$-module $\Lambda^1(\mathcal{M}_\infty^{m,n})$,

2. *there exists two control systems F on $\mathcal{M}_\infty^{m,n}$ and $\widetilde{F}$ on $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ such that, for all function $\widetilde{h} \in \mathbf{C}^\infty(\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}})$, defined on a neighborhood of $\varphi(\overline{\mathcal{X}})$,*

$$\left(L_{\widetilde{F}}\widetilde{h}\right) \circ \varphi \;=\; L_F\left(\widetilde{h} \circ \varphi\right) \quad . \tag{6.23}$$

Then φ is a local diffeomorphism at $\overline{\mathcal{X}}$, i.e. there exists a neighborhood U of $\overline{\mathcal{X}}$ in $\mathcal{M}_\infty^{m,n}$, a neighborhood V of $\varphi(\overline{\mathcal{X}})$ in $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ and a smooth mapping (a diffeomorphism) ψ from V to U such that $\psi \circ \varphi = Id_U$ and $\varphi \circ \psi = Id_V$.

Note that (6.22) is a way of expressing that the tangent map to φ is invertible with a continuous inverse, and (6.23) is a way of expressing that φ transforms the control system F into the control system $\widetilde{F}$, in a dual manner since writing $\widetilde{F} = \varphi_* F$ would presuppose that φ is a diffeomorphism.

Proof : Let $x_1, \ldots, x_n, u_1, \ldots, u_m, \dot{u}_1, \ldots, \dot{u}_m, \ldots\ldots$ be the canonical coordinates on $\mathcal{M}_\infty^{m,n}$. The first condition implies that there exist some smooth functions a_i^k, $b_i^{k,j}$, c_i^k, $d_i^{k,j}$ such that

$$\begin{array}{rcll} \mathrm{d}x_i &=& \sum_{k=1}^{\widetilde{n}} a_i^k \mathrm{d}(z_k \circ \varphi) \;+\; \sum_{j=0}^{L} \sum_{i=1}^{\widetilde{m}} b_i^{k,j} \mathrm{d}(v_k^{(j)} \circ \varphi) & i = 1, \ldots, n \\ \mathrm{d}u_i &=& \sum_{k=1}^{\widetilde{n}} c_i^k \mathrm{d}(z_k \circ \varphi) \;+\; \sum_{j=0}^{L} \sum_{i=1}^{\widetilde{m}} d_i^{k,j} \mathrm{d}(v_k^{(j)} \circ \varphi) & i = 1, \ldots, m\ . \end{array} \tag{6.24}$$

Let K be the integer such that the functions $z_1 \circ \varphi, \ldots, z_{\widetilde{n}} \circ \varphi, v_1 \circ \varphi, \ldots, v_{\widetilde{m}} \circ \varphi, \ldots, v_1^{(L)} \circ \varphi, \ldots, v_{\widetilde{m}}^{(L)} \circ \varphi$, and the functions a_i^k, $b_i^{k,j}$, c_i^k, $d_i^{k,j}$ all depend on x, u, $\dot{u}$, ..., $u^{(K)}$ only. Then $z_1 \circ \varphi, \ldots, z_{\widetilde{n}} \circ \varphi, v_1 \circ \varphi, \ldots, v_{\widetilde{m}} \circ \varphi$ are $\widetilde{n} + \widetilde{m}$ functions of the $n + (K+1)m$ variables $x_1, \ldots, x_n, u_1, \ldots, u_m, \ldots, u_1^{(K)}, \ldots, u_m^{(K)}$ which, from condition 1 in the proposition are independent because the fact the forms in (6.24) form a basis of the module of all forms implies in particular that a finite number of them has full rank *at all point* as vectors in the cotangent vector space. Hence, from the finite dimensional inverse function theorem, one may locally replace, in $x_1, \ldots, x_n, u_1, \ldots, u_m, \ldots, u_1^{(K)}, \ldots, u_m^{(K)}$, $\widetilde{n} + \widetilde{m}$ coordinates with the functions $z_1 \circ \varphi, \ldots, z_{\widetilde{n}} \circ$

$\varphi, v_1 \circ \varphi, \ldots, v_{\widetilde{m}} \circ \varphi$. In particular, there exists $n+m$ functions ξ_i and ζ_i^0 defined on a neighborhood of $(\bar{z}, \bar{v}, \dot{\bar{v}}, \ldots, \bar{v}^{(L)})$ —with $\varphi(\overline{\mathcal{X}}) = (\bar{z}, \bar{v}, \dot{\bar{v}}, \ddot{\bar{v}}, \ldots, \bar{v}^{(L)})$— and such that

$$\begin{array}{rcll} x_i &=& \xi_i(z \circ \varphi, v \circ \varphi, \ldots, v^{(L)} \circ \varphi, \mathcal{Y}) & i = 1, \ldots, n \\ u_i &=& \zeta_i^0(z \circ \varphi, v \circ \varphi, \ldots, v^{(L)} \circ \varphi, \mathcal{Y}) & i = 1, \ldots, m \end{array} \tag{6.25}$$

where $\mathcal{Y}$ represents some of the $n + (K+1)m$ variables $x, u, \dot{u}, \ldots, u^{(K)}$ (all minus $\widetilde{n} + (L+1)\widetilde{m}$ of them). $\mathrm{d}x_i$ and $\mathrm{d}u_i$ may be computed by differentiating (6.25) ; the expression involves the partial derivatives of the functions ξ_i and ζ_i and comparing with the expressions in (6.24), one may conclude that

$$\frac{\partial \xi_i}{\partial \mathcal{Y}} \;=\; 0 \;, \quad \frac{\partial \zeta_i^0}{\partial \mathcal{Y}} \;=\; 0 \;, \tag{6.26}$$

and we may write, instead of (6.25),

$$\begin{array}{rcll} x_i &=& \xi_i(z \circ \varphi, v \circ \varphi, \ldots, v^{(L)} \circ \varphi) & i = 1, \ldots, n \\ u_i &=& \zeta_i^0(z \circ \varphi, v \circ \varphi, \ldots, v^{(L)} \circ \varphi) & i = 1, \ldots, m \end{array} \tag{6.27}$$

We then define the functions ζ_i^j for $j > 0$ by

$$\zeta_i^j \;=\; L_{\widetilde{F}} \zeta_i^0 \tag{6.28}$$

(note that this makes ζ_i^j a smooth function of $z, v, \ldots, v^{(l+j)}$) and we define ψ by

$$\psi(z, v, \dot{v}, \ddot{v}, \ldots) = (x, u, \dot{u}, \ddot{u}, \ldots) \quad \text{with} \quad \begin{array}{rcl} x_i &=& \xi_i(z, v, \ldots, v^{(L)}) \,, \\ u_i^{(j)} &=& \zeta_i^j(z, v, \ldots, v^{(L+j)}) \,, \; j \geq 0 \,. \end{array} \tag{6.29}$$

Its is straightforward to check that (6.23), (6.28), (6.29) and the fact that $L_F^j u$ is $u^{(j)}$ imply that $\varphi \circ \psi = Id$ and $\psi \circ \varphi = Id$. ∎

6.4 Dynamic equivalence

The objective of the previous sections is the following definition. As announced in the introduction, it mimics the notion of equivalence, or **equivalence by endogenous dynamic feedback** given in [68] for analytic systems (analyticity plays no role at all in the definition of local equivalence), which coincides with the one given in [36, 37] when the transformations are algebraic. The present definition is more concise than in [68] and allows some simple geometric considerations, but the concept of equivalence is the same one. It also coincides with "dynamic equivalence" as defined in [55, 56], see below. It is proved in [68] that if two systems are equivalent in this sense then there exists a dynamic feedback in the sense of (6.2) which is *endogenous* and nonsingular and transforms one system into a "prolongation" of the other.

Definition 6.1 (Equivalence)**.** *Two systems F on $\mathcal{M}_\infty^{m,n}$ and $\widetilde{F}$ on $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ are* equivalent *at $\overline{\mathcal{X}} \in \mathcal{M}_\infty^{m,n}$ and $\overline{\mathcal{Y}} \in \mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ if and only if there exist a neighborhood U of $\overline{\mathcal{X}}$ in $\mathcal{M}_\infty^{m,n}$, a neighborhood V of $\overline{\mathcal{Y}}$ in $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$, and a diffeomorphism $\varphi : U \to V$ such that $\varphi(\mathcal{X}) = \mathcal{Y}$ and*

$$\widetilde{F} \;=\; \varphi_* F \tag{6.30}$$

on U. They are globally equivalent if there exists a diffeomorphism φ from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ such that (6.30) holds everywhere.

Note that in the definition of *local* equivalence, the diffeomorphism is only defined locally. This might be worrying : it is not very practical to know that something may be constructed in a region which imposes infinitely many constraints on infinitely many derivatives of the input u. This actually does not occur because a neighborhood U of a point $\mathcal{X}$ contains an open set of the form $\pi_K^{-1}(U_K)$ with U_K open in $\mathcal{M}_K^{m,n}$, so that being in U imposes some constraints on $x, u, \dot{u}, \ddot{u}, \ldots, u^{(K)}$ but none on $u^{(K+1)}, u^{(K+2)}, \ldots$.

Some notions of dynamic equivalence ("dynamic equivalence" and "dynamic feedback equivalence") are also given in [55, 56]. To describe them, let us come back to the framework of section 6.2, where $\mathcal{M}_\infty^{m,n}$ is a sub-bundle of $J^\infty(\pi)$ and $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ is a sub-bundle of $J^\infty(\widetilde{\pi})$; the transformations considered in [55, 56] have to be defined from $J^\infty(\pi)$ to $J^\infty(\widetilde{\pi})$ whereas our diffeomorphism φ is only defined on $\mathcal{M}_\infty^{m,n}$ (and maps it onto $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$) ; actually, Lie-Bäcklund transformations are usually defined, like in [55, 56], all over $J^\infty(\pi)$; this is referred to as *outer* transformations, or outer symmetries if it maps a system into itself, whereas *inner* transformations are these, like our φ, defined only "on the solutions", i.e. on $\mathcal{M}_\infty^{m,n}$. Since the transformations in [56] are required to be invertible on the solutions only, it is proved there that a transformation like our φ may be extended (at least locally) to $J^\infty(\pi)$ and therefore that local equivalence in the sense of Definition 6.1 is the same as the local version of the one called "dynamic equivalence" (and not "dynamic feedback equivalence") in [56].

It is clear that equivalence *is* an equivalence relation on systems, i.e. on vector fields of the form (6.19) because the composition of two diffeomorphisms is a diffeomorphism. There is not however a natural group acting on systems since a given diffeomorphism might transform a system F into a system G and transform another system F' into a vector field on $\mathcal{M}_\infty^{m,n'}$ which is not a system. For instance, for $p_1, \ldots, p_m$ nonnegative, the diffeomorphism $\Upsilon_{n,(p_1,\ldots,p_m)}$ defined in (6.15) transforms any system on $\mathcal{M}_\infty^{m,n}$ into a system on $\mathcal{M}_\infty^{m,n+p_1+\ldots+p_m}$ whereas the diffeomorphism $\Upsilon_{n+p_1+\ldots+p_m,(-p_1,\ldots,-p_m)}$ –its inverse– transforms most systems on $\mathcal{M}_\infty^{m,n+p_1+\ldots+p_m}$ into a vector field on $\mathcal{M}_\infty^{m,n}$ which is not a "system" because it does not have the required structure on the coordinates which are called "inputs" on $\mathcal{M}_\infty^{m,n}$. Two important questions arise : what is exactly the class of diffeomorphisms which transform at least one system into another system and what is the class of vector fields equivalent to a system by such a diffeomorphism. An element of answer to the latter question is that "non-classical" systems [33, 30], i.e. these where the right-hand side of (6.1) depends also on some time-derivatives of u, or vector fields on which the second constraint in (6.21) does not hold, *are* in this class of vector fields because they are transformed by $\Upsilon_{n,(K,\ldots,K)}$, where K is the number of derivatives of the input appearing in the system, into a (classical) system, this illustrates that generalized state-space representations [33, 30] are "natural" ; however, it is clear that the class of vector fields which may be conjugated to a "system" is much larger : the only system (classical or not) on $\mathcal{M}_\infty^{m,0}$ is C and very few systems on $\mathcal{M}_\infty^{m,n}$ are transformed into C by $\Upsilon_{n,(-n,0,\ldots,0)}$ for example. A partial answer to the former question is given by :

Theorem 6.1. *The number of inputs m is invariant under equivalence.*

Proof : For any function h, $L_{\widetilde{F}}\left(h \circ \varphi^{-1}\right) = (L_F h) \circ \varphi^{-1}$. The integer m from Proposition 6.2 is therefore preserved by a diffeomorphism φ. ∎

Further remarks on the class of diffeomorphisms which transform at least one system into another system may be done. One may restrict its attention to systems of the same dimension, i.e. to diffeomorphisms from $\mathcal{M}_\infty^{m,n}$ to itself because if φ goes from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{m,N}$ with $N > n$ and transforms a system into a system, $\Upsilon_{n,(N-n,0,\ldots,0)} \circ \varphi$ is a diffeomorphism of $\mathcal{M}_\infty^{m,N}$ that transforms a system into a system. In the single-input case ($m = 1$), as stated in section 6.6, φ must be static, which is a complete answer to the question because a static diffeomorphism transforms *any* system into a system. In the case of at least two inputs ($m > 1$), the literature ([62, Theorem 4.4.5] or [1, Theorem 3.1], but these have to be adapted since they are stated in an "outer" context) tells us that either φ is static or it does not preserve the fibers of $\pi_k : \mathcal{M}_\infty^{m,n} \to \mathcal{M}_k^{m,n}$ for any k, i.e., if φ is given by $\varphi(x, u, \dot{u}, \ddot{u}, \ldots) = (z, v, \dot{v}, \ddot{v}, \ldots)$, there is no k such that $(z, v, \dot{v}, \ldots, v^{(k)})$ is a function of $(x, u, \dot{u}, \ldots, u^{(k)})$ only. This is related to the statement [23] that, when dynamic feedback is viewed as adding some integrators plus performing a static feedback, it is inefficient to add the same number of integrators on each input.

6.5 Static equivalence

Definition 6.2 (Static equivalence)**.** *Two systems F on $\mathcal{M}_\infty^{m,n}$ and $\widetilde{F}$ on $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ are (locally/ globally) static equivalent if and only if they are (locally/globally) feedback equivalent with the diffeomorphism φ in (6.30) being a **static** diffeomorphism.*

From Proposition 6.1, we know that a static diffeomorphism really defines an invertible static feedback transformation in the usual sense, this is summed up in the following :

Theorem 6.2. *Both the number of inputs m and the dimension n of the state are invariant under static equivalence. Moreover, $\pi_{-1} \circ \varphi$ provides a local diffeomorphism in the classical state-space $\mathbb{R}^n$ and the u component of $\pi_0 \circ \varphi$ provides a nonsingular feedback transformation which together provide an invertible static feedback transformation in the usual sense.*

6.6 The single-input case

It was proved in [23, 22] that a single-input system which is "dynamic feedback linearizable" is "static feedback linearizable". The meaning of dynamic feedback linearizable was weaker that being equivalent to a linear system as meant here : "exogenous" feedbacks (see [68]) were allowed in [23] as well as singular (feedbacks which may change the number of inputs for example). The following Theorem 6.3 may be viewed as a generalization of this result to non-linearizable systems, but with a more restrictive dynamic equivalence.

It is known that the only transformations on an infinite jet bundle with only one "dependent variable" which preserves the contact structure (Lie-Bäcklund transformation in [1], $\mathcal{C}$-transformation in [62]) are infinite prolongations of transformations on first jets (Lie transformation according to [62]), see for instance [62, Theorems 6.3.7 and 4.4.5]. The following result is similar in spirit. We give the full proof, a little long but elementary : it basically consists in counting the dimensions carefully, it is complicated by the fact that we do not make any a priori regularity assumption (for instance, the functions χ_i and ψ_i defining the diffeomorphism are not assumed to depend on a locally constant number of derivatives of u).

Theorem 6.3. *Let F and $\widetilde{F}$ be two systems on $\mathcal{M}_\infty^{1,n}$ (i.e. two single input systems with the same number of states). Any (local/global) diffeomorphism φ such that $\widetilde{F} = \varphi_* F$ is static. Hence they are (locally/globally) equivalent if and only if they are (locally/globally) static equivalent.*

Proof : The second statement is a straightforward consequence of the first one. Let us consider a diffeomorphism φ such that $\widetilde{F} = \varphi_* F$ and prove that φ is static. Suppose that, in coordinates,

φ and φ^{-1} are given by $\varphi(x,\mathcal{U}) = (z,\mathcal{V})$ and $\varphi^{-1}(z,\mathcal{V}) = (x,\mathcal{U})$ with :

$$\begin{array}{rclcrcl} z &=& \chi_{-1}(x,\mathcal{U}) && x &=& \psi_{-1}(z,\mathcal{V}) \\ v &=& \chi_0(x,\mathcal{U}) && u &=& \psi_0(z,\mathcal{V}) \\ &\vdots& && &\vdots& \\ v^{(j)} &=& \chi_j(x,\mathcal{U}) && u^{(j)} &=& \psi_j(z,\mathcal{V}) \\ &\vdots& && &\vdots& \end{array} \tag{6.31}$$

Since $\widetilde{F} = \varphi_* F$, we have

$$\begin{array}{rcl} L_F\chi_{-1}(x,\mathcal{U}) &=& \widetilde{f}(\,\chi_{-1}(x,\mathcal{U})\,,\,\chi_0(x,\mathcal{U})\,) \\ L_F\chi_j(x,\mathcal{U}) &=& \chi_{j+1}(x,\mathcal{U}) \quad \text{for } j \geq 0\ . \end{array} \tag{6.32}$$

Let $\mathcal{X}$ be an arbitrary point of the domain where φ is defined. From the definition of a diffeomorphism, there is an integer $J \geq -1$ and a neighborhood U of $\mathcal{X}$ (J is $\delta(\pi_0 \circ \varphi)(\mathcal{X})$ if U is small enough) such that χ_{-1} and χ_0 depend only on $x, u, \dot{u}, \ldots, u^{(J)}$ on U and $\frac{\partial \chi_{-1}}{\partial u^{(J)}}$ and $\frac{\partial \chi_0}{\partial u^{(J)}}$ are not both identically zero on U (one might take all the open set where φ is defined –$\mathcal{M}^{m,1}_{\infty}$ in the global case– instead of U, but this might cause J to be infinite).

If J was -1, χ_{-1} and χ_0 would both depend only on x, but the dimension of x is n and the dimension of (χ_{-1}, χ_0) is $n+1$: there would be a function such that $h(\chi_{-1}, \chi_0)$ would be zero on U and this would prevent φ from being a diffeomorphism ; hence $J \geq 0$.

The first equation in (6.32), and the second one for $j = 0$, imply :

$$\begin{array}{rcl} \dfrac{\partial \chi_{-1}}{\partial x} f(x,u) + \dfrac{\partial \chi_{-1}}{\partial u}\dot{u} + \cdots + \dfrac{\partial \chi_{-1}}{\partial u^{(J)}} u^{(J+1)} &=& \widetilde{f}(\,\chi_{-1}(x \ldots u^{(J)})\,,\,\chi_0(x \ldots u^{(J)})\,)\ , \\ \dfrac{\partial \chi_0}{\partial x} f(x,u) + \dfrac{\partial \chi_0}{\partial u}\dot{u} + \cdots + \dfrac{\partial \chi_0}{\partial u^{(J)}} u^{(J+1)} &=& \chi_1(x,\mathcal{U})\ . \end{array}$$

By taking the derivative with respect to $u^{(J+1)}$ of the first equation and with respect to $u^{(j)}$ for $j \geq J+2$ of the second equation,

$$\frac{\partial \chi_{-1}}{\partial u^{(J)}} = 0 \quad \text{and} \quad 0 = \frac{\partial \chi_1}{\partial u^{(j)}} \quad \text{for } j \geq J+2\ . \tag{6.33}$$

This implies that that χ_{-1} is a function of $x, u, \ldots, u^{(J-1)}$ (x if $J = 0$) only, χ_0 is a function of $x, u, \ldots, u^{(J-1)}, u^{(J)}$ only (by definition of J), and χ_1 of x, u, …, $u^{(J-1)}$, $u^{(J)}$, $u^{(J+1)}$ only. It is then easy to deduce by induction from the second relation in (6.32) that for all $j \geq 0$, χ_j is a function of $x, u, \ldots, u^{(J+j+1)}$ on this neighborhood with

$$\frac{\partial \chi_j}{\partial u^{(J+j)}} = \frac{\partial \chi_0}{\partial u^{(J)}}\ , \quad j \geq 0\ . \tag{6.34}$$

From the first relation in (6.33) and the definition of J, $\frac{\partial \chi_0}{\partial u^{(J)}}$ is not identically zero on U. Hence, there is a point $\overline{\mathcal{X}} = (\overline{x}, \overline{u}, \dot{\overline{u}}, \ldots) \in U$ such that $\frac{\partial \chi_0}{\partial u^{(J)}}(\overline{\mathcal{X}}) = \frac{\partial \chi_0}{\partial u^{(J)}}(\overline{x}, \overline{u}, \ldots, \overline{u}^{(J)}) \neq 0$. Let K be $\delta(\pi_0 \circ \varphi^{-1})(\overline{\mathcal{X}})$ —note that it might be smaller than $\delta(\pi_0 \circ \varphi^{-1})(\mathcal{X})$— i.e. ψ_{-1} and ψ_0 locally depend only on $z, v, \ldots, v^{(K)}$, and $\frac{\partial \psi_{-1}}{\partial v^{(K)}}$ and $\frac{\partial \psi_0}{\partial v^{(K)}}$ are not both identically zero on any neighborhood of $\overline{\mathcal{X}}$. This implies, since $\frac{\partial \chi_0}{\partial u^{(J)}}$ is nonzero at $\overline{\mathcal{X}}$, that there is a neighborhood $\overline{U}$ of $\overline{\mathcal{X}}$ such that, on $\overline{U}$, $\frac{\partial \chi_0}{\partial u^{(J)}}$ does not vanish, ψ_{-1} and ψ_0 depend only on $z, v, \ldots, v^{(K)}$ and $\frac{\partial \psi_{-1}}{\partial v^{(K)}}$ and $\frac{\partial \psi_0}{\partial v^{(K)}}$ are not both identically zero. We have, on $\overline{U}$,

$$\begin{array}{rcl} x &=& \psi_{-1}(\,\chi_{-1}(x,u,\ldots,u^{(J-1)})\,,\,\chi_0(x,u,\ldots,u^{(J)})\,,\,\ldots,\chi_K(x,u,\ldots,u^{(J+K)})\,) \\ u &=& \psi_0(\,\chi_{-1}(x,u,\ldots,u^{(J-1)})\,,\,\chi_0(x,u,\ldots,u^{(J)})\,,\,\ldots,\chi_K(x,u,\ldots,u^{(J+K)})\,)\ . \end{array} \tag{6.35}$$

K cannot, for the same dimensional reasons as J, be equal to -1, hence $K \geq 0$. Now, suppose that $J \geq 1$. Then $J + K \geq 1$, and taking the derivative of both identities in (6.35) with respect to $u^{(J+K)}$ therefore yields

$$\frac{\partial \psi_{-1}}{\partial v^{(K)}} \frac{\partial \chi_K}{\partial u^{(J+K)}} = \frac{\partial \psi_0}{\partial v^{(K)}} \frac{\partial \chi_K}{\partial u^{(J+K)}} = 0 \tag{6.36}$$

identically on $\overline{U}$. This is impossible because on one hand $\frac{\partial \chi_K}{\partial u^{(J+K)}}$ does not vanish because of (6.34) and on the other hand K has been defined so that $\frac{\partial \psi_{-1}}{\partial v^{(K)}}$ and $\frac{\partial \psi_0}{\partial v^{(K)}}$ are not both identically zero on $\overline{U}$. Hence $J \geq 1$ is impossible.

We have proved that $J = 0$. Hence χ_j depends only on $x, u \ldots u^{(j)}$ (x for $j = -1$) for all $j \geq -1$ (see above) and $\frac{\partial \chi_j}{\partial u^{(j)}}$ is, for all j, nonsingular at all points (consequence of the smooth invertibility of φ). This is the definition of a static diffeomorphism. ∎

6.7 Dynamic linearization

*A **controllable linear system** is a system of the form (6.19) where the function f is linear, i.e. $f(x, u) = Ax + Bu$ with A and B constant matrices, and (Kalman rank condition) the rank of the columns of B, AB, A^2B is n.*

There is a canonical form under static feedback, known as Brunovský canonical form [17] for these systems : they may be transformed via a static diffeomorphism (from $\mathcal{M}_\infty^{m,n}$ to itself) to a linear system where A and B have the form of some "chains of integrators" of "length" $r_1, \ldots, r_m$; the diffeomorphism $\Upsilon_{n,(-r_1,\ldots,-r_m)}$ from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{m,0}$ (see (6.15)) which "cuts off" all these integrators then transforms this system into C (see (6.20)) :

Proposition 6.4 ([17]). *A controllable linear system with m inputs is globally equivalent to the canonical system C on $\mathcal{M}_\infty^{m,0}$.*

We wish to call dynamic linearizable a system which is equivalent to a *controllable linear* system. From the above proposition, this may equivalently be stated as :

Definition 6.3. *A system is (locally/globally) dynamic linearizable if and only if it is (locally/globally) equivalent to the canonical linear system C on $\mathcal{M}_\infty^{0,m}$.*

Of course this concept is the same as in [68, "analytic approach"] since the equivalence is the same. In [36, 37, 68], the notion of *linearizing outputs* or *flat outputs* is used to define *flat control systems* as these which admit such outputs. It is proved that flatness coincides with equivalence by endogenous feedback to a controllable linear system. In [55, 56] a system is called free if the differential algebra $(\mathbf{C}^\infty(\mathcal{M}_\infty^{m,n}), L_F)$ is free ; the linearizing outputs we define below are free generators of this differential algebra. The following theorem in a sense re-states the result "flat ⇔ linearizable by endogenous feedback".

Theorem 6.4 (linearizing outputs). *A system F on $\mathcal{M}_\infty^{m,n}$ is locally dynamic linearizable at a point $\mathcal{X}$ if and only if there exist m smooth functions $h_1, \ldots, h_m$ from a neighborhood of $\mathcal{X}$ in $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$ such that $(L_F^j h_k)_{1\leq k\leq m, 0\leq j}$ is a system local of coordinates at $\mathcal{X}$. It is globally dynamic linearizable and only if there exist m smooth functions $h_1, \ldots, h_m$ from $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$ such that $(L_F^j h_k)_{1\leq k\leq m, 0\leq j}$ is a global system of coordinates. These functions are called linearizing outputs.*

Proof : If F is dynamic linearizable, there exists a (local/global) diffeomorphism φ from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{m,0}$ such that $C = \varphi_* F$. Define h_k by $h_k = v_k^{(j)} \circ \varphi$ with $v_k^{(j)}$ the canonical coordinates on $\mathcal{M}_\infty^{0,m}$. Since $v_k^{(j)} = L_C^j v_k$ (the jth Lie derivative of v_k along C) and $C = \varphi_* F$, we have

$$L_F^j h_k \ = \ L_F^j\,(v_k \circ \varphi) \ = \ \left(L_{\varphi_* F}^j v_k\right) \circ \varphi \ = \ v_k^{(j)} \circ \varphi \ ,$$

so that, since φ is a diffeomorphism and $(v_k^{(j)})_{1\le k\le m, 0\le j}$ is a system of coordinates on $\mathcal{M}_\infty^{m,0}$, $(v_k^{(j)} \circ \varphi)_{1\le k\le m, 0\le j}$ is a system of coordinates on $\mathcal{M}_\infty^{m,n}$. Conversely, if there exist m functions $h_1, \ldots, h_m$ enjoying this property, then one may define the diffeomorphism φ mapping a point $(x,\mathcal{U})$ of $\mathcal{M}_\infty^{m,n}$ to the point of $\mathcal{M}_\infty^{0,m}$ whose coordinate $v_k^{(j)}$ is $L_F^j h_k(x,\mathcal{U})$. It is clear that $\varphi_* F = C$. ∎

Of course, this is far from being a solution to dynamic feedback linearization since one has to determine if linearizing outputs exist, which is not an easy task ; see Chapter 7 for bibliography and a discussion of this topic. Let us give a rather convenient way of tackling this problem by transforming it into its "infinitesimal" version. Recall that a Pfaffian system is a family of differential forms of degree 1 with constant rank ; any family of forms generating the same module (or co-distribution) defines the same Pfaffian system. The infinitesimal version of linearizing outputs is and object already defined n[4, 83] :

Definition 6.4. *A Pfaffian system $(\omega_1, \ldots, \omega_m)$ is called a* ***linearizing Pfaffian system*** *at point $\mathcal{X}$ if and only if, for a certain neighborhood U of $\mathcal{X}$, the restriction to U of the forms $L_F^j \omega_k$, $j \ge 0$, $1 \le k \le m$ form a basis of the $\mathbf{C}^\infty(U)$-module $\Lambda^1(U)$ of all differential forms on U.*

We have three comments on this definition. Firstly, this is a property of the Pfaffian system $(\omega_1, \ldots, \omega_m)$ rather than the m-uple of 1-forms since it is not changed when changing the collection of forms $\omega_1, \ldots, \omega_m$ into another collection which span the same module. Secondly, one may prove than the rank of such a Pfaffian system *must* be m (see the proof of Proposition 6.2). Finally, one should not be mislead by the terminology : **existence of a linearizing Pfaffian system does not imply linearizability** :

Theorem 6.5. *A system F on $\mathcal{M}_\infty^{m,n}$ is locally dynamic linearizable at point $\mathcal{X}$ if and only if there exists, on a neighborhood of $\mathcal{X}$, a linearizing Pfaffian system $(\omega_1, \ldots, \omega_m)$ which is locally completely integrable.*

By locally completely integrable, we mean the classical Frobenius condition $\mathrm{d}\omega_k \wedge \omega_1 \wedge \ldots \wedge \omega_m = 0$; note that the condition that $(L_F^j \omega_k)_{1\le k\le m, 0\le j}$ be a basis of $\Lambda^1(U)$ implies that the rank at all point of $(\omega_1, \ldots, \omega_m)$ is m, and is therefore constant.

Proof : The condition is obviously necessary from Theorem 6.4 by taking $\omega_k = \mathrm{d}h_k$. Conversely, one may apply the finite dimensional Frobenius theorem to $(\omega_1, \ldots, \omega_m)$ because they depend on a finite number of variables, and, as noticed above, they have constant rank m : there exists m functions $h_1 \ldots h_m$ (of the same number of variables than these appearing in $\omega_1 \ldots \omega_m$) such that $\mathrm{d}h_1, \ldots, \mathrm{d}h_m$ span the same co-distribution than $\omega_1, \ldots, \omega_m$; this implies that $(L_F^j \mathrm{d}h_k)_{1\le k\le m, 0\le j}$ is also a basis of $\Lambda^1(U)$. Define the map $\varphi : U \to \mathcal{M}_\infty^{m,0}$ as assigning to a point $(x,\mathcal{U})$ of $\mathcal{M}_\infty^{m,n}$ to the point of $\mathcal{M}_\infty^{0,m}$ whose coordinate $v_k^{(j)}$ is $L_F^j h_k(x,\mathcal{U})$. It is clear that for all function $\tilde{h} \in \mathbf{C}^\infty(\mathcal{M}_\infty^{0,m})$, $\left(L_C \tilde{h}\right) \circ \varphi \ = \ L_\varphi \left(\tilde{h} \circ \varphi\right)$, so that theorem 6.3 implies that φ is a local diffeomorphism. ∎

This result is more interesting in the light of the fact that a controllable system admits a linearizing Pfaffian system at "almost all" points. Next chapter develops further this point of view, see also [4] for a more algebraic approach.

Chapitre 7

Reproduction de l'article:

E. Aranda-Bricaire, C. H. Moog et J.-B. Pomet, "Infinitesimal Brunovský form for nonlinear systems with applications to dynamic linearization",

in *Geometry in Nonlinear Control and Differential Inclusions*, B. Jakubczyk et al. eds, *Banach Center Publications*, Vol. 32, pp. 19–33, 1995.

7.1 Introduction and Problem Statement

The purpose of this note is to present a "geometric" version of the constructions made in [4, 83]. The framework from Chapter 6 will be used; it is briefly summed up in section 7.2.

The contribution of [4, 83] was to construct a so-called "infinitesimal Brunovský form" ("non-exact Brunovský form" in [83]) for controllable nonlinear systems and to relate it to dynamic linearization; they use the linear algebraic framework introduced in [12]. The point of view on the feedback linearization problem was the one of looking for "linearizing outputs", following the idea of [36, 37, 68]. It is therefore, following the terms of [36, 37, 68], *linearization via endogenous dynamic feedback*. In [83], we relied explicitly upon the notion of *differential flatness* [36, 37, 68], whereas [4] re-defines the notion of linearizing outputs in terms of dynamic decoupling and structure at infinity.

Here, in the framework of Chapter 6, dynamic linearization is equivalence to a linear system via diffeomorphism on the extended state space manifold; linearizing outputs are functions such that these and all their "time-derivatives" are a set of local coordinates on the generalized state-space manifold. The main interest of this approach over the algebraic ones is that it is possible to give local notions, and therefore singularities are not ignored.

In section 7.3, we define the infinitesimal Brunovský form and relate it to some work on time-varying linear systems and linearized systems of nonlinear systems [34, 35]. In section 7.4, we relate this construction to existence of linearizing outputs, and explain why it provides a good framework for searching linearizing outputs.

7.2 Summary of Chapter 6

2.1. The "**infinite dimensional manifold**" $\mathcal{M}_\infty^{m,n}$ is, for short, $\mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{N}}$. A global system of coordinates is $x_1, \ldots, x_n, u_1, \ldots, u_m, \dot{u}_1, \ldots, \dot{u}_m, \ddot{u}_1, \ldots$. It is endowed with the product topology : an open set may be described by some restrictions on *a finite number* of coordinates, i.e. there is a $\tilde{k}$ such that, considered as an open set of $\mathbb{R}^n \times (\mathbb{R}^m)^{\mathbb{N}} = \mathbb{R}^n \times (\mathbb{R}^m)^{\tilde{k}} \times (\mathbb{R}^m)^{\mathbb{N}}$, it can be written $\widetilde{O} \times (\mathbb{R}^m)^{\mathbb{N}}$ with $\widetilde{O}$ an open set of $\mathbb{R}^n \times (\mathbb{R}^m)^{\tilde{k}}$.

2.2. A **smooth function** on $\mathcal{M}_\infty^{m,n}$ is one which depends only on a finite number of coordinates and is smooth as a function of these coordinates. $\mathbf{C}^\infty(U)$ stands for the algebra of smooth functions defined on an open subset U of $\mathcal{M}_\infty^{m,n}$. A **smooth mapping** from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ is a mapping whose composition with any smooth function is a smooth function. A **diffeomorphism** from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ is a bijective smooth mapping whose inverse is a smooth mapping.

2.3. A **vector field** is a possibly infinite linear combination $\sum v_i \frac{\partial}{\partial w_i}$ where the v_i's are smooth functions and the w_i's are some of the coordinates $x_1, \ldots, x_n, u_1, \ldots, u_m, \dot{u}_1, \ldots, \dot{u}_m, \ldots$. A **differential form of degree 1** (or 1-form) is, with the same conventions, a *finite* linear combination $\sum v_i \mathrm{d}w_i$. $\Lambda^1(U)$ stands for the $\mathbf{C}^\infty(U)$-module of 1-forms defined on U.

2.4. All the "formulas" from finite dimensional differential calculus involving objects like Lie brackets and Lie derivatives are valid. For instance, the Lie derivative of a form $\omega = \sum v_i \mathrm{d}w_i$ along a vector field F may be computed, in coordinates, according to $L_F\omega = \sum L_F v_i \mathrm{d}w_i + v_i \mathrm{d}(L_F w_i)$. Also, a diffeomorphism carries vector fields or differential forms from one manifold to another, we use the usual notation $\varphi_* F$ or $\varphi^*\omega$.

2.5. A smooth **control system** (6.1) :

$$\dot{x} \;=\; f(x,u) \tag{7.1}$$

with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$ is represented by a single vector field

$$F \;=\; f(x,u)\frac{\partial}{\partial x} + \dot{u}\frac{\partial}{\partial u} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \ldots \tag{7.2}$$

on $\mathcal{M}_\infty^{m,n}$. We often refer to "system F", confusing system (7.1) with vector field F.

2.6. The Lie derivative along F defined by (7.2) is simply the "time-derivative" according to (7.1) : we often write $\dot{\varphi}$ or $\dot{\omega}$ instead of $L_F\varphi$ or $L_F\omega$ for a function φ or a 1-form ω.

2.7. A diffeomorphism from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$ given by $(x,u,\dot{u},\ddot{u},\ldots) \mapsto (z,v,\dot{v},\ddot{v},\ldots)$ is said to be a **static diffeomorphism** if and only if z depends only on x, v depends only on x and u, $\dot{v}$ depends only on x, u and $\dot{u}$ ... A static diffeomorphism is nothing more than a nonsingular static transformation in the usual sense : if F is a system on $\mathcal{M}_\infty^{m,n}$ and $\widetilde{F}$ is a system on $\mathcal{M}_\infty^{\widetilde{m},\widetilde{n}}$, existence of a static diffeomorphism φ such that $\widetilde{F} = \varphi_* F$ is equivalent to $n = \tilde{n}$, $m = \tilde{m}$ and static equivalence of the control systems associated with F and $\widetilde{F}$.

2.8. Of course, $n = 0$ is not ruled out in the above definitions, coordinates on $\mathcal{M}_\infty^{m,0}$ are simply $\{u,\dot{u},\ddot{u},\ldots\}$, and the only system is the **canonical linear system** with m inputs (6.20) :

$$C \;=\; \sum_0^\infty u^{(j+1)}\frac{\partial}{\partial u^{(j)}} \;. \tag{7.3}$$

It has "no state", but one should not worry about this since $n = 0$ is obtained after "cutting all the integrators" in a canonical linear system [17] and arbitrarily renaming some states "inputs". Dynamic linearizability is conjugation via a diffeomorphism to system C :

Definition 7.1 (rephrasing of Definition 6.3). *A system F is **locally dynamic linearizable** at point $\mathcal{X} \in \mathcal{M}_\infty^{m,n}$ if and only if there exists a neighborhood U of $\mathcal{X}$ in $\mathcal{M}_\infty^{m,n}$, an open subset V of $\mathcal{M}_\infty^{m,0}$, and a diffeomorphism φ from U to V such that,*
on U, $\quad \varphi_* F \;=\; C \;.$

2.9. Consider a $\mathbf{C}^\infty(U)$-module of vector fields $\mathcal{D}$ (resp. of forms $\mathcal{H}$), defined on an open set U. The annihilator of $\mathcal{D}$ is the module of the forms which vanish on all the vector fields of $\mathcal{D}$, and vice-versa :

$$\mathcal{H}^\perp \;=\; \{\, X\,,\ \forall \omega \in \mathcal{H},\ \langle \omega, X\rangle = 0\,\}\ ; \quad \mathcal{D}^\perp \;=\; \{\, \omega\,,\ \forall X \in \mathcal{D},\ \langle \omega, X\rangle = 0\,\}\ .$$

$\mathcal{D}(\mathcal{X})$ (resp. $\mathcal{H}(\mathcal{X})$) denotes the subspace of the tangent (resp. cotangent) space to $\mathcal{M}_\infty^{m,n}$ at point $\mathcal{X} \in U$ made of all the $X(\mathcal{X})$ for $X \in \mathcal{D}$ (resp. $\omega(\mathcal{X})$ for $\omega \in \mathcal{H}$). We call the dimension of $\mathcal{D}(\mathcal{X})$ (resp. $\mathcal{H}(\mathcal{X})$) the **pointwise rank** of $\mathcal{D}$ (resp. $\mathcal{H}$) at point $\mathcal{X}$. $\mathcal{D}$ or $\mathcal{H}$ is said to be **nonsingular at point $\mathcal{X}$** if and only its the pointwise rank is constant in a neighborhood of $\mathcal{X}$; it is then equal to the rank of the module over $\mathcal{C}^\infty(U)$.

7.3 The Infinitesimal Brunovský Form

Let us define the following sequence of $\mathbf{C}^\infty(\mathcal{M}_\infty^{m,n})$-modules of vector fields :

$$\begin{aligned}
\mathcal{D}_{-j} &= \text{Span}\,\{\, \tfrac{\partial}{\partial u^{(j+1)}}\,,\ \tfrac{\partial}{\partial u^{(j+2)}}\,,\ \dots\,\} \quad j \geq 0\\
&\ \vdots\\
\mathcal{D}_0 &= \text{Span}\,\{\, \tfrac{\partial}{\partial \dot u}\,,\ \tfrac{\partial}{\partial \ddot u}\,,\ \tfrac{\partial}{\partial u^{(3)}}\,,\ \dots\,\}\\
\mathcal{D}_1 &= \text{Span}\,\{\, \tfrac{\partial}{\partial u}\,,\ \tfrac{\partial}{\partial \dot u}\,,\ \tfrac{\partial}{\partial \ddot u}\,,\ \tfrac{\partial}{\partial u^{(3)}}\,,\ \dots\,\}\\
&\ \vdots\\
\mathcal{D}_{k+1} &= \mathcal{D}_k \;+\; [\,F\,,\,\mathcal{D}_k\,]\\
&\ \vdots\\
\mathcal{D}_\infty &= \sum_k \mathcal{D}_k
\end{aligned} \tag{7.4}$$

and, since these are "infinite-dimensional", we define for each $\mathcal{D}_k$ $(k \geq 1)$ its "$\frac{\partial}{\partial x}$ part" :

$$\widehat{\mathcal{D}}_k \;=\; \mathcal{D}_k \,\cap\, \text{Span}\,\{\, \frac{\partial}{\partial x}\,\}\ , \quad k \in [1, \infty] \tag{7.5}$$

(Span$\{\frac{\partial}{\partial x}\}$ stands for the $\mathbf{C}^\infty(\mathcal{M}_\infty^{m,n})$-module generated by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$), which makes $\widehat{\mathcal{D}}_k(\mathcal{X})$ (see paragraph 2.9) finite-dimensional for all $\mathcal{X} \in \mathcal{M}_\infty^{m,n}$), and yields

$$\mathcal{D}_k \;=\; \widehat{\mathcal{D}}_k \,\oplus\, \mathcal{D}_1\ , \quad k \in [1, \infty]\ . \tag{7.6}$$

Note that (7.5) and (7.6) are both valid for $k = \infty$ and that $\widehat{\mathcal{D}}_\infty$ might as well have been defined by $\widehat{\mathcal{D}}_\infty \;=\; \sum_k \widehat{\mathcal{D}}_k$. We define also a sequence of $\mathbf{C}^\infty(\mathcal{M}_\infty^{m,n})$-modules of forms :

$$\begin{aligned}
\mathcal{H}_{-j} &= \text{Span}\,\{\, \mathrm{d}x\,,\ \mathrm{d}u\,, \dots\,,\ \mathrm{d}u^{(j)}\,\} \quad j \geq 0\\
&\ \vdots\\
\mathcal{H}_0 &= \text{Span}\,\{\, \mathrm{d}x\,,\ \mathrm{d}u\,\}\\
\mathcal{H}_1 &= \text{Span}\,\{\, \mathrm{d}x\,\}\\
&\ \vdots\\
\mathcal{H}_{k+1} &= \{\, \omega \in \mathcal{H}_k\,,\ \dot\omega = L_F\omega \in \mathcal{H}_k\,\}\\
&\ \vdots\\
\mathcal{H}_\infty &= \bigcap_k \mathcal{H}_k\ .
\end{aligned} \tag{7.7}$$

See paragraphs 2.4 and 2.6 for a definition of $\dot\omega$ or $L_F\omega$. We have the following relation between the $\mathcal{D}_k$'s and the $\mathcal{H}_k$'s :

Proposition 7.1. *All the modules $\mathcal{D}_k$ and $\mathcal{H}_k$ are invariant by static feedback, i.e. by static diffeomorphism of $\mathcal{M}_\infty^{m,n}$ (see paragraph 2.7), and, for all k,*

$$\mathcal{H}_\infty \subset \mathcal{H}_{k+1} \subset \mathcal{H}_k \quad , \quad \mathcal{D}_k \subset \mathcal{D}_{k+1} \subset \mathcal{D}_\infty \quad , \quad \mathcal{H}_k \;=\; \mathcal{D}_k^\perp \quad , \quad \mathcal{D}_k \;\subset\; \mathcal{H}_k^\perp, \tag{7.8}$$

with $\mathcal{H}_k^\perp = \mathcal{D}_k$ at points where $\widehat{\mathcal{D}}_k$ is nonsingular (see paragraph 2.9).

Proof : From (7.4) and Proposition 6.1, a static diffeomorphism φ does not change $\mathcal{D}_k$ for $k \leq 1$; since the recursive definition of $\mathcal{D}_k$ for larger k only uses Lie brackets, it is then clear that the modules built according to (7.4) from $\varphi_* F$ are exactly $\varphi_* \mathcal{D}_k$. The two first relations in (7.8) are obvious from (7.4) and (7.7) and the fourth one is a consequence of the third one because $\mathcal{D}_k \subset \left(\mathcal{D}_k^\perp\right)^\perp$, with an equality at nonsingular points. Let us prove the first one by induction. It is obvious for $k \leq 1$. Let us suppose that it is true for $k \geq 1$. From the fact that if $\langle \omega, X\rangle = 0$ then $\langle L_F\omega, X\rangle = -\langle \omega, [F,X]\rangle$, we have :

$$\begin{aligned} \omega \in \mathcal{H}_{k+1} \;&\Leftrightarrow\; \omega \in \mathcal{H}_k \;\text{ and }\; L_\varphi \omega \in \mathcal{H}_k \\ &\Leftrightarrow\; \forall X \in \mathcal{D}_k\,,\;\; \langle \omega, X\rangle \;=\; \langle L_F\omega, X\rangle \;=\; 0 \\ &\Leftrightarrow\; \forall X \in \mathcal{D}_k\,,\;\; \langle \omega, X\rangle \;=\; \langle \omega, [F,X]\rangle \;=\; 0 \;\;\Leftrightarrow\;\; \omega \in \mathcal{D}_{k+1}^\perp\,. \end{aligned}$$
∎

We shall now relate this construction to accessibility. The following Lie algebra is defined in [60], and often called the **strong accessibility Lie algebra** : this Lie algebra of vector fields on $\mathbb{R}^n$ is the Lie ideal generated by all the vector fields $f(u,.) - f(v,.)$ for all possible values of u and v in the Lie algebra generated by the vector fields $f(u,.)$ for all possible values of u. The main result on strong accessibility in [60] (see the definition there) is that it is equivalent to the strong accessibility Lie algebra having rank n. In [29], the **strong jet accessibility Lie algebra** is defined; it differs from the strong accessibility Lie algebra in that the differences $f(u,.) - f(v,.)$ are replaced by derivatives of all orders with respect to all the components of u. It is easy to see (this is actually its definition in [29]) that it is the Lie algebra generated by all the vector fields

$$\mathrm{ad}^j_{f(.,u)} g_u^K\;,\;\; j \in \mathbb{N},\;\; K = (k_1,\ldots,k_m) \in \mathbb{N}^m,\; g_u^K = \frac{\partial^{k_1+\ldots+k_m} f}{\partial u_1^{k_1} \ldots \partial u_m^{k_m}}. \tag{7.9}$$

It a priori depends on u. In the analytic case, it does not depend on u and is equal, for all value of u, to the strong accessibility Lie algebra. Of course, in the general (smooth) case, full rank for this Lie algebra is sufficient, but not necessary, for strong accessibility. A vector field on $\mathbb{R}^n$ depending on u, like these defined in (7.9) and all their iterated Lie brackets, clearly defines a vector field on $\mathcal{M}_\infty^{m,n}$ (which belongs to Span$\{\frac{\partial}{\partial x}\}$ and commutes with all the $\frac{\partial}{\partial u_k^{(j)}}$ for $j \geq 1$ but not a priori with the $\frac{\partial}{\partial u_k}$'s). Here, we call $\widehat{\mathcal{L}}$ the Lie algebra composed of the vector fields on $\mathcal{M}_\infty^{m,n}$ associated to these in the strong jet accessibility Lie algebra as defined by (7.9) (or in [29]), and we define $\mathcal{L}$ by

$$\mathcal{L} \;=\; \widehat{\mathcal{L}} \oplus \mathcal{D}_1 \;=\; \widehat{\mathcal{L}} \oplus \mathrm{Span}\{\frac{\partial}{\partial u}, \frac{\partial}{\partial \dot u}, \frac{\partial}{\partial \ddot u}, \frac{\partial}{\partial u^{(3)}}, \ldots\}\,. \tag{7.10}$$

$\mathcal{L}$ is obviously a Lie algebra because $[\frac{\partial}{\partial u_k}, \widehat{\mathcal{L}}] \subset \widehat{\mathcal{L}}$ and $[\frac{\partial}{\partial u_k^{(j)}}, \widehat{\mathcal{L}}] = \{0\}$ for $j \geq 1$. The phrase “strong jet accessibility Lie algebra” will further refer to $\mathcal{L}$ rather than to a Lie algebra of vector fields on $\mathbb{R}^n$, and $\widehat{\mathcal{L}}$ is its $\frac{\partial}{\partial x}$-component. We have :

Theorem 7.1. *For any open subset U of $\mathcal{M}_\infty^{m,n}$,*

1. *$\mathcal{L}|_U$ (restriction to U of the strong jet accessibility Lie Algebra) is the Lie Algebra generated by (i.e. the involutive closure of) $\mathcal{D}_\infty|_U$ (the restriction of $\mathcal{D}_\infty$ to U).*

2. If the $\mathbf{C}^\infty$-module $\widehat{\mathcal{D}}_\infty\Big|_U$ is finitely generated, then it is a Lie algebra, and so is $\mathcal{D}_\infty|_U$, and hence :

$$\mathcal{D}_\infty|_U \;=\; \mathcal{L}|_U \quad i.e. \quad \widehat{\mathcal{D}}_\infty\Big|_U \;=\; \widehat{\mathcal{L}}\Big|_U \;. \tag{7.11}$$

Proof : Point 1 is straightforward from (7.9) and (7.4). We only have to prove that if U is such that $\widehat{\mathcal{D}}_\infty\Big|_U$ is finitely generated, then $\mathcal{D}_\infty|_U$ is a Lie algebra ; for this, we shall prove that the module of vector fields

$$\mathcal{M} \;=\; \{\, X \in \mathcal{D}_\infty|_U \;,\; [X, \mathcal{D}_\infty|_U] \subset \mathcal{D}_\infty|_U\}$$

is equal to $\mathcal{D}_\infty|_U$. By assumption, $\mathcal{D}_\infty|_U$ is generated by the vector fields $\frac{\partial}{\partial u_k^{(j)}}$, $1 \leq k \leq m$, $j \geq 0$, plus a finite number of vector fields of $\mathrm{Span}\{\mathrm{d}x\}$ whose expressions involve only a finite number, say J, of time-derivatives of u ; $\mathcal{D}_\infty|_U$ is therefore invariant by Lie bracket by the vector fields $\frac{\partial}{\partial u_k^{(j)}}$ for $j \geq J$, which span $\mathcal{D}_{-(J-1)}$. $\mathcal{M}$ therefore contains $\mathcal{D}_{-(J-1)}$; furthermore, it is a submodule of $\mathcal{D}_\infty|_U$, invariant by F from Jacobi identity. Since it is clear that, for all k, and in particular $k = -(J-1)$, $\mathcal{D}_\infty|_U$ is the smallest module of vector fields which contains $\mathcal{D}_k$ and is invariant by Lie brackets by F, $\mathcal{M} = \mathcal{D}_\infty|_U$. ∎

For further considerations, we will avoid "singular" points in the sense of the following definition where $\mathcal{H}_k + \dot{\mathcal{H}}_k$ stands for the module over smooth functions spanned by all the forms ω and $\dot\omega$ with $\omega \in \mathcal{H}_k$. "Nonsingular" was defined in paragraph 2.9.

Definition 7.2. *A point $\mathcal{X} \in \mathcal{M}_\infty^{m,n}$ is called a **Brunovský-regular point** for system F if and only if one of the two following (equivalent) conditions is satisfied :*

(i) All the modules $\widehat{\mathcal{D}}_k$ ($k \geq 2$) are nonsingular at $\mathcal{X}$.

(ii) All the modules $\mathcal{H}_k + \dot{\mathcal{H}}_k$ ($k \geq 2$) are nonsingular at $\mathcal{X}$.

These properties are true for all $k \geq 0$ if and only if they are true for $k = 2, \ldots, n+1$. We call $\boldsymbol{\rho_k}$ the locally constant rank of $\mathcal{H}_k$. Around a Brunovský-regular point, there exists an integer $\boldsymbol{k^*}$ such that, for all $k \leq k^*$, $\rho_{k+1} \leq \rho_k - 1$ and $\mathcal{H}_k = \mathcal{H}_{k+1} = \mathcal{H}_\infty$ for $k > k^*$.

Proof of *i* ⇔ *ii* : Suppose that all the $\widehat{\mathcal{D}}_k$'s, and thus all the $\mathcal{H}_k$'s, are nonsingular at $\mathcal{X}$. For a certain k, let $\{\eta_1, \ldots, \eta_{p+q}\}$ be a basis of $\mathcal{H}_k$ with $\{\eta_1, \ldots, \eta_p\}$ a basis of $\mathcal{H}_{k+1}$. The forms $\eta_1, \ldots, \eta_{p+q}, \dot\eta_{p+1}, \ldots, \dot\eta_{p+q}$ span $\mathcal{H}_k + \dot{\mathcal{H}}_k$. On the other hand, if a linear combination $\sum_{i=1}^{p+q} \mu_i\eta_i + \sum_{i=1}^{q} \lambda_i\dot\eta_{p+i}$ vanishes at $\mathcal{X}$ then, for all vector field $X \in \mathcal{D}_k$, $\langle \sum_{i=1}^{q} \lambda_i\dot\eta_{p+i}, X\rangle$, which is equal to $\langle \sum_{i=1}^{q} \lambda_i\eta_{p+i}, [F, X]\rangle$, vanishes at $\mathcal{X}$, hence $\langle \sum_{i=1}^{q} \lambda_i\eta_{p+i}, Y\rangle(\mathcal{X}) = 0$ for all $Y \in \mathcal{D}_{k+1}$; since $\{\eta_1(\mathcal{X}), \ldots, \eta_p(\mathcal{X})\}$ is a basis of the annihilator of $\mathcal{D}_{k+1}(\mathcal{X})$ and $\{\eta_1(\mathcal{X}), \ldots, \eta_{p+q}(\mathcal{X})\}$ are independent, all the λ_i's vanish at $\mathcal{X}$; hence $\sum_{i=1}^{p+q} \mu_i\eta_i$ vanishes at $\mathcal{X}$, hence all the μ_i's also vanish at $\mathcal{X}$. Hence $\{\eta_1(\mathcal{X}), \ldots, \eta_{p+q}(\mathcal{X}), \dot\eta_{p+1}(\mathcal{X}), \ldots, \dot\eta_{p+q}(\mathcal{X})\}$ is a basis of $\mathcal{H}_k(\mathcal{X}) + \dot{\mathcal{H}}_k(\mathcal{X})$ and $\mathcal{H}_k + \dot{\mathcal{H}}_k$ is nonsingular at $\mathcal{X}$.

Conversely suppose that all the modules $\mathcal{H}_k + \dot{\mathcal{H}}_k$ are nonsingular at $\mathcal{X}$. Let $\mathcal{C}_k = \{X \in \mathcal{D}_k, [F, X] \in \mathcal{D}_k\}$ and $\widehat{\mathcal{C}}_k = \mathcal{C}_k \cap \mathrm{Span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial u}\}$. Clearly, $\mathcal{C}_k = \widehat{\mathcal{C}}_k \oplus \mathcal{D}_0$. Arguments similar to these of the end of the proof of Proposition 7.1 show that $(\mathcal{H}_k + \dot{\mathcal{H}}_k)^\perp = \mathcal{C}_k$ (equality between modules). All the $\widehat{\mathcal{C}}_k$'s are therefore nonsingular at $\mathcal{X}$. Let us prove by induction that all the modules $\widehat{\mathcal{D}}_k$ are nonsingular too. This is true for $k = 1$ ($\widehat{\mathcal{D}}_1 = \{0\}$). Suppose that it is true for $k \geq 1$, and let $\{\ldots, \frac{\partial}{\partial \ddot u}, \frac{\partial}{\partial \dot u}, X_1, \ldots, X_{p+q}\}$ be a basis of $\mathcal{D}_k$ with $\{\ldots, \frac{\partial}{\partial \ddot u}, \frac{\partial}{\partial \dot u}, X_1, \ldots, X_p\}$ a basis of $\mathcal{C}_k$. Then the same arguments as in the first part of this proof show that $\{X_1(\mathcal{X}), \ldots, X_{p+q}(\mathcal{X}), [F, X_{p+1}](\mathcal{X}), \ldots, [F, X_{p+q}](\mathcal{X})\}$ is a basis of $\mathrm{Span}\{\frac{\partial}{\partial u}\} \oplus \mathcal{D}_{k+1}(\mathcal{X})$ and $\widehat{\mathcal{D}}_{k+1}$ is nonsingular at $\mathcal{X}$. ∎

Theorem 7.2 (Infinitesimal Brunovský form)**.** *Around a Brunovský-regular point there exists ρ_∞ functions of x only $\chi_1,\ldots,\chi_{\rho_\infty}$, and m 1-forms $\omega_1,\ldots,\omega_m$, and m non-negative integers $r_1\ldots,r_m$ such that*

$$\{\mathrm{d}\chi_1,\ldots,\mathrm{d}\chi_{\rho_\infty}\}\ \text{ is a basis of } \mathcal{H}_\infty=\mathcal{H}_l,\ \text{for } l\geq k^*+1 \tag{7.12}$$

$$\{\mathrm{d}\chi_1,\ldots,\mathrm{d}\chi_{\rho_\infty}\}\cup\{\,\omega_k^{(j)}\,,\ r_k\geq l\,,\ 0\leq j\leq r_k-l\,\}\ \text{ is a basis of } \mathcal{H}_l,\ \text{for all } l\leq k^*. \tag{7.13}$$

Furthermore all the ω_k's are in $\mathcal{H}_1=\mathrm{Span}\{\mathrm{d}x\}$ —i.e. $r_k\geq 1$ for all k— if and only if, at the point (x,u) under consideration,

$$\mathit{rank}_{\mathbb{R}}\,\{\,\frac{\partial f}{\partial u_1}(x,u),\ \ldots,\ \frac{\partial f}{\partial u_m}(x,u)\,\}\ =\ m\ . \tag{7.14}$$

At a Brunovský-regular point, $\mathcal{D}_\infty$ is equal to $\mathcal{D}_{n+1}$ and is hence nonsingular and hence locally finitely generated. Hence strong accessibility implies, from Theorem 7.1, that $\rho_\infty=0$. In that case and if (7.14) is met, (7.13) implies

$$\begin{array}{l}\{\,\omega_k^{(j)}\,,\ 0\leq k\leq m\,,\ 0\leq j\leq r_k-1\,\}\ \text{ is a basis of } \mathcal{H}_1\ =\ \mathrm{Span}\{\mathrm{d}x\}\\ \{\,\omega_k^{(j)}\,,\ 0\leq k\leq m\,,\ 0\leq j\leq r_k\,\}\ \text{ is a basis of } \mathcal{H}_0\ =\ \mathrm{Span}\{\mathrm{d}x,\mathrm{d}u\}\ .\end{array} \tag{7.15}$$

Hence, with $\omega_{k,j}=\omega_k^{(j)}$, and with the $a_{i,j}$'s and $b_{i,j}$'s some functions such that the matrix $[b_{i,j}]_{i,j}$ is invertible at $\mathcal{X}$,

$$\begin{array}{rcl}\dot\chi_1 &=& \gamma_1(\chi_1,\ldots,\chi_{\rho_\infty})\\ &\vdots&\\ \dot\chi_{\rho_\infty} &=& \gamma_{\rho_\infty}(\chi_1,\ldots,\chi_{\rho_\infty})\end{array}$$
$$\left.\begin{array}{rcl}\dot\omega_{i,1} &=& \omega_{i,2}\\ \dot\omega_{i,2} &=& \omega_{i,3}\\ &\vdots&\\ \dot\omega_{i,r_i-1} &=& \omega_{i,r_1}\\ \dot\omega_{i,r_1} &=& \sum_{j=1}^{n}a_{i,j}\mathrm{d}x_j+\sum_{j=1}^{m}b_{i,j}\mathrm{d}u_j\end{array}\right\}\quad 1\leq i\leq m\ . \tag{7.16}$$

We call this "infinitesimal Brunovský form" because it looks like the canonical Brunovský form [17] for linear system ; it is not a "canonical form" for any equivalence relation : the data of the forms $\omega_1,\ldots,\omega_m$ and of (7.16) does not give a unique system.

Proof : The proof goes along the lines of [4] or [83]. Since we are at a Brunovský-regular point, $\mathcal{H}_\infty$ is nonsingular and locally spanned by exactly ρ_∞ forms. These forms depend on a finite number of variables $x,u,\ldots,u^{(K)}$. One may then project these forms, and hence $\mathcal{H}_\infty$, on the finite dimensional manifold $\mathcal{M}_K^{m,n}$ (see Section 6.3.2) and use the finite dimensional Frobenius theorem : from Theorem 7.1, $\mathcal{H}_\infty$ is completely integrable and therefore is spanned by ρ_∞ exact forms $\mathrm{d}\chi_1\ldots\mathrm{d}\chi_{\rho_\infty}$ with $\chi_1\ldots\chi_{\rho_\infty}$ some functions, which depend only on x because $\mathrm{d}\chi_i\in\mathcal{D}_\infty\subset\mathcal{D}_1$. Then the forms ω_k may be constructed recursively such that (7.13) holds :
- it holds for $l\geq k^*+1$ provided all the r_k's are no larger than k^* (it will be the case).
- chose $\omega_1,\ldots,\omega_{\rho_{k^*}}$ so that $\{\mathrm{d}\chi_1,\ldots,\mathrm{d}\chi_{\rho_\infty},\omega_1,\ldots,\omega_{\rho_{k^*}}\}$ is a basis of $\mathcal{H}_{k^*}$, and set $r_1=\ldots=r_{\rho_{k^*}}=k^*$, (7.13) is then satisfied for $l\geq k^*$ provided all the remaining r_k's are no larger than k^*-1 (it will be the case).
- Induction on ℓ, downward from $\ell=k^*$ to $\ell=0$: for $0\leq\ell\leq k^*-1$, let us suppose that (7.13) is true for $l\geq\ell+1$ (assuming that all the r_k's corresponding to ω_k's which have not yet been built are no larger than ℓ), and build some ω_k's with $r_k=\ell$ so that (7.13) is true for $l\geq\ell$. It is not difficult to prove (see [4, proof of Th. 3.5], really similar because by assumption $\mathcal{H}_{\ell+1}+\dot{\mathcal{H}}_{\ell+1}$

is nonsingular here) that $\{\mathrm{d}\chi_1, \dots, \mathrm{d}\chi_{\rho_\infty}\} \cup \{\omega_k^{(j)}, r_k \geq \ell+1, 0 \leq j \leq r_k - \ell\}$ is a set of linearly independent elements of $\mathcal{H}_\ell$, actually a basis of $\mathcal{H}_{\ell+1} + \dot{\mathcal{H}}_{\ell+1} \subset \mathcal{H}_\ell$. Add, if they do not form a basis of $\mathcal{H}_\ell$, some new ω_k's with the corresponding r_k's equal to ℓ.

After $l = 0$, no new ω_k's are needed because if there is a certain number of ω_k's such that (7.13) holds for $l = 0$ (we have not yet proved there are exactly m of them), then $\mathrm{d}u_1, \dots, \mathrm{d}u_m$ are linear combinations of the $\mathrm{d}\chi_i$'s and the $\omega_k^{(j)}$'s for $r_k \geq 0$ and $0 \leq j \leq r_k$, which immediately implies that, for $q > 0$, $\mathrm{d}u_1^{(q)}, \dots, \mathrm{d}u_m^{(q)}$ are linear combinations of the $\mathrm{d}\chi_i$'s and the $\omega_k^{(j)}$'s for $r_k \geq 0$ and $0 \leq j \leq r_k + q$, i.e. (7.13) is met for $l = -q < 0$ without any additional ω_k's ; this ends the construction of the ω_k's and proves $r_k \geq 0$ for all k. There are exactly m ω_k's because an obvious consequence of (7.13) is that $\rho_l - \rho_{l+1}$ is equal to the number of r_k's larger or equal to l ; in particular, since $\rho_l - \rho_{l+1} = m$ for $l \leq 0$ (see (7.7)), the total number of ω_k's is m. To prove the very last part of the theorem, one therefore has to prove that $\rho_1 - \rho_2 = m$ if and only if (7.14) holds, which is obvious because, from (7.4), $\mathcal{D}_2 = \mathcal{D}_1 \oplus \mathrm{Span}\{\frac{\partial f}{\partial u_1}, \dots, \frac{\partial f}{\partial u_m}\}$ and because of Brunovský-regularity. ∎

The reason for defining this "Brunovský form" in [4, 83] was to suggest a way to look for "linearizing outputs" (see theorem 7.3 below for definition and comments).

Definition 6.4 introduces, as in [4, 83], the notion of a *linearizing Pfaffian system.* Recall that one should not be mislead by the terminology : a linearizing Pfaffian system, contrary to a linearizing output, does not linearize anything unless it has more properties (integrability, see Theorem 7.3). An an immediate consequence of Theorems 7.1 and 7.2 is :

Corrolary 7.2. *If a system F is locally strongly accessible around a point $\mathcal{X}$, which is Brunovský-regular for F, then F, admits, locally around $\mathcal{X}$, a linearizing Pfaffian system $(\omega_1, \dots, \omega_m)$. A possible choice is the forms $\omega_1, \dots, \omega_m$ constructed in Theorem 7.2. If (7.14) holds, $\omega_1, \dots, \omega_m$ are in $\mathcal{H}_1 = \mathrm{Span}\{\mathrm{d}x\}$.*

Comments on this "Brunovský form"

Let us indicate the similarity between the content of this section and the algebraic framework for "time-varying" linear systems developed in [33, 34] for example.

For U an open subset of $\mathcal{M}_\infty^{m,n}$, let $\mathbf{C}^\infty(U)[L_F]$ be the algebra of differential operators which are polynomials in the Lie derivative with respect to F with coefficients in $\mathbf{C}^\infty(U)$. This is a non-commutative algebra since $(aL_F)(bL_F) = abL_F^2 + a(L_F b)L_F$. It plays the same role as the non-commutative ring $k[\frac{\mathrm{d}}{\mathrm{dt}}]$ (k is a differential field) introduced in [33] to define linear time-varying systems : a linear system is a module over this ring and it is controllable if and only if it is a free $k[\frac{\mathrm{d}}{\mathrm{dt}}]$-module (which is also a k vector space).

In the nonlinear case, in [33, 35] a system is represented by a differential field k and, via Kähler differentials, one may define the linearized system as a $k[\frac{\mathrm{d}}{\mathrm{dt}}]$-module, whose equivalent here is the $\mathbf{C}^\infty(U)[L_F]$-module $\Lambda^1(U)$.

Relying upon results from [97, 29] which state that a nonlinear system satisfying the strong accessibility condition has a controllable linear approximation along "almost any" trajectory, a nonlinear system is said to be controllable in [35] if and only if the $k[\frac{\mathrm{d}}{\mathrm{dt}}]$-module associated to the differential field k is free.

Note that the assertion "$(\omega_1, \dots, \omega_m)$ is a linearizing Pfaffian system" (or (7.13) with $\rho_\infty = 0$) is equivalent to "$(\omega_1, \dots, \omega_m)$ is a basis of the $\mathbf{C}^\infty(U)[L_F]$-module $\Lambda^1(U)$" ; hence Corollary 7.2 constructs a basis of this module, and hence establishes that it is free. We have *proved* (theorem 7.1), that, at a Brunovský-regular point (and even at a point where $\widehat{\mathcal{D}}_\infty$ is locally finitely generated), the strong accessibility rank condition implies that the module is free, or that the linearized system is controllable in the sense of [33, 35]. This is not exactly a consequence of [97, 29]. Technically, the result is contained in the fact that $\mathcal{D}_\infty$ is (around a regular point) closed under Lie bracket, which may be interpreted as : the torsion submodule of the $\mathbf{C}^\infty(U)[L_F]$-module $\Lambda^1(U)$ is "integrable".

An algebraic construction of the "canonical Brunovský form" (or of a basis of the module) for controllable time-varying linear systems, based on some filtrations, is proposed in [34]. The sequence of the $\mathcal{H}_k$'s *is* a filtration of $\Lambda^1(U)$. It does not coincide with these introduced in [35], but might certainly be interpreted in the same terms. The "well-formedness" assumption in [35] corresponds to (7.14) at the end of Theorem 7.2.

7.4 Dynamic linearization as an integrability problem

Dynamic linearizability from Definition 7.1 is actually **linearizability by endogenous dynamic feedback** as defined in [68, 36, 37]. It is proved there that this is equivalent to *flatness*, i.e. to existence of *linearizing outputs* or *flat outputs*. In the present framework, these are defined below. They are given an interpretation in terms of dynamic decoupling and structure at infinity in [4] and in [68], and they are defined as the free generators of the differential algebra $\mathbf{C}^\infty(\mathcal{M}_\infty^{m,n})$ in [55, 56].

The following is an immediate consequence of Theorems 6.4 and 6.5 :

Theorem 7.3. *Let $\mathcal{X}$ be a point of $\mathcal{M}_\infty^{m,n}$. The following assertions are equivalent :*

1. *The system F is locally dynamic linearizable at point $\mathcal{X}$.*
2. *There exist m smooth functions $h_1, \ldots, h_m$ from a neighborhood of $\mathcal{X}$ in $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$ such that $(L_F^j h_k)_{1\le k\le m, 0\le j}$ is a local system of coordinates at $\mathcal{X}$. Such m functions are called **linearizing outputs** (or simply one linearizing output) [36, 37, 68].*
3. *F admits, on a neighborhood of $\mathcal{X}$, a linearizing Pfaffian system $(\eta_1, \ldots, \eta_m)$ which is completely integrable, i.e. such that $\mathrm{d}\eta_k \wedge \eta_1 \wedge \ldots \wedge \eta_m = 0$, $k = 1 \ldots m$.*

We saw in the previous section that all strongly accessible systems admit, at Brunovský-regular points, a linearizing Pfaffian system, which, of course, may not be integrable. We therefore have to investigate what *all* linearizing Pfaffian systems are, and we may say that a system is dynamic linearizable if and only if there exists one among all these which is integrable.

For an open subset U of $\mathcal{M}_\infty^{m,n}$, let $\mathcal{A}(U)$ be the algebra of $m \times m$ matrices with entries in the algebra of differential operators $\mathbf{C}^\infty(U)[L_F]$:

$$\mathcal{A}(U) \;\stackrel{\Delta}{=}\; \mathcal{M}_{m\times m}\left(\mathbf{C}^\infty(U)[L_F]\right) . \tag{7.17}$$

A matrix in $\mathcal{A}(U)$ defines an operator on m-uples of 1-forms in a straightforward manner, and we have :

Proposition 7.3. *Let $(\omega_1, \ldots, \omega_m)$ be a linearizing Pfaffian system and let $\eta_1, \ldots, \eta_m$ be m 1-forms defined on an open set U of $\mathcal{M}_\infty^{m,n}$. $(\eta_1, \ldots, \eta_m)$ is a linearizing Pfaffian system if and only if there exists $P(L_F)$ in $\mathcal{A}(U)$ which is invertible in $\mathcal{A}(U)$ and is such that*

$$\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \end{pmatrix} = P(L_F) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix} \tag{7.18}$$

Proof : There always exists $P(L_F) \in \mathcal{A}(U)$ such that (7.18) holds because $(\omega_1, \ldots, \omega_m)$ is a linearizing Pfaffian system. If $(\eta_1, \ldots, \eta_m)$ is also a linearizing Pfaffian system, there exists $Q(L_F) \in \mathcal{A}(U)$ such that

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix} = Q(L_F) \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \end{pmatrix} .$$

Hence $Q(L_F)P(L_F)$ and $P(L_F)Q(L_F)$ transform respectively $(\omega_1, \ldots, \omega_m)$ and $(\eta_1, \ldots, \eta_m)$ into themselves. Hence $Q(L_F)P(L_F) = P(L_F)Q(L_F) = I$ because the forms $\omega_k^{(j)}$ (resp. $\eta_k^{(j)}$), $1 \leq k \leq m$, $j \geq 0$, are linearly independent. Conversely, it is obvious that (7.18) with $P(L_F)$ invertible implies that $(\eta_k^{(j)})_{1 \leq k \leq m, j \geq 0}$ is a basis of the $\mathbf{C}^\infty(U)$-module $\Lambda^1(U)$. ∎

A straightforward consequence of Theorem 7.2 and Proposition 7.3 is :

Theorem 7.4. *Let $\mathcal{X} \in \mathcal{M}_\infty^{m,n}$ be a Brunovský-regular point for system F, and let $\omega_1, \ldots, \omega_m$ be the 1-forms constructed in Theorem 7.2, defined on a certain neighborhood U of $\mathcal{X}$. System F is locally dynamic linearizable at point $\mathcal{X}$ if and only if there exists an invertible matrix $P(L_F) \in \mathcal{A}(U)$ such that*

$$\begin{pmatrix} \overline{\omega}_1 \\ \vdots \\ \overline{\omega}_m \end{pmatrix} = P(L_F) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix} \tag{7.19}$$

is a locally completely integrable Pfaffian system, i.e. $\mathrm{d}\overline{\omega}_k \wedge \overline{\omega}_1 \wedge \ldots \wedge \overline{\omega}_m = 0$ for $k = 1, \ldots, m$.

Of course, this is not per se a solution to the dynamic feedback linearization problem ; it is rather a convenient way to pose the problem of deciding whether or not linearizing outputs exist. The main difficulty comes from the fact that the degree of P may be arbitrarily large because the linearizing outputs may depend on an arbitrary number of time-derivatives of u. Let us make this number artificially finite :

Definition 7.4. *System F is said to be* $\boldsymbol{(x, u, \ldots, u^{(K)})}$***-linearizable*** *(for $K = -1$, this reads x-linearizable) at point $\mathcal{X}$ if and only if there exists some linearizing outputs function of $(x, u, \ldots, u^{(K)})$ only (on x only for $K = -1$).*

Of course, a system is dynamic feedback linearizable (in the sense of Definition 7.1, i.e. linearizable by endogenous dynamic feedback according to [36, 37, 68], or dynamic linearizable according to [55, 56]) if and only if it is $(x, u, \ldots, u^{(K)})$-linearizable for a certain K. We have the following theorem which precises Theorem 7.4.

Theorem 7.5. *Let $\mathcal{X} \in \mathcal{M}_\infty^{m,n}$ be a Brunovský-regular point for system F, and let $\omega_1, \ldots, \omega_m$, and $r_1, \ldots, r_m$ be, respectively, the 1-forms and integers constructed in Theorem 7.2. System F is $(x, u, \ldots, u^{(K)})$-linearizable at point $\mathcal{X}$ if and only if there exists an invertible matrix $P(L_F) \in \mathcal{A}(U)$ satisfying the conditions of Theorem 7.4 and such that the degree of the entries of the k-th column is at most $K + r_k$.*

Proof : The condition is necessary for $(x, u, \ldots, u^{(K)})$-linearizability because if $h_1, \ldots, h_m$ are some linearizing outputs function of $x, u, \ldots, u^{(K)}$ only, (7.19) holds with $\overline{\omega}_k = \mathrm{d}\varphi_k$ and, from (7.13), the columns of P have to satisfy the degree inequalities. Conversely, suppose that (7.19) holds with the degree of the kth column of P being at most $K + r_k$ and the system $(\overline{\omega}_1, \ldots, \overline{\omega}_m)$ completely integrable, then $(\overline{\omega}_1, \ldots, \overline{\omega}_m)$ is spanned by some exact forms $(\mathrm{d}h_1, \ldots, \mathrm{d}h_m)$; the functions h_k are linearizing outputs ; the degree inequalities imply that all the $\overline{\omega}_k$'s are in $\mathcal{H}_{-K} = \mathrm{Span}\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{(K)}\}$, and hence that the h_k's are functions of $x, u, \ldots, u^{(u)}$ only. ∎

One of the reasons why our results provide a rather convenient framework is that, outside some singular points, it is not difficult to describe invertible matrices of a prescribed degree. As noticed in [34, 35, 51], the polynomial ring $\mathbf{C}^\infty(U)[L_F]$ enjoys many interesting properties. Namely, it is possible to perform right and left Euclidean division by a polynomial whose leading coefficient does not vanish. It is well known (see for example [110]) that, in the constant coefficient case, all invertible polynomial matrices are finite products of “elementary matrices”, i.e. either diagonal invertible matrices or permutation matrices or matrices whose diagonal entries are all equal to 1 while only one of the non-diagonal entries is nonzero, and it is an arbitrary polynomial. Since the tool to get such a decomposition is only Euclidean division, this remains true in the case of coefficients in $\mathbf{C}^\infty(U)$ as long as one does not have to perform Euclidean division by a polynomial whose leading coefficient vanishes. This does not happen often, although it is not very easy in general to say which singularities the original matrix should not have for this not to happen ; in the meromorphic case ([4, 83]), this never happens since the coefficient of the polynomials then belong to a field and are therefore invertible, even if they “vanish” at a point, if they are not zero. Now, if one bounds a priori the degree of the columns of P (say one wishes to decide whether $(x, u, \ldots, u^{(K)})$-linearizability holds), then all invertible matrices satisfying these bounds may be sorted into a finite number of types of finite products of elementary matrices, each type involving a finite number of functions. In each case,

$$\mathrm{d}\left(P(L_F) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix} \right) \quad = \quad 0$$

(with d acting on each entry) is a set of partial differential equations in these functions. The solubility of these PDE’s is equivalent to the existence of a system of linearizing outputs depending only on a fixed finite number of time-derivatives of u.

7.5 Conclusion

We have developed a framework for looking for linearizing outputs which gives a convenient way for writing down a system of equations whose solubility is equivalent to the existence of a system of linearizing outputs. Some work has already been done in the direction of characterizing the cases where linearizing outputs exist. These results give either sufficient conditions or necessary and sufficient conditions for existence of linearizing outputs for some particular cases. For example, $(x, u, \ldots, u^{(K)})$-linearizability (in most cases, $K = -1$) or a prescribed “structure at infinity” (see [70, 69, 72]). A criterion for existence of a matrix P of degree zero for general two-inputs systems is given in [83]. The “sufficiency” part of the result contained in [72] is re-derived in [2] in a way that simplifies, to our opinion, the argument partly due to E. Cartan. Finally, a characterization of (x, u)-linearizability for affine systems with 4 states and 2 inputs is given in Chapter 8. These last results seem to demonstrate that “infinitesimal Brunovský form” is a convenient way to tackle the problem of looking for linearizing outputs.

Chapitre 8

Reproduction de l'article:

J.-B. Pomet, "On Dynamic Feedback Linearization of Four-dimensional Affine Control Systems with Two Inputs",

ESAIM Control Optim. Calc. Var., vol. 2 , pp. 151–230, 1997.

8.1 Introduction

A deterministic finite dimensional nonlinear control system

$$\dot{x} \;=\; f(x,u) \tag{8.1}$$

where the state x lives in $\mathbb{R}^n$, the control u lives in $\mathbb{R}^m$, and f is smooth —smooth means $\mathbf{C}^\infty$ in this article— is said to be locally *static feedback equivalent* around $(\bar{x},\bar{u})$ to another system

$$\dot{z} \;=\; \widetilde{f}(z,v) \tag{8.2}$$

around $(\bar{z},\bar{v})$ if there exists a *nonsingular feedback transformation*, i.e. two maps

$$\begin{array}{rcl} u &=& \alpha(x,v) \\ x &=& \phi(z) \end{array} \tag{8.3}$$

such that $(z,v) \mapsto (\phi(z),\alpha(z,v))$ is a local diffeomorphism sending $(\bar{z},\bar{v})$ to $(\bar{x},\bar{u})$, that transforms (8.1) into (8.2). The interest of feedback equivalence is that the transformation (8.3) allows one to convert the solution to a certain control problem for system (8.1) to the solution of a similar control problem for system (8.2). It is clear that (germs of) static feedback transformations form a group acting on (germs of) systems, and that static feedback equivalence *is* an equivalence relation. This feedback equivalence has been very much studied, see for instance [15, 13, 54]. Classification of control systems modulo this equivalence is of course a very ambitious and difficult program, almost out of reach. A more restricted problem is the one of describing the orbits of controllable linear systems, i.e. systems of the form $\dot{z} = Az + Bv$ with (controllability) the columns of $B, AB, A^2B, A^3B, \ldots$ having full rank. This problem is known as static feedback linearization, and has been completely solved : in [57, 50], explicit conditions are given for a general nonlinear system to be locally static feedback equivalent to a controllable linear system.

A dynamic feedback, or dynamic compensator, as opposed to static, is one where the "old" controls u are not computed from the "new" ones v by simply static functions (8.3), but through a dynamic system which has a certain state ξ :

$$\begin{array}{rcl} u & = & \alpha(x,\xi,v) \\ \dot{\xi} & = & \gamma(x,\xi,v) \\ z & = & \phi(x,\xi)\ , \end{array} \tag{8.4}$$

where ξ lives in $\mathbb{R}^\ell$, $\ell > 0$, and ϕ is a (local) diffeomorphism of $\mathbb{R}^{n+\ell}$. (x,v) may be viewed as the "input" of the control system, and (u,x,ξ), or (u,X) as its "output".

Clearly, (8.4) allows one to transform system (8.1) into a system like (8.2). However, contrary to the case of static feedback, the dimension of the state of the transformed system (8.2) is strictly larger than the dimension of the state of the original system (8.1), and for this reason, it a priori difficult to say what an "invertible" dynamic feedback "transformation" can be. One may however, following [23], state the problem of dynamic feedback linearization as the one of deciding when a system (8.1) can be transformed via a dynamic feedback (8.4) into a linear controllable system. The problem of deciding if a given system is dynamic feedback linearizable is much more difficult than the one for static feedback and is still open. A panorama and further references on dynamic feedback linearization from the point of view of compensators (8.4) can be found in [23]. This reference contains some sufficient conditions, that have the annoying drawback of not being invariant by static feedback, and also the following three results, that are of more general interest : a single input system ($u \in \mathbb{R}$), at a "regular" point, is dynamic feedback linearizable if and only if it is static feedback linearizable ; dynamic feedback linearizability at a rest point $(x,u) = (\bar{x},0)$ implies controllability of the linear approximation of the system at this point ; a controllable system which is affine in the control —i.e. the right-hand side of (8.1) is affine with respect to u— and such that the dimension of the state is larger than the dimension of the control by at most one is always dynamic feedback linearizable.

As seen above, the case of systems with one control is completely understood outside singularities, so that the nontrivial cases have at least two controls. The cases where the dimension of the state is less than 3 are somehow trivial (again, away from singularities), and the case where it is 3 and the system is affine in the control is covered by the above mentioned result from [23]. The smallest nontrivial cases are therefore non-affine systems with three states and two inputs, and affine systems with four states and two inputs. Section 8.6 explains how to apply the results of this paper to three-dimensional non-affine systems, but the rest of the paper is devoted to systems

$$\dot{x} \;=\; X_0(x) \;+\; u_1 X_1(x) \;+\; u_2 X_2(x) \tag{8.5}$$

where $x \in \mathbb{R}^4$ and u_1 and u_2 are in $\mathbb{R}$ ($u = (u_1,u_2)$). X_0, X_1 and X_2 are smooth vector fields in $\mathbb{R}^4$. Smooth means $\mathbf{C}^\infty$ in this article.

Of course, since it is the simplest non-trivial case, the problem of dynamic feedback linearization for the four dimensional system (8.5) has already been studied. In [66], based on the results from [23], sufficient conditions on X_0, X_1 and X_2 are given. A drawback of these results is that they are not invariant by static feedback, and are only sufficient conditions. They are contained in the results of the present paper.

Rather recently, some conceptual advances have been made on dynamic equivalence and dynamic linearization, initiated in [68, 36] (see [40] for a complete exposition). In [68], a restricted class of compensators (8.4) is studied, called *endogenous* dynamic feedbacks. They are exactly these that should be called "invertible". They are the compensators (8.4) such that, by differentiating relations (8.1) and (8.4), it is possible to express ξ and v as functions of x, u, $\dot{u}$, and a finite number of time-derivatives of u. The compensator (8.4) may then be replaced by some formulas giving z and v as functions of $(x, u, \dot{u}, \ddot{u}, \dots)$, which is "invertible" by formulas giving x and u as functions of $(z, v, \dot{v}, \ddot{v}, \dots)$. On the other hand, the notion of *differential flatness* for control systems is introduced in [68, 36, 40], as roughly speaking, existence of m —two for system (8.5)— functions of x, u, $\dot{u}$ and a finite number of time-derivatives of u which are differentially independent (the Jacobian of any finite number of these functions and their time derivatives has maximum rank) and such that both x and u can be expressed as functions of these m functions and a finite number of their time-derivatives. These functions are called *linearizing outputs*, or "flat outputs". It is proved there that differential flatness is equivalent to equivalence by *endogenous* dynamic feedback to a controllable linear system. In the differential algebraic framework of [36, 40], flatness is defined as the differential field representing the system being *non-differentially* algebraic over a purely transcendental differential extension of the base field, and the linearizing output is a transcendence basis. Of course, the linearizing outputs are then "restricted" to be algebraic. With a suitable definition of endogenous dynamic equivalence between differential fields, it is proved that differential flatness is equivalent to equivalence by *endogenous* dynamic feedback to a controllable linear system.

In [56], a notion of dynamic equivalence in terms of transformations on *solutions* of the system is studied; different types of transformations are defined there in terms of infinite jets of trajectories, for smooth systems, one of them is proved there to be exactly the one studied here. A property of "freedom" is introduced that is close to differential flatness and is proved to be equivalent to equivalence to a linear system.

See [40], [23], [4] or chapter 7 for a more complete panorama and list of references on dynamic feedback equivalence and dynamic feedback linearization, with references to recent and interesting results and points of views that we do not discuss here, like the work by Shadwick [93] (and subsequent articles) that make a link between dynamic feedback linearization and the notion of absolute equivalence defined by E. Cartan for Pfaffian systems.

There was a need to develop a *geometric* framework for the invertible transformations that represent dynamic feedback. This was done by the author in Chapter 6 and independently by the authors of [36, 40] in [38, 39]. In these papers, an (infinite dimensional) differential geometric approach, based on infinite jet spaces, is used, and the transformations described above may be seen as diffeomorphism that conjugate a system to another, they are a particular case of infinite order contact transformations, or Lie-Bäcklund transformations used in the "geometric" study of differential systems and partial differential relations.

Here, we adopt the notations and the precise definitions for linearizing outputs and dynamic linearization from Chapters 6 and 7. They are summed up in section 8.2.2 and 8.2.3.

The problem of deciding endogenous[1] dynamic linearizability is then the one of deciding existence of a system of linearizing outputs. The first difficulty is that there is no known a priori bound on the number of time-derivatives of the input the linearizing outputs should depend upon (similarly, there is no a priori bound on the dimension of ξ in a compensator (8.4) that would transform a given nonlinear system into a linear system if such a compensator exists). Even for four-dimensional systems (8.5), no such bound is known. We do not address this difficulty in the present paper. We only give necessary and sufficient conditions for existence of linearizing outputs depending on x and u. We call x-dynamic and (x, u)-dynamic linearizability existence of linearizing outputs depending on u or on (x, u). Note that the present conditions are quite explicit : a small package in Maple, described in [65], that helps in the process of checking the present conditions, will soon be available from the author.

Technically, the results in this paper amount to conditions for existence of solutions to some

[1] It is announced in [40, 39] that general dynamic feedback linearizability implies endogenous dynamic linearizability. From such a result, existence of linearizing outputs would be necessary and sufficient for general dynamic feedback linearization.

differential relations : in principle, given a system, one may write the PDEs that a pair of functions $(h_1(x,u), h_2(x,u))$ has to satisfy to be a pair of linearizing outputs, and then check whether this system of PDEs has some solutions (formal integrability, Spencer co-homology, see for instance [18] or [85]). This program reaches its limits very quickly seen the complexity of the PDEs themselves, and of the computation of compatibility conditions : even if algorithms are theoretically available, writing the PDEs for linearizing outputs for a general system is already heavy, and computing the compatibility conditions via general algorithms is overwhelming. The essence of the paper is however to compute these compatibility conditions, but in a way that uses a lot of the structure of the problem and makes them tractable. In particular, we use, for the case of linearizing outputs depending on x and u, the “infinitesimal Brunovský form” introduced in [4, 3, 83], that allows to write different PDEs : the unknowns are then some coefficients of transformations that act on pairs of differential forms —the condition is that it makes them integrable— instead of the linearizing outputs themselves. It would be interesting to know whether it is general that the use of the infinitesimal Brunovský form provides a method to write the equations for linearizing outputs in a more tractable manner. This is explained into details in section 8.2, see especially subsection 8.2.6 for a discussion of the two possible ways of writing the equations for existence of linearizing outputs, either directly or via the infinitesimal Brunovský form.

The paper is organized as follows. Section 8.2 recalls or introduces some technical material, including the precise definitions of what is intended here by feedback linearization and linearizing outputs in the geometric context of Chapters 6 and 7. Sections 8.3 and 8.4 contain the results, i.e. necessary and sufficient conditions for x-dynamic linearization (section 8.3) and for (x,u)-dynamic linearization (section 8.4). Section 8.6 shows that non affine systems in $\mathbb{R}^3$ which are dynamic feedback linearizable may be transformed into an affine system (8.5) in $\mathbb{R}^4$ by a simple dynamic extension, using a result by Rouchon [88] or Sluis [96]. Most proofs are in section 8.7, and some basic facts on Pfaffian systems used in them are recalled in the Appendix. Section 8.8 makes some remarks on the problems we leave open and on the interest and limitations of the techniques we use.

8.2 Statement of the problem

8.2.1 Static Feedback

A static feedback transformation, around a point $(\bar{x}, \bar{u})$ is a local transformation on the controls $v = \phi_2(x,u)$, defined on a neighborhood of $(\bar{x}, \bar{u})$, with $\frac{\partial \phi_2}{\partial u}$ invertible (the reason for the subscript “2” is that we shall use a local diffeomorphism ϕ_1 on x, so that $(x,u) \mapsto (\phi_1(x), \phi_2(x,u)$ is a local diffeomorphism on (x,u)).

Since we are only concerned with systems like (8.5) where the controls appear linearly, we shall only need *affine* static feedback. A local affine static feedback transformation is one of the above type where ϕ_2 is affine with respect to u. It is more convenient to write the inverse of ϕ_2 with respect to u, i.e. to write, instead of $(v_1, v_2) = \phi_2(x, u_1, u_2)$,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha(x) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \beta(x) \tag{8.6}$$

with $\alpha(x)$ an invertible 2×2 matrix and $\beta(x)$ a vector, both depending smoothly on x. It transforms system (8.5) into

$$\dot{x} = \widetilde{X}_0(x) + v_1\widetilde{X}_1(x) + v_2\widetilde{X}_2(x)$$
$$\text{with} \begin{cases} \widetilde{X}_0 = X_0 + \beta_1 X_1 + \beta_2 X_2 \\ \widetilde{X}_1 = \alpha_{11} X_1 + \alpha_{21} X_2 \\ \widetilde{X}_2 = \alpha_{12} X_1 + \alpha_{22} X_2 \end{cases} \tag{8.7}$$

A system is locally **static feedback linearizable** if and only if it may be transformed by such a transformation into a system which, in some coordinates $z = \phi_1(x)$, reads like a controllable linear system $\dot{z} = Az + Bv$ in $\mathbb{R}^4$ with two inputs; these linear systems are all of the form (a) or (b) below, up to a linear feedback —like (8.6) with α and β constant— and a linear change of coordinates :

$$\text{(a)} \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = u_1 \\ \dot{z}_4 = u_2 \end{cases} \qquad \text{(b)} \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = u_1 \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = u_2 \end{cases} \tag{8.8}$$

These are the two Brunovský canonical forms for controllable linear systems with two inputs and four states, see [17]. Static feedback linearizable systems are a particular case of x-dynamic linearizable systems because (x_1, x_4) for the form (a), and (x_1, x_3) for the form (b) may be chosen as a pair of linearizing outputs (see section 8.2.3).

Static feedback will also be used in the present paper to give some simple "normal" forms modulo this transformation and a change of coordinates on x of the systems considered for each case, or set of conditions, see (8.27), (8.28), (8.30), (8.31), (8.33), (8.44), (8.60), (8.66). The term **feedback invariant** refers to a property or an object that is invariant with respect to this equivalence relation between systems.

8.2.2 "Infinite dimensional" differential calculus and equivalence by endogenous feedback

This section is devoted to briefly recalling some notations and results from Chapters 6 and 7.

As mentioned in the introduction, similar material was also presented —independently— in [38, 39]. The content of [38, 39] is more general and more formal, and tends to give as a conclusion that systems (8.1) is not a general enough class of system for control theory, whereas Chapter 6 aims at developing the sufficient framework to use classical tools from differential calculus for the study of dynamic feedback. This infinite dimensional framework is, in any case, a rather convenient way of manipulating functions and other objects which depend on a *finite* but not a priori fixed number of variables, and it allows to say that the transformations by dynamic feedback are "diffeomorphisms".

We call *generalized state manifold* for system (8.1) with n states and m inputs the "infinite dimensional manifold" $\mathcal{M}_\infty^{m,n}$ where a set of coordinates is $(x_1, \ldots, x_n,\ u_1, \ldots, u_m,\ \dot{u}_1, \ldots, \dot{u}_m,$ $\ddot{u}_1, \ldots, \ddot{u}_m, \ldots\ldots)$. It is the projective limit of the finite dimensional manifolds $\mathcal{M}_K^{m,n}$, $K \geq -1$ with coordinates $(x_1, \ldots, x_n,\ u_1, \ldots, u_m,\ \dot{u}_1, \ldots, \dot{u}_m,\ \ldots, u_1^{(K)}, \ldots, u_m^{(K)})$ —when $K = -1$, this means $(x_1, \ldots, x_n)$— and we have the obvious projections π_K from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_K^{m,n}$:

$$\pi_K(x_1 \ldots x_n, u_1 \ldots u_m, \ \ldots\ \ldots) = (x_1 \ldots x_n, u_1 \ldots u_m, \ldots\ u_1^{(K)} \ldots u_m^{(K)}) . \tag{8.9}$$

The topology is the product topology, the least fine such that all these projections are continuous, i.e. an open set is always of the form $\pi_K^{-1}(O)$ with O a (finite-dimensional) open subset of $\mathcal{M}_K^{m,n}$. In particular when a property holds *locally around a point* $(x, u, \dot{u}, \ddot{u}, u^{(3)} \ldots\ldots)$, it means that it holds on a neighborhood of this point, i.e. for points whose first coordinates (an unknown a priori but finite number) are close to these of the original point, but with no restriction on the remaining coordinates. Actually, we will often say “in a neighborhood of $(x, u, \ldots, u^{(K)})$” to indicate that the value of $(u^{(K+1)}, u^{(K+2)}, \ldots\ldots)$ does not matter, i.e. the neighborhood is of the form $\pi_K^{-1}(O)$ with O a neighborhood of $(x, u, \ldots, u^{(K)})$ in $\mathcal{M}_K^{m,n}$.

Smooth functions are functions of a finite number of coordinates which are smooth in the usual sense. Differential forms of degree 1 are *finite* linear combinations :
$a_{-1}^1 \mathrm{d}x_1 + \cdots + a_{-1}^n \mathrm{d}x_n + a_0^1 \mathrm{d}u_1 + \cdots + a_0^m \mathrm{d}u_m + \ldots + a_J^1 \mathrm{d}u_1^{(J)} + \cdots + a_J^m \mathrm{d}u_m^{(J)}$ where the a_i^j's are smooth function. Forms of any degree may be defined similarly. Vector fields are (possibly infinite) linear combinations $b_{-1}^1 \frac{\partial}{\partial x_1} + \cdots + b_{-1}^n \frac{\partial}{\partial x_n} + b_0^1 \frac{\partial}{\partial u_1} + \cdots + b_0^m \frac{\partial}{\partial u_m} + b_1^1 \frac{\partial}{\partial \dot{u}_1} + \cdots + b_1^m \frac{\partial}{\partial \dot{u}_m} + \cdots$. Note that this infinite sum is only symbolic. There is no notion of “convergence” here since a vector field may be defined as a derivation on smooth functions, which, by definition depend only on a finite number of variables, so that the sum becomes finite when computing the Lie derivative of a smooth function along this vector field.

A diffeomorphism is a mapping φ from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ which is invertible and such that φ and φ^{-1} are smooth mappings, in the sense that, for any smooth function h from $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ to $\mathbb{R}$, $h \circ \varphi$ is a smooth function from $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$, and for any smooth function k from $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$, $k \circ \varphi^{-1}$ is a smooth function from $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ to $\mathbb{R}$.

A system $\dot{x} = f(x, u)$ with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is represented by a vector field of the form $F = f(x, u)\frac{\partial}{\partial x} + \dot{u}_1 \frac{\partial}{\partial u_1} + \dot{u}_2 \frac{\partial}{\partial u_2} + \ddot{u}_1 \frac{\partial}{\partial \dot{u}_1} + \ldots$ on the manifold $\mathcal{M}_\infty^{m,n}$. It is said to be **(locally) equivalent by endogenous dynamic feedback** to the system $\dot{z} = \tilde{f}(z, v)$ with $z \in \mathbb{R}^{\tilde{n}}$ and $v \in \mathbb{R}^{\tilde{m}}$, itself represented by the vector field $\widetilde{F} = \tilde{f}(z, v)\frac{\partial}{\partial z} + \dot{v}_1 \frac{\partial}{\partial v_1} + \dot{v}_2 \frac{\partial}{\partial v_2} + \ddot{v}_1 \frac{\partial}{\partial \dot{v}_1} + \ldots$ on $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ if and only if there exists a (local) diffeomorphism from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ that conjugates these two vector fields. This implies that $\tilde{m} = m$.

These diffeomorphism exactly mimic the transformations defined in [68]. The definition of “endogenous” as opposed to “exogenous” is explained there, or in [40].

From now on, let us focus on the small dimensional system (8.5), i.e. $n = 4$ and $m = 2$. We associate to system (8.5) the following vector field on $\mathcal{M}_\infty^{2,4}$:

$$F \;=\; X_0 \;+\; u_1 X_1 \;+\; u_2 X_2 \;+\; \dot{u}_1 \frac{\partial}{\partial u_1} \;+\; \dot{u}_2 \frac{\partial}{\partial u_2} \;+\; \ddot{u}_1 \frac{\partial}{\partial \dot{u}_1} \;+\; \ldots \quad . \tag{8.10}$$

Let us call **canonical linear system with two inputs** the vector field

$$C = \dot{v}_1 \frac{\partial}{\partial v_1} + \dot{v}_2 \frac{\partial}{\partial v_2} + \ddot{v}_1 \frac{\partial}{\partial \dot{v}_1} + \ddot{v}_2 \frac{\partial}{\partial \dot{v}_2} + v_1^{(3)} \frac{\partial}{\partial \ddot{v}_1} + \ldots$$

on the manifold $\mathcal{M}_\infty^{2,0}$ where a set of coordinates is $v_1, v_2, \dot{v}_1, \dot{v}_2, \ddot{v}_1, \ddot{v}_2, \ldots$. Any controllable linear system with 2 inputs can be (globally) transformed via a diffeomorphism into the canonical linear system on $\mathcal{M}_\infty^{2,0}$, see Chapter 7. For instance, for the first case in (8.8), the diffeomorphism is given by $v_1 = x_1$, $\dot{v}_1 = x_2$, $\ddot{v}_1 = x_3$, $v_1^{(3)} = u_1$, $v_1^{(4)} = \dot{u}_1, \ldots, v_2 = x_4$, $\dot{v}_2 = u_2$, $\ddot{v}_2 = \dot{u}_2, \ldots$. Hence, system (8.5) is said to be locally linearizable by endogenous dynamic feedback, or simply **endogenous dynamic linearizable** at $\mathcal{X} \in \mathcal{M}_\infty^{2,4}$ if and only if there is a diffeomorphism φ from an open neighborhood of $\mathcal{X}$ in $\mathcal{M}_\infty^{2,4}$ to an open set of $\mathcal{M}_\infty^{2,0}$ which transforms the vector field F defined in (8.10) into the vector field C on $\mathcal{M}_\infty^{2,0}$.

Let us discuss a few more objects that will be used in the paper. Lie Brackets, exterior derivative, Lie derivatives and all objects from usual differential calculus may be defined because they (or each of their components) may all be computed finitely and depend on a finite number of variables ; all identities from differential calculus are valid (any given such identity really involves only a finite number of variable).

We call **time-derivative** along system (8.5) the Lie derivative along the vector field F. It corresponds to the derivation defined in the differential fields in [40]. It will often be denoted $\frac{\mathrm{d}}{\mathrm{dt}}$ instead of L_F. It may be applied to functions : for a function $h(x, u, \dot{u}, \ldots, u^{(K)}$, $\dot{h}$, or $L_F h$, or $\frac{\mathrm{d}}{\mathrm{dt}}h$, is the function of $x, u, \dot{u}, \ldots, u^{(K+1)}$ obtained by applying the chain rule and substituting $X_0(x) + u_1 X_1(x) + u_2 X_2(x)$ for $\dot{x}$. This time-derivative may also be applied to forms. The time-derivative of $\omega = a^1_{-1}\mathrm{d}x_1 + \cdots + a^4_{-1}\mathrm{d}x_4 + a^1_0\mathrm{d}u_1 + a^2_0\mathrm{d}u_2 + \cdots + a^1_J\mathrm{d}u_1^{(J)} + a^2_J\mathrm{d}u_2^{(J)}$, i.e. its Lie derivative with along F, is given by

$$\begin{array}{rcl}\dot{\omega} & = & a^1_{-1}\mathrm{d}\dot{x}_1 + \cdots + a^4_{-1}\mathrm{d}\dot{x}_4 + \dot{a}^1_{-1}\mathrm{d}x_1 + \cdots + \dot{a}^4_{-1}\mathrm{d}x_4 \\ & & + a^1_0\mathrm{d}\dot{u}_1 + a^2_0\mathrm{d}\dot{u}_2 + \dot{a}^1_0\mathrm{d}u_1 + \dot{a}^2_0\mathrm{d}u_2 + \cdots \\ & & \cdots + a^1_J\mathrm{d}u_1^{(J+1)} + a^2_J\mathrm{d}u_2^{(J+1)}\dot{a}^1_J\mathrm{d}u_1^{(J)} + \dot{a}^2_J\mathrm{d}u_2^{(J)}\end{array}$$

where $\mathrm{d}\dot{x}_i$ stands for the differential of the ith component of $X_0 + u_1X_1 + u_2X_2$.

Let us mention one last notation. By $\mathrm{Span}\{\mathrm{d}\mathbf{x}\}$ or $\mathrm{Span}\{\mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{u}\}$ we mean the module over smooth functions spanned by $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$, $\mathrm{d}x_4$, or by $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$, $\mathrm{d}x_4$, $\mathrm{d}u_1$, $\mathrm{d}u_2$ respectively.

8.2.3 Linearizing Outputs

Linearizing outputs, or **flat outputs** were introduced by Fliess, Lévine, Martin and Rouchon in their work on *differential flatness*. Originally, it was a way to view the problem of dynamic feedback linearization in a more tractable way, but the systems for which there exists linearizing —or flat— outputs, i.e. differentially flat systems, possess properties that are very interesting independently from the fact that they may be rendered linear in some coordinates after adding to them a dynamic compensator : all their solutions may be parameterized "freely" by the linearizing outputs, see [40].

The following is the definition of linearizing output in the framework exposed above, taken from Theorem 7.3. It totally agrees with the one in [68, 40].

Definition 8.2.1. *A pair of functions (h_1, h_2) on $\mathcal{M}^{2,4}_\infty$ is called a pair of* ***linearizing outputs*** *on an open subset U of $\mathcal{M}^{2,4}_\infty$ if the functions $\left(L^j_F h_k\right)_{k\in\{1,2\}, j\geq 0}$ are a set of coordinates on U, i.e. if $\mathcal{X} \mapsto \left(L^j_F h_k(\mathcal{X})\right)_{k\in\{1,2\}, j\geq 0}$ is a diffeomorphism from U to an open subset of $\mathbb{R}^{2\mathbb{N}} = \mathcal{M}^{2,0}_\infty$.*

It is said to be a pair of linearizing output at point $(\bar{x}, \bar{u}, \dot{\bar{u}}, \ldots, \bar{u}^{(J)})$ with $J \geq -1$ (when $J = -1$, this stands for $\bar{x}$) if it is a pair of linearizing output on an open set U of the form $\pi_J^{-1}(U_J)$ (see (8.9)) where U_J is a neighborhood of $(\bar{x}, \bar{u}, \dot{\bar{u}}, \ldots, \bar{u}^{(J)})$ in $\mathcal{M}^{2,4}_J$, i.e. $\mathbb{R}^{2J+6}$.

The following equivalent formulation (Theorem 6.4) may appear simpler. It is closer to the definition in [68, 40].

Proposition 8.2.1. *A pair of functions (h_1, h_2) on $\mathcal{M}^{2,4}_\infty$ is a pair of linearizing outputs at point $(\bar{x}, \bar{u}, \dot{\bar{u}}, \ldots, \bar{u}^{(J)})$ with $J \geq -1$ (when $J = -1$, this stands for $\bar{x}$) if and only if there exists on open set U of the form $\pi_J^{-1}(U_J)$ (see (8.9)) such that*

1. *The differential forms $\left(\mathrm{d}h_k^{(j)}\right)_{k\in\{1,2\}, j\geq 0}$ are linearly independent at all points of U (meaning that whenever you take a finite number among these, they are linearly independent)*
2. *There exists an integer L and a smooth function ψ from an open set of $\mathbb{R}^{2L+2}$ to $\mathbb{R}^6$ such that $(x, u) = \psi(h_1, h_2, \dot{h}_1, \dot{h}_2, \ldots, h_1^{(L)}, h_2^{(L)})$ on U (this is an identity between functions of $x, u, \dot{u}, \ddot{u}, \ldots$).*

As said above, the linearizing outputs have a lot of interest in themselves, when they exist. They are also very relevant for the problem of dynamic linearization, thanks to the following equivalence, pointed out in [68, 36, 40].

Proposition 8.2.2 (Theorem 6.4). *(Local) endogenous dynamic linearizability is equivalent to existence (locally) of a pair of linearizing outputs.*

The following may illuminate the above introduced notions.

Sketch of proof : A diffeomorphism φ that conjugates the vector field F defined in (8.10) to the canonical vector field C on $\mathcal{M}_\infty^{2,0}$ defines two functions $h_1 = v_1 \circ \varphi$ and $h_2 = v_2 \circ \varphi$ on $\mathcal{M}_\infty^{2,4}$ which have the property that all their Lie derivatives $L_F^j h_k$ are transformed by the diffeomorphism into the coordinate $v_k^{(j)}$, which implies that the functions $L_F^j h_k$ are locally a set of coordinates on $\mathcal{M}_\infty^{2,4}$; conversely, if two functions exists which have this property, it is very easy to build a diffeomorphism from $\mathcal{M}_\infty^{2,4}$ to $\mathcal{M}_\infty^{2,0}$ which transforms F into C. ∎

By definition of what a smooth function is, the functions in a pair of linearizing outputs depend only on a finite number of variables among x, u, $\dot{u}$, $\ddot{u}$... In Chapter 7, we say that a system is $(x, u, \ldots, u^{(K)})$**-dynamic linearizable** when there exists a pair of functions depending only on x, u, $\ldots$, $u^{(K)}$). Clearly, from proposition 8.2.2 and above, linearizability by endogenous dynamic feedback implies $(x, u, \ldots, u^{(K)})$-dynamic linearizability for a certain K. Of course, a very interesting question is : given a system, how to determine a bound K such that if it is dynamic linearizable at all, then it is $(x, u, \dot{u}, \ddot{u}, \ldots, u^{(K)})$-dynamic linearizable ? Even for systems of the form (8.5), this is the subject of ongoing research.

As explained in the introduction, we only deal, in the present paper, with linearizing outputs depending on x only, or on x and u :

Definition 8.2.2. *System (8.5) is said to be* $\boldsymbol{(x, u)}$***-dynamically linearizable*** *at the point* $\overline{\mathcal{X}} = (\bar{x}, \bar{u}, \ldots, \bar{u}^{(J)})$ *if and only if there exists a pair of linearizing outputs* (h_1, h_2) *that depend on* x *and* u *only on an open set* $\pi_K^{-1}(\overline{\mathcal{X}})$*, a pair of linearizing outputs depending on* x *and* u *only. It is said to be* ***x-dynamically linearizable*** *if these linearizing outputs depend on* x *only.*

The present paper characterizes x-dynamic linearizability and (x, u)-dynamic linearizability for systems (8.5). Systems that are proved here not to be (x, u)-dynamic linearizable might or might not be $(x, u, \dot{u})$-dynamic linearizable, or $(x, u, \dot{u}, \ddot{u})$-dynamic linearizable, and so on...

8.2.4 Non-accessibility

Since we only work at regular points, non-accessibility always means in the present paper (and with the dimensions as in (8.5)) that there exists one function $\chi(x)$, or two functions $\chi_1(x)$ and $\chi_2(x)$, such that $\dot{\chi} = \varphi(\chi)$ for some function φ, or $\dot{\chi}_i = \varphi_i(\chi_1, \chi_2)$, $i = 1, 2$ for some functions φ_1 and φ_2.

This is an obstruction to existence of a pair of linearizing outputs. Indeed, if (h_1, h_2) is a pair of linearizing outputs, $\dot{\chi} = \varphi(\chi)$, or $\dot{\chi}_i = \varphi_i(\chi_1, \chi_2)$ implies a nontrivial relation between $h_1, h_2, \dot{h}_1, \dot{h}_2, \ldots, h_1^{(J)}, h_2^{(J)}$ for a certain $J > 0$, which cannot occur from the definition of a pair of linearizing outputs.

8.2.5 Linearizing Pfaffian systems, infinitesimal Brunovský form

An infinite set of differential forms is a basis of the space of all differential forms in the neighborhood of a point if any finite number of them are linearly independent at this point and there exists a neighborhood U of this point such that any differential form defined on U may be written as a linear combination of a finite number of the forms in the "basis" with coefficients smooth functions defined in U.

Definition 8.2.3. *Let ω_1 and ω_2 be two differential forms. We say that $\{\omega_1, \omega_2\}$ is a **linearizing Pfaffian system** at a certain point $(\bar{x}, \bar{u}, \dot{\bar{u}}, \ldots, \bar{u}^{(J)})$ if and only if ω_1, ω_2 and all their time-derivatives, i.e. $\left(\omega_k^{(j)}\right)_{k\in\{1,2\},\, j\geq 0}$ form a basis of the space of all differential forms in a neighborhood of this point.*

Note that this is a property of the Pfaffian system (or the co-distribution) $\{\omega_1, \omega_2\}$ rather than the pair of forms since this property will still hold if ω_1 and ω_2 are replaced by another basis for the same Pfaffian system.

Clearly, if (h_1, h_2) is a pair of linearizing outputs, then $\{\mathrm{d}h_1, \mathrm{d}h_2\}$ is a linearizing Pfaffian system because the function ψ in proposition 8.2.1 translates into a linear combination when differentiating. The converse is also true but requires an "infinite dimensional" local inverse theorem (Theorem 6.3) :

Proposition 8.2.3 (Theorem 6.5)**.** *A pair of functions (h_1, h_2) is a pair of linearizing outputs at a point if and only if $\{\mathrm{d}h_1, \mathrm{d}h_2\}$ is a linearizing Pfaffian system at this point.*

Since we pointed out that being a linearizing Pfaffian system does not depend on the precise choice of the basis, from Frobenius theorem, it is enough to have a linearizing Pfaffian system satisfying Frobenius condition

Proposition 8.2.4 (Theorem 6.5)**.** *There exists a pair of linearizing outputs around a point if and only if there exists a linearizing Pfaffian system $\{\omega_1, \omega_2\}$ on a neighborhood of this point satisfying Frobenius condition : $\mathrm{d}\omega_1 \wedge \omega_1 \wedge \omega_2 = \mathrm{d}\omega_2 \wedge \omega_1 \wedge \omega_2 = 0$ in a neighborhood of this point.*

We have the following —straightforward— property that describes all the possible linearizing Pfaffian systems from one :

Proposition 8.2.5 (Proposition 7.3)**.** *Let $\{\omega_1, \omega_2\}$ be a linearizing Pfaffian system at a certain point.*

Then for two forms η_1 and η_2, $\{\eta_1, \eta_2\}$ is a linearizing Pfaffian system if and only if ω_1 and ω_2 are linear combinations of η_1, η_2 and a finite number of their time derivatives on a neighborhood of this point.

Analogously, a pair of functions (h_1, h_2) is a pair of linearizing outputs at this point if and only if ω_1 and ω_2 are linear combinations of $\mathrm{d}h_1$, $\mathrm{d}h_2$ and a finite number of their time derivatives on a neighborhood of this point.

Note that the fact that $\{\omega_1, \omega_2\}$ be a linearizing Pfaffian system implies that η_1 and η_2, or $\mathrm{d}h_1$, $\mathrm{d}h_2$ are *always* linear combinations of ω_1, ω_2 and a finite number of their time derivatives.

Let us now translate this property into existence of an operator relating (ω_1, ω_2) and (η_1, η_2). For an open set U in $\mathcal{M}_\infty^{2,4}$, let $\mathcal{A}(U)$ be the $\mathcal{C}^\infty(U)$ algebra :

$$\mathcal{A}(U) \;\stackrel{\Delta}{=}\; \mathcal{M}_{2\times 2}\,(\,\mathcal{C}^\infty(U)[L_F]\,)\ . \tag{8.11}$$

of 2×2 matrices whose entries are differential operators, polynomial in the derivation along F, i.e. whose entries are of the form

$$p_0 \;+\; p_1 \frac{\mathrm{d}}{\mathrm{dt}} \;+\; p_2 \frac{\mathrm{d}}{\mathrm{dt}}^{2} \;+\; \ldots \;+\; p_K \frac{\mathrm{d}}{\mathrm{dt}}^{K} \quad ,$$

where the p_i's are smooth functions from U to $\mathbb{R}$ (recall it means they depend only on x and a finite number of time-derivatives of u). Elements of $\mathcal{A}(U)$ act in an obvious manner on pairs of functions, or on pairs of differential forms.

Proposition 8.2.6 (Theorem 7.4)**.** *Let $\{\omega_1, \omega_2\}$ be a linearizing Pfaffian system at a certain point. Then for two forms η_1 and η_2, $\{\eta_1, \eta_2\}$ is a linearizing Pfaffian system if and only if on a neighborhood U of this point, there exists $P \in \mathcal{A}(U)$ such that*

$$\begin{array}{l} \bullet \;\; P \text{ has an inverse in } \mathcal{A}(U), \\ \bullet \;\; \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \;=\; P(\frac{d}{dt}) \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} . \end{array} \tag{8.12}$$

This has some interest because it is possible, at least away from some singular points, to build a linearizing Pfaffian system for any accessible system. This is the construction of the "infinitesimal Brunovský form" in chapter 7 or in [4]. Some sequences of modules (over smooth functions) of 1-forms and of vector fields, called $\mathcal{H}_k$, $\mathcal{D}_k$ and $\widehat{\mathcal{D}}_k$ are defined in Chapter 7. Points where they have constant rank are called "Brunovský-regular", and at these points, a special linearizing Pfaffian system may be constructed. Let us recall here the minimum needed for our specific dimensions. Define the following modules of vector fields over smooth functions :

$$\begin{aligned} \widehat{\mathcal{D}}_2 &= \text{Span}\{X_1, X_2\} \\ \widehat{\mathcal{D}}_3 &= \widehat{\mathcal{D}}_2 + [F, \widehat{\mathcal{D}}_2] \\ &= \text{Span}\{X_1, X_2, [X_0, X_1] - u_2[X_1, X_2], [X_0, X_2] + u_1[X_1, X_2]\} \\ \widehat{\mathcal{D}}_4 &= \widehat{\mathcal{D}}_3 + [F, \widehat{\mathcal{D}}_3] . \\ &= \text{Span}\{X_1, X_2, [X_0, X_1] - u_2[X_1, X_2], [X_0, X_2] + u_1[X_1, X_2], \\ &\qquad [X_0, [X_0, X_1]] + u_1[X_1, [X_0, X_1]] + u_2([X_1, [X_0, X_1]] - [X_0, [X_1, X_2]]) \\ &\qquad\quad - u_1 u_2 [X_1, [X_1, X_2]] - u_2^2 [X_2, [X_1, X_2]] - \dot{u}_2 [X_1, X_2], \\ &\qquad [X_0, [X_0, X_2]] + u_1([X_1, [X_0, X_2]] + [X_1, [X_0, X_2]]) + u_2 [X_1, [X_0, X_2]] \\ &\qquad\quad + u_1^2 [X_1, [X_1, X_2]] + u_1 u_2 [X_2, [X_1, X_2]] + \dot{u}_1 [X_1, X_2]\} \end{aligned} \tag{8.13}$$

Definition 8.2.4. *A point $(x, u, \dot{u})$ where the vector fields X_1 and X_2 are not collinear is called* ***Brunovský regular*** *if and only if the three distributions $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ have constant rank in a neighborhood of this point. A point $(x, u, \dot{u}, \ddot{u}, \ldots\ldots) \in \mathcal{M}_\infty^{2,4}$ is called Brunovský regular if and only if the $(x, u, \dot{u})$ is Brunovský regular.*

The fact that Brunovský regularity depends on the value of x, u and $\dot{u}$ only comes from the fact that the vector fields in (8.13) depend on the eight variables $x, u, \dot{u}$ only (note also that they are linear combinations of the four coordinate vector fields corresponding to the x-coordinates only... they might be seen as vector fields on $\mathbb{R}^4$ parameterized by u and $\dot{u}$).

We always assume that the rank of $\widehat{\mathcal{D}}_2$ is two, then, at a Brunovský regular point, the ranks of $\widehat{\mathcal{D}}_2, \widehat{\mathcal{D}}_3, \widehat{\mathcal{D}}_4$ may only be $2, 2, 2$, $2, 3, 3$, $2, 3, 4$ or $2, 4, 4$. In the two first cases, system (8.5) is not accessible (see Chapter 7). In the two other cases, Theorem 7.2 allows one to build a linearizing Pfaffian system $\{\omega_1, \omega_2\}$ which has the peculiarity that either $\{\omega_1, \dot{\omega}_1, \omega_2, \dot{\omega}_2\}$ or $\{\omega_1, \dot{\omega}_1, \ddot{\omega}_1, \omega_2\}$ is a basis of $\text{Span}\{d\mathbf{x}\}$ (see the meaning of $\text{Span}\{d\mathbf{x}\}$ at the end of section 8.2.2). Let us make this precise, only in the case where the ranks are 2,4,4 because we will not use this process in the case 2,3,4.

Proposition 8.2.7 (Infinitesimal Brunovský Form, Theorem 7.2)**.** *Around a point where the ranks of $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and and $\widehat{\mathcal{D}}_4$ are 2, 4 and 4 respectively, and if ω_1 and ω_2 are two linearly independent 1-forms in the annihilator of $\widehat{\mathcal{D}}_2$, i.e. of $\{X_1, X_2\}$:*

$$\{\omega_1, \omega_2\} \;=\; \text{Span}\{d\mathbf{x}\} \,\cap\, \{X_1, X_2\}^\perp , \tag{8.14}$$

then $\{\omega_1, \omega_2\}$ *is a linearizing Pfaffian system, and more precisely,* $\{\omega_1, \omega_2, \dot\omega_1, \dot\omega_2\}$ *is a basis of* $\mathrm{Span}\{\mathrm{d}\mathbf{x}\}$, $\{\omega_1, \omega_2, \dot\omega_1, \dot\omega_2, \ddot\omega_1, \ddot\omega_2\}$ *is a basis of* $\mathrm{Span}\{\mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{u}\}$*, and more generally* $\{\omega_1, \omega_2, \dot\omega_1, \dot\omega_2, \ldots, \omega_1^{(J)}, \omega_2^{(J)}\}$ *is a basis of* $\mathrm{Span}\{\mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{u}, \mathrm{d}\dot{\mathbf{u}}, \ldots, \mathrm{d}\mathbf{u}^{(J-2)}\}$*. The 1-forms* ω_1 *and* ω_2 *can be chosen involving* x *only.*

This is a particular case of [3, theorem 2]. The following proof may however help the reader's understanding.

Sketch of proof : The forms ω_1 and ω_2 satisfying (8.14) may always be chosen so that they involve x only because X_1 and X_2 involve x only. We use the following identity, which is true for any form ω and any vector field X :

$$\begin{aligned}\langle \dot\omega \,,\, X\rangle &= \langle L_F\omega \,,\, X\rangle \\ &= L_F\langle \omega \,,\, X\rangle - \langle \omega \,,\, [F,X]\rangle \\ &= \frac{\mathrm{d}}{\mathrm{dt}}\langle \omega \,,\, X\rangle - \langle \omega \,,\, [F,X]\rangle \,. \end{aligned} \tag{8.15}$$

Now, on one hand the forms $\dot\omega_1$ and $\dot\omega_2$ are in $\mathrm{Span}\{\mathrm{d}\mathbf{x}\}$, i.e. have no component on $\mathrm{d}u_1$ and d_2 because (8.15) implies

$$\langle \dot\omega_k \,,\, \frac{\partial}{\partial u_i}\rangle = \langle \omega_k \,,\, [F, \frac{\partial}{\partial u_i}]\rangle = \langle \omega_k \,,\, X_i\rangle = 0$$

for $k = 1,2$ and $i = 1,2$. On the other hand, $\omega_1, \omega_2, \dot\omega_1, \dot\omega_2$ are linearly independent : if it was not that case at a point, there would exist some constants $\lambda_1, \lambda_2, \mu_1, \mu_2$, not all zero, such that $\lambda_1\dot\omega_1 + \lambda_2\dot\omega_2 + \mu_1\omega_1 + \mu_2\omega_2$ would vanish at this point ; since $\langle \omega_1, X_i\rangle = \langle \omega_2, X_i\rangle = 0$, this would imply that, for $i = 1,2$, $\langle \lambda_1\dot\omega_1 + \lambda_2\dot\omega_2, X_i\rangle$ also vanish at this point ; this in turn would imply, from identity (8.15), that

$$\langle \lambda_1\omega_1 + \lambda_2\omega_2 \,,\, [X_0 + u_1X_1 + u_2X_2, X_i]\rangle$$

vanishes at this point, i.e. that $\lambda_1\omega_1 + \lambda_2\omega_2$ is in the annihilator of $\widehat{\mathcal{D}}_3$, and hence that $\lambda_1 = \lambda_2 = 0$ because the rank of $\widehat{\mathcal{D}}_3$ is 4 and ω_1 and ω_2 are independent ; this is impossible because then $\mu_1\omega_1 + \mu_2\omega_2$ would vanish at the considered point while ω_1 and ω_2 are independent. It is easy to prove the last property for all $J \geq 2$: since $\omega_1, \omega_2, \dot\omega_1, \dot\omega_2$ are in $\mathrm{Span}\{\mathrm{d}\mathbf{x}\}$ and $\mathrm{d}u_k^{(\ell+1)}$ may only appear by taking the time-derivative of $\mathrm{d}u_k^{(\ell)}$, it is clear that $\omega_k^{(j)}$ is in $\mathrm{Span}\{\mathrm{d}x, \mathrm{d}u, \mathrm{d}\dot u, \ldots, \mathrm{d}u^{(j-2)}\}$, and the linear independence of all these is proved by using recursively identity (8.15). ∎

The term "infinitesimal Brunovský form" refers to the fact that, with the above choices of the 1-forms ω_1 and ω_2, system (8.5) implies :

$$\begin{aligned} \tfrac{d}{dt}\omega_1 &= \dot\omega_1 \\ \tfrac{d}{dt}\dot\omega_1 &= \textstyle\sum_1^4 \alpha_{1,i}\mathrm{d}x_i + \beta_{1,1}\mathrm{d}u_1 + \beta_{1,2}\mathrm{d}u_2 \\ \tfrac{d}{dt}\omega_2 &= \dot\omega_2 \\ \tfrac{d}{dt}\dot\omega_2 &= \textstyle\sum_1^4 \alpha_{2,i}\mathrm{d}x_i + \beta_{2,1}\mathrm{d}u_1 + \beta_{2,2}\mathrm{d}u_2 \end{aligned}$$

where the functions $\beta_{i,j}$ are such that the 2×2 matrix $[\beta_{i,j}]$ is invertible on a neighborhood of $(\bar x, \bar u)$. If the forms ω_1 and ω_2 were integrable, one might define z function of x and v function of x, u (static feedback transformation) by $\mathrm{d}z_1 = \omega_1$, $\mathrm{d}z_2 = \dot\omega_1$, $\mathrm{d}v_1 = \ddot\omega_1$, $\mathrm{d}z_3 = \omega_2$, $\mathrm{d}z_4 = \dot\omega_2$, $\mathrm{d}v_2 = \ddot\omega_2$, such that (8.5) reads like the Brunovský canonical form (8.8.b) —we would have obtained the form (8.8.b) if we would have considered the case where the ranks of $\widehat{\mathcal{D}}_2, \widehat{\mathcal{D}}_3, \widehat{\mathcal{D}}_4$ are 2,3,4—. It is called "infinitesimal" because it is only at the level of differential forms instead of functions (coordinates) and can give functions if the differential forms are integrable, which is false in general.

Now that we have built a special linearizing Pfaffian system, we may state the following consequence of propositions 8.2.4, 8.2.6 and 8.2.7. It is specialized to x-dynamic linearization or (x,u)-dynamic linearization, and the fact that the linearizing outputs depend on x only or on x and u only is translated into a condition on the degree of the entries of the matrix P comes from the special properties on ω_1 and ω_2 given in proposition 8.2.7. Again, this is only stated in the case where the ranks of $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ are 2,4,4 because we will not use this process in the case 2,3,4.

Proposition 8.2.8 (Theorem 7.5)**.** *Let $(\bar{x},\bar{u})$ be a point where the ranks of $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ are 2,4,4, and ω_1 and ω_2 be defined in a neighborhood of $(\bar{x},\bar{u})$ as in proposition 8.2.7 (see equation (8.14)).*

System (8.5) is x-dynamic linearizable (resp. (x,u)-dynamic linearizable) at point $(\bar{x}, \bar{u}, \ldots, \bar{u}^{(J)})$ if and only if there exists a neighborhood U of this point, and a 2×2 polynomial matrix $P \in \mathcal{A}(U)$ whose entries are polynomials of degree at most 1 (resp. at most 2), such that P has an inverse in $\mathcal{A}(U)$ and the Pfaffian system $\{\eta_1, \eta_2\}$ defined by

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = P(\frac{d}{dt})\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} . \tag{8.16}$$

is completely integrable, i.e. η_1 and η_2 satisfy $\mathrm{d}\eta_1\wedge\eta_1\wedge\eta_2 = \mathrm{d}\eta_2\wedge\eta_1\wedge\eta_2 = 0$ in a neighborhood of this point.

We shall use this property, especially for (x,u)-dynamic linearizability in section 8.4. Of course, this would be useless without a reasonable description of the invertible matrices in $\mathcal{A}(U)$ of degree at most 2. In fact, away from some singularities, invertible matrices may be described as products of "elementary matrices", like unimodular matrices in the case of polynomials with constant coefficients :

Proposition 8.2.9. *Let P be a matrix in $\mathcal{A}(U)$, which has an inverse Q in $\mathcal{A}(U)$.*

- *If the degree of P is 1 on an open dense subset of U (i.e. P has degree at most 1 everywhere, and possibly zero on a closed set of empty interior), then there is an open dense subset U_0 of U such, for that all $\mathcal{X}\in U_0$, there is a neighborhood $V_{\mathcal{X}}$, a scalar smooth function a, and two invertible matrices J_1 and J_2 of degree 0 (i.e. whose entries are smooth functions), all defined on $V_{\mathcal{X}}$, such that, on $V_{\mathcal{X}}$,*

$$P(\frac{d}{dt}) = J_1\begin{pmatrix} 1 & -a\frac{d}{dt} \\ 0 & 1 \end{pmatrix} J_2 \tag{8.17}$$

- *If the degree of P is 2 on an open dense subset of U (i.e. P has degree at most 2 everywhere, and possibly 1 or 0 on a closed set of empty interior), then there is an open dense subset U_0 of U such, for that all $\mathcal{X}\in U_0$, there is a neighborhood $V_{\mathcal{X}}$, scalar smooth functions α, λ, a and b, and an invertible matrix J_1 of degree 0 (i.e. whose entries are smooth functions), all defined on $V_{\mathcal{X}}$, such that, on $V_{\mathcal{X}}$, either*

$$P(\frac{d}{dt}) = J_1\begin{pmatrix} 1 & -a\frac{d}{dt} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b\frac{d}{dt} & 1 \end{pmatrix} J_2 \tag{8.18}$$

or

$$P(\frac{d}{dt}) = J_1\begin{pmatrix} 1 & 0 \\ -a\frac{d}{dt} - b\frac{d}{dt}^2 & 1 \end{pmatrix} J_2 \tag{8.19}$$

with

$$\textit{either}\quad J_2 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \qquad \textit{or}\quad J_2 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} . \tag{8.20}$$

Proof of proposition 8.2.9 : Although the ring of polynomials $\mathcal{C}^\infty(U)[\frac{\mathrm{d}}{\mathrm{dt}}]$ is not commutative, there is a left and right Euclidean division by polynomials whose leading coefficient does not vanish (this is because the leading coefficient of the product of two polynomials is computed as if the coefficients were constant). We also use the fact that the matrix formed with the coefficients of the terms of higher degree on each column cannot be invertible for an invertible matrix, except if it is a degree zero matrix.

For the case of degree 1, at points where not all leading coefficients vanish, there is an invertible matrix K_2 of degree zero (may be take either triangular or a permutation matrix) such that $P(\frac{\mathrm{d}}{\mathrm{dt}})K_2$ has its first column of degree zero. Then at points where not both terms of this column vanish, a Euclidean division yields a smooth function a such that $J_1 = P(\frac{\mathrm{d}}{\mathrm{dt}})K_2 \begin{pmatrix} 1 & -a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix}$ has degree zero. Take $J_2 = K_2^{-1}$. The open set U_0 is the set where the functions we had to divide by do not vanish.

For the case of degree 2, Let us distinguish different cases. In all cases, we have to divide by at most three polynomials, the points where they vanish without being zero on a neighborhood —if they are zero on an open set, then the corresponding polynomial has locally a smaller degree— is closed with empty interior, the open set U_0 is its complement.

- If both polynomials in the second column of $P(\frac{\mathrm{d}}{\mathrm{dt}})$ have degree zero, then, at any point, one of them at least does not vanish, and dividing by it the corresponding polynomial (degree 2) in the first column yields a degree two polynomial $-\alpha + a\frac{\mathrm{d}}{\mathrm{dt}} + b\frac{\mathrm{d}}{\mathrm{dt}}^2$ such that
$$j_1 = P(\frac{\mathrm{d}}{\mathrm{dt}}) \begin{pmatrix} 1 & 0 \\ -\alpha + a\frac{\mathrm{d}}{\mathrm{dt}} + b\frac{\mathrm{d}}{\mathrm{dt}}^2 & 1 \end{pmatrix}$$
has degree zero. This yields (8.19) with the second expression for J_2 in (8.20).
- If both polynomials in the second column of $P(\frac{\mathrm{d}}{\mathrm{dt}})$ have degree at most 1 but they are not both of degree zero, then, at any point where the leading coefficient of this one does not vanish, Euclidean division by this polynomial of the corresponding polynomial (degree 2) in the first column yields a degree one polynomial $-\alpha + b\frac{\mathrm{d}}{\mathrm{dt}}$ such that
$$P(\frac{\mathrm{d}}{\mathrm{dt}}) \begin{pmatrix} 1 & 0 \\ -\alpha + b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix}$$
has a first column of degree zero, and then dividing by a non-vanishing element of this first column yields a such that
$$J_1 = P(\frac{\mathrm{d}}{\mathrm{dt}}) \begin{pmatrix} 1 & 0 \\ -\alpha + b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} \begin{pmatrix} 1 & a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix}$$
has degree zero. This yields (8.18) with the second expression for J_2 in (8.20).
- If at least one of the polynomials in the second column of $P(\frac{\mathrm{d}}{\mathrm{dt}})$ has degree 2, then, at points where its leading coefficient does not vanish, dividing the corresponding polynomial in the first column by this coefficient yields a function λ such that
$$P(\frac{\mathrm{d}}{\mathrm{dt}}) \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}$$
has both entries in its second column of degree at most 1 (λ is identically zero if the first column of $P(\frac{\mathrm{d}}{\mathrm{dt}})$ had degree 1 or 0). Apply one of the two first cases to $P(\frac{\mathrm{d}}{\mathrm{dt}}) \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}$

instead of $P(\frac{\mathrm{d}}{\mathrm{dt}})$. This yields either (8.19) or (8.18), with the first expression for J_2 in (8.20). ∎

8.2.6 Two ways of writing the equations for the linearizing outputs

The most natural method for deciding if there exists some linearizing outputs depending on x and u is to write down the equations that a pair of functions has to satisfy in order to be a pair of linearizing outputs, and then to find conditions (on the system (8.5)) for these equations to have solutions. Let us describe these equations, but only for the case when the linearizing outputs are restricted to depend upon x only :

Proposition 8.2.10. *Suppose that X_1 and X_2 in (8.5) or (8.10) are linearly independent. Let $h_1(x)$ and $h_2(x)$ be smooth functions ; then (h_1, h_2) is a pair of linearizing outputs at a certain point if and only if*

$$rank \begin{pmatrix} \frac{\partial \dot h_1}{\partial u_1} & \frac{\partial \dot h_1}{\partial u_2} \\ \frac{\partial \dot h_2}{\partial u_1} & \frac{\partial \dot h_2}{\partial u_2} \end{pmatrix} \leq 1 \tag{8.21}$$

$$rank \begin{pmatrix} \frac{\partial \dot h_1}{\partial u_1} & \frac{\partial \dot h_1}{\partial u_2} & 0 & 0 \\ \frac{\partial \dot h_2}{\partial u_1} & \frac{\partial \dot h_2}{\partial u_2} & 0 & 0 \\ \frac{\partial \ddot h_1}{\partial u_1} & \frac{\partial \ddot h_1}{\partial u_2} & \frac{\partial \ddot h_1}{\partial \dot u_1} & \frac{\partial \ddot h_1}{\partial \dot u_2} \\ \frac{\partial \ddot h_2}{\partial u_1} & \frac{\partial \ddot h_2}{\partial u_2} & \frac{\partial \ddot h_2}{\partial \dot u_1} & \frac{\partial \ddot h_2}{\partial \dot u_2} \end{pmatrix} \leq 2 \tag{8.22}$$

on a neighborhood of this point, and the forms $\mathrm{d}h_1$, $\mathrm{d}h_2$, $\mathrm{d}\dot h_1$, $\mathrm{d}\dot h_2$, $\mathrm{d}\ddot h_1$, $\mathrm{d}\ddot h_2$ *are independent at this point.*

Proof : Let us prove necessity. If (h_1, h_2) is a pair of linearizing outputs, the six mentioned forms have to be independent by definition. If the rank in (8.21) was 2, it is clear that the only linear combinations of the $\mathrm{d}h_k^{(j)}$'s which would also be linear combinations of $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$, $\mathrm{d}x_4$, would have all their coefficients zero except the coefficients of $\mathrm{d}h_1$ and $\mathrm{d}h_2$, which would contradict the fact that $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$ and $\mathrm{d}x_4$ are linear combinations of the $\mathrm{d}h_k^{(j)}$'s. This proves that (8.21) is necessary. If the rank in (8.22) was 3 (cannot be 4 from (8.21)), the only linear combinations of the $\mathrm{d}h_k^{(j)}$'s which would also be also linear combinations of $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$, $\mathrm{d}x_4$, would be linear combinations of $\mathrm{d}h_1$, $\mathrm{d}h_2$ and $\lambda_1 \mathrm{d}\dot h_1 + \lambda_2 \mathrm{d}\dot h_2$ with the line $(\lambda_1\,,\,\lambda_2)$ in the right kernel of the matrix in (8.22), impossible from the fact that contradict the fact that $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$ and $\mathrm{d}x_4$ are independent linear combinations of the $\mathrm{d}h_k^{(j)}$'s. This proves that (8.22) is necessary. which are also linear combinations of $\mathrm{d}x_1$, $\mathrm{d}x_2$, Sufficiency follows from solving for $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$ and $\mathrm{d}x_4$ as linear combinations of $\mathrm{d}h_1$, $\mathrm{d}h_2$, $\mathrm{d}\dot h_1$, $\mathrm{d}\dot h_2$, $\mathrm{d}\ddot h_1$ and $\mathrm{d}\ddot h_2$. ∎

Conditions (8.21)-(8.22) are better related to the vector fields defining system (8.5) using :

$$\frac{\partial \ddot h_i}{\partial \dot u_k} = \frac{\partial \dot h_i}{\partial u_k} = L_{X_k} h_i \tag{8.23}$$

and

$$\frac{\partial \ddot h_i}{\partial u_k} = L_{X_0} L_{X_k} h_i + L_{X_k} L_{X_0} h_i + 2\, u_k L^2_{X_k} h_i + u_{k'} \left(L_{X_{k'}} L_{X_k} h_i + L_{X_k} L_{X_{k'}} h_i \right) \tag{8.24}$$

where $k' = 2$ if $k = 1$ and $k' = 1$ if $k = 2$.

The two equations (8.21)-(8.22) give a system of PDEs in h_1 and h_2 (some determinants being zero), and the independence condition an inequality (a nonzero determinant). These have solutions of and only if the system is x-dynamic linearizable.

Some similar conditions on functions of x and u may be written, and existence of solution would be equivalent for (x,u)-dynamic linearizability.

A different possibility is to use the material introduced in section 8.2.5 : under non-singularity conditions (being at a "Brunovský regular" point), there exists two differential forms such that $\{\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2\}$ (or $\{\omega_1, \dot{\omega}_1, \ddot{\omega}_1, \omega_2\}$ but let us consider the first case only) is a basis of $\mathrm{Span}\{\mathrm{d}\mathbf{x}\}$, these forms may be constructed explicitly, and, from proposition 8.2.8, the system is x-linearizable or (x,u)-dynamic linearizable if and only if there exists an invertible polynomial matrix such that

$$P(\frac{\mathrm{d}}{\mathrm{dt}})\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

is made of two exact one-forms, with some bounds on the degree of the entries of P. We then translate the fact that these forms are exact into a system of PDEs in the coefficients of the matrix, using the decomposition from proposition 8.2.9. The system is x-dynamic or (x,u)-dynamic linearizable if and only if these PDEs have solutions.

These two methods —writing directly the PDEs a pair of functions has to satisfy to be a pair of linearizing outputs or writing the PDEs the coefficients of the elementary matrices in the decomposition of P have to satisfy for the Pfaffian system $P(\frac{\mathrm{d}}{\mathrm{dt}})\,(\omega_1, \omega_2)^T$ to be integrable— are obviously equivalent, although they lead to different equations.

One drawback of the second method is that it only works at "Brunovský-regular" points, while Brunovský-regularity is not necessary for dynamic feedback linearization, see the example in section 8.5. Although Brunovský-regular points form an open dense set, one cannot neglect this weakness. Note however that in the example of section 8.5, we conclude even at points which are not Brunovský-regular, by density. In general, this second method seems to yield equations that may be considered more geometrically, and it proves to be very useful in our proofs.

For the simplest cases (cases 1 to 5 in theorem 8.3.1), we have used the first (direct) method, or even no particular method from these when we simply exhibit some pairs of linearizing outputs. Case 6 in theorem 8.3.1 is not elementary; it contains a necessary condition that we prove using the first (direct) method; the proof is natural; it would also be in a sense simpler using the infinitesimal Brunovský form, but this case would then be split into two because depending whether (8.52) holds or not, the infinitesimal Brunovský form is different, and points on the boundary are not Brunovský-regular while the present proof has no problem at these points. We give as an alternative a proof based on the infinitesimal Brunovský form, outside singularities (section 8.7.1). To test for (x,u)-linearizability, we were not able to use the direct method, and we *had to* use the second one based on infinitesimal Brunovský form. It turns out that the first one yield rather huge PDEs in the linearizing outputs, and we found no obvious way to handle them naturally as in the case of x-dynamic linearization, while the second one gives some PDEs that, though very heavy computations are needed, may be handled by elementary methods.

8.3 x-dynamic linearizability

We define the following distributions

$$\begin{aligned}
\Delta_2 &= \mathrm{Span}\{\, X_1\,,\, X_2\,\} \\
\mathcal{M}_0 &= \Delta_2 + [\Delta_2, \Delta_2] = \mathrm{Span}\{\, X_1\,,\, X_2\,,\, [X_1, X_2]\,\} \\
\mathcal{M}_1 &= \mathcal{M}_0 + [\mathcal{M}_0, \mathcal{M}_0] \\
&= \mathrm{Span}\{\, X_1\,,\, X_2\,,\, [X_1, X_2]\,,\, [X_1,[X_1, X_2]]\,,\, [X_2,[X_1, X_2]]\,\} \\
\Delta_3 &= \mathrm{Span}\{\, X_1\,,\, X_2\,,\, [X_1, X_2]\,,\, [X_0, X_1]\,,\, [X_0, X_2]\,\}
\end{aligned} \tag{8.25}$$

We will only study the situation in the neighborhood of points where the rank these distributions are constant, and the vector fields X_1 and X_2 are linearly independent and we define the integers m_0, m_1, δ_3 by :

$$\operatorname{rank}\Delta_2 = 2 \qquad \begin{array}{rcl} \delta_3 & \triangleq & \operatorname{rank}\Delta_3 \\ m_0 & \triangleq & \operatorname{rank}\mathcal{M}_0 \\ m_1 & \triangleq & \operatorname{rank}\mathcal{M}_1 \ . \end{array} \tag{8.26}$$

These ranks and the distributions in (8.25) are obviously feedback invariant from their definition and (8.7).

At a point where these ranks are constant, the only possible values for (m_0, m_1, δ_3) are $(2,2,2)$, $(2,2,3)$, $(2,2,4)$, $(3,3,3)$, $(3,3,4)$, $(3,4,3)$ and $(3,4,4)$. Actually, we will not distinguish between cases $(3,4,3)$ and $(3,4,4)$, so that when $(m_0, m_1) = (3,4)$, the rank of Δ_3 need not be constant.

The following theorem allows one, in each of the cases depending on the different possible values of the above ranks, to decide whether system (8.5) is x-dynamic linearizable or not. When it is not only x-dynamic linearizable, but static feedback linearizable, this is mentioned. In addition, for each case, we give a normal form for system (8.5) up to a nonsingular *static* feedback transformation (see (8.6)) and a change of coordinates. The proof is given in section 8.7.1. A small package written in Maple that makes the needed computations, as well as these corresponding to theorem 8.4.1 if needed, will soon be available from the author ; it is described in [65].

Theorem 8.3.1. *Let $\bar{x}$ be such that the distributions spanned by the modules Δ_2, $\mathcal{M}_0$, $\mathcal{M}_1$ and Δ_3 have constant rank in a neighborhood of $\bar{x}$, with Δ_2 of rank 2, as in (8.26). Actually, if $(m_0, m_1) = (3,4)$, we do not require that the rank of Δ_3 be constant.*

1. ***If $m_0 = m_1 = 2$ and $\delta_3 = 2$**, system (8.5) is locally **non accessible** and therefore **non linearizable by endogenous feedback**. Locally around $\bar{x}$, after a preliminary nonsingular feedback transformation and in appropriate coordinates, it has the following form, where a_1 and a_2 are smooth functions :*

$$\begin{array}{rcl} \dot{z}_1 & = & a_1(z_1, z_2) \\ \dot{z}_2 & = & a_2(z_1, z_2) \\ \dot{z}_3 & = & v_1 \\ \dot{z}_4 & = & v_2 \ . \end{array} \tag{8.27}$$

2. ***If $m_0 = m_1 = 2$ and $\delta_3 = 3$**, there are three sub-cases :*
   a) ***If Δ_3 is not involutive** (i.e. if there are points x arbitrarily close to $\bar{x}$ such that $[\Delta_3, \Delta_3](x) \not\subset \Delta_3(x)$, even if $[\Delta_3, \Delta_3](\bar{x}) \subset \Delta_3(\bar{x})$), system (8.5) is **not linearizable by endogenous dynamic feedback**. It has locally, around $\bar{x}$, after a preliminary nonsingular feedback transformation and in appropriate coordinates, the following form :*

$$\begin{array}{rcl} \dot{z}_1 & = & a(z_1, z_2, z_3) \\ \dot{z}_2 & = & z_3 \\ \dot{z}_3 & = & v_1 \\ \dot{z}_4 & = & v_2 \end{array} \tag{8.28}$$

   where a is a smooth function such that

$$\frac{\partial^2 a}{\partial z_3^2} \text{ is not identically zero on any neighborhood of } \bar{x}. \tag{8.29}$$

b) ***If*** $\boldsymbol{\Delta_3}$ ***involutive and the rank of*** $\boldsymbol{\Delta_3 + [X_0, \Delta_3]}$ ***is 3*** *in a neighborhood of* $\bar{x}$*, system (8.5) is locally* ***non accessible*** *and therefore* ***non linearizable by endogenous feedback****. Locally around* $\bar{x}$*, after a preliminary nonsingular feedback transformation and in appropriate coordinates, it has the following form, with a a smooth :*

$$\begin{array}{rcl} \dot{z}_1 &=& a(z_1) \\ \dot{z}_2 &=& z_3 \\ \dot{z}_3 &=& v_1 \\ \dot{z}_4 &=& v_2 \end{array} \tag{8.30}$$

c) ***If*** $\boldsymbol{\Delta_3}$ ***involutive and the rank of*** $\boldsymbol{\Delta_3 + [X_0, \Delta_3]}$ ***is 4*** *at point* $\bar{x}$ *(and therefore in a neighborhood), system (8.5) is locally* ***static feedback linearizable****. It has, after a preliminary nonsingular feedback transformation and in appropriate coordinates, the form (8.8.a).*

3. ***If*** $\boldsymbol{m_0 = m_1 = 2}$ ***and*** $\boldsymbol{\delta_3 = 4}$*, system (8.5) is locally* ***static feedback linearizable****. It has, after a preliminary nonsingular feedback transformation and in appropriate coordinates, the form (8.8.b).*

4. ***If*** $\boldsymbol{m_0 = m_1 = 3}$ ***and*** $\boldsymbol{\delta_3 = 3}$*, system (8.5) is locally* ***non accessible*** *and therefore* ***non linearizable by endogenous feedback****. Locally around* $\bar{x}$*, after a preliminary nonsingular feedback transformation and in appropriate coordinates, it has the following form, where* a_1 *and* a_3 *are smooth functions :*

$$\begin{array}{rcl} \dot{z}_1 &=& a_1(z_1) \\ \dot{z}_2 &=& v_1 \\ \dot{z}_3 &=& a_3(z_1, z_2, z_3, z_4) \ + \ z_4 v_1 \\ \dot{z}_4 &=& v_2 \ . \end{array} \tag{8.31}$$

5. ***If*** $\boldsymbol{m_0 = m_1 = 3}$ ***and*** $\boldsymbol{\delta_3 = 4}$*, system (8.5) is locally* x***-dynamic linearizable*** *at a point* $(\bar{x}, \bar{u}_1, \bar{u}_2, \ldots)$ *if and only if*

$$\begin{array}{r} rank_{\mathbb{R}} \{ X_1(\bar{x}) \, , \, X_2(\bar{x}) \, , \, [X_0, X_1](\bar{x}) \ - \ \bar{u}_2[X_1, X_2](\bar{x}) \, , \\ [X_0, X_2](\bar{x}) \ + \ \bar{u}_1[X_1, X_2](\bar{x}) \} \ = \ 4 \ . \end{array} \tag{8.32}$$

This condition is satisfied on an open dense set of any open set where $m_0 = m_1 = 3$ *and* $\delta_3 = 4$*.*

After a preliminary nonsingular feedback transformation and in appropriate coordinates, the system has the following form :

$$\begin{array}{rcl} \dot{z}_1 &=& z_2 \\ \dot{z}_2 &=& v_1 \\ \dot{z}_3 &=& a_3(z_1, z_2, z_3, z_4) \ + \ z_4 v_1 \\ \dot{z}_4 &=& v_2 \end{array} \tag{8.33}$$

with a is a smooth function. A possible choice of linearizing outputs is given, in these coordinates, by $h_1 = z_1$*,* $h_2 = z_3$*. Condition (8.32) reads :*

$$v_1 \ + \ \frac{\partial a_3}{\partial z_4} \ \neq \ 0 \ . \tag{8.34}$$

6. ***If*** $\boldsymbol{m_0 = 3}$ ***and*** $\boldsymbol{m_1 = 4}$*, there exists a unique (up to a nonzero multiplicative function) linear combination of* X_1 *and* X_2 *:* $\widetilde{X} = \lambda_1 X_1 + \lambda_2 X_2$ *such that*

$$[\widetilde{X}, [X_1, X_2]] \ \in \ \text{Span}\,\{X_1, X_2, [X_1, X_2]\} \tag{8.35}$$

(this is the characteristic vector field, or characteristic direction of the distribution spanned by the independent vector fields X_1, X_2 and $[X_1, X_2]$).
System (8.5) is ***x-dynamic linearizable at*** $(\bar{x}, \bar{u})$ ***if and only if***

$$[\,\widetilde{X}\,,\, X_0] \quad\in\quad \mathrm{Span}\,\{\, X_1\,,\, X_2\,,\, [X_1, X_2]\,\} \tag{8.36}$$

on a neighborhood of $\bar{x}$ *and*

$$rank_{\mathbb{R}}\,\{\, X_1(\bar{x})\,,\, X_2(\bar{x})\,,\, [X_0, \widetilde{X}](\bar{x}) + \bar{u}_1[X_1, \widetilde{X}](\bar{x}) + \bar{u}_2[X_2, \widetilde{X}](\bar{x})\,\} \;=\; 3 \tag{8.37}$$

$$\begin{aligned} rank_{\mathbb{R}}\,\{\, & X_1(\bar{x})\,,\, X_2(\bar{x})\,,\, [X_1, X_2](\bar{x})\,,\, [X_0, X_1](\bar{x})\,,\, [X_0, X_2](\bar{x})\,, \\ & [X_0, [X_1, X_2]](\bar{x}) + \bar{u}_1[X_1, [X_1, X_2]](\bar{x}) + \bar{u}_2[X_2, [X_1, X_2]](\bar{x})\,\} \;=\; 4\,. \end{aligned} \tag{8.38}$$

Given any open set in $\mathbb{R}^4 \times \mathbb{R}^2$ *such that for all* $(\bar{x}, \bar{u})$ *in this open set,* $(m_0, m_1) = (3, 4)$ *and (8.36) is satisfied at* $\bar{x}$*, the set of* $(\bar{x}, \bar{u})$*'s in this open set where (8.37) and (8.38) are satisfied is open and dense.*
These conditions may also be formulated using differential forms instead of vector fields. Since $m_0 = 3$*, one may take a —unique up to a nonzero multiplicative function— differential form in the four variables* x *only annihilating* X_1*,* X_2 *and* $[X_1, X_2]$ *:*

$$\omega_1 \quad\in\quad \{X_1, X_2, [X_1, X_2]\}^{\perp}\;; \tag{8.39}$$

then $\mathrm{d}\omega_1 \wedge \omega_1$ *is a form of degree 3 that does not vanish because* $m_1 = 4$*. System (8.5) is* ***x-dynamic linearizable at*** $(\bar{x}, \bar{u})$ ***if and only if***

$$\mathrm{d}\omega_1 \wedge \omega_1 \wedge \dot{\omega}_1 \quad=\quad 0 \tag{8.40}$$

on a neighborhood of $\bar{x}$ *and*

$$rank_{\mathbb{R}}\,\{\, \omega_1(\bar{x})\,,\, \eta_1(\bar{x})\,,\, \eta_2(\bar{x})\,,\, \dot{\eta}_1(\bar{x}, \bar{u})\,,\, \dot{\eta}_2(\bar{x}, \bar{u})\,\} \;=\; 5\,, \tag{8.41}$$

$$rank_{\mathbb{R}}\,\{\, \omega_1(\bar{x})\,,\, \dot{\omega}_1(\bar{x}, \bar{u})\,\} \;=\; 2\,, \tag{8.42}$$

where η_1 *and* η_2 *are forms of degree 1 such that, for a certain 1-form* Γ*,*

$$\mathrm{d}\omega_1 \quad=\quad \omega_1 \wedge \Gamma \;+\; \eta_1 \wedge \eta_2 \tag{8.43}$$

or in other words $\mathrm{d}\omega_1 \wedge \omega_1 = \eta_1 \wedge \eta_2 \wedge \omega_1$ *(*$\{\omega_1, \eta_1, \eta_2\}$ *is the characteristic system of* ω_1*, it is the annihilator of the vector field* $\widetilde{X}$ *defined in (8.35)), and the "dot" is the time-derivative along the system, i.e. the Lie derivative along the vector field* F *(8.10). The two conditions (8.41) and (8.42) are satisfied on an open dense set of any open set where* $m_0 = 3$ *and* $m_1 = 4$*.*
When these conditions are met, all pairs of linearizing outputs may be obtained as follows : take for h_1 *a first integral of the vector field* $\widetilde{X}$ *(i.e.* $L_{\widetilde{X}} h_1 = 0$*) such that* $\mathrm{d}h$*,* ω_1 *and* $\dot{\omega}_1$ *are linearly independent. Then the Pfaffian system* $\{\mathrm{d}h_1, \omega_1\}$ *is integrable. Take for* h_2 *a second first integral of this Pfaffian system.*
Around a point where $(m_0, m_1) = (3, 4)$*, after a preliminary static feedback transformation (8.6) and in appropriate coordinates, system (8.5) has the form :*

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= f_2(z_1, z_2, z_3, z_4) \;+\; z_3 v_1 \\ \dot{z}_3 &= f_3(z_1, z_2, z_3, z_4) \;+\; z_4 v_1 \\ \dot{z}_4 &= v_2 \end{aligned} \tag{8.44}$$

Condition (8.36) or (8.40) is equivalent to f_2 being independent of z_4 :

$$\frac{\partial f_2}{\partial z_4} \;=\; 0\;, \tag{8.45}$$

and conditions (8.37) and (8.38), or (8.41) and (8.42), translate into :

$$v_1 + \frac{\partial f_3}{\partial z_4} \;\neq\; 0 \tag{8.46}$$

and

$$\left(\, v_1 + \frac{\partial f_2}{\partial z_3}\;,\; f_3 - \frac{\partial f_2}{\partial z_1} - z_3\frac{\partial f_2}{\partial z_2} + z_4 v_1 \,\right) \;\neq\; (0,0) \tag{8.47}$$

at the point under consideration. A pair of linearizing outputs is, for instance, given by (z_1, z_2) at a point where $v_1 + \frac{\partial f_2}{\partial z_3}$ does not vanish, and by $(z_3, z_2 - z_1 z_3)$ at a point where $f_3 - \frac{\partial f_2}{\partial z_1} - z_3\frac{\partial f_2}{\partial z_2} + z_4 v_1$ does not vanish.

Note that this theorem does not say anything about the situation around points $\bar{x}$ where
– either one of the distributions spanned by Δ_2, $\mathcal{M}_0$ or $\mathcal{M}_1$ is singular,
– or they are regular, $(m_0, m_1) \neq (3, 4)$ and the distribution spanned by Δ_3 is singular,
– or $(m_0, m_1, \delta_3) = (2, 2, 3)$, the distribution spanned by Δ_3 —i.e. by $\{X_1,\ X_2,\ [X_0, X_1],\ [X_0, X_2]\}$ since $(m_0, m_1) = (2, 2)$— has rank 3 and is integrable, but the distribution spanned by $\{X_1, X_2, [X_0, [X_0, X_1]], [X_0, [X_0, X_2]]\}$ is singular.

8.4 (x, u)-dynamic linearizability

8.4.1 Problem statement

Let us examine the situations in which theorem 8.3.1 concludes that there exist no pair of linearizing outputs depending on x only for system (8.5), without ruling out existence of linearizing outputs depending on more variables (u, $\dot{u}$, $\ddot{u}$...). This occurs
– in case 5 when (8.32) fails,
– in case 6 when (8.36) fails,
– in case 6 when (8.36) is satisfied but (8.37) or (8.38) fails.
The first and third situations are singularities because (see theorem 8.3.1) in case 5, (8.32) is met on an open dense set, and in case 6 if (8.36) is satisfied, (8.37) or (8.38) are met on an open dense set. We will not study these two situations. The second situation does not correspond to a singularity since X_1, X_2, $[X_1, X_2]$, and $[\widetilde{X}, X_0]$ may very well be linearly independent (this is even generic) on an open set where $(m_0, m_1) = (3, 4)$. We shall study this situation in the present section. We make one more non-singularity assumption : we rule out the points where the rank of $X_1, X_2, [X_1, X_2], [\widetilde{X}, X_0]$ drops to 3 while being 4 at arbitrarily close points. Furthermore, the techniques that we will use require to be at a Brunovský-regular point (see definition 8.2.4). Brunovský-regularity translates into condition (8.51) below. It is clear that, on an open set where $(m_0, m_1) = (3, 4)$ and X_1, X_2, $[X_1, X_2]$, and $[\widetilde{X}, X_0]$ are linearly independent, Brunovský-regular points form an open and dense set. Hence Brunovský-regularity is one more non-singularity assumption. It is needed for technical reasons, but the example in section 8.5 shows that it is not necessary. To sum up :

Rank assumptions made all over the present section :
($\widetilde{X}$ is defined by (8.35))

$$\begin{aligned} \operatorname{rank}\{X_1\,,\,X_2\,\} &= 2 && (8.48)\\ \operatorname{rank}\{X_1\,,\,X_2\,,\,[X_1,X_2]\,\} &= 3 && (8.49)\\ \operatorname{rank}\{X_1\,,\,X_2\,,\,[X_1,X_2]\,,\,[X_1,[X_1,X_2]]\,,\,[X_2,[X_1,X_2]]\,\} &= 4 && (8.50)\\ \operatorname{rank}\{\,X_1\,,\,X_2\,,\,[X_1,X_2]\,,\,[X_0,\widetilde{X}]\} &= 4 && (8.51)\\ \operatorname{rank}\{X_1\,,\,X_2\,,\,[X_0,X_1]-u_2[X_1,X_2]\,,\,[X_0,X_2]+u_1[X_1,X_2]\,\} &= 4 && (8.52)\end{aligned}$$

From (8.48)-(8.49)-(8.50), we are in case 6 of theorem 8.3.1. (8.51) indicates that (8.36) does not hold, and hence from theorem 8.3.1, there exist no pair of linearizing outputs depending on x only, i.e. system (8.5) is not x-dynamic linearizable. The purpose of this section 8.4 is to characterize the cases where system (8.5) is (x,u)-dynamic linearizable, i.e. where there exists a pair of linearizing outputs depending on x and u (but not on $\dot u$, $\ddot u$.....).

8.4.2 Main result

Let us now proceed with some preparation for our characterization of (x,u)-dynamic linearizability. The following proposition provides a particular choice of ω_1 and ω_2 (basis of $\mathcal{H}_2$) such that the expressions of $\mathrm{d}\omega_1$ and $\mathrm{d}\omega_2$ are convenient and “canonical”.

Proposition 8.4.1. *Let $(\bar x,\bar u)$ be such that the rank conditions (8.48)-(8.49)-(8.50)-(8.51)-(8.52) are satisfied. Let ω_1 and ω_2 to be two differential forms of degree 1, linear combinations of* $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$, $\mathrm{d}x_4$, *such that none of these forms vanish at $(\bar x,\bar u)$ and*

$$\begin{aligned} \omega_1 &\in \{X_1,X_2,[X_1,X_2]\}^{\perp}\\ \omega_2 &\in \{X_1,X_2,[X_0+u_1X_1+u_2X_2,\widetilde{X}]\}^{\perp}\ . \end{aligned} \tag{8.53}$$

Then $\{\omega_1,\omega_2,\dot\omega_1,\dot\omega_2\}$ is a basis of $\operatorname{Span}\{\mathrm{d}x\,\}$ *and there exist uniquely defined functions $\delta^k_{i,j}$ and γ such that γ and $\delta^2_{1,2}$ do not vanish at $(\bar x,\bar u)$ and*

$$\mathrm{d}\omega_1 \;\equiv\; \delta^2_{1,2}\,\omega_2\wedge\dot\omega_2 \quad \textit{modulo } \omega_1\ , \tag{8.54}$$

$$\mathrm{d}\omega_2 \;\equiv\; \omega_1\wedge\left(\delta^1_{2,1}\dot\omega_1+\delta^2_{2,1}\dot\omega_2-\gamma\ddot\omega_2\right)\;+\;\gamma\,\dot\omega_1\wedge\dot\omega_2 \quad \textit{modulo } \omega_2\ . \tag{8.55}$$

Note that it is clear from (8.53) that, in general, ω_1 can be chosen so as to involve x only, but ω_2 involves x and u, i.e. it is a linear combination of $\mathrm{d}x_1,\mathrm{d}x_2,\mathrm{d}x_3,\mathrm{d}x_4$ with coefficients depending both on x and u. The functions γ and $\delta^k_{i,j}$ a priori depend on x, u and a certain number of time-derivatives of u.

Proof of proposition 8.4.1 : Suppose that ω_1 and ω_2 are chosen according to (8.53). Then (8.51) and (8.35) imply that the rank of $\{X_1,X_2,[X_1,X_2],[X_0+u_1X_1+u_2X_2,\widetilde{X}]\}$ is 4, and hence that $\{\omega_1,\omega_2\}$ is a basis of the annihilator of $\{X_1,X_2\}$.

The fact that ω_1 in the orthogonal of $\{X_1,X_2,[X_1,X_2]\}$ implies that it is in the first derived system of the Pfaffian system $\{\omega_1,\omega_2\}$ –see the Appendix– and hence that

$$\mathrm{d}\omega_1 \;=\; \omega_1\wedge\Gamma_{1,1} \;+\; \omega_2\wedge\Gamma_{1,2} \tag{8.56}$$

for some forms $\Gamma_{1,1}$ and $\Gamma_{1,2}$. Now the forms ω_1, ω_2 and $\Gamma_{1,2}$ must be linearly independent from (8.50), and then the Cartan characteristic system of $\{\omega_1\}$ is $\{\omega_1,\omega_2,\Gamma_{1,2}\}$ —see the Appendix (8.191)—, but, by definition of $\widetilde{X}$, this characteristic system is the annihilator of $\widetilde{X}$, and a basis of the annihilator of $\widetilde{X}$ is $\{\omega_1,\omega_2,\dot\omega_2\}$ because, from(8.15),

$$0 \;=\; \frac{\mathrm{d}}{\mathrm{dt}}\langle\omega_2,\widetilde{X}\rangle \;=\; \langle\dot\omega_2,\widetilde{X}\rangle \;+\; \langle\omega_2,[X_0+u_1X_1+u_2X_2,\widetilde{X}]\rangle$$

and hence $\langle \dot\omega_2, \widetilde{X}\rangle$ is zero; this proves that $\Gamma_{1,2}$ must be a linear combination of ω_1, ω_2 and $\dot\omega_2$, which, substituted in (8.56), yields (8.54) with $\delta^2_{1,2}$ does not vanish because ω_1, ω_2 and $\Gamma_{1,2}$ are linearly independent.

On the other hand, $\{\omega_1, \omega_2\}$ is the annihilator of $\{X_1, X_2\}$ and therefore has a basis that can be written with the variable x only; this implies —see (8.192) in the Appendix— that its characteristic system is at most Span$\{\mathrm{d}\mathbf{x}\}$; since $\{\omega_1, \omega_2, \dot\omega_1, \dot\omega_2\}$ is a basis of Span$\{\mathrm{d}\mathbf{x}\}$, this implies

$$\mathrm{d}\omega_2 \;=\; \omega_1\wedge\Gamma_{2,1} \;+\; \omega_2\wedge\Gamma_{2,2} \;+\; \gamma\,\dot\omega_1\wedge\dot\omega_2 \tag{8.57}$$

for some forms $\Gamma_{2,1}$ and $\Gamma_{2,2}$. But we have seen above that $\{\omega_1, \omega_2, \dot\omega_2\}$ is the Cartan characteristic system of $\{\omega_1\}$. It is therefore completely integrable, and this implies that $\mathrm{d}\dot\omega_2 \equiv 0$ modulo $\{\omega_1, \omega_2, \dot\omega_2\}$; but taking the time derivative of (8.57) yields $\mathrm{d}\dot\omega_2 \equiv \dot\omega_1 \wedge (\Gamma_{2,1} + \gamma\ddot\omega_2))$ modulo $\{\omega_1, \omega_2, \dot\omega_2\}$; $\Gamma_{2,1} \equiv -\gamma\ddot\omega_2$, which does imply, together with (8.57), the relation (8.55). ∎

We are now ready to state the theorem that characterizes (x,u)-linearizability. Its proof is given in section 8.7.2.

Theorem 8.4.1. *Let $(\bar x, \bar u)$ be a point where conditions (8.48)-(8.49)-(8.50)-(8.51)-(8.52) are met, and let the forms ω_1 and ω_2 be defined according to (8.53) and the functions $\delta^1_{2,1}$ and γ be defined by (8.55). System (8.5) is (x,u)-dynamically linearizable at point $\overline{\mathcal{X}} = (\bar x, \bar u, \dot{\bar u}, \ldots\ldots)$ if and only if the function $\delta^1_{2,1}$ —or equivalently the form of degree 5 $\mathrm{d}\omega_2\wedge\omega_2\wedge\dot\omega_2\wedge\ddot\omega_2$— does not vanish at $\overline{\mathcal{X}}$ and the first derived system of the Pfaffian system $\{\,\omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot\omega_2\,,\,\omega_2\}$ has rank 1 and is integrable, i.e. there exists a function α, defined on a neighborhood of $\overline{\mathcal{X}}$, such that*

$$\mathrm{d}\left(\omega_1 + \alpha\omega_2 - \frac{2\,\gamma}{\delta^1_{2,1}}\dot\omega_2\right)\;\wedge\;\left(\omega_1 + \alpha\omega_2 - \frac{2\,\gamma}{\delta^1_{2,1}}\dot\omega_2\right) \;=\; 0\;. \tag{8.58}$$

When these conditions are met, all the possible pairs of linearizing outputs depending on x and u may be described as follows. Let $\Omega_3 = \omega_1 + \alpha\omega_2 - \frac{2\gamma}{\delta^1_{2,1}}\dot\omega_2$, and $\dot\Omega_3$ be the time-derivative of this differential form (i.e. its Lie derivative along the dynamics F of the system). The Pfaffian system $\{\omega_2, \Omega_3, \dot\Omega_3\}$ is completely integrable. A pair of functions (h_1, h_2) depending on (x,u) is a pair of linearizing outputs if and only if $\{\mathrm{d}h_1, \mathrm{d}h_2\} \subset \{\omega_2, \Omega_3, \dot\Omega_3\}$ with $\Omega_3 \in \{\mathrm{d}h_1, \mathrm{d}h_2\}$ and $\dot\Omega_3 \notin \{\mathrm{d}h_1, \mathrm{d}h_2\}$. A possible construction is as follows : since $\mathrm{d}\Omega_3\wedge\Omega_3 = 0$, take h_1 such that $\mathrm{d}h_1$ does not vanish and $\mathrm{d}h_1 = k\Omega_3$ (k non-vanishing function); take for h_2 another first integral of $\{\omega_2, \Omega_3, \dot\Omega_3\}$ such that the coefficient of ω_2 when expressing $\mathrm{d}h_2$ as a linear combination of ω_2, Ω_3 and $\dot\Omega_3$ does not vanish (i.e. the rank of $\{\mathrm{d}h_2, \mathrm{d}h_1, \mathrm{d}\dot h_1\}$ does not drop to 2).

This theorem is stated in terms of the forms ω_1 and ω_2. These forms are only defined up to a non-vanishing multiplicative function by relation (8.53). However, the condition does not depend on the particular choice of ω_1 and ω_2. In a sense this is a consequence of the theorem itself since (x,u)-dynamic linearizability is clearly static feedback invariant and does not depend on the choice of ω_1 and ω_2, but the following proposition asserts that a priori these conditions *are* static feedback invariant.

Proposition 8.4.2. *The conditions of theorem 8.4.1 are invariant by static feedback and do not depend on the particular choice of ω_1 and ω_2 in (8.53). Indeed the Pfaffian system $\{\omega_2, \omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot\omega_2\}$ does not depend on this particular choice.*

Proof : It can be checked from (8.55) that if one changes ω_1 into $\lambda_1\omega_1$ and ω_2 into $\lambda_2\omega_2$, where λ_1 and λ_2 are non-vanishing functions, then $\delta^1_{2,1}$ is changed into $\frac{\lambda_2}{\lambda_1^2}\delta^1_{2,1}$ and γ into $\frac{1}{\lambda_1}\gamma$. This implies the proposition since (8.53) defines ω_1 and ω_2 up to a nonzero multiplicative function in a feedback invariant way. ∎

Let us make a remark on "singular" points, i.e. points where the ranks considered in (8.48)-(8.49)-(8.50)-(8.51)-(8.52) are not constant. We do not study the situation at these points, in particular at points which are not Brunovský-regular, i.e. points where the rank in (8.52) drops. As illustrated by the example in section 8.5, this singularity is usually not a singularity of (x,u)-dynamic linearization, but only of the proofs given here : the linearizing outputs are well defined at these points too, enjoy the property of being linearizing outputs. On the contrary, points where $\delta^1_{2,1}$, or the form $\mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2$, vanish are, according to the theorem, actual singularities of (x,u)-dynamic linearizability : in a domain where the rank assumptions (8.48)-(8.49)-(8.50)-(8.51)-(8.52) hold, there exists no linearizing outputs function of x and u in the neighborhood of a point where $\delta^1_{2,1}$ vanishes. It is interesting, with this respect, to notice that, under the —generic— assumptions (8.48)-(8.49)-(8.50)-(8.51)-(8.52), it is *impossible* to build an example where (x,u)-dynamic feedback linearization would be everywhere nonsingular since for any value of x and u, there is a value of $\dot{u}$ where $\delta^1_{2,1}$ vanishes.

8.4.3 How to check the conditions

We claim that the conditions of theorem 8.4.1 are completely explicit. Let us explain how to check them on a system (8.5) given by the expression of the vector fields X_0, X_1 and X_2 in some coordinates x_1, x_2, x_3, x_4 :

1. Compute ω_1 and ω_2 according to (8.53). This involves the computation of Lie brackets, and then finding the annihilator of some families of vectors, which in coordinates is common linear algebra (Gauss elimination).
2. Compute $\dot{\omega}_1$, $\dot{\omega}_2$ and $\ddot{\omega}_2$. The time-derivatives are Lie derivatives along the vector field (8.10).
3. To compute $\delta^1_{2,1}$ and γ, use the following identities, consequence of (8.55) :

$$\begin{aligned}
\mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2 &= \delta^1_{2,1}\, \omega_1 \wedge \dot{\omega}_1 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2 \\
\mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_1 \wedge \dot{\omega}_2 &= -\gamma\, \omega_1 \wedge \ddot{\omega}_2 \wedge \omega_2 \wedge \dot{\omega}_1 \wedge \dot{\omega}_2 \\
&= -\gamma\, \omega_1 \wedge \dot{\omega}_1 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2 \\
\mathrm{d}\omega_2 \wedge \omega_1 \wedge \omega_2 &= \gamma\, \dot{\omega}_1 \wedge \dot{\omega}_2 \wedge \omega_1 \wedge \omega_2 \ .
\end{aligned} \tag{8.59}$$

   Hence one may for instance compute the forms of degree 5 $\mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2$ and $\mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_1 \wedge \dot{\omega}_2$, check that the first one does not vanish, they appear to be of the form $\rho_1 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}u_1 + \rho_2 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}u_2$ and $\rho_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}u_1 + \rho_4 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}u_2$ respectively, with ρ_1, ρ_2, ρ_3 and ρ_4 some functions of x, u and $\dot{u}$, with $\rho_1\rho_4 - \rho_2\rho_3 = 0$, then

$$\frac{2\,\gamma}{\delta^1_{2,1}} = -\frac{2\,\rho_3}{\rho_1} = -\frac{2\,\rho_4}{\rho_2} \ .$$

4. The Pfaffian system $\{\, \omega_1 - \frac{2\gamma}{\delta^1_{2,1}} \dot{\omega}_2 \,,\, \omega_2 \}$ is then known
5. Use usual procedure to compute its first derived system : the forms $\mathrm{d}\left(\omega_1 - \frac{2\gamma}{\delta^1_{2,1}} \dot{\omega}_2\right)$ and $\mathrm{d}\omega_2$ must be proportional modulo $\{\, \omega_1 - \frac{2\gamma}{\delta^1_{2,1}} \dot{\omega}_2 \,,\, \omega_2 \}$; if it is the case, this yields α such that $\mathrm{d}\left(\omega_1 - \frac{2\gamma}{\delta^1_{2,1}} \dot{\omega}_2 + \alpha\omega_2\right)$ is zero modulo $\{\, \omega_1 - \frac{2\gamma}{\delta^1_{2,1}} \dot{\omega}_2 \,,\, \omega_2 \}$.

6. Check whether $\mathrm{d}\left(\omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2 + \alpha\omega_2\right)$ is also zero modulo $\omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2 + \alpha\omega_2$.

Note that a small package written in Maple that makes the above computations, as well as these corresponding to theorem 8.3.1, will soon be available from the author ; it is described in [65].

8.4.4 The result in particular coordinates

Let us now give a "normal form" for the systems we are studying in this section, i.e. these meeting conditions (8.48)-(8.49)-(8.50)-(8.51)-(8.52). It basically consists, as in "case 6" of theorem 8.3.1, in taking some coordinates (they exist from (8.48)-(8.49)-(8.50)) in which the control distribution is in "Engel's normal form", and use a feedback to annihilate two components of the drift, then the coordinates are slightly changed to emphasize condition (8.51) :

Proposition 8.4.3. *If the rank conditions (8.48)-(8.49)-(8.50)-(8.51)-(8.52) hold around a point $(\bar{x},\bar{u})$, there exists a system of coordinates around this point, and a static feedback defined around this point which give the following form to system (8.5) :*

$$\left.\begin{array}{rcl} \dot{z}_1 &=& v_1 \\ \dot{z}_2 &=& z_4 \;+\; z_3 v_1 \\ \dot{z}_3 &=& f(z_1,z_2,z_3,z_4) \;+\; g(z_1,z_2,z_3,z_4)v_1 \\ \dot{z}_4 &=& v_2 \end{array}\right\} \tag{8.60}$$

where

$$\frac{\partial g}{\partial z_4} \tag{8.61}$$

and

$$\begin{array}{rcl} D_1 &=& \dfrac{\partial g}{\partial z_4}(v_2 - f v_1) \;+\; z_4\dfrac{\partial g}{\partial z_2} \;+\; f\dfrac{\partial g}{\partial z_3} \\ && -\;\left(\dfrac{\partial f}{\partial z_1} + z_3\dfrac{\partial f}{\partial z_2} + g\dfrac{\partial f}{\partial z_3} + f\dfrac{\partial f}{\partial z_4}\right) \end{array} \tag{8.62}$$

do not vanish at $(\bar{x},\bar{u})$.

Proof of proposition 8.4.3 : From lemma 8.7.4 (section 8.7.1), using the feedback (8.119) yields the (8.44). Condition (8.51) implies that $\frac{\partial f_2}{\partial z_4}$ does not vanish. One may therefore take as new coordinates $(z_1,z_2,z_3,f_2(z_1,z_2,z_3,z_4))$ instead of (z_1,z_2,z_3,z_4), and this yields the normal form (8.60), changing also v_2. Relations (8.61) are simply a translation of (8.50) and (8.52). ∎

Proposition 8.4.4. *System (8.60) —which is system (8.5) written in appropriate coordinates— is (x,u)-dynamic linearizable around a point $\overline{\mathcal{X}}$ if and only if the functions f and g have, in a neighborhood of $\overline{\mathcal{X}}$, the form*

$$f \;=\; \frac{a_0 \;+\; a_1 z_4 \;+\; a_2 z_4^2}{c_0 \;+\; c_1 z_4} \quad ; \quad g \;=\; \frac{b_0 \;+\; b_1 z_4}{c_0 \;+\; c_1 z_4} \tag{8.63}$$

where a_0, a_1, a_2, b_0, b_1, c_0 and c_1 are functions of z_1,z_2,z_3 only, which satisfy the following PDE :

$$\mathrm{d}\Gamma\wedge\Gamma \;=\; 0 \quad \textit{with} \quad \Gamma \;=\; (b_1 - z_3 a_2)\,\mathrm{d}z_1 \;+\; a_2\mathrm{d}z_2 \;-\; c_1\mathrm{d}z_3 \tag{8.64}$$

and $\delta^1_{2,1}$ does not vanish at this point ($c_0 \;+\; c_1 z_4$ should obviously not vanish either).

Remarks :

1- The system of PDEs (8.64) reads :

$$z_3\left(c_1\frac{\partial a_2}{\partial z_2}-a_2\frac{\partial c_1}{\partial z_2}\right)+c_1\frac{\partial a_2}{\partial z_1}-a_2\frac{\partial c_1}{\partial z_1}+b_1\frac{\partial c_1}{\partial z_2}-c_1\frac{\partial b_1}{\partial z_2}-a_2\frac{\partial b_1}{\partial z_3}+b_1\frac{\partial a_2}{\partial z_3}+{a_2}^2\;=\;0 \qquad (8.65)$$

2- There is an explicit formula for $\delta^1_{2,1}$ using the a_i, b_i and c_i but it is quite long, and does not really matter here.

This proposition gives a simple way to check whether the system is (x,u)-dynamic linearizable provided one has found coordinates where it is in the normal form (8.60) —of course finding these coordinates involves solving some linear PDEs, so that the really explicit test is given by theorem 8.4.1 which only involves some differentiations, and some algebraic manipulations—. Actually, the coordinates in which a given system meeting conditions (8.48)-(8.49)-(8.50)-(8.51)-(8.52) is in the form (8.60) are not unique, and the expression of f and g, for the same system, may depend on the choice of coordinates, among all these that yield a form like (8.60)). Naturally, the fact that these f and g meet or not the conditions of the proposition does not depend on this choice. It however raises the question of finding, among all the coordinates that produce a normal form like (8.60), these which produce the "simplest" f and g. Let us give an answer only for the special case when the conditions of the proposition are met (i.e. in the (x,u)-linearizable case). It is obvious that if f and g are affine in z_4 (special case of (8.63) : $a_2=c_1=0$, $c_0=1$), the PDE (8.58) is met, because Γ is simply $b_1\mathrm{d}z_1$; it turns out that the converse is true : if f and g are not affine, but of the form (8.63) with $a_2\neq 0$ or $c_1\neq 0$, and with the PDE (8.58), then some "better" coordinates may be found, in which f and g are affine in the fourth coordinate :

Proposition 8.4.5. *There exists coordinates where the system, after a static feedback transformation, is in the form (8.60) with f and g satisfying the conditions of proposition 8.4.4, if and only if there is another set of coordinates $(\zeta_1,\zeta_2,\zeta_3,\zeta_4)$, and another static feedback transformation which yields a normal form (8.60) with f and g affine with respect to the fourth coordinate :*

$$\left.\begin{array}{rcl}\dot\zeta_1&=&w_1\\ \dot\zeta_2&=&\zeta_4\;+\;\zeta_3\,w_1\\ \dot\zeta_3&=&p_0(\zeta_1,\zeta_2,\zeta_3)+\zeta_4p_1(\zeta_1,\zeta_2,\zeta_3)\;+\;\left(q_0(\zeta_1,\zeta_2,\zeta_3)+\zeta_4q_1(\zeta_1,\zeta_2,\zeta_3)\right)w_1\\ \dot\zeta_4&=&w_2\end{array}\right\} \qquad (8.66)$$

and $\delta^1_{2,1}$ does not vanish if and only if the following quantity does not vanish :

$$\begin{array}{l}q_1\dot w_1\;+\;w_1\left(p_1+w_1q_1\right)^2\;+\;w_1\frac{\partial}{\partial\zeta_1}\left(p_1+w_1q_1\right)\\ \quad-\;\frac{\partial}{\partial\zeta_2}\left[(p_0+w_1q_0)-\zeta_3w_1(p_1+w_1q_1)\right]\;-\;\left(p_1+w_1q_1\right)^2\frac{\partial}{\partial\zeta_3}\frac{p_0+w_1q_0}{p_1+w_1q_1}\;.\end{array} \qquad (8.67)$$

In these coordinates, a pair of linearizing outputs is given by
$h_1=\zeta_1$, $h_2=\zeta_3-(p_1-w_1q_1)\,\zeta_2$.

Proof of proposition 8.4.5 : The expression (8.67) is obtained by computing $\mathrm{d}\omega_2\wedge\omega_2\wedge\dot\omega_2\wedge\ddot\omega_2$ and checking that it vanishes if and only if (8.67) vanishes, at least at points where (8.52) holds, i.e. where $\omega_1\wedge\omega_2\wedge\dot\omega_2\neq 0$. This is left to the reader. Use the simplest choice :

$$\begin{array}{rcl}\omega_1&=&\mathrm{d}\zeta_2-\zeta_3\mathrm{d}\zeta_1\\ \omega_2&=&\mathrm{d}\zeta_3-q_1(\zeta_1,\zeta_2,\zeta_3)\mathrm{d}\zeta_1\;-\;\left(p_1(\zeta_1,\zeta_2,\zeta_3)+w_1q_1(\zeta_1,\zeta_2,\zeta_3)\right)\omega_1\;.\end{array}$$

The "if" part of the proposition is obvious because, as noticed just above the proposition, (8.66) is a particular case of (8.60)-(8.58), and (8.67) ensures that $\delta^1_{2,1}\neq 0$. Let us prove the "only if" part. We suppose that the conditions of proposition 8.4.4 hold, and we build an invertible transformation $(z_1,z_2,z_3,z_4)\mapsto(\zeta_1,\zeta_2,\zeta_3,\zeta_4)$, and an invertible static feedback transformation $(z_1,z_2,z_3,z_4,v_1,v_2)\mapsto(z_1,z_2,z_3,z_4,w_1,w_2)$, that transforms (8.60) into (8.66). Condition (8.64) implies that there exists a function $\psi_1(z_1,z_2,z_3)$ and a non-vanishing function $k(z_1,z_2,z_3)$ such that

$$\mathrm{d}\psi_1 \;=\; k\,\Gamma\;. \tag{8.68}$$

Now, ω_1 may be chosen $\omega_1=\mathrm{d}z_2-z_3\mathrm{d}z_1$ and then Γ defined in (8.64) is also equal to : $\Gamma=b_1\mathrm{d}z_1+a_2\omega_1-c_1\mathrm{d}z_3$. Since the rank of $\{\mathrm{d}z_1,\mathrm{d}z_3,\omega_1\}$ is 3 and b_1 and c_1 do not vanish simultaneously (this would cause $\frac{\partial g}{\partial z_4}$ to vanish), the rank of $\{\omega_1,\Gamma\}$ is locally constant, equal to 2, and this Pfaffian system is therefore completely integrable, because these two forms involve only three variables (z_1, z_2, z_3) ; hence there exists three functions ψ_2, k', k'', such that

$$\mathrm{d}\psi_2 \;=\; k'\omega_1\;+\;k''\Gamma\;,\quad k'\neq 0\;\;. \tag{8.69}$$

Let us then define

$$\begin{aligned} w_1 \;&=\; \dot{\psi}_1 \;=\; k\,\langle\,\Gamma\;,\;X_0\;+\;u_1X_1\;+\;u_2X_2\,\rangle \\ &=\; k\,\frac{(c_0b_1-c_1b_0)v_1\;-\;c_1a_0\;+\;(c_0a_2-a_1c_1)z_4}{c_0\;+\;c_1z_4}\;. \end{aligned} \tag{8.70}$$

From this equation, one may express v_1 as a function of w_1. Substituting v_1 for this expression in (8.60)-(8.63), one obtains the following expressions for $\dot z_1$, $\dot z_2$, $\dot z_3$, which are now linear with respect to z_4 :

$$\dot z_1 \;=\; \frac{1}{c_0b_1-c_1b_0}\left(\frac{c_0+c_1z_4}{k}w_1\;+\;c_1a_0\;+\;(a_1c_1-c_0a_2)z_4\right) \tag{8.71}$$

$$\dot z_2 \;=\; \frac{1}{c_0b_1-c_1b_0}\left(\frac{z_3}{k}(c_0+c_1z_4)w_1\;+\;z_3c_1a_0\;+\;(c_0b_1-c_1b_0+a_1c_1-c_0a_2)z_4\right) \tag{8.72}$$

$$\dot z_3 \;=\; \frac{b_0+b_1z_4}{k\,(c_0b_1-c_1b_0)}\,w_1\;+\;a_0b_1\;+\;(a_1b_1-a_2b_0)z_4 \tag{8.73}$$

Let us then define

$$\zeta_1 \;=\; \psi_1(z_1,z_2,z_3) \tag{8.74}$$

$$\zeta_2 \;=\; \psi_2(z_1,z_2,z_3) \tag{8.75}$$

$$\zeta_3 \;=\; \frac{k''(z_1,z_2,z_3)}{k(z_1,z_2,z_3)} \tag{8.76}$$

$$\zeta_4 \;=\; k'(z_1,z_2,z_3)\,z_4 \tag{8.77}$$

Let us see that in these coordinates, and with w_1 given by (8.70), we have (8.66) :

- $\dot\zeta_1=w_1$ is a consequence of (8.74) and (8.70),
- From (8.75), $\dot\zeta_2=\langle\,\mathrm{d}\psi_2\,,\,X_0+u_1X_1+u_2X_2\,\rangle$, which is also equal, from (8.70) and (8.69), to $\frac{k''}{k}w_1+k'\langle\,\omega_1\,,\,X_0+u_1X_1+u_2X_2\,\rangle$, which, since $\langle\,\omega_1\,,\,X_0+u_1X_1+u_2X_2\,\rangle=z_4$, and considering (8.76) and (8.77), yields $\dot\zeta_2=\zeta_4+\zeta_3w_1$.
- In the expressions for $\dot z_1$, $\dot z_2$ and $\dot z_3$ given by (8.71), (8.72) and (8.73), all the functions of (z_1,z_2,z_3) may be expressed as functions of $(\zeta_1,\zeta_2,\zeta_3)$, and z_4 may be substituted for $\frac{\zeta_4}{k'}$ (see (8.77)) ; therefore, $\dot z_1$, $\dot z_2$ and $\dot z_3$ are polynomials in ζ_4 and w_1 with coefficients function of $(\zeta_1,\zeta_2,\zeta_3$ with one term of degree zero, one term of degree 1 in ζ_4, one term of degree 1 in w_1 and one term of degree 2 in ζ_4w_1 ; since ζ_3 is a function of (z_1,z_2,z_3), $\dot\zeta_3$ is also such a polynomial, which allows one to define functions p_o, p_1, q_0 and q_1 such that $\dot\zeta_3$ is as in (8.66).
- $\dot\zeta_4$ is equal to $k'(\zeta_1,\zeta_2,\zeta_3)v_2$ plus some terms which depend only on ζ_1, ζ_2, ζ_3, ζ_4 and v_1. Since k' does not vanish, calling all this expression w_2 defines a nonsingular feedback that yields the required form.

∎

8.5 An example

Let us consider the following system, which is given as example 2 in [23] :

$$\begin{array}{rcl} \dot{x}_1 &=& x_2 + x_3 u_2 \\ \dot{x}_2 &=& x_3 + x_1 u_2 \\ \dot{x}_3 &=& u_1 + x_2 u_2 \\ \dot{x}_4 &=& u_2 \,. \end{array} \tag{8.78}$$

The transformation $z_1 = x_4$, $z_2 = x_2$, $z_3 = x_1$, $z_4 = x_3$, $v_1 = u_2$, $v_2 = u_1 + x_2 u_2$ puts it into the form (8.66), known to be (x,u)-dynamic linearizable. Let us however follow the general method. We have :

$$\widetilde{X} = X_1 = \tfrac{\partial}{\partial x_3}\,, \quad X_2 = x_3 \tfrac{\partial}{\partial x_1} + x_1 \tfrac{\partial}{\partial x_2} + x_2 \tfrac{\partial}{\partial x_3} + \tfrac{\partial}{\partial x_4}\,, \quad [X_1, X_2] = \tfrac{\partial}{\partial x_1}\,,$$

$$[X_0, \widetilde{X}] = -\tfrac{\partial}{\partial x_2}\,, \quad [X_0 + u_1 X_1 + u_2 X_2, \widetilde{X}] = -\tfrac{\partial}{\partial x_2} - u_2 \tfrac{\partial}{\partial x_1}\,.$$

Brunovský-regular points are points where (8.52) holds, i.e. points where

$$x_1 - u_1 \neq 0 \,. \tag{8.79}$$

The simplest choice for ω_1 and ω_2 is (see (8.53)) :

$$\begin{array}{rcl} \omega_1 &=& \mathrm{d}x_2 - x_1 \mathrm{d}x_4 \,, \\ \omega_2 &=& \mathrm{d}x_1 - u_2 \mathrm{d}x_2 + (u_2 x_1 - x_3)\mathrm{d}x_4 \,. \end{array} \tag{8.80}$$

By expressing $\mathrm{d}\omega_2 = -\mathrm{d}u_2 \wedge \mathrm{d}x_2 + \mathrm{d}(u_2 x_1 - x_3) \wedge \mathrm{d}x_4$ in the basis $\{\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2, \ddot{\omega}_2\}$ (at points where (8.79) holds), with

$$\begin{array}{rcl} \dot{\omega}_1 &=& \mathrm{d}x_3 + u_2 \mathrm{d}x_1 - (x_2 + u_2 x_3)\mathrm{d}x_4 \\ \dot{\omega}_2 &=& -u_2^2 \mathrm{d}x_1 + (1 - \dot{u}_2)\mathrm{d}x_2 + (-u_1 + x_3 u_2^2 + x_1 \dot{u}_2)\mathrm{d}x_4 \,. \\ \ddot{\omega}_2 &=& (x_1 - u_1)\mathrm{d}u_2 + (\cdots)\mathrm{d}x_1 + (\cdots)\mathrm{d}x_2 + (\cdots)\mathrm{d}x_3 + (\cdots)\mathrm{d}x_4 \,, \end{array} \tag{8.81}$$

one obtains an expression like (8.55) with :

$$\delta^1_{2,1} = \frac{2\,(\dot{u}_2 + u_2^3 - 1)}{x_1 - u_1} \quad ; \quad \gamma = -\frac{1}{x_1 - u_1} \,, \tag{8.82}$$

so that $\delta^1_{2,1} \neq 0$ is equivalent to $\dot{u}_2 + u_2^3 - 1 \neq 0$. Then the form $\omega_3 = \omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2$ may be explicitly computed. $\mathrm{d}\omega_2 \wedge \omega_2 \wedge \omega_3$ and $\mathrm{d}\omega_3 \wedge \omega_2 \wedge \omega_3$ are collinear :

$$\mathrm{d}\omega_3 \wedge \omega_2 \wedge \omega_3 = -\alpha\, \mathrm{d}\omega_2 \wedge \omega_2 \wedge \omega_3 \quad \text{with} \quad \alpha = \frac{u_2^2}{\dot{u}_2 + u_2^3 - 1} \,. \tag{8.83}$$

A basis of the derived system of $\{\omega_2, \omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2\}$ is therefore

$$\Omega_3 = \omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2 + \alpha\omega_2 = \frac{x_1 - u_1}{\dot{u}_2 + u_2^3 - 1}\mathrm{d}x_4 \,. \tag{8.84}$$

It is obviously integrable, condition (8.58) of theorem 8.4.1 is satisfied, hence the system is (x,u)-dynamic linearizable at points where $\delta^1_{2,1}$ does not vanish. Since x_4 is a first integral of $\{\Omega_3\}$, and a basis (at points where (8.78) holds) for the Pfaffian system $\{\omega_2, \Omega_3, \dot{\Omega}_3\}$ is $\{\mathrm{d}x_1 - u_2 \mathrm{d}x_2, \mathrm{d}x_4, \mathrm{d}u_2\}$ —it is indeed integrable, and three independent first integrals are x_4,

u_2 and $x_1 - x_2u_2$—, theorem 8.4.1 implies that two functions $(h_1(x,u), h_2(x,u))$ form a pair of linearizing outputs if and only if h_1 and h_2 are two independent functions of x_4, u_2 and $x_1 - x_2u_2$ such that $\mathrm{d}h_1, \mathrm{d}h_2, \mathrm{d}u_2$ are independent but $\mathrm{d}x_4$ is a linear combination of $\mathrm{d}h_1$ and $\mathrm{d}h_2$. The simplest choice is

$$h_1 = x_4 \quad ; \quad h_2 = x_1 - u_2x_2 \ . \tag{8.85}$$

Let us illustrate on this example the invertible transformations on pairs of differential forms introduced in section 8.2.5 (following [4, 3]). The functions h_1 and h_2 given by (8.85) are related to the forms ω_1 and ω_2 defining the "infinitesimal Brunovský form" by :

$$\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} = \begin{pmatrix} 1 & -x_2\frac{\mathrm{d}}{\mathrm{dt}} - (u_2x_1 - x_3) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\dot{u}_2 + u_2^3 - 1}{x_1 - u_1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{8.86}$$

with $b = \frac{2\gamma}{\delta_{2,1}^1}$ (this may be re-arranged into an expression like (8.143) with some scalar function a and matrix function J_1). Indeed from Proposition 8.2.3 and 8.2.6, and since (ω_1, ω_2) is a linearizing Pfaffian system and the matrices in the right-hand side of (8.86) are all invertible, this is enough to prove that (h_1, h_2) is a pair of linearizing outputs at Brunovský-regular points. Note that the expressions in (8.86) are indeed singular at "Brunovský-singular points" —points that are not Brunovský-regular— so that the ideas based on the infinitesimal Brunovský form fail at these points, while linearizing outputs h_1 and h_2 may obviously be continued at these points, and it may be checked directly that they continue to be linearizing outputs at these points ; indeed, since

$$\begin{array}{rclrcl} \dot{h}_1 & = & u_2\,, & \dot{h}_2 & = & x_2 - x_1u_2^2 - x_2\dot{u}_2\,, \\ \ddot{h}_1 & = & \dot{u}_2 & \ddot{h}_2 & = & x_3 + x_1u_2 - x_2u_2^2 - x_3u_2^3 \\ h_1^{(3)} & = & \ddot{u}_2\,, & & & -(x_3 + x_1u_2)\dot{u}_2 - x_2\ddot{u}_2\,, \end{array} \tag{8.87}$$

One may solve for x_1, x_2, x_3, x_4, u_2, $\dot{u}_2$ and $\ddot{u}_2$ in (8.85)-(8.87) and express them as (rational) functions of $h_1, \dot{h}_1, \ddot{h}_1, h_1^{(3)}, h_2, \dot{h}_2, \ddot{h}_2$ at all points where $\dot{u}_2 + u_2^3 - 1 \neq 0$. It is clear on this example that the requirement of Brunovský-regularity is purely technical, and the singularities of dynamic feedback linearization are not related to the singularities of the "infinitesimal Brunovský form". The singularity $\delta_{2,1}^1 = 0$, on the other hand is really a singularity of (x,u)-dynamic linearization.

The conclusion for this system is :

- It is not x-dynamic linearizable at any point, as a consequence of theorem 8.3.1, case 6.
- It is (x,u)-dynamic linearizable at all points where $\dot{u}_2 + u_2^3 - 1 \neq 0$.
  This is a consequence of theorem 8.4.1 at points where $x_1 - u_1 \neq 0$. At points where $\dot{u}_2 + u_2^3 - 1 \neq 0$ and $x_1 - u_1 = 0$, it is not a consequence of theorem 8.4.1, but is clear from (8.87).
- It is not (x,u)-dynamic linearizable at points where $\dot{u}_2 + u_2^3 - 1 = 0$.
  This is a consequence of theorem 8.4.1 at points where $x_1 - u_1 \neq 0$. At points where $x_1 - u_1 = \dot{u}_2 + u_2^3 - 1 = 0$, this is not a consequence of theorem 8.4.1, but may be proved as follows. Suppose that there is a pair of linearizing outputs (h_1, h_2) in an open neighborhood of such a point. Points where $x_1 - u_1 \neq 0$ are dense on this neighborhood, and (h_1, h_2) is still a pair of linearizing outputs at these points (if the neighborhood is small enough). Hence (see above) h_1 and h_2 are functions of x_4, u_2 and $x_1 - u_2x_2$: $h_i(x_1, x_2, x_3, x_4, u_1, u_2) = \chi_i(x_4, u_2, x_1 - u_2x_2)$. Because the rank of $\mathrm{d}x_4, \mathrm{d}u_2\mathrm{d}(x_1 - u_2x_2)$ is 3, the smooth functions χ_i are unique and may be prolonged at the point under consideration (where $u_1 - x_1$ vanishes). Computing the time-derivatives of the functions h_i from these identities, it can be seen that their partial derivative with respect to x_2 all vanish at points where $\dot{u}_2 + u_2^3 - 1 = 0$. This prevents x_2 from being, around such a point, a smooth function of

$h_1, h_2, \dot{h}_1, \dot{h}_2, \ddot{h}_1, \ddot{h}_2, \ldots\ldots$, and hence (h_1, h_2) from being a pair of linearizing outputs at these points.

Note that the singularity $\dot{u}_2 + u_2^3 - 1 = 0$ does not correspond to a singularity of the linear approximation. Consider for instance the solution

$$u_1(t) = -1\ , \quad u_2(t) = 1\ , \quad x_1(t) = x_2(t) = 1\ , \ x_3(t) = -1\ , \ x_4(t) = t\ .$$

Clearly $\dot{u}_2 + u_2^3 - 1$ is zero along this solution, while the linear approximation $\delta\dot{x} = A\delta x + B\delta u$, with

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} ,$$

is controllable. An example where this occurs at an equilibrium instead of a nontrivial solution is obtained by replacing x_2 with x_4 in $\dot{x}_1$, the singularity $\delta^1_{2,1} = 0$ then occurs when $\dot{u}_2 + u_2^3 = 0$ while the linear approximation at $(x, u) = (0, 0)$ is controllable.

8.6 Non-affine systems in $\mathbb{R}^3$

Consider a system

$$\dot{\xi} = f(\xi, w_1, w_2) \tag{8.88}$$

where ξ lives in $\mathbb{R}^3$. A system of the form (8.5) can always be brought to this form at a point where one of the control vector fields does not vanish by finding coordinates in which this control vector field is the first coordinate vector field, dropping the corresponding control and taking this first coordinate as a new control. The converse is not correct in general.

However a necessary condition for feedback linearization, that can be found in [88] or in [96] implies that if system (8.88) linearizable by dynamic feedback (even in a more general sense than endogenous), it has a dynamic extension of dimension 4 which is affine in the control. The following proposition is a consequence of theorem 1 in [88], except the regularity of γ, but this is automatic if one wants the linearizing outputs to be smooth :

Proposition 8.6.1 ([88]). *At a point $(\bar{\xi}, \bar{w}_1, \bar{w}_2)$ where rank$\{\frac{\partial f}{\partial w_1}, \frac{\partial f}{\partial w_2}\}$ is 2, a necessary condition for system (8.88) to be dynamic feedback linearizable is that there exist, locally around $(\bar{\xi}, \bar{w}_1, \bar{w}_2)$, a static feedback transformation $(w_1, w_2) = \gamma(\xi, v_1, v_2)$ such that $f(\xi, \gamma(\xi, v_1, v_2))$ be affine with respect to v_1 : $f(\xi, \gamma(\xi, v_1, v_2)) = a(\xi, v_2) + v_1 b(\xi, v_2)$.*

In the case of system (8.88), an explicit condition for existence of this static feedback transformation may be given, but this is outside the scope of the present paper. It is clear that the necessary condition for dynamic linearization given in proposition 8.6.1 is exactly the condition needed to transform system (8.88) into an affine 4-dimensional system. This is summed up in the following result, which allows one to apply to 3-dimensional non-affine systems (8.88) all the results obtained in the previous sections for 4-dimensional affine systems.

Proposition 8.6.2. *At a point $(\bar{\xi}, \bar{w}_1, \bar{w}_2)$ where rank$\{\frac{\partial f}{\partial w_1}, \frac{\partial f}{\partial w_1}\}$ is 2, either system (8.88) is not dynamic feedback linearizable or one may construct a static feedback transformation $(w_1, w_2) = \gamma(\xi, v_1, v_2)$ such that dynamic feedback linearization of (8.88) is equivalent to dynamic feedback linearization of*

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} &= a(x_1, x_2, x_3, x_4) \ + \ u_1\, b(x_1, x_2, x_3, x_4) \\ \dot{x}_4 &= u_2\ . \end{aligned} \tag{8.89}$$

8.7 Proofs

All over these proofs, some known facts about Pfaffian systems (derived systems, characteristic system...) are used. They are briefly recalled in the Appendix.

8.7.1 Proof of theorem 8.3.1

Case 1 ($\mathbf{m_0 = m_1 = 2}$, $\delta_3 = 2$)

$m_0 = 2$ means that the distribution spanned by the control vector fields X_1 and X_2 is involutive. Frobenius theorem yields a set of coordinates (z_1, z_2, z_3, z_4) such that $\{\frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4}\}$ is a basis of this distribution, then

$$\begin{array}{rcl} v_1 & = & L_{X_0} z_3 \;+\; u_1 L_{X_1} z_3 \;+\; u_2 L_{X_2} z_3 \\ v_2 & = & L_{X_0} z_4 \;+\; u_1 L_{X_1} z_4 \;+\; u_2 L_{X_2} z_4 \end{array}$$

is a nonsingular static feedback because X_1 and X_2 are independent at point $\bar{x}$. System (8.5) reads, in the above coordinates as

$$\begin{array}{rclcrcl} \dot{z}_1 & = & a_1(z_1, z_2, z_3, z_4) & \qquad & \dot{z}_3 & = & v_1 \\ \dot{z}_2 & = & a_2(z_1, z_2, z_3, z_4) & \qquad & \dot{z}_4 & = & v_2 \;. \end{array}$$

Δ_3 is then spanned by $\frac{\partial}{\partial z_3}$, $\frac{\partial}{\partial z_4}$, $\frac{\partial a_1}{\partial z_3}\frac{\partial}{\partial z_1} + \frac{\partial a_2}{\partial z_3}\frac{\partial}{\partial z_2}$ and $\frac{\partial a_1}{\partial z_4}\frac{\partial}{\partial z_1} + \frac{\partial a_2}{\partial z_4}\frac{\partial}{\partial z_2}$. $\delta_3 = 2$ implies that a_1 and a_2 do not depend on z_3 and z_4. This yields (8.27).

Case 2.a ($\mathbf{m_0 = m_1 = 2}$, $\delta_3 = 3$)

Since $\{X_1, X_2\}$ is integrable of rank 2, there exists two independent functions constant along X_1 and X_2, and one of them at least has either its Lie derivative along $[X_0, X_1]$ or its Lie derivative along $[X_0, X_2]$ that does not vanish at $\bar{x}$ because if not the rank of Δ_3 would drop to two ; let z_2 be this one, and z_1 be the other one, and define $z_3 = L_{X_0} z_2$. $L_{X_1} z_3$ or $L_{X_2} z_3$ does not vanish at $\bar{x}$ (because they are equal to $L_{[X_1,X_0]} z_2$ and $L_{[X_2,X_0]} z_2$) and hence z_3 is independent from z_1 and z_2, let z_4 be a fourth function, such that (z_1, z_2, z_3, z_4) is a system of coordinates. The nonsingular feedback

$$\begin{array}{rcl} v_1 & = & L^2_{X_0} z_2 \;+\; u_1 L_{X_1} L_{X_0} z_2 \;+\; u_2 L_{X_2} L_{X_0} z_2 \\ v_2 & = & L_{X_0} z_4 \;+\; u_1 L_{X_1} z_4 \;+\; u_2 L_{X_2} z_4 \end{array} \tag{8.90}$$

transforms system (8.5) into

$$\begin{array}{rcl} \dot{z}_1 & = & a(z_1, z_2, z_3, z_4) \\ \dot{z}_2 & = & z_3 \\ \dot{z}_3 & = & v_1 \\ \dot{z}_4 & = & v_2 \;, \end{array} \tag{8.91}$$

with a a certain smooth function. Since Δ_3 spans a distribution of rank 3 and :

$$\Delta_3 \;\; = \;\; \mathrm{Span}\{ \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4}, \frac{\partial}{\partial z_2} + \frac{\partial a}{\partial z_3}\frac{\partial}{\partial z_1}, \frac{\partial a}{\partial z_4}\frac{\partial}{\partial z_1} \} \;,$$

the function a cannot depend on z_4, and then

$$\Delta_3 + [\Delta_3, \Delta_3] \;\; = \;\; \mathrm{Span}\{ \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4}, \frac{\partial}{\partial z_2} + \frac{\partial a}{\partial z_3}\frac{\partial}{\partial z_1}, \frac{\partial^2 a}{\partial z_3^2}\frac{\partial}{\partial z_1} \}$$

so that the assumption on Δ_3 is equivalent to $\frac{\partial^2 a}{\partial z_3^2}$ being identically zero on no neighborhood of $\bar{x}$. This proves that system (8.5) has the form (8.28) with the condition (8.29), after the change of coordinates and the nonsingular feedback transformation we just introduced. There remains to prove that system (8.28) cannot be linearizable by endogenous feedback under condition (8.29). This is a consequence of the following lemma 8.7.1 because if system (8.28) was linearizable by endogenous feedback on a neighborhood of a point $\bar{x}$, then there would exist a pair of linearizing outputs on a neighborhood of this point, and hence the system would also be linearizable by endogenous feedback around any point of that neighborhood, including these, given by condition (8.29), where $\frac{\partial^2 a}{\partial z_3^2}$ is non zero.

Lemma 8.7.1. *System (8.28) is not linearizable by endogenous dynamic feedback in any neighborhood of a point $\bar{\bar{z}} = (\bar{\bar{z}}_1, \bar{\bar{z}}_2, \bar{\bar{z}}_3, \bar{\bar{z}}_4)$ such that*

$$\frac{\partial^2 a}{\partial z_3^2}(\bar{\bar{z}}_1, \bar{\bar{z}}_2, \bar{\bar{z}}_3) \neq 0 .$$

Proof of lemma 8.7.1 : Suppose that there exists two linearizing outputs h_1 and h_2, smooth functions of a finite number of variables among z_1, z_2, z_3, z_4, v_1, v_2, $\dot{v}_1$, $\dot{v}_2$, $\ddot{v}_1$, $\ddot{v}_2$, ... $v_1^{(L)}$, $v_2^{(L)}$, with L a non negative integer, defined on an open subset $O \subset \mathbb{R}^{2L+6}$ containing a point $(\bar{\bar{z}}_1, \bar{\bar{z}}_2, \bar{\bar{z}}_3, \bar{\bar{z}}_4, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_1^{(L)}, \bar{v}_2^{(L)})$ for some $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_1^{(L)}, \bar{v}_2^{(L)})$. All variables may be recovered from h_1, h_2 and all their time derivatives so that in particular there exists smooth functions ψ_1 and ψ_2 such that

$$z_1 = \psi_1(h_1, \dot{h}_1, \ldots, h_1^{(K_{1,1})}, h_2, \dot{h}_2, \ldots, h_2^{(K_{1,2})}) \tag{8.92}$$

$$z_2 = \psi_2(h_1, \dot{h}_1, \ldots, h_1^{(K_{2,1})}, h_2, \dot{h}_2, \ldots, h_2^{(K_{2,2})}) . \tag{8.93}$$

This holds in the open set O, which may be restricted so that $\frac{\partial^2 a}{\partial z_3^2}(z_1, z_2, z_3)$ does not vanish on O. The integer $K_{i,j}$ is the one such that ψ_i does not depend on $h_j^{(K_{i,j}+1)}$, but does depend on $h_j^{(K_{i,j})}$ on O, i.e. $\frac{\partial \psi_i}{\partial h_j^{(K_{i,j})}}$ is not identically zero. Then, since (8.28) implies $\dot{z}_1 = a(z_1, z_2, \dot{z}_2)$, one has, by substitution,

$$\begin{aligned} & \frac{\partial \psi_1}{\partial h_1}\dot{h}_1 + \cdots + \frac{\partial \psi_1}{\partial h_1^{(K_{1,1})}} h_1^{(K_{1,1}+1)} + \frac{\partial \psi_1}{\partial h_2}\dot{h}_2 + \cdots + \frac{\partial \psi_1}{\partial h_2^{(K_{1,2})}} h_2^{(K_{1,2}+1)} \\ = \;& a(\, \psi_1 \, , \, \psi_2 \, , \, \frac{\partial \psi_2}{\partial h_1}\dot{h}_1 + \cdots + \frac{\partial \psi_2}{\partial h_1^{(K_{2,1})}} h_1^{(K_{2,1}+1)} + \frac{\partial \psi_2}{\partial h_2}\dot{h}_2 + \cdots + \frac{\partial \psi_2}{\partial h_2^{(K_{2,2})}} h_2^{(K_{2,2}+1)} \,) . \end{aligned} \tag{8.94}$$

One must have

$$K_{1,1} = K_{2,1} \quad , \qquad K_{1,2} = K_{2,2} \tag{8.95}$$

because the left-hand side in (8.94) depends only on $h_1, \dot{h}_1, \ldots, h_1^{(K_{1,1}+1)}, h_2, \dot{h}_2, \ldots,$ $h_2^{(K_{1,2}+1)}$ and does depend on $h_1^{(K_{1,1}+1)}$ and $h_2^{(K_{2,1}+1)}$, and the right-hand side depends only on $h_1, \dot{h}_1, \ldots,$ $h_1^{(K_{2,1}+1)}, h_2, \dot{h}_2, \ldots, h_2^{(K_{2,2}+1)}$ and does depend on $h_1^{(K_{2,1}+1)}$ and $h_2^{(K_{2,2}+1)}$ because, since $\frac{\partial^2 a}{\partial z_3^2}$ does not vanish on O, $\frac{\partial a}{\partial z_3}$ is not identically zero on any open subset of O.

Differentiating two times both sides of (8.94) with respect to $h_j^{(K_{1,j}+1)}$, and keeping in mind that, from (8.95), $K_{1,j} = K_{2,j}$, one has (note that neither ψ_1 nor ψ_2 nor the partial derivatives of them depend on $h_j^{(K_{1,j}+1)}$) :

$$\begin{aligned} 0 = \left(\frac{\partial \psi_2}{\partial h_j^{(K_{1,j})}}\right)^2 \frac{\partial^2 a}{\partial z_3^2}(\, \psi_1 \, , \, \psi_2 \, , \, \frac{\partial \psi_2}{\partial h_1}\dot{h}_1 + \cdots + \frac{\partial \psi_2}{\partial h_1^{(K_{1,1})}} h_1^{(K_{1,1}+1)} + \frac{\partial \psi_2}{\partial h_2}\dot{h}_2 + \cdots \\ \cdots + \frac{\partial \psi_2}{\partial h_2^{(K_{1,2})}} h_2^{(K_{1,2}+1)} \,) , \end{aligned} \tag{8.96}$$

for $j \in \{1,2\}$, and hence $\frac{\partial \psi_2}{\partial h_j^{(K_1,j)}}$ is identically zero on O which contradicts the fact that it was precisely chosen (small enough) not to be identically zero on O. ∎

Case 2.b ($\mathbf{m_0 = m_1 = 2}$, $\boldsymbol{\delta_3 = 3}$)

Since Δ_3 is integrable of rank 3, and $\{X_1, X_2\}$ is integrable of rank 2, and contained in Δ_3, there are two independent functions z_1 and z_2 such that z_1 and z_2 are constant along X_1 and X_2 and z_1 constant along the vector fields of Δ_3. Let z_3 be given by $z_3 = L_{X_0} z_2$ and z_4 be such that (z_1, z_2, z_3, z_4) is a system of coordinates. The nonsingular feedback (8.90) transforms system (8.5) into a system of the form (8.91) above, where a depends on z_1 only because, since $L_{X_1} z_1 = L_{X_2} z_1 = 0$, one has $a = \dot z_1 = L_{X_0} z_1$, and $L_{X_0} z_1$ is constant along Δ_3 because z_1 is and $[X_0, \Delta_3] \subset \Delta_3$. $\dot z_1 = a(z_1)$ clearly implies non-accessibility.

Case 2.c ($\mathbf{m_0 = m_1 = 2}$, $\boldsymbol{\delta_3 = 3}$)

Static feedback linearizability follows from classical results, see [57, 50]. Let us however describe the coordinates in which the system has the form (8.8.a). Since Δ_3 is integrable of rank 3, there is a function z_1 such that $\mathrm{d}z_1$ is the annihilator of Δ_3. Let z_2 and z_3 be given by $z_2 = L_{X_0} z_1$ and $z_3 = L^2_{X_0} z_1$, the rank of $\{\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3\}$ is 3 because $\delta_3 = 3$. Let z_4 be any function such that $\{z_1, z_2, z_3, z_4\}$ is a system of coordinates. The nonsingular feedback

$$\begin{array}{rcl} v_1 &=& L^3_{X_0} z_1 \;+\; u_1 L_{X_1} L^2_{X_0} z_1 \;+\; u_2 L_{X_2} L^2_{X_0} z_1 \\ v_2 &=& L_{X_0} z_4 \;+\; u_1 L_{X_1} z_4 \;+\; u_2 L_{X_2} z_4 \end{array}$$

transforms system (8.5) into (8.8.a).

Case 3 ($\mathbf{m_0 = m_1 = 2}$, $\boldsymbol{\delta_3 = 4}$)

As in case 2.c, static feedback linearization follows from classical results, see [57, 50], but we however describe the coordinates in which the system has the form (8.8.b). Because $m_0 = 2$, X_1 and X_2 span an integrable distribution of rank 2, let z_1 and z_3 be two independent functions that annihilate X_1 and X_2, and let z_2 and z_4 be defined by $z_2 = L_{X_0} z_1$ and $z_4 = L_{X_0} z_3$. $\delta_3 = 4$ implies that (z_1, z_2, z_3, z_4) is a system of coordinates, and the following nonsingular feedback

$$\begin{array}{rcl} v_1 &=& L^2_{X_0} z_1 \;+\; u_1 L_{X_1} L_{X_0} z_1 \;+\; u_2 L_{X_2} L_{X_0} z_1 \\ v_2 &=& L^2_{X_0} z_3 \;+\; u_1 L_{X_1} L_{X_0} z_3 \;+\; u_2 L_{X_2} L_{X_0} z_3 \end{array}$$

transforms system (8.5) into (8.8.b).

Cases 4 and 5 ($\mathbf{m_0 = m_1 = 3}$, $\boldsymbol{\delta_3 = 3}$ or 4)

Since $m_0 = m_1 = 3$, $\mathcal{M}_0 = \mathcal{M}_1$ spans an integrable distribution of rank 3. Let z_1 be a first integral of this distribution. In case 5 ($\delta_3 = 4$), define z_2 by

$$z_2 \;=\; L_{X_0} z_1 \; . \tag{8.97}$$

One then has, for $i \in \{1,2\}$, $L_{X_i} z_2 = -L_{[X_0, X_i]} z_1$ because $L_{X_i} z_1 \equiv 0$, $i = 1, 2$, and hence $\delta_3 = 4$ prevents $L_{X_1} z_2$ and $L_{X_2} z_2$ from both vanishing at $\bar x$. Up to a permutation of the two controls, we may suppose that

$$L_{X_1} z_2(\bar x) \;\neq\; 0 \; . \tag{8.98}$$

In case 4 ($\delta_3 = 3$), pick any z_2 such that (8.98) holds, it is possible since X_1 does not vanish. Since $L_{X_1} z_1 = 0$, the rank of $\{\mathrm{d}z_1, \mathrm{d}z_2\}$ is 2 at point $\bar{x}$. The vector field

$$(L_{X_2} z_2)\, X_1 \;-\; (L_{X_1} z_2)\, X_2 \tag{8.99}$$

does not vanish at point $\bar{x}$, z_1 and z_2 are two independent functions constant along it, let z_3 be a third independent first integral of this vector field, and z_4 be given by

$$z_4 \;=\; \frac{L_{X_1} z_3}{L_{X_1} z_2}\ .$$

(z_1, z_2, z_3, z_4) is a system of coordinates because z_1, z_2 and z_3 are constant along the vector field (8.99) while the Lie derivative of z_4 along it does not vanish at $\bar{x}$ (a simple computation shows that if it would vanish, the rank of $\mathcal{M}_0$ would drop to 2). Defining v_1 and v_2 according to the nonsingular feedback transformation

$$\begin{array}{rcl} v_1 &=& L_{X_0} z_2 \;+\; u_1 L_{X_1} L_{X_0} z_1 \;+\; u_2 L_{X_2} L_{X_0} z_1 \\ v_2 &=& L_{X_0} z_4 \;+\; u_1 L_{X_1} z_4 \;+\; u_2 L_{X_2} z_4 \end{array}$$

(with a possible permutation of the indices 1 and 2 in the right-hand sides, if needed to get (8.98)) yields, in the above defined coordinates, the normal form (8.31) in case 4, and (8.33) in case 5. In both cases, a_3 is given by

$$a_3 \;=\; L_{X_0} z_3 \;-\; \frac{L_{X_1} z_3}{L_{X_1} z_2} L_{X_0} z_2\ ,$$

$\dot{z}_3$ is obtained because

$$L_{X_2} z_3 \;=\; \frac{L_{X_2} z_2}{L_{X_1} z_2} L_{X_1} z_3\ ,$$

and (in case 4) $a_1 = L_{X_0} z_1$ depends only on z_1 because $\delta_3 = 3$ implies that $\Delta_3 = \mathcal{M}_0$ and hence that $L_{X_0} z_1$ is a first integral of the three dimensional integrable distribution spanned by $\mathcal{M}_0$.

In case 4, non-accessibility follows immediately from the normal form (8.31). In case 5, let us prove that system (8.33) is x-dynamic linearizable around $(\bar{z}, \bar{v})$ if and only if $\frac{\partial a}{\partial z_4}(\bar{z}) + \bar{v}_1 \neq 0$. Let (h_1, h_2) be a pair of linearizing outputs, depending on z only.

Lemma 8.7.2. *Let h_1, h_2 be two functions depending on z only such that (h_1, h_2) is a pair of linearizing outputs for system (8.33) on a neighborhood of $(\bar{z}, \bar{v})$.*
Then the rank of $\{\mathrm{d}z_1, \mathrm{d}h_1, \mathrm{d}h_2\}$ is 2 on a neighborhood of $\bar{z}$.

Proof : If it was not the case, there would be points $\bar{\bar{z}}$, arbitrarily close to $\bar{z}$, where this rank would be 3, and where (h_1, h_2) would still be a pair of linearizing outputs. z_1 is constant along both control vector fields, and since (h_1, h_2) would still be a pair of linearizing outputs, there is, from (8.21)-(8.23), a nonzero linear combination of X_1 and X_2, say Z, along which both h_1 and h_2 are constant. It is impossible that $L_{X_i} h_j$ vanishes at $\bar{\bar{x}}$ for all $i, j \in \{1, 2\}$, so that up to a permutation, we may suppose that $L_{X_1} h_1 \neq 0$. This yields, following the same construction as above —construction of coordinates where the system has form (8.33)— a set of coordinates

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \;=\; (z_1, h_1, h_2, \frac{L_{X_1} h_2}{L_{X_1} h_1})$$

and a nonsingular feedback $w_1 = \dot{h}_1$, $w_2 = \dot{\zeta}_4$ such that the system is also of the form (8.33) with ζ instead of z and w instead of v :

$$\begin{array}{rclcrcl} \dot{\zeta}_1 &=& \zeta_2 & \qquad & \dot{\zeta}_3 &=& a_3(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \;+\; \zeta_4 w_1 \\ \dot{\zeta}_2 &=& w_1 & & \dot{\zeta}_4 &=& w_2 \end{array}$$

where (ζ_2, ζ_3) should be a pair of linearizing outputs. This is impossible from (8.22) because

$$\begin{pmatrix} \frac{\partial \dot\zeta_2}{\partial w_1} & \frac{\partial \dot\zeta_2}{\partial w_2} & 0 & 0 \\ \frac{\partial \dot\zeta_3}{\partial w_1} & \frac{\partial \dot\zeta_3}{\partial w_2} & 0 & 0 \\ \frac{\partial \ddot\zeta_2}{\partial w_1} & \frac{\partial \ddot\zeta_2}{\partial w_2} & \frac{\partial \ddot\zeta_2}{\partial \dot w_1} & \frac{\partial \ddot\zeta_2}{\partial \dot w_2} \\ \frac{\partial \ddot\zeta_3}{\partial w_1} & \frac{\partial \ddot\zeta_3}{\partial w_2} & \frac{\partial \ddot\zeta_3}{\partial \dot w_1} & \frac{\partial \ddot\zeta_3}{\partial \dot w_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \zeta_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\partial a_3}{\partial \zeta_2} + \zeta_4 \frac{\partial a_3}{\partial \zeta_3} + w_2 & \frac{\partial a_3}{\partial \zeta_4} + w_1 & \zeta_4 & 0 \end{pmatrix}$$

and hence $\frac{\partial a_3}{\partial \zeta_4} + w_1$ should be identically zero on an open set, which is absurd because its derivative with respect to w_1 is 1. ∎

From this lemma 8.7.2, z_1 is a function of the two linearizing functions, and therefore one may replace h_1 or h_2 by z_1 in (h_1, h_2) and still have a pair of linearizing outputs. Let for instance $h_1 = z_1$, then (8.21) is automatically satisfied, and (8.22) implies that h_2 must depend on z_1, z_2, z_3 only because

$$\begin{pmatrix} \frac{\partial \dot h_1}{\partial v_1} & \frac{\partial \dot h_1}{\partial v_2} & 0 & 0 \\ \frac{\partial \dot h_2}{\partial v_1} & \frac{\partial \dot h_2}{\partial v_2} & 0 & 0 \\ \frac{\partial \ddot h_1}{\partial v_1} & \frac{\partial \ddot h_1}{\partial v_2} & \frac{\partial \ddot h_1}{\partial \dot v_1} & \frac{\partial \ddot h_1}{\partial \dot v_2} \\ \frac{\partial \ddot h_2}{\partial v_1} & \frac{\partial \ddot h_2}{\partial v_2} & \frac{\partial \ddot h_2}{\partial \dot v_1} & \frac{\partial \ddot h_2}{\partial \dot v_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & \frac{\partial h_2}{\partial z_4} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ * & * & * & \frac{\partial h_2}{\partial z_4} \end{pmatrix} ,$$

and the independence condition in proposition 8.2.10 implies (8.34). Conversely, if (8.34) is satisfied, system (8.33) is x-dynamic linearizable with (z_1, z_3) as a pair of linearizing outputs, because z_2 is $\dot z_1$, and z_4 is (inverse function theorem) a function of $\dot z_3$, z_1, z_2, z_3, v_1, i.e. of $\dot z_3$, z_1, $\dot z_1$, z_3, $\ddot z_1$.

Case 6 ($\mathbf{m_0 = 3,\ m_1 = 4}$)

Let us first clarify the correspondence between the conditions in terms of differential forms and these in terms of vector fields. Since the form ω_1 is defined by (8.39) and involves only the four variables x, $\mathrm{d}\omega_1$ must be of the form (8.43) because there is only four variables. Let us prove that, as written just after (8.43),

$$\widetilde{X} \ \in\ \{\omega_1, \eta_1, \eta_2\}^\perp \ . \tag{8.100}$$

From the definitions of ω_1 and $\widetilde{X}$, one has $\langle \omega_1, [\widetilde{X}, Y]\rangle = 0$ for $Y = X_1$ and for $Y = X_2$ and for $Y = [X_1, X_2]$, but one also has $\langle \omega_1, \widetilde{X}\rangle = \langle \omega_1, Y\rangle = 0$ for these Y's, and hence, from the classical formula [100, II-(1.10)] linking Lie Bracket and exterior derivative, $\langle \omega_1, [\widetilde{X}, Y]\rangle = 0$ implies $\mathrm{d}\omega_1(\widetilde{X}, Y) = 0$, but from (8.43), and using again $\langle \omega_1, \widetilde{X}\rangle = \langle \omega_1, Y\rangle = 0$, this reads : $\langle \eta_1, \widetilde{X}\rangle\langle \eta_2, Y\rangle - \langle \eta_1, Y\rangle\langle \eta_2, \widetilde{X}\rangle = 0$. Since the three vectors $X_1, X_2, [X_1, X_2]$ are linearly independent ($m_0 = 3$) and the two differential forms η_1 and η_2 are also linearly independent ($m_1 = 4$ implies that $\mathrm{d}\omega_1 \wedge \omega_1 = \eta_1 \wedge \eta_2 \wedge \omega_1$ does not vanish), the last equality implies $\langle \eta_1, \widetilde{X}\rangle = \langle \eta_2, \widetilde{X}\rangle = 0$, and this proves (8.100).

We then have the following

Lemma 8.7.3. *Condition (8.36) is equivalent to condition (8.40). If (8.36) or (8.40) holds, condition (8.37) is equivalent to condition (8.41) and condition (8.38) is equivalent to condition (8.42).*

Proof of lemma 8.7.3 : From the definition of ω_1, (8.36) may be written $\langle \omega_1, [X_0, \widetilde{X}]\rangle = 0$, or also $\langle \omega_1, [F, \widetilde{X}]\rangle = 0$ because $[F, \widetilde{X}] = [X_0, \widetilde{X}] + u_1[X_1, \widetilde{X}] + u_2[X_2, \widetilde{X}]$ and the last two terms vanish on ω_1. From the classical identity (8.15) and the fact that $\langle \omega_1, \widetilde{X}\rangle$ is zero, $\langle \omega_1, [F, \widetilde{X}]\rangle = 0$

is equivalent to $\langle \dot{\omega}_1, \widetilde{X} \rangle = 0$, which is equivalent, from (8.100) to $\dot{\omega}_1$ being a linear combination of ω_1, η_1 and η_2. This is (8.40).

Let us proceed to prove that (8.37) is equivalent to (8.41) if (8.40) holds. Consider the three vector fields $\widetilde{X}$, $\frac{\partial}{\partial u_1}$, $\frac{\partial}{\partial u_2}$ in the six variables x, u. Their annihilator is $\{\omega_1, \eta_1, \eta_2\}$. Now consider the six vector fields obtained by adding the Lie brackets of these by F : $\{\widetilde{X}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, [F, \widetilde{X}], [F, \frac{\partial}{\partial u_1}], [F, \frac{\partial}{\partial u_2}]\}$. From the classical identity (8.15), a form ω annihilates all these at a point if and only if ω and $\dot{\omega}$ annihilate the original three at this point, i.e. if and only if both ω and $\dot{\omega}$ are linear combinations of ω_1, η_1 and η_2 at this point. It is the case of ω_1 because (8.40) holds, and the rank of these six vector fields therefore cannot be more than 5 ; it is equal to 5 exactly at points where the time-derivative of any linear combination of η_1 and η_2 is linearly independent from ω_1, η_1 and η_2, i.e. at points where (8.41) holds. Now this rank is 5 exactly at points where (8.37) holds because $[F, \frac{\partial}{\partial u_i}] = X_i$ and therefore these six vector fields have the same rank as :

$$\{X_1, X_2, [F, \widetilde{X}], \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\} .$$

Let us proceed to prove that (8.38) is equivalent to (8.42) if (8.40) holds. Consider the five vector fields $\{X_1, X_2, [X_1, X_2], \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\}$ in the six variables x, u. Their annihilator is $\{\omega_1\}$. Now consider the ten vector fields obtained by adding the Lie brackets of these by F :

$$\{X_1, X_2, [X_1, X_2], \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, [F, X_1], [F, X_2], [F, [X_1, X_2]], [F, \frac{\partial}{\partial u_1}], [F, \frac{\partial}{\partial u_2}]\} .$$

A form that annihilates all these vector fields at a point must be collinear to ω_1 at this point because it has to annihilate at least the five original ones. The form ω_1 vanishes on all these vector fields exactly at the point where ω_1 *and* $\dot{\omega}_1$ vanish on the five original vector fields, i.e. (since these five are linearly independent) exactly at points where the rank of $\{\omega_1, \dot{\omega}_1\}$ drops to 1. Therefore, the rank of the ten vector fields is 6 at points where (8.42) holds, and 5 at points where it does not hold. But these ten vector have the same rank as :

$$\{X_1, X_2, [X_1, X_2], \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, [X_0, X_1], [X_0, X_2], [X_0 + u_1 X_1 + u_2 X_2, [X_1, X_2]]\},$$

and this has rank 6 if and only if (8.38) holds. ∎

Let us now prove necessity of the conditions (8.36)-(8.37)-(8.38), or (8.40)-(8.41)-(8.42). Lemma 8.7.3, that we have now proved, allows us to simply prove that (8.36) is necessary first, and then to prove that (8.41) and (8.42) are necessary.

Suppose that there exists a pair of linearizing outputs (h_1, h_2) with h_1 and h_2 depending on x only. We use conditions (8.21) and (8.22) from proposition 8.2.10 to derive the necessary condition (8.36). We have

$$\dot{h}_i \;=\; L_{X_0} h_i \;+\; u_1 L_{X_1} h_i \;+\; u_2 L_{X_2} h_i \; . \tag{8.101}$$

Equation (8.21) implies that the rank of

$$\begin{pmatrix} L_{X_1} h_1 & L_{X_2} h_1 \\ L_{X_1} h_2 & L_{X_2} h_2 \end{pmatrix}$$

is one. Since $m_1 = 4$, the functions $L_{X_1} h_1$ and $L_{X_2} h_1$ cannot vanish together ; without loss of generality, suppose that $L_{X_1} h_1$ does not vanish at the point under consideration. Then, with λ the function given by $\lambda = L_{X_2} h_1 / L_{X_1} h_1$, and defining the vector field Z_2 by

$$Z_2 \;=\; X_2 \;-\; \lambda\, X_1 \tag{8.102}$$

one has

$$L_{Z_2}h_1 \;=\; L_{Z_2}h_2 \;=\; 0\,. \tag{8.103}$$

Define the vector fields Z_0 and Z_1 by

$$Z_0 \;=\; X_0 - \frac{L_{X_0}h_1}{L_{X_1}h_1}\,X_1\;, \qquad Z_1 \;=\; \frac{1}{L_{X_1}h_1}\;X_1\,. \tag{8.104}$$

The systems (8.5) then reads

$$\dot{x} \;=\; Z_0 \,+\, w_1\,Z_1 \,+\, u_2\,Z_2\;, \tag{8.105}$$

with w_1 defined as follows($(u_1, u_2) \mapsto (w_1, u_2)$ defines a regular static feedback :

$$w_1 \;=\; \dot{h}_1 \;=\; L_{X_0}h_1 \,+\, u_1 L_{X_1}h_1 \,+\, u_2 L_{X_2}h_1 \;=\; L_{X_0}h_1 \,+\, (u_1+\lambda u_2)L_{X_1}h_1\,. \tag{8.106}$$

Then $\dot{h}_1$ and $\dot{h}_2$ may be written :

$$\begin{aligned} \dot{h}_1 &= w_1 \\ \dot{h}_2 &= L_{Z_0}h_2 \,+\, w_1\,L_{Z_1}h_2\;. \end{aligned} \tag{8.107}$$

The second time-derivatives are then given by :

$$\begin{aligned} \ddot{h}_1 &= \dot{w}_1 \\ \ddot{h}_2 &= L^2_{Z_0}h_2 \,+\, w_1\,(L_{Z_1}L_{Z_0} \,+\, L_{Z_0}L_{Z_1})\,h_2 \,+\, w_1\,L^2_{Z_1}h_2 \,+\, \dot{w}_1\,L_{Z_1}h_2 \\ &\quad +\,(L_{Z_2}L_{Z_0}h_2 \,+\, w_1\,L_{Z_2}L_{Z_1}h_2)\,u_2\;. \end{aligned} \tag{8.108}$$

The function $\ddot{h}_2$ must not depend on u_2 —this is (8.22)— and hence

$$L_{Z_2}L_{Z_0}h_2 \;=\; L_{Z_2}L_{Z_1}h_2 \;=\; 0\,. \tag{8.109}$$

Now, on one hand, from (8.103)-(8.107)-(8.105), $L_{Z_1}h_1$ is identically equal to 1, and $L_{Z_2}h_1$, $L_{Z_0}h_1$ and $L_{Z_2}h_1$ are identically zero, so that $L_{[Z_2,Z_1]}h_1$ and $L_{[Z_2,Z_0]}h_1$ are obviously zero, and on the other hand, since $L_{Z_2}h_2$ is identically zero from (8.103), $L_{Z_2}L_{Z_1}h_2$ is equal to $L_{[Z_2,Z_1]}h_2$ and $L_{Z_2}L_{Z_0}h_2$ is equal to $L_{[Z_2,Z_0]}h_2$; this and (8.109) above implies :

$$L_{[Z_2,Z_1]}h_1 \;=\; L_{[Z_2,Z_1]}h_2 \;=\; 0\;, \tag{8.110}$$

$$L_{[Z_2,Z_0]}h_1 \;=\; L_{[Z_2,Z_0]}h_2 \;=\; 0\;. \tag{8.111}$$

The two independent functions h_1 and h_2 are, from (8.110) and (8.103), constant along the vector fields Z_2 and $[Z_1, Z_2]$, which are linearly independent because $m_1 = 3$. This implies that the distribution spanned by these two vector fields is integrable, and therefore that the Lie Bracket $[Z_2, [Z_1, Z_2]]$ is a linear combination of Z_2 and $[Z_1, Z_2]$. From (8.102) and (8.104), the Lie bracket $[Z_2, [Z_1, Z_2]]$ is equal to $\lambda[Z_2, [X_1, X_2]] + (L_{X_1}\lambda)[X_1, X_2] + ((L_{X_1}\lambda)^2 - L_{X_2}L_{X_1}\lambda)X_1$. Hence, $[Z_2, [X_1, X_2]]$ must be a linear combination of X_1, X_2 and $[X_1, X_2]$. This implies, from the definition of the characteristic vector field $\widetilde{X}$ —see (8.35)—that Z_2 is collinear to $\widetilde{X}$:

$$Z_2 \;=\; \alpha\,\widetilde{X} \tag{8.112}$$

with α a nonzero function. Since the vector fields annihilating $\mathrm{d}h_1$ and $\mathrm{d}h_2$ are the linear combinations of Z_2 and $[Z_2, Z_1]$, (8.110) and (8.111) imply that $[Z_2, Z_0]$ is a linear combination of Z_2 and $[Z_2, Z_1]$, which implies in particular that it is a linear combination of X_1, X_2 and $[X_1, X_2]$. From (8.112), this implies condition (8.36). We have proved the necessity of condition (8.36).

Let us now prove that existence of h_1 and h_2 with the above properties imply (8.41)-(8.42). From above, $\mathrm{d}h_1$ and $\mathrm{d}h_2$ vanish on $\widetilde{X}$ and $[Z_2, \widetilde{X}]$, and are linearly independent because (h_1, h_2) is a pair of linearizing outputs ; hence, from the definition of ω_1 and the fact that both $\widetilde{X}$ and $[Z_2, \widetilde{X}]$ are linear combinations of X_1, X_2 and $[X_1, X_2]$, the form ω_1 is a linear combination of $\mathrm{d}h_1$ and $\mathrm{d}h_2$, i.e. there exists some functions λ_1 and λ_2 such that

$$\omega_1 \ = \ \lambda_1 \,\mathrm{d}h_1 \ + \ \lambda_2 \,\mathrm{d}h_2 \ . \tag{8.113}$$

Computing the time-derivative of this yields

$$\dot\omega_1 \ = \ \lambda_1 \,\mathrm{d}\dot h_1 \ + \ \lambda_2 \,\mathrm{d}\dot h_2 \ + \ \dot\lambda_1 \,\mathrm{d}h_1 \ + \ \dot\lambda_2 \,\mathrm{d}h_2 \ . \tag{8.114}$$

This implies (8.42) because on one hand the two functions λ_1 and λ_2 do not vanish simultaneously because ω_1 does not vanish, and on the other hand $\mathrm{d}h_1$, $\mathrm{d}h_2$, $\mathrm{d}\dot h_1$ and $\mathrm{d}\dot h_2$ are linearly independent because (h_1, h_2) is a pair of linearizing outputs. Condition (8.40), already proved because it is equivalent to (8.36) from lemma 8.7.3, implies :

$$\dot\omega_1 \ = \ \mu_0\,\omega_1 \ + \ \mu_1\,\eta_1 \ + \ \mu_2\,\eta_2$$

for some functions μ_0, μ_1, μ_2. (8.42) implies that μ_1 and μ_2 do not vanish simultaneously. Let η be η_1 if μ_2 does not vanish, and η_2 if μ_2 vanishes. Then $\{\omega_1, \dot\omega_1, \eta\}$ is another basis for the annihilator of $\widetilde{X}$. Since $\mathrm{d}h_1$ and $\mathrm{d}h_2$ are in the annihilator of $\{\widetilde{X}, [Z_2, \widetilde{X}]\}$, they are linear combinations of ω_1, $\dot\omega_1$ and η. Since $\{\mathrm{d}h_1, \mathrm{d}h_2\}$ is a linearizing Pfaffian system, this implies, from Proposition 8.2.5, that $\{\omega_1, \eta\}$ is a linearizing Pfaffian system, and hence that ω_1, η and all their time derivatives are linearly independent, and in particular $\omega_1, \dot\omega_1, \ddot\omega_1, \eta, \dot\eta$ has rank 5, but from the above construction, it is also the rank of $\omega_1, \eta_1, \dot\eta_1, \eta_2, \dot\eta_2$. This proves (8.41).

According to the remarks just after the proof of lemma 8.7.3, we have now proved the necessity of either (8.36)-(8.37)-(8.38), or (8.40)-(8.41)-(8.42). Let us prove sufficiency, and at the same time validity of the way of building linearizing outputs given in the theorem. Again, from lemma 8.7.3, it is enough to prove sufficiency of (8.40)-(8.41)-(8.42).

From (8.40) and (8.42), equation (8.43) implies

$$\mathrm{d}\omega_1 \ = \ \omega_1 \wedge \Gamma' \ + \ k\dot\omega_1 \wedge \eta \tag{8.115}$$

where k is a non-vanishing function and η is either η_1 or η_2, and then (8.41) implies

$$\mathrm{rank}\{\omega_1, \dot\omega_1, \ddot\omega_1, \eta, \dot\eta\} \ = \ 5 \ . \tag{8.116}$$

Let α_1 and α_2 be some (non vanishing simultaneously) functions such that $\{\omega_1, \alpha_1\dot\omega_1 + \alpha_2\eta\}$ is a basis of the annihilator of $\{X_1, X_2\}$. Then $\{\omega_1, \dot\omega_1, \eta, \alpha_1\ddot\omega_1 + \alpha_2\dot\eta\}$ is a basis of $\mathrm{Span}\{\mathrm{d}\mathbf{x}\}$ —all four are in $\mathrm{Span}\{\mathrm{d}\mathbf{x}\}$ because $\alpha_1\dot\omega_1 + \alpha_2\eta$ vanishes on X_1 and X_2, and they are independent from (8.116)— and $\{\omega_1, \dot\omega_1, \ddot\omega_1, \eta, \dot\eta, \alpha_1\omega_1^{(3)} + \alpha_2\ddot\eta\}$ is a basis of $\mathrm{Span}\{\mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{u}\}$ because of (8.116) and the fact that X_1 and X_2 are supposed to have rank 2, and then an easy induction shows that

$$\{\omega_1, \dot\omega_1, \ddot\omega_1, \ldots, \omega_1^{(j+2)}, \eta, \dot\eta, \ddot\eta, \ldots, \eta^{(j+1)}, \alpha_1\omega_1^{(j+3)} + \alpha_2\eta^{(j+2)}\}$$

is a basis of $\mathrm{Span}\{\mathrm{d}x, \mathrm{d}u, \mathrm{d}\dot u, \ldots, \mathrm{d}u^{(j)}\}$ for all $j \geq 0$. This implies that $\{\omega_1, \eta\}$ is a linearizing Pfaffian system (see definition 8.2.3).

Let us now build a pair of linearizing outputs as explained in the theorem. If h_1 is built as indicated, i.e. such that

$$L_{\widetilde{X}}h_1 \ = \ 0 \quad \text{and} \quad \mathrm{rank}\{\omega_1, \dot\omega_1, \mathrm{d}h_1\} \ = \ 3 \ , \tag{8.117}$$

the Pfaffian system $\{\mathrm{d}h_1, \omega_1\}$ is integrable because (8.115) and the fact that $\mathrm{d}h_1$ is a linear combination of $\omega_1, \dot{\omega}_1, \eta$ imply $\mathrm{d}\omega_1 \wedge \omega_1 \wedge \mathrm{d}h_1 = 0$. Let h_2 be a second function such that $\{\mathrm{d}h_1, \mathrm{d}h_2\}$ is another basis for $\{\omega_1, \mathrm{d}h_1\}$. These $\mathrm{d}h_1$ and $\mathrm{d}h_2$ are obviously linear combinations of ω_1, $\dot{\omega}_1$ and η, but this may be inverted : ω_1 is a linear combination of $\mathrm{d}h_1$ and $\mathrm{d}h_2$, and η is, from (8.117) a linear combination of ω_1, $\dot{\omega}_1$, $\mathrm{d}h_1$, and hence of $\mathrm{d}h_1$, $\mathrm{d}h_2$, $\mathrm{d}\dot{h}_1$ and $\mathrm{d}\dot{h}_2$. Since $\{\omega_1, \eta\}$ is a linearizing Pfaffian system, $(\mathrm{d}h_1, \mathrm{d}h_2)$ is, from Proposition 8.2.5, also a linearizing Pfaffian system, and (h_1, h_2) is a pair of linearizing outputs from Proposition 8.2.3. This completes the proof of sufficiency.

Let us now prove the assertions concerned with the "normal form". The normal form itself is a consequence of the following lemma :

Lemma 8.7.4 ("Engel's normal form")**.** *Let X_1 and X_2 be two vector fields in $\mathbb{R}^4$ and let $\bar{x} \in \mathbb{R}^4$ be such that*

$$\begin{aligned} rank_{\mathbb{R}}\,\{\, X_1(\bar{x})\,,\, X_2(\bar{x})\,\} &= 2\,, \\ rank_{\mathbb{R}}\,\{\, X_1(\bar{x})\,,\, X_2(\bar{x})\,,\, [X_1, X_2](\bar{x})\,\} &= 3\,, \\ rank_{\mathbb{R}}\,\{\, X_1(\bar{x})\,,\, X_2(\bar{x})\,,\, [X_1, X_2](\bar{x})\,,\, [X_1,[X_1, X_2]](\bar{x})\,,\, [X_2,[X_1, X_2]](\bar{x})\,\} &= 4\,. \end{aligned}$$

Then there exists four functions α_{11}, α_{12}, α_{21}, α_{22}, and a set of coordinates $(z_1,\, z_2,\, z_3,\, z_4)$ such that the matrix $\begin{pmatrix} \alpha_{11}(\bar{x}) & \alpha_{12}(\bar{x}) \\ \alpha_{21}(\bar{x}) & \alpha_{22}(\bar{x}) \end{pmatrix}$ is invertible, and, locally around $\bar{x}$,

$$\alpha_{11}X_1 + \alpha_{21}X_2 \;=\; \frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_2} + z_4\frac{\partial}{\partial z_3}\;; \quad \alpha_{12}X_1 + \alpha_{22}X_2 \;=\; \frac{\partial}{\partial z_4}\,. \tag{8.118}$$

The proof is very classical, see for example [18]. Now, by assumption, the vector fields X_1 and X_2 satisfy these assumptions, and the feedback

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} L_{X_0}z_1 \\ L_{X_0}z_4 \end{pmatrix} \tag{8.119}$$

yields the equations (8.44) in the coordinates given by lemma 8.7.4. The fact that the coordinate-free and feedback invariant conditions (8.40), (8.41), (8.42) translate into (8.45), (8.46), (8.47) respectively is a routine computation from

$$\begin{aligned} \omega_1 &= \mathrm{d}z_2 - z_3\mathrm{d}z_1 & \dot{\omega}_1 &= \mathrm{d}f_2 + v_1\mathrm{d}z_3 - (f_3 + z_4v_1)\mathrm{d}z_1 \\ \eta_1 &= \mathrm{d}z_1\,, & \dot{\eta}_1 &= \mathrm{d}v_1\,, \\ \eta_2 &= \mathrm{d}z_3\,, & \dot{\eta}_2 &= \mathrm{d}f_3 + v_1\mathrm{d}z_4v_1 + z_4\mathrm{d}v_1\,. \end{aligned}$$

Alternative proof of Case 6

Here we suppose in addition that we are at Brunovský-regular point, i.e. the rank condition (8.52) holds, and we give a proof for case 6 based on the infinitesimal Brunovský form. To give a thorough treatment of case 6, one should consider the case when the rank in (8.52) is three in a neighborhood —then there is a different infinitesimal Brunovský form, as in the second point of proposition 8.2.7, and also points where it three, while being 4 in an open dense set of a neighborhood —at such points, an infinitesimal Brunovský form does not exist but one might conclude by density.

Condition (8.52) implies, see proposition 8.2.7, that if two forms ω_1 and ω_2 make up a basis of $\widehat{\mathcal{D}}_2^{\perp}$, then $\{\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2\}$ is a basis of $\mathrm{Span}\{\mathrm{d}\mathbf{x}\}$. In addition, ω_1 may be taken in $\mathcal{M}_0^{\perp}$ (i.e. $\{\omega_1\}$ is the first derived system of the Pfaffian system $\{\omega_1, \omega_2\}$). Then we have :

$$\begin{aligned} \mathrm{d}\omega_1 &\equiv \omega_2 \wedge (\delta_1\dot{\omega}_1 + \delta_2\dot{\omega}_2) && \text{modulo } \omega_1 \\ \mathrm{d}\omega_2 &\equiv \gamma\, \dot{\omega}_1 \wedge \dot{\omega}_2 && \text{modulo } \{\omega_1, \omega_2\}\,. \end{aligned} \tag{8.120}$$

Since on one hand the rank of $\mathcal{M}_1$ is constant equal to 4, and on the other hand the rank of $\mathcal{M}_0$ is constant equal to 3,

$$\left.\begin{array}{ll}\delta_1 \text{ and } \delta_2 & \text{do not vanish simultaneously,}\\ \gamma & \text{does not vanish.}\end{array}\right\} \tag{8.121}$$

A computations shows that :

$$\mathrm{Span}\{\widetilde{X}\} \;=\; \{\omega_1\,,\,\omega_2\,,\,\delta_1\dot\omega_1+\delta_2\dot\omega_2\,\}^{\perp}\,. \tag{8.122}$$

The proof of characterization (8.36) relies on the following lemma, proved further :

Lemma 8.7.5. *The following three properties are equivalent :*

(i) There exist two invertible matrices J_1 and J_2 of degree zero and three functions a, h_1 and h_2, all defined on a neighborhood of the point $\mathcal{X}$, such that

$$\begin{pmatrix}\mathrm{d}h_1\\ \mathrm{d}h_2\end{pmatrix} \;=\; J_1\begin{pmatrix}1 & -a\,\frac{d}{dt}\\ 0 & 1\end{pmatrix} J_2\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \tag{8.123}$$

(ii) $\delta_2 \;=\; 0$ on a neighborhood of $\mathcal{X}$.

(iii) (8.36) holds on a neighborhood of $\mathcal{X}$.

This is enough to conclude. Indeed, sufficiency in case 6 of theorem 8.3.1 is obvious because, from proposition 8.2.8, point (i) implies x-dynamic linearizability. Let us prove necessity : if system (8.5) is x-dynamic linearizable in a neighborhood of a point $\overline{\mathcal{X}}$, then from propositions 8.2.8 and 8.2.9, there is an open set U_0, dense in a neighborhood of $\overline{\mathcal{X}}$, such that point (i) holds for all $\mathcal{X}\in U_0$. From the lemma, this implies that δ_2 is zero on U_0. Hence it is zero on a neighborhood of $\overline{\mathcal{X}}$. This completes the proof of case 6 of theorem 8.3.1, the normal form being proved the same way as in the first proof.

Proof of lemma 8.7.5 :

(ii)⇔(iii) : We have, from (8.122) and identity (8.15),

$$\begin{aligned}0 \;=\; \frac{\mathrm{d}}{\mathrm{dt}}\langle\delta_1\omega_1+\delta_2\omega_2\,,\,\widetilde{X}\rangle \;&=\; \langle\dot\delta_1\omega_1+\dot\delta_2\omega_2+\delta_1\dot\omega_1+\delta_2\dot\omega_2\,,\,\widetilde{X}\rangle\\ &\quad+\langle\delta_1\omega_1+\delta_2\omega_2\,,\,[X_0+u_1X_1+u_2X_2,\widetilde{X}]\rangle\,,\end{aligned}$$

which, from the fact that $\langle\omega_i,\widetilde{X}\rangle$ and $\langle\omega_1,[X_i,\widetilde{X}]\rangle$ are identically zero for $i=1,2$, yields

$$\delta_1\,\langle\omega_1\,,\,[X_0,\widetilde{X}]\rangle \;+\; \delta_2\,\langle\omega_2\,,\,[X_0+u_1X_1+u_2X_2,\widetilde{X}]\rangle \;=\; 0 \tag{8.124}$$

which implies, since δ_1 and δ_2 do not vanish simultaneously and $[X_0+u_1X_1+u_2X_2,\widetilde{X}]$ does not vanish, that $\delta_2=0$ is equivalent to $\langle\omega_1,[X_0,\widetilde{X}]\rangle=0$, i.e. to (8.36).

(ii)⇒(i) : Since $\delta_2=0$, (8.120) implies that $\{\omega_1,\omega_2,\dot\omega_1\}$ is the characteristic system of ω_1 and therefore is integrable. In particular, there exists a function h_2 such that

$$\mathrm{d}h_2 \;=\; \lambda_0\omega_1 \;+\; \lambda_1\dot\omega_1 \;+\; \lambda_2\omega_2$$

with a non-vanishing λ_2 ; then

$$\mathrm{d}\omega_1 \;\equiv\; \widetilde{\delta}_1\,\mathrm{d}h_2\;\wedge\dot\omega \quad \text{modulo}\,\omega_1$$

which implies that $\{\omega_1,\mathrm{d}h_2\}$ is integrable and in particular that there exists a function h_1 such that

$$\begin{pmatrix}\mathrm{d}h_1\\ \mathrm{d}h_2\end{pmatrix} \;=\; \begin{pmatrix}\mu_1 & \mu_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}\omega_1\\ \mathrm{d}h_2\end{pmatrix} \;=\; \begin{pmatrix}\mu_1 & \mu_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \lambda_0+\lambda_1\frac{\mathrm{d}}{\mathrm{dt}} & \lambda_2\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

where $\mu_1\lambda_1$ does not vanish. This is point (i).

(i)⇒(ii) : Let Ω_1, Ω_2 and Ω_3 be defined by

$$\begin{pmatrix}\Omega_1\\ \Omega_2\end{pmatrix} \;=\; J_2\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\,, \tag{8.125}$$

$$\begin{pmatrix} \Omega_3 \\ \Omega_2 \end{pmatrix} = \begin{pmatrix} 1 & -a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = \begin{pmatrix} \Omega_1 - a\dot{\Omega}_2 \\ \Omega_2 \end{pmatrix} . \tag{8.126}$$

then (8.123) implies that $\{\Omega_3, \Omega_2\}$ is integrable and hence, for some 1-forms $\Gamma_{i,j}$,

$$\begin{array}{rcl} \mathrm{d}\Omega_2 &=& \Omega_2 \wedge \Gamma_{2,2} + \Omega_3 \wedge \Gamma_{2,3} = \Omega_1 \wedge \Gamma_{2,3} + \Omega_2 \wedge \Gamma_{2,2} - a\,\dot{\Omega}_2 \wedge \Gamma_{2,3} \\ \mathrm{d}\Omega_3 &=& \Omega_2 \wedge \Gamma_{3,2} + \Omega_3 \wedge \Gamma_{3,3} = \Omega_1 \wedge \Gamma_{2,3} + \Omega_2 \wedge \Gamma_{2,2} - a\,\dot{\Omega}_2 \wedge \Gamma_{2,3} \end{array} \tag{8.127}$$

Taking the time-derivative of the second equation yields

$$\begin{array}{rcl} \mathrm{d}\dot{\Omega}_2 &=& \Omega_1 \wedge \dot{\Gamma}_{2,3} + \dot{\Omega}_1 \wedge \Gamma_{2,3} \\ && + \Omega_2 \wedge \dot{\Gamma}_{2,2} + \dot{\Omega}_2 \wedge \Big(\Gamma_{2,2} - a\dot{\Gamma}_{2,3} - \dot{a}\Gamma_{2,3}\Big) - a\,\ddot{\Omega}_2 \wedge \Gamma_{2,3} \end{array} \tag{8.128}$$

and finally, since $\mathrm{d}\Omega_1 = \mathrm{d}(\Omega_3 + a\dot{\Omega}_2) = \mathrm{d}\Omega_3 + a\mathrm{d}\dot{\Omega}_2 - \dot{\Omega}_2 \wedge \mathrm{d}a$,

$$\begin{array}{rcl} \mathrm{d}\Omega_1 &=& \Omega_1 \wedge \Big(\Gamma_{2,3} + a\dot{\Gamma}_{2,3}\Big) + \Omega_2 \wedge \Big(\Gamma_{2,2} + a\dot{\Gamma}_{2,2}\Big) \\ && + a\,\dot{\Omega}_1 \wedge \Gamma_{2,3} + \dot{\Omega}_2 \wedge \Big(-a\Gamma_{2,3} + a\Gamma_{2,2} - a^2\dot{\Gamma}_{2,3} - a\dot{a}\Gamma_{2,3} - \mathrm{d}a\Big) \\ && - a^2\,\ddot{\Omega}_2 \wedge \Gamma_{2,3} \\ \mathrm{d}\Omega_2 &=& \Omega_1 \wedge \Gamma_{2,3} + \Omega_2 \wedge \Gamma_{2,2} - a\,\dot{\Omega}_2 \wedge \Gamma_{2,3} \end{array} \tag{8.129}$$

From (8.125), $\{\Omega_1, \Omega_2\}$ is the same differential system as $\{\omega_1, \omega_2\}$ and therefore, from (8.120), $\mathrm{d}\Omega_i \equiv \lambda_i\dot{\Omega}_1 \wedge \dot{\Omega}_2$ modulo $\{\Omega_1, \Omega_2\}$ for $i = 1, 2$ and λ_i certain functions; from the second equation in (8.129), this implies that $\Gamma_{2,3}$ is a linear combination of $\Omega_1, \Omega_2, \dot{\Omega}_1, \dot{\Omega}_2$; from the first equation in (8.129), it is actually a linear combination of $\Omega_1, \Omega_2, \dot{\Omega}_2$ because the $\dot{\Omega}_1$-term would produce a $\ddot{\Omega}_2 \wedge \dot{\Omega}_1$-term in the last term of $\mathrm{d}\Omega_1$ (it cannot be canceled by another term because there is no $\ddot{\Omega}_2$ in $\gamma_{2,3}$) ; this implies, if $\Gamma_{2,3} = \lambda_1\Omega_1 + \lambda_1\Omega_2 + \lambda_0\dot{\Omega}_2$,

$$\mathrm{d}\Omega_2 = \Omega_2 \wedge \widetilde{\Gamma}_{2,2} + (\lambda_0 + a\lambda_1)\Omega_1 \wedge \dot{\Omega}_2 \tag{8.130}$$

where $\widetilde{\Gamma}_{2,2}$ contains $\Gamma_{2,2}$ plus other terms. This implies in particular that $\mathrm{d}\Omega_2 \equiv 0$ modulo $\{\Omega_1, \Omega_2\}$ which implies that Ω_2 is in the first derived system of $\{\Omega_1, \Omega_2\}$ (i.e. in the annihilator of $\{X_1, X_2, [X_1, X_2]\}$) and therefore that it is collinear to ω_1, or in other terms that matrix J_2 is triangular :

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = \begin{pmatrix} \beta_1 & \beta_2 \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{8.131}$$

where $\alpha\beta_2$ does not vanish. Then (8.120) yields

$$\mathrm{d}\Omega_2 \equiv \frac{1}{\beta_2}\Omega_1 \wedge \left((\delta_1 - \frac{\beta_1\delta_2}{\beta_2})\dot{\Omega}_2 + \delta_2\frac{\alpha}{\beta_2}\dot{\Omega}_1\right) \quad \text{modulo } \Omega_2 \tag{8.132}$$

By comparing this and (8.130), we see that $\delta_2\alpha = 0$ which implies that δ_2 is identically zero because α does not vanish.

8.7.2 Proof of the results on (x, u)-dynamic linearizability

In this section, we prove theorem 8.4.1 and proposition 8.4.4. They are proved together because we are not able to prove the intrinsic condition of theorem 8.4.1 without the help of the coordinates of the normal form (8.60). In the course of the proof, we will need the four following technical lemmae (lemmae 8.7.6, 8.7.7, 8.7.8 and 8.7.9), that are proved further.

Lemma 8.7.6. *Let ω_1 and ω_2 be chosen according to (8.53). Let b and J_2 be respectively a scalar smooth function and a 2×2 invertible matrix (of degree zero) with entries smooth functions, defined on a neighborhood of a point $\mathcal{Y}$ and let Ω_1 and Ω_3 be defined by*

$$\begin{pmatrix} \Omega_1 \\ \Omega_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b\frac{d}{dt} & 1 \end{pmatrix} J_2 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} . \tag{8.133}$$

The forms Ω_1 and Ω_3 satisfy the following relations :

$$\mathrm{d}\Omega_1 \equiv 0 \quad \textit{modulo } \{\, \Omega_1 \,,\, \Omega_3 \,,\, \dot{\Omega}_3 \,\} \tag{8.134}$$

$$\mathrm{d}\Omega_3 \equiv 0 \quad \textit{modulo } \{\, \Omega_3 \,,\, \Omega_1 \wedge \dot{\Omega}_3 \,\} \tag{8.135}$$

on a neighborhood of $\mathcal{Y}$ if and only if there exist smooth functions h_1, h_2 and a, and a 2×2 invertible matrix J_1 with entries smooth functions defined on a neighborhood of $\mathcal{Y}$, such that

$$\begin{aligned}\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} &= J_1 \begin{pmatrix} 1 & -a\frac{d}{dt} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b\frac{d}{dt} & 1 \end{pmatrix} J_2 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \\ &= J_1 \begin{pmatrix} 1 & -a\frac{d}{dt} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_3 \end{pmatrix} .\end{aligned} \tag{8.136}$$

Lemma 8.7.7. *Let ω_1 and ω_2 be some 1-forms satisfying (8.53), and hence (8.55), around a point where γ and $\delta^2_{1,2}$ do not vanish (implied by the rank assumptions (8.48)-(8.49)-(8.50)-(8.51)-(8.52)).*

1. *There cannot exist functions a, b, h_1, h_2 and two invertible 2×2 matrices of degree zero J_1 and J_2, all defined on a neighborhood of the considered point, such that*

$$J_1 \begin{pmatrix} 1 & 0 \\ -a\frac{d}{dt} - b\frac{d}{dt}^2 & 1 \end{pmatrix} J_2 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} . \tag{8.137}$$

2. *There cannot exist functions α, a, b, h_1, h_2 and an invertible 2×2 matrix of degree zero J_1, all defined on a neighborhood of the considered point, such that*

$$J_1 \begin{pmatrix} 1 & -a\frac{d}{dt} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b\frac{d}{dt} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} . \tag{8.138}$$

Lemma 8.7.8. *Let ω_1 and ω_2 be some 1-forms satisfying (8.53), and hence (8.55). If, for some functions λ_1, λ_2 and λ_3, one has*

$$\begin{aligned} \mathrm{d}\omega_2 &\equiv 0 \quad \textit{modulo } \{\omega_2, \Omega, \dot{\Omega}\} \\ \textit{with } \Omega &= \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\dot{\omega}_2 \,, \end{aligned} \tag{8.139}$$

then λ_1 and λ_3 are related to the functions appearing in (8.55) by :

$$\lambda_3 \left(2\gamma\,\lambda_1 + \delta^1_{2,1}\,\lambda_3\right) = 0 . \tag{8.140}$$

Lemma 8.7.9. *Let f and g be two smooth functions from an open subset $O \subset \mathbb{R}^4$ to $\mathbb{R}$. The following two assertions are equivalent :*

- *$\frac{\partial g}{\partial z_4}$ does not vanish on O and f and g are solutions of the following equations on O :*

$$2\,\frac{\partial g}{\partial z_4}\,\frac{\partial^3 g}{\partial z_4^3} - 3\left(\frac{\partial^2 g}{\partial z_4^2}\right)^2 = 0 \,, \tag{8.141}$$

$$2\,\frac{\partial g}{\partial z_4}\,\frac{\partial^3 f}{\partial z_4^3} - 3\,\frac{\partial^2 g}{\partial z_4^2}\,\frac{\partial^2 f}{\partial z_4^2} = 0 \,. \tag{8.142}$$

- *There exists a_0, a_1, a_2, b_0, b_1, c_0 and c_1, seven smooth functions of z_1, z_2, z_3 defined on O (i.e. on its projection on $\mathbb{R}^3$) such that*

$$c_0(z_1,z_2,z_3) + z_4 c_1(z_1,z_2,z_3) \quad \textit{and} \quad \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix} (z_1,z_2,z_3)$$

do not vanish on O and f and g are given by (8.63) on O.

Proof of theorem 8.3.1 and proposition 8.4.4 : Let us consider a point $\overline{\mathcal{X}} = (\bar{x}, \bar{u}, \dot{\bar{u}}, \ldots\ldots)$ such that conditions (8.48)-(8.49)-(8.50)-(8.51)-(8.52) hold at $(\bar{x}, \bar{u})$. Let the forms ω_1 and ω_2 be defined according to (8.53) and the functions $\delta^1_{2,1}$ and γ by (8.55). Let also $(z_1,\, z_2,\, z_3,\, z_4)$ be some coordinates in which system (8.5) has the form (8.60) (they exist from Proposition 8.4.3), and the functions f and g be defined accordingly from an open subset of $\mathbb{R}^4$ to $\mathbb{R}$.

We have to prove the following :

1. The following three properties are equivalent :
   - (x, u)-dynamic linearizability of system (8.5) at point $\overline{\mathcal{X}}$, or of (8.60) at the corresponding point in terms of $(z, v, \dot{v}, \ldots)$,
   - conditions of proposition 8.4.4 on the functions f and g,
   - condition in terms of Pfaffian systems of theorem 8.4.1.
2. When they are satisfied, the possible pairs of linearizing outputs depending on x and u are these described in theorem 8.4.1.

The (easy) proof of the second point will be given at the very end when equivalence is totally understood.

From proposition 8.2.8, (x, u)-dynamic linearizability is equivalent to existence of a matrix $P(\frac{\mathrm{d}}{\mathrm{dt}})$ whose entries are polynomials in $\frac{\mathrm{d}}{\mathrm{dt}}$ of degree at most 2, which has an inverse of the same type (except we do not need to know whether the degree of the entries of the inverse is also at most 2), and transforms the pair of forms (ω_1, ω_2) into a pair that defines an integrable Pfaffian system. Now use the second point of proposition 8.2.9 ; it allows four possible decompositions of the matrix $P(\frac{\mathrm{d}}{\mathrm{dt}})$, but only on an open dense set of points of the neighborhood of $\overline{\mathcal{X}}$ where $P(\frac{\mathrm{d}}{\mathrm{dt}})$ is defined, and $\overline{\mathcal{X}}$ might not belong to this open dense set. Around these points though, lemma 8.7.7 states that three of the four decompositions proposed by proposition 8.2.9 are impossible due to the form of $\mathrm{d}\omega_1$ and $\mathrm{d}\omega_2$ given by (8.54)-(8.55), so that only the last one is possible. If the decomposition of proposition 8.2.9 was available at all points, item 1 of the following lemma would be equivalent to (x, u)-dynamic linearizability, and the following lemma would end the proof.

Lemma 8.7.10. *Let ω_1 and ω_2 be chosen according to (8.53), the functions $\delta^1_{2,1}$ and γ be defined by (8.55), and some coordinates z_1, z_2, z_3, z_4 be fixed according to Proposition 8.4.3, in which system (8.5) has the form (8.60), and the functions f and g be defined accordingly from an open subset of $\mathbb{R}^4$ to $\mathbb{R}$. The following four assertions are equivalent :*

1. *There exists an invertible matrix J_1 of degree zero and six functions α, λ, a, b, h_1 and h_2, all defined on a neighborhood of the point $\mathcal{Y}$, such that b does not vanish on this neighborhood and*

$$\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} = J_1 \begin{pmatrix} 1 & -a\,\frac{d}{dt} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b\,\frac{d}{dt} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{8.143}$$

2. *There exist three functions α, λ and b, all defined on a neighborhood of $\mathcal{Y}$, such that b does not vanish on this neighborhood and, with*

$$\begin{pmatrix} \Omega_1 \\ \Omega_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b\,\frac{d}{dt} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \quad , \tag{8.144}$$

   one has

$$\mathrm{d}\Omega_1 \equiv 0 \qquad \textit{modulo}\ \{\, \Omega_1 \,,\, \Omega_3 \,,\, \dot{\Omega}_3 \,\} \tag{8.145}$$

$$\mathrm{d}\Omega_3 \equiv 0 \qquad \textit{modulo}\ \{\, \Omega_3 \,,\, \Omega_1 \wedge \dot{\Omega}_3 \,\} \tag{8.146}$$

3. *$\delta^1_{2,1}$ does not vanish at $\mathcal{Y}$ and the first derived system of the Pfaffian system $\{\omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2\,,\,\omega_2\}$ has rank 1 and is integrable, i.e. there exists a (unique) function α such that (8.58) is satisfied.*
4. *The function $\delta^1_{2,1}$ does not vanish at $\mathcal{Y}$ and, in the normal form (8.60), the functions f and g are, on a neighborhood of $\mathcal{Y}$, of the form (8.63) where a_0, a_1, a_2, b_0, b_1, c_0 and c_1 are functions of z_1, z_2, z_3 only, which satisfy (8.64).*

If one of these conditions is met (and therefore all of them), λ, α and b in (8.143) and (8.144) are uniquely defined :

$$\lambda \;=\; 0\,, \quad b \;=\; \frac{2\gamma}{\delta^1_{2,1}}\,, \quad \alpha \textit{ is uniquely defined by (8.58)}\,. \tag{8.147}$$

This lemma contains the real technical difficulties of the paper. The proof is given further (page 201), let us however sketch it. Equivalence between 1 and 2 is given by lemma 8.7.6, it is a manipulation on Pfaffian systems, and only needs the fact that the Pfaffian system $\{\omega_1, \omega_2\}$ may be written in four variables (the coordinates of x). It is very simple to prove that 3 implies 2, but the converse is not obvious : since we were not able to prove it directly, we used the coordinates z of the normal form, and instead of proving that 2 implies 3, we prove that 2 implies 4 by writing (8.145)-(8.146) in the coordinates $(z, v, \dot{v}, \ldots)$ of the normal form (8.60) as some differential relations on the functions α, λ and b with the functions f and g as parameters, eliminating the unknowns α, λ and b, and obtaining some PDEs on f and g that imply the form of f and g given by point 4 above (or by proposition 8.4.4), these computations have been conducted with the computer algebra system "Maple" (version 5.2). The fact that 4 implies 2 is a simple computation in coordinates, made easier by proposition 8.4.5.

Unfortunately, the conclusion of proposition 8.2.9 is not valid at all point, so that the results we want to prove do not follow from the above lemma 8.7.10, proposition 8.2.9 and lemma 8.7.7. Let us however prove that (x, u)-dynamic linearizability at point $\overline{\mathcal{X}}$ is equivalent to one of the four equivalent conditions of lemma 8.7.10 being satisfied at point $\overline{\mathcal{X}}$. This will end the proof that (x, u)-dynamic linearizability, the conditions of proposition 8.4.4 on the functions f and g (item 4 of lemma 8.7.10) and the condition in terms of Pfaffian systems of theorem 8.4.1 (item 3 of lemma 8.7.10) are equivalent.

Point 1 of lemma 8.7.10 implies (x, u)-dynamic linearizability from proposition 8.2.8 because the matrix applied to (ω_1, ω_2) in (8.143) is obviously invertible and of degree 2. Conversely, suppose that there exists a pair of linearizing outputs (h_1, h_2) depending only on x and u, defined around $\overline{\mathcal{X}}$, and let us prove that item 3 of lemma 8.7.10 holds on a neighborhood of $\overline{\mathcal{X}}$. From proposition 8.2.8, there exists $P(\frac{\mathrm{d}}{\mathrm{dt}}) \in \mathcal{A}(U)$, with U a neighborhood $\overline{\mathcal{X}}$, such that

$$\left.\begin{array}{l} P(\frac{\mathrm{d}}{\mathrm{dt}}) \text{ is invertible in } \mathcal{A}(U) \\ \deg P \;\leq\; 2 \;\text{ on } U \\ P(\frac{\mathrm{d}}{\mathrm{dt}}) \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \;=\; \begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} \end{array}\right\} \tag{8.148}$$

Since γ does not vanish at $\overline{\mathcal{X}}$ and the rank assumptions (8.48)-(8.49)-(8.50)-(8.51)-(8.52) hold at $\overline{\mathcal{X}}$, we may suppose, by possibly restricting U, that

$$\left.\begin{array}{l} \gamma \text{ does not vanish on } U\;\;, \\ \text{(8.48)-(8.49)-(8.50)-(8.51)-(8.52) hold on } U\;\;. \\ \deg P \;=\; 2 \quad \text{on an open dense subset of } U\;\;. \end{array}\right\} \tag{8.149}$$

The last statement is implied by the second one because if $\deg P$ is strictly less than 2 on an open set, the system is x-dynamic linearizable and this contradicts (8.51) from theorem 8.3.1.

Then, from proposition 8.2.9, there is an open dense subset U_0 of U such that, for all $\mathcal{Y} \in U_0$, the matrix $P(\frac{\mathrm{d}}{\mathrm{dt}})$ may be decomposed according to one of the four forms (8.18)-(8.19)-(8.20). From lemma 8.7.7 three of these four forms are forbidden, because conditions (8.48)-(8.49)-(8.50)-(8.51)-(8.52) hold at point $\mathcal{Y}$. Hence, around each point $\mathcal{Y} \in U_0$, there exists functions α, λ, a, b, and a matrix J_1, defined on a neighborhood of $\mathcal{Y}$ such that (8.143) is true on a neighborhood of $\mathcal{Y}$. By restricting possibly the open sense set U_0, we may suppose that b does not vanish on U_0 (b cannot vanish on an open set, because then P would have degree at most 1 on this open set, and therefore the linearizing outputs would depend on x only, and this would, from theorem 8.3.1, contradict (8.51). Then the conditions of point 1 of lemma 8.7.10 are satisfied on U_0. By applying lemma 8.7.10 at each point $\mathcal{Y}$ in U_0, one has, for all $\mathcal{Y} \in U_0$, a neighborhood of $\mathcal{Y}$ such that

- $\delta^1_{2,1}$ does not vanish on this neighborhood,
- there is a unique function $\alpha_{\mathcal{Y}}$ defined on this neighborhood such that

$$\mathrm{d}\left(\omega_1 + \alpha_{\mathcal{Y}}\omega_2 - \frac{2\,\gamma}{\delta^1_{2,1}}\dot{\omega}_2\right) \;\wedge\; \left(\omega_1 + \alpha_{\mathcal{Y}}\omega_2 - \frac{2\,\gamma}{\delta^1_{2,1}}\dot{\omega}_2\right) \;=\; 0\;, \tag{8.150}$$

- there are a smooth scalar function $a_{\mathcal{Y}}$ and an invertible matrix $J_{1,\mathcal{Y}}$ with entries some smooth functions, all defined on this neighborhood, so that, on this neighborhood,

$$\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} = J_{1,\mathcal{Y}} \begin{pmatrix} 1 & -a_{\mathcal{Y}}\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2\gamma}{\delta^1_{2,1}}\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \alpha_{\mathcal{Y}} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{8.151}$$

The last point is obtained by substituting the functions λ and b by the value they must have from (8.147). The second point implies in particular, by making the wedge product of both sides by ω_2 and multiplying by $(\delta^1_{2,1})^2$, that

$$\mathrm{d}\left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \omega_2 \;+\; \alpha_{\mathcal{Y}}\delta^1_{2,1}\,\mathrm{d}\omega_2 \wedge \left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \omega_2 \;=\; 0\,. \tag{8.152}$$

but on the other hand, the differential form of degree 4 $\mathrm{d}\omega_2 \wedge \left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \omega_2$ is, from (8.55), given by

$$\mathrm{d}\omega_2 \wedge \omega_2 \wedge (\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2) \;=\; \omega_1 \wedge (\frac{1}{2}\delta^1_{2,1}\dot{\omega}_1 - \gamma\ddot{\omega}_2) \wedge \omega_2 \wedge (-2\gamma\dot{\omega}_2)\;,$$

and therefore does not vanish on U. Existence of $\alpha_{\mathcal{Y}}$ satisfying (8.152) may be translated in some determinants made with the coefficients of the two differential forms of degree 4 being zero, but if these determinants are zero on an open dense subset U_0, they are zero all over U, and therefore, since $\mathrm{d}\omega_2 \wedge \left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \omega_2$ does not vanish, there is a function ν, uniquely defined all over U, such that

$$\mathrm{d}\left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \omega_2 \;+\; \nu\,\mathrm{d}\omega_2 \wedge \left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \omega_2 \;=\; 0\,. \tag{8.153}$$

Of course, since on the neighborhood of each point $\mathcal{Y}$, the function $\alpha_{\mathcal{Y}}$ is uniquely defined, it must coincide with $\frac{\nu}{\delta^1_{2,1}}$ where it is defined.

Then, let us define the form ω_3 by

$$\omega_3 \;=\; \delta^1_{2,1}\omega_1 \;+\; \nu\omega_2 \;-\; 2\gamma\dot{\omega}_2\;; \tag{8.154}$$

equation (8.151) reads

$$\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} \;=\; J_{1,\mathcal{Y}} \begin{pmatrix} \omega_2 \;-\; a\left(\frac{1}{\delta^1_{2,1}}\dot{\omega}_3 \;-\; \frac{\dot{\delta}^1_{2,1}}{(\delta^1_{2,1})^2}\omega_3\right) \\ \frac{1}{\delta^1_{2,1}}\omega_3 \end{pmatrix} \tag{8.155}$$

and therefore, $\mathrm{d}h_1$ and $\mathrm{d}h_2$ are linear combinations of ω_2, ω_3 and $\dot\omega_3$ on a neighborhood of each point $\mathcal{Y} \in U_0$. This implies that the rank of $\{\mathrm{d}h_1, \mathrm{d}h_2, \omega_2, \omega_3, \dot\omega_3\}$ is at most 3 on the open dense U_0, it is therefore also at most 3 on all U. Since the rank of $\{\omega_2, \omega_3, \dot\omega_3\}$ is three all over U (because γ does not vanish on U, see (8.149)), there are six functions $\mu_{i,j}$, (uniquely) defined all over U, such that

$$\mathrm{d}h_i \;=\; \mu_{i,1}\omega_2 \;+\; \mu_{i,2}\omega_3 \;+\; \mu_{i,3}\dot\omega_3$$

for $i = 1, 2$, or in other words

$$\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} \;=\; \begin{pmatrix} \mu_{1,1} & \mu_{1,2} + \mu_{1,3}\frac{\mathrm{d}}{\mathrm{dt}} \\ \mu_{2,1} & \mu_{2,2} + \mu_{2,3}\frac{\mathrm{d}}{\mathrm{dt}} \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix} \tag{8.156}$$

which implies, from (8.154),

$$\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} \;=\; \begin{pmatrix} \mu_{1,1} & \mu_{1,2} + \mu_{1,3}\frac{\mathrm{d}}{\mathrm{dt}} \\ \mu_{2,1} & \mu_{2,2} + \mu_{2,3}\frac{\mathrm{d}}{\mathrm{dt}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \delta^1_{2,1} & \nu \;-\; 2\gamma\frac{\mathrm{d}}{\mathrm{dt}} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

which implies, from (8.148), and because ω_1, ω_2 and all their time-derivatives are linearly independent, that

$$P(\frac{\mathrm{d}}{\mathrm{dt}}) \;=\; \begin{pmatrix} \mu_{1,1} & \mu_{1,2} + \mu_{1,3}\frac{\mathrm{d}}{\mathrm{dt}} \\ \mu_{2,1} & \mu_{2,2} + \mu_{2,3}\frac{\mathrm{d}}{\mathrm{dt}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \delta^1_{2,1} & \nu \;-\; 2\gamma\frac{\mathrm{d}}{\mathrm{dt}} \end{pmatrix}$$

This implies that $\delta^1_{2,1}$ must not vanish on U because $P(\frac{\mathrm{d}}{\mathrm{dt}})$ could not be invertible in the neighborhood of the zeroes of $\delta^1_{2,1}$, where the first column of the second factor would vanish.

Since $\delta^1_{2,1}$ does not vanish on U, the function $\alpha \stackrel{\Delta}{=} \frac{\nu}{\delta^1_{2,1}}$ is defined all over U, coincides with each $\alpha_{\mathcal{Y}}$ where these are defined ; this and (8.150) imply that (8.58) is satisfied on U_0 with this definition of α ; since U_0 is dense in U, (8.58) is even satisfied all over U. This proves that (x, u)-linearizability implies item 3 of lemma 8.7.10, and ends the proof of equivalence between (x, u)-dynamic linearizability, the conditions of proposition 8.4.4 on the functions f and g and the condition in terms of Pfaffian systems of theorem 8.4.1.

To end the proof of theorem 8.4.1 and proposition 8.4.4, there only remains to prove that the possible pairs of linearizing outputs depending on x and u only are these described in theorem 8.4.1. We have proved above that an arbitrary pair of linearizing output has to satisfy (8.155) around all points $\mathcal{Y}$ in an open and dense subset of a neighborhood of $\overline{\mathcal{X}}$, with $\omega_3 = \delta^1_{2,1}\Omega_3$ (compare (8.154), the fact that $\alpha = \nu/\delta^1_{2,1}$ as noticed just after (8.153), and the definition of Ω_3 in theorem 8.4.1). This implies that $\mathrm{d}h_1$ and $\mathrm{d}h_2$ are two independent linear combinations of Ω_3 and $\omega_2 - a\dot\Omega_3$ for a certain function a. This is exactly the form of a pair of linearizing outputs described in theorem 8.4.1. ∎

We now prove the four technical lemmae (lemmae 8.7.6, 8.7.7, 8.7.8 and 8.7.9) and then proceed with the proof of lemma 8.7.10 that was the cornerstone of the above proof.

Proof of lemma 8.7.6 : Suppose that Ω_1 and Ω_3 satisfy the identities (8.134)-(8.135) on a neighborhood of $\mathcal{Y}$. Then the Pfaffian system $\{\Omega_1, \Omega_3, \dot\Omega_3\}$ is completely integrable because (8.134)-(8.135) obviously imply that $\mathrm{d}\Omega_1$ and $\mathrm{d}\Omega_3$ are zero modulo $\{\Omega_1, \Omega_3, \dot\Omega_3\}$, and (8.135) implies that, for a certain 1-form Γ_3 and a certain function k, $\mathrm{d}\Omega_3 = \Omega_3 \wedge \Gamma_3 + k\Omega_1 \wedge \dot\Omega_3$, but taking the time-derivative of both sides yields

$$\mathrm{d}\dot\Omega_3 \;=\; \dot\Omega_3 \wedge \Gamma_3 \;+\; \Omega_3 \wedge \dot\Gamma_3 \;+\; \dot k\Omega_1 \wedge \dot\Omega_3 \;+\; k\dot\Omega_1 \wedge \dot\Omega_3 \;+\; k\Omega_1 \wedge \ddot\Omega_3$$

which obviously implies that $\mathrm{d}\dot\Omega_3$ is zero modulo $\{\Omega_1, \Omega_3, \dot\Omega_3\}$. Integrability of this Pfaffian system implies that there exists a function h_1 defined on a neighborhood of $\mathcal{Y}$ such that

$$\mathrm{d}h_1 \;=\; \lambda_1\Omega_1 \;+\; \lambda_2\Omega_3 \;+\; \lambda_3\dot\Omega_3$$

with λ_1, λ_2, λ_3 some functions, λ_1 nonzero at $\mathcal{Y}$. Then $\{\Omega_3, \mathrm{d}h_1\}$ is integrable because (8.135) implies that $\mathrm{d}\Omega_3$ is zero modulo $\{\Omega_3, \mathrm{d}h_1 \wedge \dot{\Omega}_3\}$, and hence modulo $\{\Omega_3, \mathrm{d}h_1\}$. Hence there is a second function h_2 such that

$$\mathrm{d}h_2 = \mu_1 \mathrm{d}h_1 + \mu_2 \Omega_3$$

with μ_1, μ_2 some functions, μ_2 nonzero at $\mathcal{Y}$. The functions h_1, h_2 built above, together with $a = -\lambda_3/\lambda_1$ and $J_1 = \begin{pmatrix} 1 & 0 \\ \mu_1 & \mu_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 0 & 1 \end{pmatrix}$ satisfy (8.136).

Conversely, suppose that (8.136) holds. Let us define $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ by

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = J_2 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} , \tag{8.157}$$

$$\begin{pmatrix} \Omega_4 \\ \Omega_3 \end{pmatrix} = \begin{pmatrix} 1 & -a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \tag{8.158}$$

i.e.

$$\Omega_3 = \Omega_2 - b\,\dot{\Omega}_1 \tag{8.159}$$

$$\Omega_4 = \Omega_1 - a\dot{\Omega}_3 = \Omega_1 - a\left(\dot{\Omega}_2 - \dot{b}\,\dot{\Omega}_1 - b\,\ddot{\Omega}_1\right) \tag{8.160}$$

We shall use the following basis (over smooth functions) for the space of all 1-forms :

$$\begin{aligned} \{\Omega_1, \Omega_2, \Omega_3, \dot{\Omega}_2, \Omega_4, \dot{\Omega}_4,& \\ \ddot{\Omega}_2, \Omega_2^{(3)}, \Omega_2^{(4)}, \ldots\ldots &, \\ \Omega_1^{(4)}, \Omega_1^{(5)}, \Omega_1^{(6)}, \ldots\ldots &\} \end{aligned} \tag{8.161}$$

where, in addition, $\{\Omega_1, \Omega_2, \Omega_3, \dot{\Omega}_2\}$ is a basis of $\mathrm{Span}\{\mathrm{d}\mathbf{x}\}$.

Then (8.143) implies that the Pfaffian system (Ω_3, Ω_4) is completely integrable and therefore that there exists some 1-forms $\Gamma_{i,j}$ such that

$$\left.\begin{aligned} \mathrm{d}\Omega_3 &= \Omega_3 \wedge \Gamma_{3,3} + \Omega_4 \wedge \Gamma_{3,4} \\ \mathrm{d}\Omega_4 &= \Omega_3 \wedge \Gamma_{4,3} + \Omega_4 \wedge \Gamma_{4,4} \end{aligned}\right\} \tag{8.162}$$

It is possible to express the 1-forms $\Gamma_{i,j}$ in (8.162) as (finite) linear combinations of the forms in (8.161), and it is always possible to choose them such that, for $i = 3, 4$,

$$\left.\begin{aligned} &\Gamma_{i,3} \text{ has no } \Omega_3 \text{ term,} \\ &\Gamma_{i,4} \text{ has no } \Omega_3 \text{ term and no } \Omega_4 \text{ term.} \end{aligned}\right\} \tag{8.163}$$

Taking the exterior derivative of (8.160) yields

$$\mathrm{d}\Omega_1 = \mathrm{d}\Omega_4 + a\,\mathrm{d}\dot{\Omega}_3 - \dot{\Omega}_3 \wedge \mathrm{d}a ,$$

and taking the time-derivative of the first equation in (8.162) yields

$$\mathrm{d}\dot{\Omega}_3 = \Omega_3 \wedge \dot{\Gamma}_{3,3} + \dot{\Omega}_3 \wedge \Gamma_{3,3} + \Omega_4 \wedge \dot{\Gamma}_{3,4} + \dot{\Omega}_4 \wedge \Gamma_{3,4}$$

and finally, the two above equations yield, since $\dot{\Omega}_3 = \frac{\Omega_1 - \Omega_4}{a}$,

$$\begin{aligned} \mathrm{d}\Omega_1 = {} & \Omega_1 \wedge \left(\Gamma_{3,3} - \frac{\mathrm{d}a}{a}\right) + \Omega_3 \wedge \left(\Gamma_{4,3} + a\,\dot{\Gamma}_{3,3}\right) \\ & + \Omega_4 \wedge \left(\Gamma_{4,4} - \Gamma_{3,3} + a\,\dot{\Gamma}_{3,4} + \frac{\mathrm{d}a}{a}\right) + a\,\dot{\Omega}_4 \wedge \Gamma_{3,4} . \end{aligned} \tag{8.164}$$

On the other hand, since $(\Omega_1, \Omega_2) = (X_1, X_2)^{\perp}$, the Pfaffian system defined by (Ω_1, Ω_2) can be defined with the help of the variable x (i.e. the four coordinates of x) only, and therefore (see (8.192) in the Appendix), its Cartan characteristic system is at most Span$\{\mathrm{d}\mathbf{x}\}$, i.e. at most $\{\Omega_1, \Omega_2, \Omega_3, \dot{\Omega}_2\}$, which implies that, for some functions k_1 and k_2,

$$\left.\begin{array}{rcl} \mathrm{d}\Omega_1 & \equiv & k_1\,\Omega_3 \wedge \dot{\Omega}_2 \\ \mathrm{d}\Omega_2 & \equiv & k_2\,\Omega_3 \wedge \dot{\Omega}_2 \end{array}\right\} \text{modulo } \{\Omega_1, \Omega_2\}\ . \tag{8.165}$$

The first equation above implies, from (8.164) and (8.163), and using the fact that the 1-forms in (8.161) are a basis for all 1-forms, that $\Gamma_{4,3} + a\dot{\Gamma}_{3,3}$ is a linear combination of Ω_1, Ω_2, Ω_3, $\dot{\Omega}_2$ and Ω_4, $\Gamma_{4,4} - \Gamma_{3,3} + a\dot{\Gamma}_{3,4} + \frac{\mathrm{d}a}{a}$ is a linear combination of Ω_1, Ω_2, Ω_3 and Ω_4, and $\Gamma_{3,4}$ is a linear combination of Ω_1, Ω_2 and $\dot{\Omega}_4$, with the coefficient of Ω_4 in $\Gamma_{4,3} + a\dot{\Gamma}_{3,3}$ equal to the coefficient of Ω_3 in $\Gamma_{4,4} - \Gamma_{3,3} + a\dot{\Gamma}_{3,4} + \frac{\mathrm{d}a}{a}$:

$$\Gamma_{4,3} + a\,\dot{\Gamma}_{3,3} = c_1\Omega_1 + c_2\Omega_2 + c_3\Omega_3 + c_4\dot{\Omega}_2 + d_3\Omega_4 \tag{8.166}$$

$$\Gamma_{4,4} - \Gamma_{3,3} + a\,\dot{\Gamma}_{3,4} + \frac{\mathrm{d}a}{a} = d_1\Omega_1 + d_2\Omega_2 + d_3\Omega_3 + d_4\Omega_4 \tag{8.167}$$

$$\Gamma_{3,4} = e_1\Omega_1 + e_2\Omega_2 + e_3\dot{\Omega}_4 \tag{8.168}$$

and finally, (8.164) yields

$$\left.\begin{array}{l} \mathrm{d}\Omega_1 = \Omega_1 \wedge \Delta_1 + \Omega_2 \wedge \Delta_2 + c_4\,\Omega_3 \wedge \dot{\Omega}_2 \\ \text{with} \left\{\begin{array}{rcl} \Delta_1 & = & \Gamma_{3,3} - \frac{\mathrm{d}a}{a} - c_1\Omega_3 - d_1\Omega_4 - ae_1\dot{\Omega}_4 \\ \Delta_2 & = & -\,c_2\Omega_3 - d_2\Omega_4 - ae_2\dot{\Omega}_4 \end{array}\right. \end{array}\right\} \tag{8.169}$$

Now, from (8.159),

$$\mathrm{d}\Omega_2 = \mathrm{d}\Omega_3 + b\,\mathrm{d}\dot{\Omega}_1 + \mathrm{d}b \wedge \dot{\Omega}_1\ ,$$

which allows, getting $\mathrm{d}\Omega_3$ from (8.162) and $\mathrm{d}\dot{\Omega}_1$ from (8.169)'s time-derivative, and using the fact that $\dot{\Omega}_1 = \frac{\Omega_2 - \Omega_3}{b}$ and $\dot{\Omega}_3 = \frac{\Omega_1 - \Omega_4}{a}$, to compute $\mathrm{d}\Omega_2$ and, forgetting the exterior products starting with Ω_1, Ω_2 or Ω_3, to obtain

$$\mathrm{d}\Omega_2 \equiv b\,(\frac{c_4}{a} - d_2)\,\dot{\Omega}_2 \wedge \Omega_4 - a\,b\,e_2\dot{\Omega}_2 \wedge \dot{\Omega}_4 + e_3\,\Omega_4 \wedge \dot{\Omega}_4 \ \text{ modulo } \{\Omega_1, \Omega_2, \Omega_3\} \tag{8.170}$$

which, since the second identity in (8.165) implies $\mathrm{d}\Omega_2 \equiv 0$ modulo $\{\Omega_1, \Omega_2, \Omega_3\}$, yields

$$c_4 = a\,d_2 \quad \text{and} \quad e_2 = e_3 = 0\ . \tag{8.171}$$

We get (8.134) from (8.169) with $e_2 = 0$ after substituting Ω_4 for $\Omega_1 - a\dot{\Omega}_3$. The same substitution in (8.160)-(8.168) with $e_2 = e_3 = 0$ yields (8.135). ∎

Proof of lemma 8.7.7 :

Point 1 : First, let us notice that b cannot be identically zero around the considered point, because this would imply x-dynamic linearizability, which, from theorem 8.3.1, contradicts (8.51). Define Ω_1 and Ω_2 by $(\Omega_1, \Omega_2)^T = J_2(\omega_1, \omega_2)^T$; then (8.137) implies that the Pfaffian system $\{\Omega_1, \Omega_2 - a\dot{\Omega}_1 - b\ddot{\Omega}_1\}$ is completely integrable, which implies

$$\mathrm{d}\Omega_1 = \Omega_1 \wedge \Gamma_1 + (\Omega_2 - a\dot{\Omega}_1 - b\ddot{\Omega}_1) \wedge \Gamma_2,$$

for some 1-forms Γ_1 and Γ_2. On the other hand, because $\{\Omega_1, \Omega_2\}$ span the annihilator of $\{X_1, X_2\}$, the characteristic system of this Pfaffian system is included in Span$\{\mathrm{d}\mathbf{x}\}$ (see (8.192) in the Appendix), and hence one must have $\mathrm{d}\Omega_1 \equiv k\eta_1 \wedge \eta_2$ modulo $\{\Omega_1, \Omega_2\}$ with k a function and η_1 and η_2 two form in Span$\{\mathrm{d}\mathbf{x}\}$. This implies, since b does not vanish and $\ddot{\Omega}_1$ is not in

Span{d$\mathbf{x}$}, that, in the above relation, Γ_2 is a linear combination of Ω_1, Ω_2 and $a\dot{\Omega}_1 + b\ddot{\Omega}_1$, which in turn implies, for a certain function k,

$$\mathrm{d}\Omega_1 \;\equiv\; k\,\Omega_2 \wedge (a\dot{\Omega}_1 + b\ddot{\Omega}_1) \;\text{ modulo } \Omega_1 \;.$$

This implies that Ω_1 is in the derived system of the Pfaffian system $\{\Omega_1, \Omega_2\}$, and therefore, from (8.54)-(8.55), that Ω_1 is collinear to ω_1. The above relation with Ω_1 collinear to ω_1 contradicts (8.54) because $a\dot{\Omega}_1 + b\ddot{\Omega}_1$ is not a linear combination of ω_1, ω_2 and $\dot{\omega}_2$.

Point 2 : Suppose that (8.137) holds. From, lemma 8.7.6, the identities (8.134)-(8.135) must hold locally with

$$\begin{array}{rcl} \Omega_1 &=& \omega_1 \\ \Omega_3 &=& \omega_2 \,+\, \alpha\,\omega_1 \,-\, b\,\dot{\omega}_1 \end{array}$$

and in particular, this would imply that

$$\mathrm{d}\omega_1 \;\equiv\; 0 \;\text{ modulo } \{\omega_1, \omega_2 - b\dot{\omega}_1, \dot{\omega}_2 + (\alpha - \dot{b})\dot{\omega}_1 - b\ddot{\omega}_1\}$$

which is impossible, because, from (8.54),

$$\mathrm{d}\omega_1 \wedge \omega_1 \wedge (\omega_2 - b\dot{\omega}_1) \wedge \Big(\dot{\omega}_2 + (\alpha - \dot{b})\dot{\omega}_1 - b\ddot{\omega}_1\Big) \;=\; b^2 \delta^2_{1,2}\, \omega_2 \wedge \dot{\omega}_2 \wedge \omega_1 \wedge \dot{\omega}_1 \wedge \ddot{\omega}_1 .$$ ∎

Proof of lemma 8.7.8 : The expression for Ω in (8.139) implies

$$\begin{array}{rcl} \lambda_1\omega_1 &=& \Omega\lambda_2\omega_2 \;-\; \lambda_3\dot{\omega}_2 \;, \\ \lambda_1\dot{\omega}_1 &=& -\,\dot{\lambda}_1\omega_1 \;+\; \dot{\Omega} \;-\; \dot{\lambda}_2\omega_2 \;-\; (\lambda_3 + \lambda_2)\dot{\omega}_2 \;-\; \lambda_3\ddot{\omega}_2 \;. \end{array} \tag{8.172}$$

Using the above relations in (8.55), one obtains that $\lambda_1^2 \mathrm{d}\omega_2$ is equal to $\lambda_3\,(\delta^1_{2,1}\lambda_3 + 2\gamma\lambda_1)\,\dot{\omega}_2 \wedge \dot{\omega}_2$ modulo $\{\omega_2, \Omega, \dot{\Omega}\}$. This proves the lemma. ∎

Proof of lemma 8.7.9 : By simple substitution, it is clear that the forms of f and g given in (8.63) satisfy equations (8.141)-(8.142). Let us prove the converse. Since $\frac{\partial g}{\partial z_4} \neq 0$, one may define $h = 1/\frac{\partial g}{\partial z_4}$. Equation (8.141) then yields

$$\left(\frac{\partial h}{\partial z_4}\right)^2 \;-\; 2\,h\,\frac{\partial^2 h}{\partial z_4^2} \;=\; 0\;,$$

whose non-vanishing solutions are exactly the squares, and opposite of squares of nonzero polynomials in z_4 of degree at most 1, with coefficients function of z_1, z_2 and z_3 ; if the degree is 0, g is affine in z_4, if it is 1, g is homographic in z_4, still with coefficients function of z_1, z_2 and z_3, this yields the form for g given in (8.63). Substituting g for its expression given by (8.63) in equation (8.142) yields $(c_0 + c_1 z_4)\frac{\partial^3 f}{\partial z_4^3} + 3c_1\frac{\partial^2 f}{\partial z_4^2} = 0$, which states that $(c_0 + c_1 z_4)f$ is a polynomial of degree at most 2 in z_4 and therefore implies that f is of the form given in (8.63). ∎

Proof of lemma 8.7.10 :

<u>1⇔2</u> : This is an obvious consequence of lemma 8.7.6 because (8.145)-(8.146) are identical to (8.134)-(8.135).

<u>3⇒2</u> : Let b be defined by (8.147) :

$$b \;=\; \frac{2\,\gamma}{\delta^1_{2,1}}\;,$$

and α be the one from relation (8.58). Define Ω_1 and Ω_3 as in (8.144), with $\lambda = 0$. Relation (8.58) implies $\mathrm{d}\Omega_3 \equiv 0$ modulo Ω_3, so it implies a fortiori (8.146). Now (8.145) is equivalent here to

$$\mathrm{d}\omega_2 \equiv 0 \quad \text{modulo } \{\,\omega_2\,,\, \omega_1 - b\dot{\omega}_2\,,\, \dot{\omega}_1 + (\alpha - \dot{b})\dot{\omega}_2 - b\ddot{\omega}_2\,\}$$

but a simple computation from (8.55) show that this is true when $b = \frac{2\gamma}{\delta^1_{2,1}}$.

4⇒3 : From proposition 8.4.5, if point 4 is true, then some other coordinates may be found where the system has the simpler form (8.66). We shall compute in these coordinates with the following choice

$$\begin{aligned} \omega_1 &= \mathrm{d}\zeta_2 - \zeta_3\mathrm{d}\zeta_1 \\ \omega_2 &= \mathrm{d}\zeta_3 - (q_0 + \zeta_4 q_1)\,\mathrm{d}\zeta_1 - (p_1 + w_1 q_1)\,\omega_1 \end{aligned} \tag{8.173}$$

On one hand, one has

$$\mathrm{d}\omega_2 \equiv 0 \text{ modulo } \{\omega_2, \mathrm{d}\zeta_1, \mathrm{d}w_1\} \tag{8.174}$$

by computing the exterior derivative of ω_2 given by (8.173) and replacing $\mathrm{d}\zeta_1$ and $\mathrm{d}w_1$ with zero and $\mathrm{d}\zeta_3$ with $(p_1 + w_1 q_1)\,\mathrm{d}\zeta_2$.

On the other hand, from (8.52), $\{\omega_1, \omega_2, \dot\omega_2\}$ is a basis of $\{\mathrm{d}\zeta_1, \mathrm{d}\zeta_2, \mathrm{d}\zeta_3\}$ and hence one has

$$\mathrm{d}\zeta_1 = \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\dot\omega_2$$

for some functions λ_1, λ_2 and λ_3. Applying lemma 8.7.8 for $\Omega = \mathrm{d}\zeta_1$, and noticing that λ_3 cannot vanish because $\omega_1 \wedge \omega_2 \wedge \mathrm{d}\zeta_1$ does not vanish from (8.173) yields, from (8.174) :

$$2\gamma\,\lambda_1 + \delta^1_{2,1}\,\lambda_3 = 0\,.$$

The above two relation imply, since by assumption $\delta^1_{2,1}$ does not vanish, that $\mathrm{d}\zeta_1$ is a linear combination of ω_2 and $\omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot\omega_2$, and this clearly implies point 3.

2⇒4 : This is the long and difficult part of the proof. It is all done using the symbolic computation system Maple, version 5, release 3, with the package "liesymm" to manipulate differential forms, in the coordinates of the normal form (8.60).

We are now working in coordinates, with system (8.60) for some f and g. We make the following choice for ω_1 and ω_2 :

$$\begin{aligned} \omega_1 &= \mathrm{d}z_2 - z_3\,\mathrm{d}z_1 \\ \omega_2 &= \mathrm{d}z_3 - g\,\mathrm{d}z_1 - \left(\tfrac{\partial f}{\partial z_4} + v_1\tfrac{\partial g}{\partial z_4}\right)\omega_1\ . \end{aligned} \tag{8.175}$$

The idea of the proof is quite straightforward : We suppose that there exists functions α, λ and b satisfying (8.145)-(8.146), we write these equations explicitly in terms of α, λ and b, and we eliminate α, λ and b to obtain the conditions on f and g are as described in point 4.

Step 1. *With the choice (8.175) for ω_1 and ω_2, we have the following decomposition of $\mathrm{d}\omega_1$ and $\mathrm{d}\omega_2$, more precise than (8.54) and (8.55) in proposition 8.4.1 :*

$$\mathrm{d}\omega_1 = \omega_1 \wedge \left(\delta^0_{1,1}\omega_2 + \delta^2_{1,1}\dot\omega_2\right) + \delta^2_{1,2}\,\omega_2 \wedge \dot\omega_2\ , \tag{8.176}$$

$$\begin{aligned} \mathrm{d}\omega_2 = \omega_1 \wedge \left(\delta^0_{2,1}\omega_2 + \delta^1_{2,1}\dot\omega_1 + \delta^2_{2,1}\dot\omega_2 - \gamma\ddot\omega_2\right) &+ \omega_2 \wedge \left(\delta^1_{2,2}\dot\omega_1 + \delta^2_{2,2}\dot\omega_2\right) \\ &+ \gamma\,\dot\omega_1 \wedge \dot\omega_2 \end{aligned} \tag{8.177}$$

for some functions $\delta^k_{i,j}$ and γ that may be computed explicitly using f, g and some of their partial derivatives.

Indeed, (8.54) reads

$$\mathrm{d}\omega_1 = \omega_1 \wedge \Gamma_1 + \delta^2_{1,2}\omega_2 \wedge \dot\omega_2\ ,$$

for some form Γ_1, but $\mathrm{d}\omega_1 = \mathrm{d}z_1 \wedge \mathrm{d}z_3$ and $\{\omega_1, \omega_2, \dot\omega_2\}$ is a basis of $\{\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3\}$ —because it is the characteristic system of $\{\omega_1\}$ from the above equation— so that Γ_1 must be a linear combination of ω_1, ω_2 and $\dot\omega_2$. This implies (8.176).

Also, (8.55) reads

$$d\omega_2 \;=\; \omega_2\wedge\Gamma_2 \;+\; \omega_1\wedge\big(\delta_{2,1}^0\omega_2+\delta_{2,1}^1\dot\omega_1+\delta_{2,1}^2\dot\omega_2-\gamma\ddot\omega_2\big) \;+\; \gamma\dot\omega_1\wedge\dot\omega_2\;,$$

for a certain form Γ_2, but

$$d\omega_2 \;=\; dz_1\wedge dg \;-\; (\frac{\partial f}{\partial z_4}+v_1\frac{\partial g}{\partial z_4})\,dz_1\wedge dz_3 \;-\; d(\frac{\partial f}{\partial z_4}+v_1\frac{\partial g}{\partial z_4})\wedge\omega_1$$

and hence $d\omega_2$ is, modulo $\{\omega_1\}$, a linear combination of dz_1, dz_2, dz_3 and dz_4, i.e. of ω_1, ω_2, $\dot\omega_1$ and $\dot\omega_2$; this implies that Γ_2 must be a linear combination of ω_1, ω_2, $\dot\omega_1$ and $\dot\omega_2$, and therefore (8.177).

Step 2. *If α, λ and b satisfy (8.145)-(8.146), then*

$$\begin{array}{l}\lambda \textit{ may depend on } z_1,\, z_2,\, z_3,\, z_4,\, v_2-v_1 f(z_1,z_2,z_3,z_4) \textit{ only,}\\ \alpha \textit{ and } b \textit{ may depend on } z_1,\, z_2,\, z_3,\, z_4,\, v_1,\, v_2,\, \dot v_1,\, \dot v_2 \textit{ only.}\end{array}\qquad(8.178)$$

Relations (8.157) and (8.159) imply :

$$\Omega_1 \;=\; \omega_2 \,+\, \lambda\,\omega_1 \qquad (8.179)$$

$$\Omega_2 \;=\; \omega_1+\alpha\Omega_1 \;=\; (1+\alpha\lambda)\,\omega_1 \;+\; \alpha\,\omega_2 \qquad (8.180)$$

$$\Omega_3 \;=\; \omega_1+\alpha\Omega_1-b\dot\Omega_1 \;=\; \alpha\,(\omega_2+\lambda\omega_1) \;-\; b\left(\dot\omega_2 \,+\, \lambda\dot\omega_1 \,+\, (\dot\lambda-\frac{1}{b})\,\omega_1\right) \qquad (8.181)$$

$$\begin{aligned}\dot\Omega_3 \;=\;& \left(\dot\lambda(\alpha-\dot b)-b\ddot\lambda\right)\omega_1 \;+\; \dot\alpha\,(\omega_2+\lambda\omega_1)\\ &+\,(1-2b\dot\lambda)\,\dot\omega_1 \;+\; (\alpha-\dot b)\,(\dot\omega_2\,+\,\lambda\dot\omega_1) \;-\; b\,(\ddot\omega_2\,+\,\lambda\ddot\omega_1)\end{aligned} \qquad (8.182)$$

Taking the exterior derivative of (8.181) and (8.179) yields

$$d\Omega_3 \;=\; d\omega_1 \;+\; \alpha d\Omega_1 \;-\; b d\dot\Omega_1 \;+\; d\alpha\wedge\Omega_1 \;-\; db\wedge\dot\Omega_1 \qquad (8.183)$$

$$\text{with}\quad d\Omega_1 \;=\; d\omega_2 \;+\; \lambda\,d\omega_1 \;+\; d\lambda\wedge\omega_1 \qquad (8.184)$$

$$d\dot\Omega_1 \;=\; d\dot\omega_2 \;+\; \dot\lambda\,d\omega_1 \;+\; \lambda\,d\dot\omega_1 \;+\; d\dot\lambda\wedge\omega_1 \;+\; d\lambda\wedge\dot\omega_1 \qquad (8.185)$$

Relation (8.183) implies :

$$\begin{aligned}d\Omega_3 \;=\;& (1+\alpha\lambda-b\dot\lambda)d\omega_1+\alpha d\omega_2+\alpha d\lambda\wedge\omega_1-b\,(d\dot\omega_2+\lambda d\dot\omega_1+d\dot\lambda\wedge\omega_1+d\lambda\wedge\dot\omega_1)\\ &+\,d\alpha\wedge(\omega_2+\lambda\omega_1) \;+\; db\wedge\frac{\Omega_3-\omega_1-\alpha(\omega_2+\lambda\omega_1)}{b}\end{aligned}$$

Taking the time-derivative of both sides in (8.176) and (8.177), we have

$$\left.\begin{aligned}d\dot\omega_1 \;=\;& \omega_1\wedge\left(\dot\delta_{1,1}^0\omega_2 \,+\, (\delta_{1,1}^0+\dot\delta_{1,1}^2)\dot\omega_2+\delta_{1,1}^2\ddot\omega_2\right)\\ &+\,\omega_2\wedge\left(-\,\delta_{1,1}^0\dot\omega_1 \,+\, \dot\delta_{1,2}^2\dot\omega_2 \,+\, \delta_{1,2}^2\ddot\omega_2\right)\\ &+\,\delta_{1,1}^2\,\dot\omega_1\wedge\dot\omega_2\\ d\dot\omega_2 \;=\;& \omega_1\wedge\left(\dot\delta_{2,1}^0\omega_2+\dot\delta_{2,1}^1\dot\omega_1+(\delta_{2,1}^0+\dot\delta_{2,1}^2)\dot\omega_2+\delta_{2,1}^1\ddot\omega_1+(\delta_{2,1}^2-\dot\gamma)\ddot\omega_2-\gamma\omega_2^{(3)}\right)\\ &+\,\omega_2\wedge\left((\dot\delta_{2,2}^1-\delta_{2,1}^0)\dot\omega_1+\dot\delta_{2,2}^2\dot\omega_2+\delta_{2,2}^1\ddot\omega_1+\delta_{2,2}^2\ddot\omega_2\right)\\ &+\,\dot\omega_2\wedge\left((-\delta_{2,1}^2+\delta_{2,2}^1-\dot\gamma)\dot\omega_1 \,-\, \gamma\ddot\omega_1\right)\end{aligned}\right\} \qquad (8.186)$$

Equation (8.146) implies in particular that $d\Omega_3\equiv 0$ modulo $\{\omega_1,\Omega_1,\Omega_3\}$, i.e. —see (8.181)— modulo $\{\omega_1,\omega_2,\dot\omega_2+\lambda\dot\omega_1\}$. Equations (8.176), (8.177) and (8.186) imply

$$\left.\begin{array}{llllll} d\omega_1 & \equiv & 0 & d\dot\omega_1 & \equiv & 0\\ d\omega_2 & \equiv & 0 & d\dot\omega_2 & \equiv & \lambda\gamma\,\dot\omega_1\wedge\ddot\omega_1\end{array}\right\}\ \text{modulo}\ \{\omega_1,\omega_2,\dot\omega_2+\lambda\dot\omega_1\}\,.$$

Then, from (8.183), (8.184), (8.185),

$$\mathrm{d}\Omega_3 \equiv -b\,(\lambda\gamma\,\dot\omega_1\wedge\ddot\omega_1+\mathrm{d}\lambda\wedge\dot\omega_1)\ \text{modulo}\ \{\omega_1,\omega_2,\dot\omega_2+\lambda\dot\omega_1\}\,,$$

which in turn implies

$$\mathrm{d}\lambda\,+\,\lambda\gamma\ddot\omega_1 \equiv 0\ \ \text{modulo}\ \{\omega_1,\omega_2,\dot\omega_1,\dot\omega_2\}\,.$$

Since $\{\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3,\mathrm{d}z_4\}$ is another basis for $\{\omega_1,\omega_2,\dot\omega_1,\dot\omega_2\}$ and, from (8.175) and (8.60),

$$\begin{array}{rcl}\dot\omega_1 &=& \mathrm{d}z_4\,+\,v_1\mathrm{d}z_3\,-\,(f+v_1g)\,\mathrm{d}z_1\ ,\\ \ddot\omega_1 &=& \mathrm{d}\,(v_2-fv_1)\,+\,v_1\mathrm{d}f\,+\,\dot v_1\mathrm{d}z_3\,+\,\left(\dot f\,+\,v_1\dot g\,+\,\dot v_1 g\right)\mathrm{d}z_1\ ,\end{array}\tag{8.187}$$

that $\mathrm{d}\lambda$ is a linear combination of $\mathrm{d}\,(v_2-fv_1)$, $\mathrm{d}z_1$, $\mathrm{d}z_2$, $\mathrm{d}z_3$ and $\mathrm{d}z_4$; this proves the statement on λ in (8.178).

Replacing $\dot\omega_1$ and $\ddot\omega_1$ with zero and $\dot\omega_2$ with $(\frac{1+\alpha\lambda}{b}-\dot\lambda)\omega_1+\frac{\alpha}{b}\omega_2$ in the expression of $\Omega_1\wedge\dot\Omega_3$ obtained from (8.179) and (8.182) obviously yields only some terms in $\omega_1\wedge\omega_2$, $\omega_1\wedge\ddot\omega_2$ and $\omega_2\wedge\ddot\omega_2$, hence

$$\Omega_1\wedge\dot\Omega_3 \equiv 0\ \ \text{modulo}\ \{\Omega_3,\dot\omega_1,\ddot\omega_1,\omega_1\wedge\omega_2,\omega_1\wedge\ddot\omega_2,\omega_2\wedge\ddot\omega_2\}\ .$$

Therefore, Equation (8.146) implies in particular that $\mathrm{d}\Omega_3\equiv 0$ modulo $\{\Omega_3,\dot\omega_1,\ddot\omega_1,\omega_1\wedge\omega_2,\omega_1\wedge\ddot\omega_2,\omega_2\wedge\ddot\omega_2\}$, i.e. modulo $\{b\dot\omega_2-\alpha(\omega_2+\lambda\omega_1)+(b\dot\lambda-1)\omega_1,\dot\omega_1,\ddot\omega_1,\omega_1\wedge\omega_2,\omega_1\wedge\ddot\omega_2,\omega_2\wedge\ddot\omega_2\}$. From (8.176), (8.177) and (8.186), we have :

$$\begin{array}{rclcrcl}\mathrm{d}\omega_1 &\equiv& 0 && \mathrm{d}\dot\omega_1 &\equiv& 0\\ \mathrm{d}\omega_2 &\equiv& 0 && \mathrm{d}\dot\omega_2 &\equiv& -\gamma\,\omega_1\wedge\omega_2^{(3)}\end{array}$$

modulo $\{b\dot\omega_2-\alpha(\omega_2+\lambda\omega_1)+(b\dot\lambda-1)\omega_1,\dot\omega_1,\ddot\omega_1,\omega_1\wedge\omega_2,\omega_1\wedge\ddot\omega_2,\omega_2\wedge\ddot\omega_2\}$. Hence, from (8.183), (8.184), (8.185)

$$\begin{array}{rcl}\mathrm{d}\Omega_3 &\equiv& \omega_1\wedge\left(b\gamma\,\omega_2^{(3)}\,-\,\alpha\mathrm{d}\lambda\,+\,b\,d\mathrm{d}\dot\lambda\,+\,\dfrac{\mathrm{d}b}{b}\right)\\ && +\,(\omega_2+\lambda\omega_1)\wedge\left(\dfrac{\alpha}{b}\mathrm{d}b\,-\,\mathrm{d}\alpha\right)\end{array}$$

This implies in particular that

$$\left.\begin{array}{l}b\gamma\,\omega_2^{(3)}\,-\,\alpha\mathrm{d}\lambda\,+\,b\,d\mathrm{d}\dot\lambda\,+\,\dfrac{\mathrm{d}b}{b}\ \equiv\ 0\\ \dfrac{\alpha}{b}\mathrm{d}b\,-\,\mathrm{d}\alpha\ \equiv\ 0\end{array}\right\}\ \text{modulo}\ \{\omega_1,\omega_2,\dot\omega_1,\dot\omega_2,\ddot\omega_1,\ddot\omega_2\}\,.\tag{8.188}$$

We have already shown above that $\mathrm{d}\lambda$ is a linear combination of $\omega_1,\omega_2,\dot\omega_1,\dot\omega_2,\ddot\omega_1$; hence $\mathrm{d}\dot\lambda$ is a linear combination of $\omega_1,\omega_2,\dot\omega_1,\dot\omega_2,\ddot\omega_1,\ddot\omega_2,\omega_1^{(3)}$. This and the above equations imply that $\mathrm{d}b$ and $\mathrm{d}\alpha$ are linear combinations of ω_1, ω_2, $\dot\omega_1$, $\dot\omega_2$, $\ddot\omega_1$, $\ddot\omega_2$, $\omega_1^{(3)}$, $\omega_2^{(3)}$. This yields the second statement in (8.178) for $\{\omega_1,\omega_2,\dot\omega_1,\dot\omega_2,\ddot\omega_1,\ddot\omega_2,\omega_1^{(3)},\omega_2^{(3)}\}$ is another basis for $\{\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3,\mathrm{d}z_4,\mathrm{d}v_1,\mathrm{d}v_2,\mathrm{d}\dot v_1,\mathrm{d}\dot v_2\}$.

Note that the second relation in (8.188) actually implies that $\alpha\mathrm{d}b-b\mathrm{d}\alpha$ is a linear combination of $\omega_1,\omega_2,\dot\omega_1,\dot\omega_2,\ddot\omega_1,\ddot\omega_2$, i.e. that α/b depends on z_1,z_2,z_3,z_4,v_1,v_2 only (and not on $\dot v_1,\dot v_2$).

Step 3. *If α, λ and b satisfy (8.145)-(8.146) with b non vanishing, then λ must be identically zero.*

We prove, in coordinates, that (8.145)-(8.146) implies $\lambda = 0$, in a symbolic computation session conducted with *Maple V, release 3*, using the package `liesymm`. We are omitting the Maple session here ; it is in the original paper.

Step 4. *If α, λ and b satisfy (8.145) with λ identically zero, then $\delta^1_{2,1}$ cannot vanish and b must be given by*

$$b \;=\; \frac{2\,\gamma}{\delta^1_{2,1}}\;. \tag{8.189}$$

Since $\lambda = 0$, we have $\Omega_1 = \omega_2$, $\Omega_3 = \omega_1 \;-\; b\,\dot\omega_2 \;+\; \alpha\,\omega_2$, but from lemma 8.7.6, $\mathrm{d}\Omega_1$ satisfies (8.134), i.e.

$$\mathrm{d}\omega_2 \;\equiv\; 0 \quad \text{modulo}\;\{\,\omega_2\;,\;\Omega_3\;,\;\dot\Omega_3\,\}$$

From lemma 8.7.8 with $\Omega = \Omega_3$, $\lambda_1 = 1$, $\lambda_2 = \alpha$ and $\lambda_3 = -b$, the above relation implies that $b\,(b\delta^1_{2,1} - 2\gamma)$ is identically zero on U, but we assume here that b does not vanish, hence $b\delta^1_{2,1} - 2\gamma$ must be identically zero, and therefore $\delta^1_{2,1}$ does not vanish (because γ does not vanish), and b is given by (8.189).

Step 5. *If α, λ and b satisfy (8.145)-(8.146) with λ identically zero and b is given by $b = \dfrac{2\gamma}{\delta^1_{2,1}}$, there is a unique possible value for α, and f and g must be of the form (8.63)-(8.64)*

This computation was in the same *Maple V, release 3* session, that we are omitting. Substituting λ and b, (8.145)-(8.146) yields three equations. One is linear with respect to α and yields α ; substituting in the two others yields two equations that simplify into the PDEs (8.141)-(8.142).

8.8 Conclusion

The present paper provides, for the 4-dimensional affine system (8.5), some new necessary and sufficient conditions for existence of linearizing outputs depending on x and u. These conditions are easily computable. They also allow one to treat 3-dimensional non-affine systems. This is very much related to dynamic feedback linearization, or flatness, as explained in section 8.2, but this paper is however not a general answer to dynamic feedback linearizability of 4-dimensional systems with 2 inputs, for the following three reasons that are subjects for future research to end the study of this small dimension.

One restriction comes from the regularity assumptions. The example presented in section 8.5 shows that they are not necessary. A thorough treatment of singularities, or at least a clear identification of the real singularities of dynamic feedback linearization is therefore not achieved.

We also restrict our attention to "endogenous feedback". See [68, 40] for a discussion of general dynamic feedback and endogenous dynamic feedback. Note that the authors of this latter reference have announced a proof of the fact that general dynamic linearizability would imply linearizability by endogenous dynamic feedback, at least away from some singularities.

We have further restricted the class of dynamic linearization by requiring that the linearizing output depend on x and u only. The natural follow-up to this work is to decide whether systems which are not (x, u)-dynamic feedback linearizable are simply not dynamic feedback linearizable (at least endogenously), or if some are $(x, u, \dot u)$-dynamic linearizable for example... In fact, no example of a system of these dimensions which admits no pair of linearizing outputs depending on x and u but admits some depending on more time-derivatives of u has ever been exhibited. Since these dimensions are usually these of academic examples —because it is the smallest non-trivial ones— and have been studied a lot, it may seem reasonable to conjecture that the systems that are proved in the present paper to be non (x, u)-dynamic linearizable are indeed not linearizable by endogenous dynamic feedback.

Let us finally make a remark on the method of the proofs. In a sense, the present results amount to giving conditions for some nonlinear partial differential equations to have solutions (see section 8.2.6). Since the PDEs are high order —see (8.21)-(8.22), and for (x, u)-dynamic linearization, the order is higher— one might think that some sophisticated tools for checking integrability, like Spencer cohomology, should be involved. It turns out however that the proofs are all elementary, and never make use of more sophisticated tools than Frobenius theorem. Actually, when using the infinitesimal Brunovský form and writing the equations for the coefficients of decomposition in elementary transformations of the invertible transformation "$P(\frac{\mathrm{d}}{\mathrm{dt}})$"' instead of writing directly the equations for the linearizing outputs, as in the proof of theorem 8.4.1 or the "alternative" proof of case 6 in theorem 8.3.1, we use Frobenius theorem to write the equations in a convenient way (like the equation (8.144)-(8.145)-(8.146) for theorem 8.4.1), but then the arguments used to give conditions for existence of solutions to these equations are in a sense even not first order like Frobenius theorem, but "zeroth order", i.e. the solutions (α, λ and b in the case (8.144)-(8.145)-(8.146) may be explicitly computed (expression involving functions in the equations of the system) from part of the equations, and the compatibility conditions are obtained by substituting these expressions in the remaining equations. It is of course tempting to ask whether in general when using the infinitesimal Brunovský form to test for existence of linearizing outputs depending on a pre-defined number of time-derivatives of the inputs, this feature always appears —the equations for the coefficients of the invertible transformation contain enough non-differential equations to obtain them solving non-differential equations— or if this is particular to the small dimensions considered here.

Appendix : Some facts on Pfaffian systems

In this section, we recall some very basic definitions on Pfaffian systems, and some precise facts we are going to use. For details or proofs, see e.g. [100] or [18].

A **Pfaffian system** I of rank r around a point can be defined as a module (over smooth functions) of differential 1-forms which is generated by r 1-forms which are point-wisely linearly independent around this point, or also as an ideal of differential forms (of arbitrary degrees, with the exterior product as "multiplication"), which has the peculiarity of being generated by independent 1-forms. It is defined by giving r independent 1-forms. r 1-forms which generate the same module define the same Pfaffian system.

A **congruence** like $\Omega_1 \equiv \Omega_2$ modulo $\{\eta_1, \eta_2, \ldots\}$ where the Ω_i's are 2-forms and the η_j's are 1-forms (we only need this) means modulo the ideal generated by $\{\eta_1, \eta_2, \ldots\}$, i.e. it means that there exists some forms α_j such that $\Omega_1 - \Omega_2 = \eta_1 \wedge \alpha_1 + \eta_2 \wedge \alpha_2 + \ldots$; it is equivalent to $(\Omega_1 - \Omega_2) \wedge \eta_1 \wedge \eta_2 \wedge \ldots = 0$

A Pfaffian system also defines an "orthogonal distribution", spanned by the vector fields which annihilate these 1-forms.

We will only be interested in the case $m = 1$ or $m = 2$, and we therefore speak of the Pfaffian system $I = \{\omega\}$ or $I = \{\omega_1, \omega_2\}$.

It is **completely integrable** if it is, locally, generated by 1 (resp 2) exact 1-forms, or equivalently, by Frobenius theorem, if $\mathrm{d}\omega \equiv 0$ modulo $\{\omega\}$ (resp. $\mathrm{d}\omega_i \equiv 0$ modulo $\{\omega_1, \omega_2\}$ for $i = 1, 2$), or also if the orthogonal distribution being closed under Lie brackets. We call **first integral** of the Pfaffian system, or of the orthogonal distribution a function h such that $\mathrm{d}h \neq 0$ and $\mathrm{d}h \in I$.

Derived System

For a given Pfaffian system I, consider the module made of the forms of degree 1 which are in I and whose exterior derivative (form of degree 2) is also in I ; at points where it has constant rank, this module defines a Pfaffian system called the **derived system** $I^{(1)}$ of I. Iterating this process, one ends either with the zero Pfaffian system or with an integrable one. A Pfaffian system is equal to its first derived system if and only if it is integrable. In the case of a Pfaffian system of rank 1, either it is integrable or its derived system is zero ; in the case of a Pfaffian system of rank 2 $\{\omega_1, \omega_2\}$, either it is integrable, or there exists (non both zero) functions λ_1 and λ_2 such that

$$\lambda_1 \mathrm{d}\omega_1 \; + \; \lambda_2 \mathrm{d}\omega_2 \quad \equiv \quad 0 \quad \text{modulo } \{\omega_1, \omega_2\} \ ,$$

and in this case the first derived system is $\{\lambda_1\omega_1 + \lambda_2\omega_2\}$ or there exists no such functions (i.e. the restrictions of $\mathrm{d}\omega_1$ and $\mathrm{d}\omega_2$ to the annihilator of $\{\omega_1, \omega_2\}$ are two linearly independent bilinear forms), and then the derived system is zero. The orthogonal distribution to the derived system of a given Pfaffian system is spanned by the orthogonal distribution to this system plus all the Lie brackets between two vector fields in this distribution :

$$(I^{(1)})^\perp \;\; = \;\; I^\perp \; + \; [\, I^\perp \, , \, I^\perp \,] \;\; .$$

Cartan Characteristic System

The Cartan characteristic system $\mathcal{C}(I)$ of a given Pfaffian system I may be defined through the vectors that it annihilates :

$$\mathcal{C}(I)^\perp \;\; = \;\; \{\, X \in I^\perp \, / \, [X, I^\perp] \subset I^\perp \,\} \; . \tag{8.190}$$

It is always integrable if it has constant rank, and a Pfaffian system is integrable if and only if it is equal to its Cartan characteristic system.

There is a basis of the Pfaffian system whose elements are linear combination of some $\mathrm{d}\psi_i$, with coefficients functions of the on ψ_i's only, where the ψ_i's are all first integrals of $\mathcal{C}(I)$, and $\mathcal{C}(I)$ is the smallest Pfaffian system having this property.

For a non-integrable system of rank 1 $\{\omega\}$, it is always possible, where the rank of the characteristic system is constant, to find $2p$ independent 1-forms η_i such that the rank of $\{\omega, \eta_1, \ldots, \eta_{2p}\}$ is $2p+1$ and

$$\mathrm{d}\omega \;\; \equiv \;\; \eta_1 \wedge \eta_2 \; + \; \eta_3 \wedge \eta_4 \; + \; \ldots \; + \; \eta_{2p-1} \wedge \eta_{2p} \quad \text{modulo } \{\omega\} \tag{8.191}$$

and the characteristic system is then $\{\omega, \eta_1, \ldots, \eta_{2p}\}$ (and this is automatically completely integrable).

For a non-integrable system of rank 2 $\{\omega_1, \omega_2\}$, all we need is the following : if it is possible to express this Pfaffian system with 4 variables $\chi_1, \chi_2, \chi_3, \chi_4$ (i.e. there exists a basis of this Pfaffian system made of two 1-forms which are linear combinations of $\mathrm{d}\chi_1, \mathrm{d}\chi_2, \mathrm{d}\chi_3, \mathrm{d}\chi_4$ with coefficients functions of $\chi_1, \chi_2, \chi_3, \chi_4$ only), then its characteristic system is $\{\mathrm{d}\chi_1, \mathrm{d}\chi_2, \mathrm{d}\chi_3, \mathrm{d}\chi_4\}$, and for any forms η_1 and η_2 such that $\{\omega_1, \omega_2, \eta_1, \eta_2\}$ spans the same module as $\{\mathrm{d}\chi_1, \mathrm{d}\chi_2, \mathrm{d}\chi_3, \mathrm{d}\chi_4\}$, we have

$$\mathrm{d}\omega_k \;\; = \;\; \omega_1 \wedge \Gamma_{k,1} \; + \; \omega_2 \wedge \Gamma_{k,2} \; + \; \lambda_k \, \eta_1 \wedge \eta_2 \tag{8.192}$$

for some 1-forms $\Gamma_{k,j}$ and some functions λ_k.

Chapitre 9

Reproduction de l'article:

D. Avanessoff et J.-B. Pomet, "Flatness and Monge parameterization of two-input systems, control-affine with 4 states or general with 3 states",

ESAIM Control Optim. Calc. Var., vol. 13 , pp. 237–264, 2007.

Abstract. This paper studies Monge parameterization, or differential flatness, of control-affine systems with four states and two controls. Some of them are known to be flat, and this implies admitting a Monge parameterization. Focusing on systems outside this class, we describe the only possible structure of such a parameterization for these systems, and give a lower bound on the order of this parameterization, if it exists. This lower-bound is good enough to recover the known results about "(x, u)-flatness" of these systems, with much more elementary techniques.

9.1 Introduction

In control theory, after a line of research on exact linearization by dynamic state feedback [53, 22, 23], the concept of differential flatness was introduced in 1992 in[37] (see also [40, 41]). Flatness is equivalent to exact linearization by dynamic state feedback of a special type, called "endogenous" [37], but, as pointed out in that reference, it has its own interest, maybe more important than linearity. An interpretation and framework for that notion is also proposed in Chapters 6 and 7 or in [104] ; see [71] for a recent review.

The *Monge problem* (see the the survey article [111], published in 1932, that mentions the prominent contributions [48] and [20], and others) is the one of finding explicit formulas giving the "general solution" of an under-determined system of ODEs as functions of some arbitrary functions of time and a certain number of their time-derivatives (in fact [111] allows to change the independent variable, but we keep it to be time). Let us call such formulas a *Monge parameterization*, its *order* being the number of time-derivatives.

The authors of [37] already made the link with the above mentioned work on under-determined systems of ODEs dating back from the beginning of 20^{th} century ; for instance, they used [48, 20] to obtain, in [88, 73] some results on flatness or linearizability of control systems.

Let us precise the relation between flatness and Monge parameterizability : flatness is existence of some functions —we call this collection of functions a *flat output*— of the state, the controls and a certain number j of time-derivatives of the control, that "invert" the formulas of a Monge parameterization, i.e. a solution $t \mapsto (x(t), u(t))$ of the control system corresponds to only one choice of the arbitrary functions of time appearing in the parameterization, given by these functions. Let us call j the *order* of the flat output.

Characterizing differential flatness, or dynamic state feedback linearizability is still an open problem [42], apart from the case of single-input systems [22, 20]. The main difficulty is that the order of a parameterization or a flat output, if there exists any, is not known beforehand : for a given system, if one can construct a parameterization, or a flat output, it has a definite order, but if, for some integer j, one prove that there is no parameterization of order j, then it might admit a parameterization of higher order, and we do not know any a priori bound on the possible j's. In the present paper, we consider systems of the smallest dimensions for which the answer is not known; we do not really overcome the above mentioned "main difficulty", in the sense that we only say that our class of systems does not admit a parameterization of order less than some numbers, but the description of the parameterization that we give, and the resulting system of PDEs is valid at any order.

Consider a general control-affine system in $\mathbb{R}^4$ with two controls, where $\xi \in \mathbb{R}^4$ is the state, $\widetilde{w}_1$ and $\widetilde{w}_2$ are the two scalar controls and X_0, X_1 and X_2 are three smooth vector fields :

$$\dot{\xi} = X_0(\xi) + \widetilde{w}_1 X_1(\xi) + \widetilde{w}_2 X_2(\xi) \ .$$

In Chapter 8, one can find a necessary and sufficient condition on X_0, X_1, X_2 for this system to admit a flat output depending on the state and control only ($j = 0$ according to the above notations). Systems who do *not* satisfy this conditions may or may not admit flat outputs depending also on some time-derivatives of the control ($j > 0$). This is recalled and commented in section 9.2.4 and 9.5.

Instead of the above control system, we study a reduced equation (9.3); let us briefly explain why it represents, modulo a possibly dynamic feedback transformation, all the relevant cases. Systems for which the iterated Lie brackets of X_1 and X_2 do not have maximum rank can be treated in a rather simple manner, see the first cases of Theorem8.3.1; if on the contrary iterated Lie brackets do have maximum rank, it is well known (Engel normal form for distributions of rank 2 in $\mathbb{R}^4$, see [18]) that, after a nonsingular feedback ($\widetilde{w}_i = \beta^{i,0}(\xi) + \beta^{i,1}(\xi)w_1 + \beta^{i,2}(\xi)w_2$, $i = 1, 2$, with $\beta^{1,1}\beta^{2,2} - \beta^{1,2}\beta^{2,1} \neq 0$), there are coordinates such that the system reads

$$\dot{\xi}_1 = w_1 \ , \quad \dot{\xi}_2 = \gamma(\xi_1, \xi_2, \xi_3, \xi_4) + \xi_3 w_1 \ , \quad \dot{\xi}_3 = \delta(\xi_1, \xi_2, \xi_3, \xi_4) + \xi_4 w_1 \ , \quad \dot{\xi}_4 = w_2 \tag{9.1}$$

with some smooth functions γ and δ. One can eliminate w_1 and w_2 and, renaming $\xi_1, \xi_2, \xi_3, \xi_4$ as x, y, z, w, obtain the two following relations between these four functions of time :

$$\dot{y} = \gamma(x, y, z, w) + z\dot{x} \ , \quad \dot{z} = \delta(x, y, z, w) + w\dot{x} \tag{9.2}$$

(this can also be seen as a control system with state (x, y, z) and controls w and $\dot{x}$). If γ does not depend on w, this system is always parameterizable, and even flat (see Chapter 8 or Example 9.2.5 below). If, on the contrary, γ does depend on its last argument, one can, around a point where the partial derivative is nonzero, invert γ with respect to w, i.e. transform the first equation into $w = g(x, y, z, \dot{y} - z\dot{x})$ for some function g, and obtain, substituting into the last equation, a single differential relation between x, y, z written as (9.3) in next section.

Note that (9.3) also represents the general (non-affine) systems in $\mathbb{R}^3$ with two controls that satisfy the necessary condition given in [88, 96], i.e. they are "ruled" ; we do not develop this here, see [7] or a future publication.

The paper technically focuses on Monge parameterizations of (9.3). The problem is unsolved if g and h are such that system (9.1) does not satisfy the above mentioned necessary and sufficient condition. We do not give a complete solution, but our results are more general than —and imply— these of Chapter 8. The techniques used in the present paper, derived from the original proof of non-parameterizability of some special systems in [48] (see also [87]), are much simpler and elementary that these of Chapter 8 : recovering the results from that paper in this way has some interest in itself.

9.2 Problem statement

9.2.1 The systems under consideration

This paper studies the solutions $t \mapsto (x(t), y(t), z(t))$ of the scalar differential equation

$$\dot{z} \;=\; h(x,y,z,\lambda) \;+\; g(x,y,z,\lambda)\,\dot{x} \quad \text{with} \quad \lambda = \dot{y} - z\dot{x} \tag{9.3}$$

where g and h are two real analytic functions $\Omega \to \mathbb{R}$, Ω being an open connected subset of $\mathbb{R}^4$. We assume that g does depend on λ ; more precisely, associating to g a map $G : \Omega \to \mathbb{R}^4$ defined by $G(x,y,z,\lambda) = (x,y,z,g(x,y,z,\lambda))$, and denoting by g_4 the partial derivative of g with respect to its fourth argument,

$$g_4 \text{ does not vanish on } \Omega \quad \text{and} \quad G \text{ defines a diffeomorphism } \Omega \to G(\Omega)\ . \tag{9.4}$$

We denote by $\widehat{\Omega}$ the open connected subset of $\mathbb{R}^5$ defined from Ω by :

$$(x,y,z,\dot{x},\dot{y}) \in \widehat{\Omega} \;\Leftrightarrow\; (x,y,z,\dot{y}-z\dot{x}) \in \Omega\ . \tag{9.5}$$

From g and h one may define γ and δ, two real analytic functions $G(\Omega) \to \mathbb{R}$, such that $G^{-1}(x,y,z,w) = (x,y,z,\gamma(x,y,z,w))$ and $\delta = h \circ G^{-1}$, i.e.

$$w = g(x,y,z,\lambda) \Leftrightarrow \lambda = \gamma(x,y,z,w)\ , \tag{9.6}$$

$$h(x,y,z,\lambda) = \delta(x,y,z,g(x,y,z,\lambda))\ , \quad \text{i.e.} \quad \delta(x,y,z,w) = h(x,y,z,\gamma(x,y,z,w))\ . \tag{9.7}$$

Then, one may associate to (9.3) the control-affine system (9.1) in $\mathbb{R}^4$ with two controls, that can also be written as (9.2) ; our interest however focuses on system (9.3) defined by g and h as above. Let us set some conventions :

The functions γ and δ when using the notations γ and δ, *it is not assumed that they are related to g and h* by (9.6) and (9.7), unless this is explicitly stated.

Notations for the derivatives We denote partial derivatives by subscript indexes. For functions of many variables, like $\varphi(u,\ldots,u^{(k)},v,\ldots,v^{(\ell)})$ in (9.10), we use the name of the variable as a subscript : $\varphi_{v^{(\ell)}}$ means $\partial\varphi/\partial v^{(\ell)}$; $p_{xu^{(k-1)}}$ means $\partial^2 p/\partial x\partial u^{(k-1)}$ in (9.16-b). Since the arguments of g, h, γ, δ and a few other functions will sometimes be intricate functions of other variables, we use numeric subscripts for their partial derivatives : h_2 stands for $\partial h/\partial y$, or $g_{4,4,4}$ for $\partial^3 g/\partial\lambda^3$. To avoid confusions, we will not use numeric subscripts for other purposes than partial derivatives, except the subscript 0, as in $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0)$ for a reference point.

The dot denotes, as usual, derivative with respect to time, and $^{(j)}$ the j^{th} time-derivatives.

The following elementary lemma —we do write it for the argument is used repeatedly throughout the paper— states that no differential equation independent from (9.3) can be satisfied identically by *all* solutions of (9.3) :

Lemma 9.2.1. *For $M \in \mathbb{N}$, let W be an open subset of $\mathbb{R}^{3+2M}$ and $R : W \to \mathbb{R}$ a smooth function. If* any *solution $I \to \mathbb{R}^3$, $t \mapsto (x(t), y(t), z(t))$, with I some time interval, of system (9.3) such that $(z(t), x(t), \ldots, x^{(M)}(t), y(t), \ldots\ y^{(M)}(t))$ is in W for all t in I, satisfies $R(z(t), y(t), \ldots, y^{(M)}(t), x(t), \ldots, x^{(M)}(t)) = 0$ identically, then R is identically zero on W.*

Proof. For *any* $\mathcal{X} \in W$ there is a germ of solution of (9.3) such that $(z(0), x(0), \ldots, x^{(M)}(0), y(0), \ldots, y^{(M)}(0)) = \mathcal{X}$. Indeed, take e.g. for $x(.)$ and $y(.)$ the polynomials in t of degree M that have these derivatives at time zero ; Cauchy-Lipschitz theorem then yields a (unique) $z(.)$ solution of (9.3) with the prescribed $z(0)$. □

9.2.2 The notion of parameterization

In order to give rigorous definitions without taking care of time-intervals of definition of the solutions, we consider germs of solutions at time 0, instead of solutions themselves. For O an open subset of $\mathbb{R}^n$, the notation $\mathcal{C}_0^\infty(\mathbb{R}, O)$ stands for the set of germs at $t = 0$ of smooth functions of one variable with values in O, see e.g. [43].

Let k, ℓ, L be some non negative integers, U an open subset of $\mathbb{R}^{k+\ell+2}$ and V an open subset of $\mathbb{R}^{2L+3}$. We denote by $\mathcal{U} \subset \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^2)$ (resp. $\mathcal{V} \subset \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^3)$) the set of germs of smooth functions $t \mapsto (u(t), v(t))$ (resp. $t \mapsto (x(t), y(t), z(t))$) such that their jets at $t = 0$ to the order precised below are in U (resp. in V) :

$$\begin{aligned}
\mathcal{U} &= \{(u,v) \in \mathcal{C}_0^\infty(\mathbb{R},\mathbb{R}^2) | (u(0), \dot{u}(0), \ldots, u^{(k)}(0), v(0), \ldots, v^{(\ell)}(0)) \in U\}, && (9.8)\\
\mathcal{V} &= \{(x,y,z) \in \mathcal{C}_0^\infty(\mathbb{R},\mathbb{R}^3) | (x(0), y(0), z(0), \dot{x}(0), \dot{y}(0), \ldots, x^{(L)}(0), y^{(L)}(0)) \in V\}. && (9.9)
\end{aligned}$$

These are open sets for the Whitney $\mathcal{C}^\infty$ topology [43, p. 42].

Definition 9.2.2 (Monge parameterization). *Let k, ℓ, L be non negative integers, $L > 0$, $k \leq \ell$, and $\mathcal{X} = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ldots, x_0^{(L)}, y_0^{(L)})$ be a point in $\widehat{\Omega} \times \mathbb{R}^{2L-2}$ ($\widehat{\Omega}$ is defined in (9.5)). A* parameterization of order (k, ℓ) at $\mathcal{X}$ *for system (9.3) is defined by*

- *a neighborhood V of $\mathcal{X}$ in $\widehat{\Omega} \times \mathbb{R}^{2L-2}$,*
- *an open subset $U \subset \mathbb{R}^{k+\ell+2}$ and*
- *three real analytic functions $U \to \mathbb{R}$, denoted φ, ψ, χ,*

such that, with $\mathcal{U}$ and $\mathcal{V}$ defined from U and V according to (9.8)-(9.9), and $\Gamma : \mathcal{U} \to \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^3)$ the map that assigns to $(u, v) \in \mathcal{U}$ the germ $\Gamma(u, v)$ at $t = 0$ of

$$t \mapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \varphi(u(t), \dot{u}(t), \ldots, u^{(k)}(t), v(t), \dot{v}(t), \ldots, v^{(\ell)}(t)) \\ \psi(u(t), \dot{u}(t), \ldots, u^{(k)}(t), v(t), \dot{v}(t), \ldots, v^{(\ell)}(t)) \\ \chi(u(t), \dot{u}(t), \ldots, u^{(k)}(t), v(t), \dot{v}(t), \ldots, v^{(\ell)}(t)) \end{pmatrix}, \qquad (9.10)$$

the following three properties hold :

1. *for all (u, v) belonging to $\mathcal{U}$, $\Gamma(u, v)$ is a solution of system (9.3),*
2. *the map Γ is open and $\Gamma(\mathcal{U}) \supset \mathcal{V}$,*
3. *the two maps $U \to \mathbb{R}^3$ defined by the triples $(\varphi_{u^{(k)}}, \psi_{u^{(k)}}, \chi_{u^{(k)}})$ and $(\varphi_{v^{(\ell)}}, \psi_{v^{(\ell)}}, \chi_{v^{(\ell)}})$ are identically zero on no open subset of U.*

Remark 9.2.3 (On ordering the pairs (k, ℓ)). *Since u and v play a symmetric role, they can always be exchanged, and there is no lack of generality in assuming $k \leq \ell$. This convention is useful only when giving bounds on (k, ℓ). For instance, $k \geq 2$ means that both integers are no smaller than 2.*

Example 9.2.4. *Consider the equation $\dot{z} = y + (\dot{y} - z\dot{x})\dot{x}$, i.e. (9.3) with $g = \lambda$, $h = y$ (and $\widehat{\Omega} = \mathbb{R}^3$). At any $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ddot{x}_0, \ddot{y}_0)$ such that $\ddot{x}_0 + \dot{x}_0^3 \neq 1$, a parameterization of order $(1, 2)$ is given by :*

$$x = v\,, \quad y = \frac{\dot{v}^2 u + \dot{u}}{\ddot{v} + \dot{v}^3 - 1}\,, \quad z = \frac{(1 - \ddot{v})u + \dot{v}\dot{u}}{\ddot{v} + \dot{v}^3 - 1}\,. \qquad (9.11)$$

It is easy to check that (x, y, z) *given by these formulas does satisfy the equation, point 2 is true because the above formulas can be "inverted" by* $u = -z + y\dot{x}$, $v = x$ *(this gives the "flat output" see section 9.7), point 3 is true because* $\psi_{\dot{u}}$, $\psi_{\ddot{v}}$, $\chi_{\dot{u}}$ *and* $\chi_{\ddot{v}}$ *are nonzero rational functions. Here,* $L = 2$ *and* V *can be taken the whole set of* $(x, y, z, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}) \in \mathbb{R}^7$ *such that* $\ddot{x} + \dot{x}^3 \neq 1$ *and* U *the whole set of* $(u, \dot{u}, v, \dot{v}, \ddot{v}) \in \mathbb{R}^5$ *such that* $\ddot{v} + \dot{v}^3 \neq 1$.

Example 9.2.5. *Suppose that the function* γ *in (9.2) depends on* x, y, z *only (this is treated by case 6 of Theorem8.3.1). For such systems, eliminating* w *does not lead to (9.3), but to the simpler relation* $\dot{y} - z\dot{x} = \gamma(x, y, z)$. *One can easily adapt the above definition replacing (9.3) by this relation. This system* $\dot{y} - z\dot{x} = \gamma(x, y, z)$ *admits a parameterization of order (1,1) at any* $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0)$ *such that* $\dot{x}_0 + \gamma_3(x_0, y_0, z_0) \neq 0$.
Proof. In a neighborhood of such a point,, the map $(x, \dot{x}, y, z) \mapsto (x, \dot{x}, y, \gamma(x, y, z) + z\dot{x})$ *is a local diffeomorphism, whose inverse can be written as* $(x, \dot{x}, y, \dot{y}) \mapsto (x, \dot{x}, y, \chi(x, \dot{x}, y, \dot{y}))$, *thus defining a map* χ. *Then* $x = u, y = v, z = \chi(u, \dot{u}, v, \dot{v})$ *defines a parameterization of order (1,1) in a neighborhood of these points.*

Remark 9.2.6. *The integer* L *characterizes the number of derivatives needed to describe the open set where the parameterization is valid. For instance, in Examples 9.2.4 and 9.2.5, L must be taken no smaller than 2 and 1 respectively. Obviously, a parameterization of order* (k, ℓ) *at* $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ldots, x_0^{(L)}, y_0^{(L)})$ *is also, for* $L' > L$ *and* any $(x_0^{(L+1)}, y_0^{(L+1)}, \ldots, x_0^{(L')}, y_0^{(L')})$, *a parameterization of the same order at* $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ldots, x_0^{(L')}, y_0^{(L')})$.

The above definition is local around some jet of solutions of (9.3). In general, the idea of a global parameterization, meaning that Γ would be defined globally, is not realistic ; it is not realistic either to require that there exists a parameterization around all jets (this would be "everywhere local" rather than "global") : the systems in example 9.2.5 admit a local parameterization around "almost every" jets, meaning jets outside the zeroes of a real analytic function (namely jets such that $\dot{x} + \gamma_3(x, y, z) \neq 0$). We shall not define more precisely the notion of "almost everywhere local" parameterizability, but rather the following (sloppier) one.

Definition 9.2.7. *We say that system (9.3)* admits a parameterization of order (k, ℓ) somewhere in Ω *if there exist an integer* L *and at least one jet* $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ldots, x_0^{(L)}, y_0^{(L)}) \in \widehat{\Omega} \times \mathbb{R}^{2L-2}$ *with a parameterization of order* (k, ℓ) *at this jet in the sense of Definition 9.2.2*

In a colloquial way this is a "somewhere local" property. Using real analyticity, "somewhere local" should imply "almost everywhere local", but we do not investigate this.

9.2.3 The functions S, T and J

Given g, h, let us define three functions S, T and J, to be used to discriminate different cases. They were already more or less present in Chapter 8. The most compact way is as follows : let ω, ω^1 and η be the following differential forms in the variables x, y, z, λ :

$$\begin{aligned} \omega^1 \;=\; \mathrm{d}y - z\,\mathrm{d}x\;, \quad \omega \;&=\; -2\,{g_4}^2\,\mathrm{d}x + (g_{4,4}\,h_4 - g_4\,h_{4,4})\,\omega^1 - g_{4,4}\,(\,\mathrm{d}z - g\,\mathrm{d}x)\;, \\ \eta \;&=\; \mathrm{d}z - g\,\mathrm{d}x - h_4\,\omega^1\;. \end{aligned} \tag{9.12}$$

From (9.4), $\omega\wedge\omega^1\wedge\eta = 2{g_4}^2\,\mathrm{d}x\wedge\,\mathrm{d}y\wedge\,\mathrm{d}z \neq 0$. Decompose $\mathrm{d}\omega\wedge\omega$ on the basis $\omega, \omega^1, \eta, \mathrm{d}\lambda$, thus defining the functions S, T and J (we say more on their expression and meaning in section 9.4) :

$$\mathrm{d}\omega \wedge \omega \;=\; -\left(\frac{S}{2g_4}\,\mathrm{d}\lambda\wedge\eta + \frac{T}{2}\,\mathrm{d}\lambda\wedge\omega^1 \;+\; J\,\omega^1\wedge\eta\right)\wedge\omega\;. \tag{9.13}$$

Example 9.2.8. *Le us illustrate the computation of S, T and J on the following three particular cases of (9.3). For each of them, the table below gives the differential forms ω and η, the decomposition of $d\omega\wedge\omega$ on ω^1,ω,η, $d\lambda$ and the resulting S,T,J according to (9.13). System (a) was already studied in Example 9.2.4.*

$$(a):\dot z = y+(\dot y-z\dot x)\dot x\ ,\qquad (b):\dot z = y+(\dot y-z\dot x)(\dot y-(z-1)\dot x)\ ,\qquad (c):\dot z = y+(\dot y-z\dot x)^2\dot x\ . \quad (9.14)$$

system (9.14)	$g(x,y,z,\lambda)$	$h(x,y,z,\lambda)$	$-\omega/2$ η	$d\omega\wedge\omega$	S,T,J
(a)	λ	y	dx $dz-\lambda\,dx$	0	0, 0, 0
(b)	λ	$y+\lambda^2$	$dy-(z-1)\,dx$ $dz-\lambda\,dx-2\lambda\omega^1$	$\omega^1\wedge\eta\wedge\omega$	0, 0, -1
(c)	λ^2	y	$dz+3\lambda^2\,dx$ $dz-\lambda^2\,dx$	$\frac{3}{\lambda}\,d\lambda\wedge\eta\wedge\omega$	-12, 0, 0

9.2.4 Contributions and organization of the paper

If $S=T=J=0$, i.e. $\mathrm{d}\omega\wedge\omega=0$, system (9.3) admits a parameterization of order (1,2), at all points except some singularities. This is stated further as Theorem 9.4.3, but was already contained in Chapter 8. We conjecture that these systems are the only parameterizable ones of these dimensions, i.e. system (9.3) admits no parameterization of any order if $(S,T,J)\neq(0,0,0)$, i.e. if $\mathrm{d}\omega\wedge\omega\neq 0$.

This is unfortunately still a conjecture, but we give the following results, valid if $(S,T,J)\neq(0,0,0)$ (recall that $k\leq\ell$, see Remark 9.2.3) :

- system (9.3) admits no parameterization of order (k,ℓ) with $k\leq 2$ or $k=\ell=3$ (Theorem 9.5.4),
- a parameterization of order (k,ℓ) must come from a solution of the system of PDEs $\mathcal{E}^{\gamma,\delta}_{k,\ell}$ (Theorem 9.5.1),
- since a solution of this system of PDEs is also sufficient to construct a parameterization (Theorem 9.3.7), the conjecture can be entirely re-formulated in terms of this system of partial differential relations.

Note that this allows one to recover the results from Chapter 8 on (x,u)-flatness[1]. See Remark 9.5.6 for details.

The paper is organized as follows. Section 9.3 is about the above mentioned partial differential system $\mathcal{E}^{\gamma,\delta}_{k,\ell}$. Section 9.4 is devoted to some special constructions for the case where $S=T=0$, and geometric interpretations. The main results are stated in Section 9.5, based on sufficient conditions obtained in Sections 9.3 and 9.4, and necessary conditions stated and proved in Section 9.6. Section 9.7 and 9.8 comment on flatness versus Monge parameterization and then give a conclusion and some perspectives.

9.3 A system of partial differential equations

This section can profitably be skipped or overlooked in a first reading ; the reader will come back when needed to this material that might appear, at first sight, somehow disconnected from the thread of the paper.

It defines $\mathcal{E}^{\gamma,\delta}_{k,\ell}$ and its “regular solutions”, proves that a regular solution induces a parameterization of order (k,ℓ), and that no regular solution exists unless $k\geq 3$ and $\ell\geq 4$.

[1]The term “dynamic linearizable” in Chapter 8 is synonymous to “flat” here.

9.3.1 The equation $\mathcal{E}_{k,\ell}^{\gamma,\delta}$, regular solutions

For k and ℓ some positive integers, we define a partial differential system in $k+\ell+1$ independent variables and one dependent variable, i.e. the unknown is one function of $k+\ell+1$ variables. The dependent variable is denoted by p and the independent variables by $u,\dot{u},\ldots,u^{(k-1)},x,v,\dot{v},\ldots,v^{(\ell-1)}$. Although the names of the variables may suggest "time-derivatives", time is *not* a variable here.

In $\mathbb{R}^{k+\ell+1}$ with the independent variables as coordinates, let F be the differential operator of order 1

$$F=\sum_{i=0}^{k-2}u^{(i+1)}\frac{\partial}{\partial u^{(i)}}+\sum_{i=0}^{\ell-2}v^{(i+1)}\frac{\partial}{\partial v^{(i)}}\ , \tag{9.15}$$

where the first sum is zero if $k\leq 1$ and the second one is zero if $\ell\leq 1$.

Let $\widetilde{\Omega}$ be an open connected subset of $\mathbb{R}^4$ and γ,δ two real analytic functions $\widetilde{\Omega}\to\mathbb{R}$ such that γ_4 (partial derivative of γ with respect to its 4^{th} argument, see end of section 9.2.1) does not vanish on $\widetilde{\Omega}$. Consider the system of two partial differential equations and three inequations :

$$\mathcal{E}_{k,\ell}^{\gamma,\delta}\ \left\{\begin{array}{ll} p_{u^{(k-1)}}\big(Fp_x-\delta(x,p,p_x,p_{xx})\big)-p_{xu^{(k-1)}}\big(Fp-\gamma(x,p,p_x,p_{xx})\big)=0\ , & \text{(a)}\\ p_{u^{(k-1)}}\,p_{xv^{(\ell-1)}}-p_{xu^{(k-1)}}\,p_{v^{(\ell-1)}}=0\ , & \text{(b)}\\ p_{u^{(k-1)}}\neq 0\ , & \text{(c)}\\ p_{v^{(\ell-1)}}\neq 0\ , & \text{(d)}\\ \gamma_1+\gamma_2\,p_x+\gamma_3\,p_{xx}+\gamma_4\,p_{xxx}-\delta\neq 0\ . & \text{(e)}\end{array}\right. \tag{9.16}$$

To any p satisfying $\mathcal{E}_{k,\ell}^{\gamma,\delta}$, we associate two functions σ and τ, and a vector field E :

$$\sigma=-\frac{p_{v^{(\ell-1)}}}{p_{u^{(k-1)}}}\ ,\qquad \tau=\frac{-Fp+\gamma(x,p,p_x,p_{xx})}{p_{u^{(k-1)}}}\ ,\qquad E=\sigma\frac{\partial}{\partial u^{(k-1)}}+\frac{\partial}{\partial v^{(\ell-1)}}\ . \tag{9.17}$$

We also introduce the differential operator D (see Remark 9.3.2 on the additional variables $\dot{x},\ldots,x^{(k+\ell-1)}$) :

$$D=F+\tau\frac{\partial}{\partial u^{(k-1)}}+\sum_{i=0}^{k+\ell-2}x^{(i+1)}\frac{\partial}{\partial x^{(i)}}\ . \tag{9.18}$$

Definition 9.3.1 (Regular solutions of $\mathcal{E}_{k,\ell}^{\gamma,\delta}$)**.** *A* regular *solution of system* $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ *is a real analytic function* $p:O\to\mathbb{R}$*, with* O *a connected open subset of* $\mathbb{R}^{k+\ell+1}$*, such that the image of* O *by* (x,p,p_x,p_{xx}) *is contained in* $\widetilde{\Omega}$*, (9.16-a,b) are identically satisfied on* O*, the left-hand sides of (9.16-c,d,e) are not identically zero, and, for at least one integer* $K\in\{1,\ldots,k+\ell-2\}$*,*

$$ED^Kp\neq 0 \tag{9.19}$$

(not identically zero, as a function of $u,\ldots,u^{(k-1)},x,v,\ldots,v^{(\ell-1)},\dot{x},\ldots,x^{(K)}$ *on* $O\times\mathbb{R}^K$*). We call it* K-regular *if* K *is the smallest such integer, i.e. if* $ED^ip=0$ *for all* $i\leq K-1$*.*

Remark 9.3.2 (on the additionnal variables $\dot{x},\ldots,x^{(k+\ell-1)}$ in D)**.** *These variables appear in the expression (9.18). Note that* D *is only applied (recursively) to functions of* $u,\ldots,u^{(k-1)},x,v,\ldots,v^{(\ell-1)}$ *only; hence we view it as a vector field in* $\mathbb{R}^{k+\ell+1}$ *with these variables as parameters. In fact,* D *is only used in* ED^ip*,* $1\leq i\leq k+\ell-1$*. This is a polynomial with respect to the variables* $\dot{x},\ddot{x},\ldots,x^{(i)}$ *with coefficients depending on* $u,\ldots,u^{(k-1)},x,v,\ldots,v^{(\ell-1)}$ *via the functions* p*,* γ*,* δ *and their partial derivatives. Hence* $ED^ip=0$ *means that all these coefficients are zero, i.e. it encodes a collection of differential relations on* p*, where the spurious variables* $\dot{x},\ddot{x},\ldots,x^{(i)}$ *no longer appear. Likewise,* $ED^ip\neq 0$ *means that one of these relations is not satisfied.*

Definition 9.3.3. *We say that system* $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ admits a regular (resp. K-regular) solution somewhere in $\widehat{\Omega}$ *if there exist at least an open connected* $O \subset \mathbb{R}^{k+\ell+1}$ *and a regular (resp. K-regular) solution* $p : O \to \mathbb{R}$.

Remark 9.3.4. *It is easily seen that p is solution of* $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ *if and only if there exist σ and τ such that (p, σ, τ) is a solution of*

$$\begin{array}{rlll} Fp + \tau p_{u^{(k-1)}} &= \gamma(x, p, p_x, p_{xx}) & & Ep = 0\,,\ \sigma_x = 0\,, \\ Fp_x + \tau p_{x,u^{(k-1)}} &= \delta(x, p, p_x, p_{xx}) & & p_{u^{(k-1)}} \neq 0\,,\ \tau_x \neq 0\,,\ \sigma \neq 0 \end{array} \tag{9.20}$$

Indeed, (9.16) does imply the above relations with σ and τ given by (9.17); in particular, $\tau_x \neq 0$ is equivalent to (e) and $\sigma \neq 0$ to (d); conversely, eliminating σ and τ in (9.20), one recovers $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ *. Note also that, with g and h related to γ and δ by (9.6) and (9.7), any solution of the above equations and inequations satisfies*

$$Dp_x = h(x, p, p_x, Dp - p_x\dot{x}) + g(x, p, p_x, Dp - p_x\dot{x})\dot{x}\ . \tag{9.21}$$

The following will be used repeatedly in the paper :

Lemma 9.3.5. *If p is a solution of system* $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ *and*

1. *either it satisfies a relation of the type $p_x = \alpha(x, p)$ with α a function of two variables,*
2. *or it satisfies a relation of the type $p_{xx} = \alpha(x, p, p_x)$ with α a function of three variables,*
3. *or it satisfies two relations of the type $p_{xxx} = \alpha(x, p, p_x, p_{xx})$ and $Fp_{xx} + \tau p_{xxu^{(k-1)}} = \psi(x, p, p_x, p_{xx})$, with ψ and α two functions of four variables,*

then it satisfies $ED^i p = 0$ for all $i \geq 0$ and hence is not *a regular solution of* $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ *.*

Proof. Point 1 implies point 2 because differentiating the relation $p_x = \alpha(x, p)$ with respect to x yields $p_{xx} = \alpha_x(x, p) + p_x\alpha_p(x, p)$. Likewise, point 2 implies point 3 : differentiating the relation $p_{x,x} = \alpha(x, p, p_x)$ with respect to x yields $p_{xxx} = \alpha_x(x, p, p_x) + p_x\alpha_p(x, p, p_x) + p_{xx}\alpha_{p_x}(x, p, p_x)$ while differentiating it along the vector field $F + \tau\,\partial/\partial u^{(k-1)}$ and using (9.20) yields $Fp_{xx} + \tau p_{xxu^{(k-1)}} = \gamma(x, p, p_x, p_{xx})\,\alpha_p(x, p, p_x) + \delta(x, p, p_x, p_{xx})\,\alpha_{p_x}(x, p, p_x)$.

Let us prove that point 3 implies $ED^i p = ED^i p_x = ED^i p_{xx} = 0$ for all $i \geq 0$, hence the lemma. It is indeed true for $i = 0$ and the following three relations

$$Dp = \gamma(x, p, p_x, p_{xx}) + \dot{x}\,p_x\ ,\quad Dp_x = \delta(x, p, p_x, p_{xx}) + \dot{x}\,p_{xx}\ ,\quad Dp_{xx} = \psi(x, p, p_x, p_{xx}) + \dot{x}\,\alpha(x, p, p_x, p_{xx})\ ,$$

that are implied by (9.18), (9.20) and the two relations in point 3 allow one to go from i to $i+1$ ($ED^i x = Ex^{(i)} = 0$ and $ED^i\dot{x} = Ex^{(i+1)} = 0$ from the very definition of D and E). □

9.3.2 The relation with Monge parameterizations

Let us now explain how a Monge parameterization for system (9.3) can be deduced from a regular solution $p : O \to \mathbb{R}$ of $\mathcal{E}_{k,\ell}^{\gamma,\delta}$. This may seem anecdotal but it is not, for we shall prove (cf. sections 9.5 and 9.6) that all Monge parameterizations are of this type, except when g and h are such that $\mathrm{d}\omega \wedge \omega = 0$ (see (9.12)-(9.13)).

We saw in Remark 9.3.4 that (9.16-e) is equivalent to $\tau_x \neq 0$; let $(u_0, \ldots, u_0^{(k-1)}, x_0, v_0, \ldots, v_0^{(\ell-1)}) \in O$ be such that $\tau_x(u_0, \ldots, u_0^{(k-1)}, x_0, v_0, \ldots, v_0^{(\ell-1)}) \neq 0$. Choose any $(u_0^{(k)}, v_0^{(\ell)}) \in \mathbb{R}^2$ (for instance with $v_0^{(\ell)} = 0$) such that

$$u_0^{(k)} - \sigma(u_0, \ldots, u_0^{(k-1)}, v_0, \ldots, v_0^{(\ell-1)})\,v_0^{(\ell)} = \tau(u_0, \ldots, u_0^{(k-1)}, x_0, v_0, \ldots, v_0^{(\ell-1)})\ . \tag{9.22}$$

Then, the implicit function theorem provides a neighborhood V of $(u_0,\ldots,u_0^{(k)},v_0,\ldots,v_0^{(\ell)})$ in $\mathbb{R}^{k+\ell+2}$ and a real analytic map $\varphi : V \to \mathbb{R}$ such that $\varphi(u_0,\ldots,u_0^{(k)},v_0,\ldots,v_0^{(\ell)}) = x_0$ and

$$\tau(u,\ldots,u^{(k-1)},\varphi(u\cdots v^{(\ell)}),v,\ldots,v^{(\ell-1)}) \;=\; u^{(k)} - \sigma(u,\ldots,u^{(k-1)},v,\ldots,v^{(\ell-1)})\,v^{(\ell)} \tag{9.23}$$

identically on V. Two other maps $V \to \mathbb{R}$ may be defined by

$$\psi(u,\ldots,u^{(k)},v,\ldots,v^{(\ell)}) \;=\; p(u,\ldots,u^{(k-1)},\varphi(\cdots),v,\ldots,v^{(\ell-1)}), \tag{9.24}$$
$$\chi(u,\ldots,u^{(k)},v,\ldots,v^{(\ell)}) \;=\; p_x(u,\ldots,u^{(k-1)},\varphi(\cdots),v,\ldots,v^{(\ell-1)}). \tag{9.25}$$

From these φ, ψ and χ, one can define a map Γ as in (9.10) that is a candidate for a parameterization. We prove below that, if p is a regular solution of $\mathcal{E}_{k,\ell}^{\gamma,\delta}$, then this Γ is indeed a parameterization, at least away from some singularities. The following lemma describes these singularities; it is proved at the end of the paper, page 231.

Lemma 9.3.6. *Let O be an open connected subset of $\mathbb{R}^{k+\ell+1}$ and $p : O \to \mathbb{R}$ be a K-regular solution of system $\mathcal{E}_{k,\ell}^{\gamma,\delta}$, see (9.16). Define the map $\pi : O \times \mathbb{R}^K \to \mathbb{R}^{K+2}$ by*

$$\pi(u\cdots u^{(k-1)},x,v\cdots v^{(\ell-1)},\dot{x}\cdots x^{(K)}) = \begin{pmatrix} p_x(u\cdots u^{(k-1)},x,v\cdots v^{(\ell-1)}) \\ p(u\cdots u^{(k-1)},x,v\cdots v^{(\ell-1)}) \\ Dp(u\cdots u^{(k-1)},x,v\cdots v^{(\ell-1)},\dot{x}) \\ \vdots \\ D^K p(u\cdots u^{(k-1)},x,v\cdots v^{(\ell-1)},\dot{x}\cdots x^{(K)}) \end{pmatrix}. \tag{9.26}$$

There exist two non-negative integers $i_0 \le k$ and $j_0 \le \ell$ such that $i_0 + j_0 = K + 2$ and

$$\det\left(\frac{\partial\pi}{\partial u^{(k-i_0)}},\ldots,\frac{\partial\pi}{\partial u^{(k-1)}},\frac{\partial\pi}{\partial v^{(\ell-j_0)}},\ldots,\frac{\partial\pi}{\partial v^{(\ell-1)}}\right) \tag{9.27}$$

is a nonzero real analytic function on $O \times \mathbb{R}^K$.

We can now state precisely the announced sufficient condition. Its interest is discussed in Remark 9.5.5.

Theorem 9.3.7. *Let $p : O \to \mathbb{R}$, with $O \subset \mathbb{R}^{k+\ell+1}$ open, be a K-regular solution of system $\mathcal{E}_{k,\ell}^{\gamma,\delta}$, and i_0, j_0 be given by Lemma 9.3.6. Then, the maps φ,ψ,χ constructed above define a parameterization Γ of system (9.3) of order (k,ℓ) (see Definition 9.2.2) at any jet of solutions $(x_0,y_0,z_0,\dot{x}_0,\ldots,x_0^{(K)},\dot{y}_0,\ldots,y_0^{(K)})$ such that, for some $u_0,\ldots,u_0^{(k-1)},v_0,\ldots,v_0^{(\ell-1)}$,*

$$\left.\begin{array}{l} (u_0,\ldots,u_0^{(k-1)},x_0,v_0,\ldots,v_0^{(\ell-1)}) \in O\,, \\ z_0 = p_x(u_0,\ldots,u_0^{(k-1)},v_0,\ldots,v_0^{(\ell-1)},x_0)\,, \\ y_0^{(i)} = D^i p(u_0,\ldots,u_0^{(k-1)},v_0,\ldots,v_0^{(\ell-1)},x_0,\ldots,x_0^{(i)}) \quad 0 \le i \le K\,, \end{array}\right\} \tag{9.28}$$

the left-hand sides of (9.16-c,d,e) are all nonzero at $(u_0,\ldots,u_0^{(k-1)},x_0,v_0,\ldots,v_0^{(\ell-1)})$, and the function $ED^K p$ and the determinant (9.27) are nonzero at point $(u_0,\ldots,u_0^{(k-1)},x_0,\ldots,x_0^{(K)},v_0,\ldots,v_0^{(\ell-1)}) \in O \times \mathbb{R}^K$.

Proof. Let us prove that Γ given by (9.10), with the maps φ,ψ,χ constructed above, satisfies the three points of Definition 9.2.2. Differentiating (9.23) with respect to $u^{(k)}$ and $v^{(\ell)}$ yields $\varphi_{u^{(k)}}\tau_x = 1$, $\varphi_{v^{(\ell)}}\tau_x = -\sigma$, hence the point 3 ($\sigma \neq 0$ from (9.20)). To prove point 1, let $u(.), v(.)$

be arbitrary and $x(.), y(.), z(.)$ be defined by (9.10). Differentiating (9.10) with respect to time, using relations (9.24) and (9.25), taking $u^{(k)}(t)$ from (9.23), one has

$$\dot{y}(t) = Fp + \tau p_{u^{(k-1)}} + v^{(\ell)}(t)Ep + \dot{x}(t)\,z(t)\,, \quad \dot{z}(t) = Fp_x + \tau p_{x,u^{(k-1)}} + v^{(\ell)}(t)Ep_x + \dot{x}(t)\,p_{xx}\,,$$

where F is given by (9.15) and the argument $(u(t)\dots u^{(k-1)}(t), x(t), v(t)\dots v^{(\ell-1)}(t))$ for Fp, Fp_x Ep, Ep_x, τ, $p_{x,u^{(k-1)}}$, $p_{u^{(k-1)}}$ and p_{xx} is omitted. Then, (9.20) implies, again omitting the arguments of p_{xx}, one has $\dot{y}(t) = \gamma(x(t), y(t), z(t), p_{xx}) + z(t)\dot{x}(t)$, and $\dot{z}(t) = \delta(x(t), y(t), z(t), p_{xx}) + p_{xx}\dot{x}(t)$. The first equation yields $p_{xx} = g(x(t), y(t), z(t), \dot{y}(t) - z(t)\dot{x}(t))$ with g related to γ by (9.6), and then the second one yields (9.3), with h related to δ by (9.7). This proves point 1. The rest of the proof is devoted to point 2.

Let $t \mapsto (x(t), y(t), z(t))$ be a solution of (9.3). We may consider $\Gamma(u,v) = (x,y,z)$ (see (9.10)) as a system of three ordinary differential equations in two unknown functions u, v :

$$\begin{aligned} u^{(k)} - \sigma(u,\dots,u^{(k-1)},v,\dots,v^{(\ell-1)})v^{(\ell)} - \tau(u,\dots,u^{(k-1)},x,v,\dots,v^{(\ell-1)}) &= 0, & (9.29)\\ p(u,\dots,u^{(k-1)},x,v,\dots,v^{(\ell-1)}) &= y, & (9.30)\\ p_x(u,\dots,u^{(k-1)},x,v,\dots,v^{(\ell-1)}) &= z. & (9.31)\end{aligned}$$

Differentiating (9.30) $K+1$ times, substituting $u^{(k)}$ from (9.29), and using the fact that $ED^ip = 0$ for $i \leq K$ (see Definition 9.3.1), we get

$$D^ip\,(u(t),\dots,u^{(k-1)}(t),v(t),\dots,v^{(\ell-1)}(t),x(t),\dots,x^{(i)}(t)) = \frac{d^iy}{dt^i}(t)\,, \quad 1 \leq i \leq K, \quad (9.32)$$

$$\begin{aligned} &v^{(\ell)}(t)\;ED^Kp\,(u(t),\dots,u^{(k-1)}(t),v(t),\dots,v^{(\ell-1)}(t),x(t),\dots,x^{(K)}(t))\\ &\quad + D^{K+1}p\,(u(t),\dots,u^{(k-1)}(t),v(t),\dots,v^{(\ell-1)}(t),x(t),\dots,x^{(K+1)}(t)) = \frac{d^{K+1}y}{dt^{K+1}}(t)\,. \quad (9.33)\end{aligned}$$

Equations (9.30)-(9.31)-(9.32) can be written

$$\pi(u,\dots,\dots,u^{(k-1)},x,v,\dots,v^{(\ell-1)},\dot{x},\dots,x^{(K)}) = \begin{pmatrix} z\\ y\\ \dot{y}\\ \vdots\\ y^{(K)}\end{pmatrix} \qquad (9.34)$$

with π given by (9.26). From the implicit function theorem, since the determinant (9.27) is nonzero, (9.30)-(9.31)-(9.32) yields $u^{(k-i_0)},\dots,u^{(k-1)}$, $v^{(\ell-j_0)},\dots,v^{(\ell-1)}$ as explicit functions of $u,\dots,u^{(k-i_0-1)}$, $v,\dots,v^{(\ell-j_0-1)}$, $x,\dots,x^{(K)}$, $y,\dots,y^{(K)}$ and z. Let us single out these giving the lowest order derivatives :

$$\begin{aligned} u^{(k-i_0)} &= f^1(u,\dots,u^{(k-1-i_0-1)},v,\dots,v^{(\ell-j_0-1)},x,\dots,x^{(K)},z,y,\dots,y^{(K)}),\\ v^{(\ell-j_0)} &= f^2(u,\dots,u^{(k-1-i_0-1)},v,\dots,v^{(\ell-j_0-1)},x,\dots,x^{(K)},z,y,\dots,y^{(K)}).\end{aligned} \qquad (9.35)$$

Let us prove that, provided that (x,y,z) is a solution of (9.3), system (9.35) is equivalent to (9.29)-(9.30)-(9.31), i.e. to $\Gamma(u,v) = (x,y,z)$. It is obvious that any $t \mapsto (u(t), v(t), x(t), y(t), z(t))$ that satisfies (9.3), (9.29), (9.30) and (9.31) also satisfies (9.35), because these equations were obtained from consequences of those. Conversely, let $t \mapsto (u(t), v(t), x(t), y(t), z(t))$ be such that (9.3) and (9.35) are satisfied ; differentiating (9.35) and substituting each time $\dot{z}$ from (9.3) and $(u^{(k-i_0)}, v^{(\ell-j_0)})$ from (9.35), one obtains

$$\begin{aligned} u^{(k-i_0+i)} &= f^{1,i}(u,\dots,u^{(k-1-i_0-1)},v,\dots,v^{(\ell-j_0-1)},x,\dots,x^{(K+i)},z,y,\dots,y^{(K+i)}),\ i \in \mathbb{N},\\ v^{(\ell-j_0+j)} &= f^{2,j}(u,\dots,u^{(k-1-i_0-1)},v,\dots,v^{(\ell-j_0-1)},x,\dots,x^{(K+j)},z,y,\dots,y^{(K+j)}),\ j \in \mathbb{N}.\end{aligned} \qquad (9.36)$$

Now, substitute the values of $u^{(k-i_0)},\ldots,u^{(k)}$, $v^{(\ell-j_0)},\ldots,v^{(\ell)}$ from (9.36) into (9.29), (9.30) and (9.31) ; either the obtained relations are identically satisfied, and hence it is true that any solution of (9.3) and (9.35) also satisfies (9.29)-(9.30)-(9.31), or one obtains at least one relation of the form (recall that $k\leq\ell$) :

$$R(u,\ldots,u^{(k-1-i_0-1)},v,\ldots,v^{(\ell-j_0-1)},x,\ldots,x^{(K+\ell)},z,y,\ldots,y^{(K+\ell)})=0.$$

This relation has been obtained (indirectly) by differentiating and combining (9.3)-(9.29)-(9.30)-(9.31). This is absurd because (9.29)-(9.30)-(9.31)-(9.32)-(9.33) are the only independent relations of order k,ℓ obtained by differentiating and combining[2] (9.29)-(9.30)-(9.31) because, on the one hand, since $D^K p\neq 0$, differentiating more (9.33) and (9.29) will produce higher order differential equations in which higher order derivatives cannot be eliminated, and on the other hand, differentiating (9.31) and substituting $\dot z$ from (9.3), $u^{(k)}$ from (9.29) and $\dot y$ from (9.32) for $i=1$ yields the trivial $0=0$ because p is a solution of $\mathcal{E}^{\gamma,\delta}_{k,\ell}$, see the proof of point 1 above.

We have now established that, for (x,y,z) a solution of (9.3), $\Gamma(u,v)=(x,y,z)$ is equivalent to (9.35). Using Cauchy Lipschitz theorem with continuous dependence on the parameters, one can define a continuous map $s:\mathcal{V}\to\mathcal{U}$ mapping a germ (x,y,z) to the unique germ of solution of (9.35) with fixed initial condition $(u,\ldots,u^{(k-i_0-1)},v,\ldots,v^{(\ell-j_0-1)})=(u_0,\ldots,u_0^{(k-i_0-1)},v_0,\ldots,v_0^{(\ell-j_0-1)})$. Then s is a continuous right inverse of Γ, i.e. $\Gamma\circ s=Id$. This proves point 2. □

9.3.3 On (non-)existence of regular solutions of system $\mathcal{E}^{\gamma,\delta}_{k,\ell}$

Conjecture 9.3.8. *For any real analytic functions γ and δ (with $\gamma_4\neq 0$), and any integers k,ℓ, the partial differential system $\mathcal{E}^{\gamma,\delta}_{k,\ell}$ (see (9.16)) does not admit any regular solution p.*

An equivalent way of stating this conjecture is : "the equations $ED^ip=0$, for $1\leq i\leq k+\ell-2$, are consequences of (9.16)". Note that "$ED^ip=0$" in fact encodes several partial differential relations on p; see Remark 9.3.2. If γ and δ are polynomials, this can be easily phrased in terms of the differential ideals in the set of polynomials with respect to the variables $u,\ldots,u^{(k-1)},x,v,\ldots,v^{(\ell-1)}$ with $k+\ell+1$ commuting derivatives (all the partial derivatives with respect to these variables).

This is still a conjecture for general integers k and ℓ, but we prove it for "small enough" k,ℓ, namely if one of them is smaller than 3 or if $k=\ell=3$. The following statements assume $k\leq\ell$ (see remark 9.2.3).

Proposition 9.3.9. *If system $\mathcal{E}^{\gamma,\delta}_{k,\ell}$, with $k\leq\ell$, admits a regular solution, then $k\geq 3$, $\ell\geq 4$ and the determinant*

$$\begin{vmatrix} p_{u^{(k-1)}} & p_{u^{(k-2)}} & p_{u^{(k-3)}}\\ p_{xu^{(k-1)}} & p_{xu^{(k-2)}} & p_{xu^{(k-3)}}\\ p_{xxu^{(k-1)}} & p_{xxu^{(k-2)}} & p_{xxu^{(k-3)}}\end{vmatrix} \tag{9.37}$$

is a nonzero real analytic function.

Proof. Straightforward consequence of Lemma 9.3.5 and the three following lemmas, proved pages 232 through 237. □

[2] In other words, (9.29)-(9.30)-(9.31)-(9.32)-(9.33), as a system of ODEs in u and v, is formally integrable (see e.g. [18, Chapter IX]). This means, for a systems of ODEs with independent variable t, that no new independent equation of the same orders (k with respect to u and ℓ with respect to v) can be obtained by differentiating and combining these equations. It is known [18, Chapter IX] that a sufficient condition is that this is true when differentiating only once and the system allows one to express the highest order derivatives as functions of the others. Formal integrability also means that, given any initial condition $(u(0),\ldots,u^{(k)}(0),v(0),\ldots,v^{(\ell)}(0))$ that satisfies these relations, there is a solution of the system of ODEs with these initial conditions.

Lemma 9.3.10. *If p is a solution of system $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ and either $k=1$ or* $\begin{vmatrix} p_{u^{(k-1)}} & p_{u^{(k-2)}} \\ p_{xu^{(k-1)}} & p_{xu^{(k-2)}} \end{vmatrix} = 0$, *then around each point such that $p_{u^{(k-1)}} \neq 0$, there exists a function α of two variables such that a relation $p_x = \alpha(x,p)$ holds identically on a neighborhood of that point.*

Lemma 9.3.11. *Suppose that p is a solution of $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ with*

$$\ell \geq k \geq 2\ , \quad p_{u^{(k-1)}} \neq 0\ , \quad \begin{vmatrix} p_{u^{(k-1)}} & p_{u^{(k-2)}} \\ p_{xu^{(k-1)}} & p_{xu^{(k-2)}} \end{vmatrix} \neq 0\ . \tag{9.38}$$

If either $k=2$ or the determinant (9.37) is identically zero, then, around any point where the two quantities in (9.38) are nonzero, there exists a function α of three variables such that a relation $p_{x,x} = \alpha(x,p,p_x)$ holds identically on a neighborhood of that point.

Lemma 9.3.12. *Let $k=\ell=3$. For any solution p of $\mathcal{E}_{3,3}^{\gamma,\delta}$, in a neighborhood of any point where the determinant (9.37) is nonzero, there exist two functions α and ψ of four variables such that $p_{xxx} = \alpha(x,p,p_x,p_{xx})$ and $F p_{xx} + \tau p_{xxu^{(k-1)}} = \psi(x,p,p_x,p_{xx})$ identically on a neighborhood of that point.*

9.4 Remarks on the case where $S=T=0$.

9.4.1 Geometric meaning of the differential form $\boldsymbol{\omega}$ and the condition $\boldsymbol{S=T=0}$

For (x,y,z) such that the set $\Lambda = \{\lambda \in \mathbb{R}, (x,y,z,\lambda) \in \Omega\}$ is nonempty, (9.3) defines, by varying λ in Λ and $\dot{x}$ in $\mathbb{R}$, a surface Σ in [the tangent space at (x,y,z) to] $\mathbb{R}^3$. Fixing λ in Λ and varying $\dot{x}$ in $\mathbb{R}$ yields a straight line S_λ (direction $(1,z,g(x,y,z,\lambda))$). Obviously, $\Sigma = \bigcup_{\lambda\in\Lambda} S_\lambda$; Σ is a ruled surface. For each $\lambda \in \Lambda$, let P_λ be the *osculating hyperbolic paraboloid to Σ along S_λ*, i.e. the unique[3] such quadric that contains S_λ and has a contact of order 2 with Σ at *all* points of S_λ. Its equation is, omitting the argument (x,y,z,λ) of h and g,

$$(\dot{y}-z\dot{x}-\lambda)\left(\dot{x}+\frac{(h_{44}g_4-g_{44}h_4)}{2\,{g_4}^2}(\dot{y}-z\dot{x}-\lambda)+\frac{g_{44}}{2\,{g_4}^2}(\dot{z}-g\dot{x}-h)\right)-\frac{\dot{z}-g\dot{x}-h}{g_4}+\frac{h_4}{g_4}(\dot{y}-z\dot{x}-\lambda)=0\,.$$

With ω, ω^1, η defined in (9.12) and $\dot{\xi}$ the vector with coordinates $\dot{x},\dot{y},\dot{z}$, the above equation reads

$$-\left(\langle\omega^1,\dot{\xi}\rangle-\lambda\right)\frac{\langle\omega,\dot{\xi}\rangle+(h_{44}g_4-g_{44}h_4)\,\lambda+g_{44}h}{2{g_4}^2}-\frac{\langle\eta,\dot{\xi}\rangle-h}{g_4}=0\ ,$$

that can in turn be rewritten $\langle\omega^1,\dot{\xi}\rangle\langle\omega,\dot{\xi}\rangle-\langle\omega^3,\dot{\xi}\rangle-a^0=0$, with ω^3 and a^0 some differential form and function ; ω, ω^3 and a^0 are uniquely defined up to multiplication by a non-vanishing function ; they encode how the "osculating hyperbolic paraboloid" depends on x,y,z and λ.

We will have to distinguish the case when S and T, whose explicit expressions derive from (9.12) and (9.13) :

$$S = 2\,g_4\,g_{4,4,4}\ -\ 3\,{g_{4,4}}^2\ ,\quad T = 2\,g_4\,h_{4,4,4}\ -\ 3\,g_{4,4}\,h_{4,4}\ , \tag{9.39}$$

[3]General hyperbolic paraboloid : $\left(a^{11}\dot{x}+a^{12}Y+a^{13}Z\right)\left(a^{21}\dot{x}+a^{22}Y+a^{23}Z\right)+a^{31}\dot{x}+a^{32}Y+a^{33}Z+a^0=0$, where the matrix $[a^{ij}]$ is invertible and Y,Z stand for $\dot{y}-z\dot{x}-\lambda$, $\dot{z}-g\dot{x}-h$. It contains S_λ if and only if $a^{11}=a^{31}=a^0=0$. Contact at order 2 means $a^{13}=0$, $a^{33}=-a^{12}a^{21}/g_4$, $a^{32}=-h_4a^{33}$, $a^{22}=\frac{1}{2}a^{21}(g_4h_{44}-g_{44}h_4)/{g_4}^2$, $a^{23}=\frac{1}{2}a^{21}g_{44}/{g_4}^2$. Normalization : $a^{12}=a^{21}=1$.

are zero. From (9.13), it means that the Lie derivative of ω along $\partial/\partial\lambda$ is co-linear to ω, and this is classically equivalent to a decomposition $\omega = k\,\hat{\omega}^2$ where $k \neq 0$ is a function of the four variables x, y, x, λ but $\hat{\omega}^2$ is a differential form in the *three* variables x, y, z, the first integrals of $\partial/\partial\lambda$. Then, one can prove that the form $\hat{\omega}^3 = \omega^3/k$ and the function $\hat{a}^0 = a^0/k$ also involve the variables x, y, z only. From ω's expression, one can take for instance $k = g_{44}$ or $k = gg_{44} - 2{g_4}^2$ (they do not vanish simultaneously because g_4 does not vanish). Hence $S = T = 0$ if and only if, for each fixed (x, y, z), the osculating hyperbolic paraboloid P_λ in fact does not depend on λ i.e. the surface Σ itself is a hyperbolic paraboloid, its equation being

$$\langle\omega^1, \dot{\xi}\rangle\langle\hat{\omega}^2, \dot{\xi}\rangle + \langle\hat{\omega}^3, \dot{\xi}\rangle + \hat{a}^0 = 0 \ , \tag{9.40}$$

where $\dot{\xi}$ is the vector of coordinates $\dot{x}, \dot{y}, \dot{z}$. This yields the following proposition[4], where the functions $\hat{a}^0$, $\hat{a}^1$, $\hat{a}^2$, $\hat{b}^0$, $\hat{b}^1$, $\hat{c}^0$, $\hat{c}^1$ of x, y, z are defined by

$$\hat{\omega}^2 = \frac{\omega}{k} = \hat{b}^1\,\mathrm{d}x + \hat{a}^2\omega^1 - \hat{c}^1\,\mathrm{d}z, \quad \hat{\omega}^3 = \frac{\omega^3}{k} = \hat{b}^0\,\mathrm{d}x + \hat{a}^1\omega^1 - \hat{c}^0\,\mathrm{d}z, \quad \hat{a}^0 = \frac{a^0}{k} \ . \tag{9.41}$$

Proposition 9.4.1. *If S and T, given by (9.12)-(9.13), or (9.39), are identically zero on Ω, then, for any $(x_0, y_0, z_0, \lambda_0)$ in Ω, there exist an open set $W \subset \mathbb{R}^3$, an open interval $I \subset \mathbb{R}$, with $(x_0, y_0, z_0, \lambda_0) \in W \times I \subset \Omega$, and seven smooth functions $W \to \mathbb{R}$ denoted by $\hat{a}^0$, $\hat{a}^1$, $\hat{a}^2$, $\hat{b}^0$, $\hat{b}^1$, $\hat{c}^0$, $\hat{c}^1$ such that $\hat{c}^0 + \hat{c}^1\lambda$ does not vanish on $W \times I$, $\hat{c}^1\hat{b}^0 - \hat{b}^1\hat{c}^0$ does not vanish on W, and, for $(x, y, z, \lambda) \in W \times I$, $\dot{x} \in \mathbb{R}$ and $\dot{z} \in \mathbb{R}$, equation (9.3) is equivalent to*

$$\lambda\left(\hat{b}^1(x,y,z)\dot{x} + \hat{a}^2(x,y,z)\lambda - \hat{c}^1(x,y,z)\dot{z}\right) + \left(\hat{b}^0(x,y,z)\dot{x} + \hat{a}^1(x,y,z)\lambda - \hat{c}^0(x,y,z)\dot{z}\right) + \hat{a}^0(x,y,z) = 0.$$

9.4.2 A parameterization of order $(1, 2)$ if $S = T = J = 0$

It is known Chapter 8 that system (9.3) is (x, u)-flat (see section 9.7) if $S = T = J = 0$. For the sake of completeness, let re-state this result in terms of parameterization. We start with the following particular case of (9.3) :

$$\dot{z} = \kappa(x,y,z)\,\dot{x}\,\lambda + a(x,y,z)\,\lambda + b(x,y,z)\,\dot{x} + c(x,y,z) \quad \text{with} \quad \lambda = \dot{y} - z\dot{x} \tag{9.42}$$

where κ does not vanish on the domain where it is defined. Note that Example 9.2.4 was of this type with $\kappa = 1$, $a = b = 0$, $c = y$. For short, define the following vector fields :

$$X^0 = c\frac{\partial}{\partial z} \ , \ X^1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + b\frac{\partial}{\partial z} \ , \ X^2 = \frac{\partial}{\partial y} + a\frac{\partial}{\partial z} \ , \ X^3 = \kappa\frac{\partial}{\partial z} \ .$$

Note that, for h an arbitrary smooth function of x, y and z, X^0h, X^1h, X^2h, X^3h also depend on x, y, z only.

Lemma 9.4.2. *System (9.42) admits a parameterization of order (1,2) at any $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ddot{x}_0, \ddot{y}_0)$ such that*

$$\kappa\,\ddot{x}_0 + \kappa^2\dot{x}_0^3 + \left(X^1\kappa - X^3b + 2a\kappa\right)\dot{x}_0^2 + \left(X^1a + X^0\kappa - X^3c - X^2b + a^2\right)\dot{x}_0 + X^0a - X^3c \neq 0. \tag{9.43}$$

[4] We introduced the osculating hyperbolic paraboloid because it gives some geometric insight on ω, S and T, but it is not formally *needed* : Proposition 9.4.1 can be stated without it, and proved as follows, based on (9.39) (see also [7]) : the general solution of $S = 0$ is a linear fractional expression $g = (\hat{b}^0 + \hat{b}^1\lambda)/(\hat{c}^0 + \hat{c}^1\lambda)$ where $\hat{b}^0$, $\hat{b}^1$, $\hat{c}^0$, $\hat{c}^1$ are functions of x, y, z only —this is known, for $S/(g_4)^2$ is the Schwartzian derivative of g with respect to its 4th argument, but anyway elementary— and $g_4 \neq 0$ translates into $\hat{b}^0\hat{c}^1 - \hat{b}^1\hat{c}^0 \neq 0$; then $T = 0$ yields $h = (\hat{a}^0 + \hat{a}^1\lambda + \hat{a}^2\lambda^2)/(\hat{c}^0 + \hat{c}^1\lambda)$ with $\hat{a}^0$, $\hat{a}^1$, $\hat{a}^2$ functions of x, y, z. With such g and h, multiplying both sides of (9.3) by $\hat{c}^0 + \hat{c}^1\lambda$ yields the equation in Proposition 9.4.1.

Proof. From (9.43), the two vector fields $Y = X^2 + \dot{x}X^3$ and $Z = [\,X^0 + \dot{x}X^1\,,\; X^2 + \dot{x}X^3\,] + \ddot{x}X^3$ are linearly independent at point $(x_0, y_0, z_0, \dot{x}_0, \ddot{x}_0)$. Let then h be a function of $(x, y, z, \dot{x})$ such that $Yh = 0$ and $Zh \neq 0$; its "time-derivative along system (9.42)", given by $\dot{h} = X^0 h + \big(X^1 h\big)\dot{x} + \big(Yh\big)\lambda + \big(\partial h/\partial\dot{x}\big)\ddot{x}$, does not depend on λ : it is a function of $(x, y, z, \dot{x}, \ddot{x})$; also, since $Yh = 0$, one has $Y\dot{h} = Zh$; finally, $Zh \neq 0$ implies that $\mathrm{d}h \wedge \mathrm{d}\dot{h} \wedge \mathrm{d}x \wedge \mathrm{d}\dot{x} \wedge \mathrm{d}\ddot{x} \neq 0$. In turn, this implies that $(x, y, z, \dot{x}, \ddot{x}) \mapsto \big(h(x, y, z, \dot{x}), \dot{h}(x, y, z, \dot{x}, \ddot{x}), x, \dot{x}, \ddot{x}\big)$ defines a local diffeomorphism at $(x_0, y_0, z_0, \dot{x}_0, \ddot{x}_0)$. Let ψ and χ be the two functions of five variables such that the inverse of that local diffeomorphism is $(u, \dot{u}, v, \dot{v}, \ddot{v}) \mapsto (v, \psi(u, \dot{u}, v, \dot{v}, \ddot{v}), \chi(u, \dot{u}, v, \dot{v}, \ddot{v}), \dot{v}, \ddot{v})$. The parameterization (9.10) is given by : $x = v, y = \psi(u, \dot{u}, v, \dot{v}, \ddot{v}), z = \chi(u, \dot{u}, v, \dot{v}, \ddot{v})$. □

Theorem 9.4.3. *If $S = T = J = 0$, then system (9.3) admits a parameterization of order $(1, 2)$ at any $(x_0, y_0, z_0, \dot{x}_0,\, \dot{y}_0, \ddot{x}_0, \ddot{y}_0) \in \big(\widehat{\Omega} \times \mathbb{R}^2\big) \setminus F$, where $F \subset \widehat{\Omega} \times \mathbb{R}^2$ is closed with empty interior.*

Proof. From Proposition 9.4.1, (9.3) and (9.40) are identical. Since $\mathrm{d}\hat{\omega}^2 \wedge \hat{\omega}^2 = 0$ (see (9.13)-(9.41)), there is a local change of coordinates $(\tilde{x}, \tilde{y}, \tilde{z}) = P(x, y, z)$ such that $\hat{\omega}^2 = k'\,\mathrm{d}\tilde{x}$ and $\omega^1 = k''\big(\mathrm{d}\tilde{y} - \tilde{z}\,\mathrm{d}\tilde{x}\big)$ with $k' \neq 0$, $k'' \neq 0$. Hence P transforms (9.40) into (9.42), for some κ, a, b, c. Lemma 9.4.2 gives φ, ψ, χ defining a parameterization of order (1,2) for this system. Then $P^{-1} \circ \varphi, P^{-1} \circ \psi, P^{-1} \circ \chi$ define one for the original system (9.3), or (9.40); $\big(\widehat{\Omega} \times \mathbb{R}^2\big) \setminus F$ is the inverse image by P of the set defined by (9.43). □

9.4.3 A normal form if $S = T = 0$ and $J \neq 0$

Proposition 9.4.4. *Assume that the functions g and h defining system (9.3) are such that S and T defined by (9.13) or (9.39) are identically zero on Ω, and let $(x_0, y_0, z_0, \lambda_0) \in \Omega$ be such that $J(x_0, y_0, z_0, \lambda_0) \neq 0$.*

There exist an open set $W \subset \mathbb{R}^3$ and an open interval $I \subset \mathbb{R}$ such that $(x_0, y_0, z_0, \lambda_0) \in W \times I \subset \Omega$, a smooth diffeomorphism P from W to $P(W) \subset \mathbb{R}^3$ and six smooth functions $P(W) \to \mathbb{R}$ denoted κ, α, β, a, b, c such that, with the change of coordinates $(\tilde{x}, \tilde{y}, \tilde{z}) = P(x, y, z)$, system (9.3) reads

$$\dot{\tilde{z}} = \kappa(\tilde{x}, \tilde{y}, \tilde{z})\,\big(\dot{\tilde{y}} - \alpha(\tilde{x}, \tilde{y}, \tilde{z})\,\dot{\tilde{x}}\big)\,\big(\dot{\tilde{y}} - \beta(\tilde{x}, \tilde{y}, \tilde{z})\,\dot{\tilde{x}}\big) \;+\; a(\tilde{x}, \tilde{y}, \tilde{z})\,\dot{\tilde{x}} + b(\tilde{x}, \tilde{y}, \tilde{z})\,\dot{\tilde{y}} + c(\tilde{x}, \tilde{y}, \tilde{z}) \quad (9.44)$$

and none of the functions κ, $\alpha - \beta$, α_3 and β_3 vanish on W.

Proof. From Lemma 9.4.1, we consider system (9.40). Let P^1, P^2 be a pair of independent first integrals of the vector field $\hat{c}^1\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\right) + \hat{b}^1\frac{\partial}{\partial z}$; from (9.41), $\omega^1, \hat{\omega}^2$ span the annihilator of this vector field, and hence are independent linear combinations of $\mathrm{d}P^1$ and $\mathrm{d}P^2$: possibly interchanging P^1 and P^2 or adding one to the other, there exist smooth functions k^1, k^2, f^1, f^2 such that $\hat{\omega}^i = k^i\left(\mathrm{d}P^2 - f^i\,\mathrm{d}P^1\right)$, $f^1 - f^2 \neq 0$, $k^i \neq 0$, $i = 1, 2$. Now, take for P^3 any function such that $\mathrm{d}P^1 \wedge \mathrm{d}P^2 \wedge \mathrm{d}P^3 \neq 0$; decomposing $\hat{\omega}^3$, we get three smooth functions p^0, p^1, p^2 such that $\hat{\omega}^3 = p^0\left(-\,\mathrm{d}P^3 + p^1\,\mathrm{d}P^1 + p^2\,\mathrm{d}P^2\right)$, $p^0 \neq 0$. The change of coordinates $P = (P^1, P^2, P^3)$ does transform system (9.40) into (9.44) with

$$\kappa = \frac{k^1 k^2}{p^0} \circ P^{-1}, \quad \alpha = f^1 \circ P^{-1}, \quad \beta = f^2 \circ P^{-1}, \quad a = p^1 \circ P^{-1}, \quad b = p^2 \circ P^{-1}, \quad c = \frac{\hat{a}^0}{p^0} \circ P^{-1}\,.$$

κ and $\alpha - \beta$ are nonzero because $f^1 - f^2$, k^1 and k^2 are. α_3 and β_3 are nonzero because the inverse images of $\alpha_3\,\mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y} \wedge \mathrm{d}\tilde{z}$ and $\beta_3\,\mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y} \wedge \mathrm{d}\tilde{z}$ by P are $\mathrm{d}P^1 \wedge \mathrm{d}P^2 \wedge \mathrm{d}f^i$ for $i = 1, 2$, that are equal, by construction, to $\mathrm{d}\omega^1 \wedge \omega^1/(k^1)^2$ and $\mathrm{d}\hat{\omega}^2 \wedge \hat{\omega}^2/(k^2)^2$, which are both nonzero (the second one because $J \neq 0$). □

Note that (9.44) is not in the form (9.3) unless $\alpha = \tilde{z}$ or $\beta = \tilde{z}$. This suggests, since $\alpha_3 \neq 0$ and $\beta_3 \neq 0$, the following local changes of coordinates A and B, that both turn (9.44) to a new system of the form (9.3) :

$$(\tilde{x}, \tilde{y}, \tilde{z}) \mapsto A(\tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{x}, \tilde{y}, \alpha(\tilde{x}, \tilde{y}, \tilde{z})) \quad \text{and} \quad (\tilde{x}, \tilde{y}, \tilde{z}) \mapsto B(\tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{x}, \tilde{y}, \beta(\tilde{x}, \tilde{y}, \tilde{z}))\ . \tag{9.45}$$

These two systems of the form (9.3) correspond to two choices h^1, g^1 and h^2, g^2 instead of the original h, g, and they yield, according to (9.6) and (9.7), two possible sets of functions γ and δ. These will be used in Theorem 9.6.5 ; let us give their explicit expression :

$$\gamma^i(x, y, z, w) = \frac{w - m^{i,0}(x, y, z)}{m^{i,1}(x, y, z)}\ , \quad \delta^i = n^{i,0} + n^{i,1}\gamma + n^{i,2}\gamma^2\ , \quad i \in \{1, 2\} \tag{9.46}$$

with (these are obtained from each other by interchanging α and β) :

$$\begin{array}{l} m^{1,0} = (\alpha_1 + \alpha\,\alpha_2 + (a + b\,\alpha)\,\alpha_3) \circ A^{-1}\ , \quad m^{1,1} = (\kappa\,\alpha_3\,(\alpha - \beta)) \circ A^{-1}\ , \\ n^{1,0} = \alpha_3 \circ A^{-1}\ , \quad n^{1,1} = (\alpha_2 + b\,\alpha_3) \circ A^{-1}\ , \quad n^{1,2} = (\kappa\,\alpha_3) \circ A^{-1}\ , \\ m^{2,0} = (\beta_1 + \beta\,\beta_2 + (a + b\,\beta)\,\beta_3) \circ B^{-1}\ , \quad m^{2,1} = (\kappa\,\beta_3\,(\beta - \alpha)) \circ B^{-1}\ , \\ n^{2,0} = \beta_3 \circ B^{-1}\ , \quad n^{2,1} = (\beta_2 + b\,\beta_3) \circ B^{-1}\ , \quad n^{2,2} = (\kappa\,\beta_3) \circ B^{-1}\ . \end{array} \tag{9.47}$$

Example 9.4.5. *System (9.14-b) in Example 9.2.8 is already as in (9.44). The above choices are, for this system :*

$$\gamma^1(x, y, z, w) = w\ , \quad \delta^1(x, y, z, w) = y + w^2\ , \quad \gamma^2(x, y, z, w) = -w\ , \quad \delta^2(x, y, z, w) = y + w^2\ . \tag{9.48}$$

9.5 Main results

We gather here our main results in a synthetic manner. They rely on precise *local* results from other sections : sufficient (sections 9.4 and 9.3.2) or necessary (section 9.6) conditions for parameterizability, results on solutions of the partial differential system $\mathcal{E}^{\gamma,\delta}_{k,\ell}$ (section 9.3.3) and on the relation between flatness and parameterizability (section 9.7). We are not able to give local precise necessary and sufficient conditions at a given point (jet) because singularities are not the same for necessary and for sufficient conditions ; instead, we use the "somewhere" notion as in Definitions 9.3.3 and 9.2.7.

Theorem 9.5.1. *System (9.3) admits a parameterization of order (k, ℓ) somewhere in Ω if and only if*

1. *either $S = T = J = 0$ on Ω (in this case, one can take $(k, \ell) = (1, 2)$),*
2. *or $S = T = 0$ on Ω and one of the two systems $\mathcal{E}^{\gamma^1,\delta^1}_{k,\ell}$ or $\mathcal{E}^{\gamma^2,\delta^2}_{k,\ell}$ with γ^i, δ^i given by (9.46)-(9.47), admits a regular solution somewhere in $\widehat{\Omega}$.*
3. *or S and T are not both identically zero, and the system $\mathcal{E}^{\gamma,\delta}_{k,\ell}$ with γ and δ defined from g and h according to (9.6) and (9.7) admits a regular solution somewhere in $\widehat{\Omega}$.*

Proof. Sufficiency : the parameterization is provided, away from an explicitly described set of singularities, by Theorem 9.4.3 if point 1 holds, and by Theorem 9.3.7 if one of the two other points holds. For necessity, assume that there is a parameterization of order (k, ℓ) at a point $(x, y, z, \dot{x}, \dot{y}, \ldots, x^{(L)}, y^{(L)})$ in $\big(\widehat{\Omega} \times \mathbb{R}^{2L-2}\big)\backslash F$. From Theorems 9.6.2 and 9.6.5, it implies that one of the three points holds. □

Example 9.5.2. *Consider again systems (a), (b) and (c) in (9.14). From point 1 of the theorem, system (a) admits a parameterization of order (1,2), see also Example 9.2.4. System (b) is concerned by point 2 of the theorem : it has a parameterization of order k, ℓ if and only one of the two systems of PDEs*

$$\begin{array}{l} p_{u^{(k-1)}}\big(Fp_x - p - {p_{xx}}^2)\big) - p_{xu^{(k-1)}}\big(Fp \pm p_{xx})\big) = p_{u^{(k-1)}}\, p_{xv^{(\ell-1)}} - p_{xu^{(k-1)}}\, p_{v^{(\ell-1)}} = 0\;, \\ p_{u^{(k-1)}} \neq 0\;, \quad p_{v^{(\ell-1)}} \neq 0\;, \quad p + {p_{xx}}^2 \pm p_{xxx} \neq 0 \end{array} \tag{9.49}$$

admits a "regular solution". Point 3 of the theorem is relevant to system (c) because $S \neq 0$: (c) admits a parameterization of order k, ℓ if and only there is a "regular solution" p to

$$\begin{array}{l} p_{u^{(k-1)}}\big(Fp_x - p)\big) - p_{xu^{(k-1)}}\big(Fp - \sqrt{p_{xx}})\big) = p_{u^{(k-1)}}\, p_{xv^{(\ell-1)}} - p_{xu^{(k-1)}}\, p_{v^{(\ell-1)}} = 0\;, \\ p_{u^{(k-1)}} \neq 0\;, \quad p_{v^{(\ell-1)}} \neq 0\;, \quad p - p_{xxx}/2\sqrt{p_{xx}} \neq 0\;. \end{array} \tag{9.50}$$

If Conjecture 9.3.8 is true, neither system (b) nor system (c) admits a parameterization of any order.

Theorem 9.5.1 gives a central role to the system of PDEs $\mathcal{E}_{k,\ell}^{\gamma,\delta}$. It makes Conjecture 9.3.8 equivalent to Conjecture 9.5.3 below. Theorem 9.5.4 states that the conjecture is true for k, ℓ "small enough".

Conjecture 9.5.3. *If $d\omega \wedge \omega$ (or (S,T,J)) is not identically zero on Ω, then system (9.3) does not admit a parameterization of any order at any point (jet of any order).*

Theorem 9.5.4. *If system (9.3) admits a parameterization of order (k,ℓ), with $k \leq \ell$, at some jet, then either $S = T = J = 0$ or $k \geq 3$ and $\ell \geq 4$.*

Proof. This is a simple consequence of Theorem 9.5.1 and Proposition 9.3.9. □

Remark 9.5.5. *If our Conjecture 9.3.8 is correct, the systems $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ never have any regular solutions, and the sufficiency part of Theorem 9.5.1 (apart from case 1) is essentially void, and so is Theorem 9.3.7. However, Conjecture 9.3.8 is still a conjecture, and the interest of the sufficient conditions above is to make this conjecture, that only deals with a set of partial differential equalities and inequalities, equivalent to Conjecture 9.5.3 below. For instance, if one comes up with a regular solution of some of these systems $\mathcal{E}_{k,\ell}^{\gamma,\delta}$, this will yield a new class of systems that admit a parameterization.*

Remark 9.5.6 (on recovering the results of Chapter 8)**.** *The main result in that reference can be phrased :*

" (9.1) is (x,u)-dynamic linearizable (i.e. (x,u)-flat) if and only if $S = T = J = 0$ ".

Sufficiency is elementary in Chapter 8 ; Theorem 9.4.3 implies it. The difficult part is to prove that $S = T = J = 0$ is necessary ; that proof is very technical in Chapter 8 : it relies on some simplifications performed via computer algebra. From our Proposition 9.7.4, (x,u)-flatness implies existence of a parameterization of some order (k,ℓ) with $k \leq 3$ and $\ell \leq 3$. Hence Theorem 9.5.1 does imply the above statement.

9.6 Necessary conditions

9.6.1 The case where S and T are not both zero

The following lemma is needed to state the theorem.

Lemma 9.6.1. *If $(S, T, J) \neq (0, 0, 0)$ and system (9.3) admits a parameterization (φ, ψ, χ) of order (k, ℓ) at point $(x_0, y_0, z_0, \ldots, x_0^{(L)}, y_0^{(L)}) \in \mathbb{R}^{2L+3}$, then $\varphi_{u^{(k)}}$ is a nonzero real analytic function.*

Proof. Assume a parameterization where φ does not depend on $u^{(k)}$. Substituting in (9.3) yields

$$\dot{\chi} = h(\varphi, \psi, \chi, \dot{\psi} - \chi\dot{\varphi}) + g(\varphi, \psi, \chi, \dot{\psi} - \chi\dot{\varphi})\dot{\varphi} \ .$$

Since $\dot{\varphi}$ does not depend on $u^{(k+1)}$, differentiating twice with respect to $u^{(k+1)}$ yields

$$\chi_{u^{(k)}} = \psi_{u^{(k)}}(h_4 + g_4\dot{\varphi}) \ , \quad 0 = {\psi_{u^{(k)}}}^2(h_{4,4} + g_{4,4}\dot{\varphi}) \ .$$

If $\psi_{u^{(k)}}$ was zero, then, from the first relation, $\chi_{u^{(k)}}$ would too, and this would contradict point 3 in Definition 9.2.2; hence the second relation implies that $h_{4,4} + g_{4,4}\dot{\varphi}$ is identically zero. From point 2 in the same definition, it implies that *all* solutions of (9.3) satisfy the relation : $h_{4,4}(x, y, z, \dot{y} - z\dot{x}) + g_{4,4}(x, y, z, \dot{y} - z\dot{x})\dot{x} = 0$. From Lemma 9.2.1, this implies that $h_{4,4}$ and $g_{4,4}$ are the zero function of four variables, and hence $S = T = J = 0$. This proves the lemma. □

Theorem 9.6.2. *Assume that either S or T is not identically zero on Ω, and that system (9.3) admits a parameterization of order (k, ℓ) at $\mathcal{X} = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ldots, x_0^{(L)}, y_0^{(L)}) \in \widehat{\Omega} \times \mathbb{R}^{2L-2}$, with k, ℓ, L some integers and φ, ψ, χ defined on $U \subset \mathbb{R}^{k+\ell+2}$.*

Then $k \geq 1$, $\ell \geq 1$ and, for any point $(u_0, \ldots, u_0^{(k)}, v_0, \ldots, v_0^{(\ell)}) \in U$ (not necessarily sent to $\mathcal{X}$ by the parameterization) such that

$$\varphi_{u^{(k)}}(u_0, \ldots, u_0^{(k)}, v_0, \ldots, v_0^{(\ell)}) \neq 0,$$

there exist a neighborhood O of $(u_0, \ldots, u_0^{(k-1)}, \varphi(u_0 \cdots v_0^{(\ell)}), v_0, \ldots, v_0^{(\ell-1)})$ in $\mathbb{R}^{k+\ell+1}$ and a regular solution $p : O \to \mathbb{R}$ of $\mathcal{E}_{k,\ell}^{\gamma,\delta}$, related to φ, ψ, χ by (9.23), (9.24) and (9.25), the functions γ and δ being related to g and h by (9.6) and (9.7).

Remark 9.6.3. *The regular solution p is K-regular for some positive integer $K \leq k + \ell - 2$. If $L > K$, Theorem 9.3.7 implies, possibly away from some singular values of $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ldots, x_0^{(K)}, y_0^{(K)})$, that system (9.3) also admits a parameterization of order (k, ℓ) at $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ldots, x_0^{(K)}, y_0^{(K)})$. See also Remark 9.2.6.*

Proof. Assume that system (9.3) admits a parameterization (φ, ψ, χ) of order (k, ℓ) at $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ldots, x_0^{(L)}, y_0^{(L)})$. Since $\varphi_{u^{(k)}}$ does not vanish, one can apply the inverse function theorem to the map

$$(u, \dot{u}, \ldots, u^{(k)}, v, \dot{v}, \ldots, v^{(\ell)}) \mapsto (u, \ldots, u^{(k-1)}, \varphi(u, \ldots, u^{(k)}, v, \ldots, v^{(\ell)}), v, \ldots, v^{(\ell)})$$

and define locally a function r of $k + \ell + 2$ variables such that

$$\varphi(u, \dot{u}, \ldots, u^{(k)}, v, \dot{v}, \ldots, v^{(\ell)}) = x \ \Leftrightarrow \ r(u, \dot{u}, \ldots, u^{(k-1)}, x, v, \dot{v}, \ldots, v^{(\ell)}) = u^{(k)} \ . \qquad (9.51)$$

Defining two functions p, q by substitution of $u^{(k)}$ in ψ, χ, the parameterization can be re-written implicitly as

$$\left\{ \begin{array}{l} y = p(u, \dot{u}, \ldots, u^{(k-1)}, x, v, \dot{v}, \ldots, v^{(\ell)}), \\ z = q(u, \dot{u}, \ldots, u^{(k-1)}, x, v, \dot{v}, \ldots, v^{(\ell)}), \\ u^{(k)} = r(u, \dot{u}, \ldots, u^{(k-1)}, x, v, \dot{v}, \ldots, v^{(\ell)}). \end{array} \right. \qquad (9.52)$$

We now work with this form of the parameterization and $u, \dot{u}, \ldots, u^{(k-1)}, x, \dot{x}, \ddot{x}, \ldots\ v, \dot{v}, \ldots, v^{(\ell)}$, $v^{(\ell+1)}, \ldots$ instead of $u, \dot{u}, \ldots, u^{(k-1)}, u^{(k)}, \ldots\ v, \dot{v}, \ldots, v^{(\ell)}, v^{(\ell+1)}, \ldots$. In order to simplify notations, let us agree that, if $k=0$, the list $u, \dot{u}, \ldots, u^{(k-1)}$ is empty and any term involving the index $k-1$ is zero (same with $\ell-1$ if $\ell=0$). Let us also define $\mathcal{P}$ and $\mathcal{Q}$ by

$$\mathcal{P} = Fp + rp_{u^{(k-1)}} + v^{(\ell)}p_{v^{(\ell-1)}} + v^{(\ell+1)}p_{v^{(\ell)}} , \quad \mathcal{Q} = Fq + rq_{u^{(k-1)}} + v^{(\ell)}q_{v^{(\ell-1)}} + v^{(\ell+1)}q_{v^{(\ell)}} , \tag{9.53}$$

with F given by (9.15). $\mathcal{P}$ and $\mathcal{Q}$ depend on $u, \dot{u}, \ldots, u^{(k-1)}, x, v, \dot{v}, \ldots, v^{(\ell+1)}$ but not on $\dot{x}$; Fp and Fq depend neither on $\dot{x}$ nor on $v^{(\ell+1)}$. When substituting (9.52) in (9.3), using $\dot{y} = \mathcal{P} + \dot{x}p_x$ and $\dot{z} = \mathcal{Q} + \dot{x}q_x$, one obtains :

$$\mathcal{Q} + \dot{x}q_x = h(x,p,q,\lambda) + g(x,p,q,\lambda)\dot{x} \quad \text{with} \quad \lambda = \mathcal{P} + \dot{x}(p_x - q). \tag{9.54}$$

Differentiating each side three times with respect to $\dot{x}$, one obtains :

$$q_x = (h_4(x,p,q,\lambda) + g_4(x,p,q,\lambda)\dot{x})\,(p_x-q) + g(x,p,q,\lambda), \tag{9.55}$$
$$0 = (h_{4,4}(x,p,q,\lambda) + g_{4,4}(x,p,q,\lambda)\dot{x})\,(p_x-q)^2 + 2g_4(x,p,q,\lambda)(p_x-q), \tag{9.56}$$
$$0 = (h_{4,4,4}(x,p,q,\lambda) + g_{4,4,4}(x,p,q,\lambda)\dot{x})\,(p_x-q)^3 + 3g_{4,4}(x,p,q,\lambda)(p_x-q)^2. \tag{9.57}$$

Combining (9.56) and (9.57) to cancel the first term in each equation, one obtains (see S and T in (9.39)) :

$$\Big(T(x,p,q,\lambda) + S(x,p,q,\lambda)\dot{x}\Big)\,(p_x-q)^2 = 0. \tag{9.58}$$

The second factor must be zero because, if $T + S\dot{x}$ was identically zero as a function of $u, \ldots, u^{(k-1)}, x, v, \ldots\ v^{(\ell-1)}$, then, by Definition 9.2.2 (point 2), *all* solutions $(x(t), y(t), z(t))$ of (9.3) would satisfy $T(x,y,z,\dot{y}-z\dot{x}) + \dot{x}S(x,y,z,\dot{y}-z\dot{x}) = 0$ identically, and this would imply that S and T are identically zero functions of 4 variables, but we supposed the contrary. The relation $q = p_x$ implies

$$\lambda = \mathcal{P} = Fp + rp_{u^{(k-1)}} + v^{(\ell)}p_{v^{(\ell-1)}} + v^{(\ell+1)}p_{v^{(\ell)}} \tag{9.59}$$

and (9.55) then yields $p_{xx} = g(x,p,p_x,\lambda)$, or, with γ defined by (9.6),

$$\lambda = \gamma(x,p,p_x,p_{xx}) . \tag{9.60}$$

Since neither p nor Fp nor r depend on $v^{(\ell+1)}$, (9.59) and (9.60) yield $p_{v^{(\ell)}} = 0$, i.e. p is a function of $u, \ldots, u^{(k-1)}, x,\ v, \ldots, v^{(\ell-1)}$ only. Then (9.59) and (9.60) imply (9.109) with $f = \gamma$. Furthermore, since $\varphi_{v^{(\ell)}} \neq 0$ (point 3 of Definition 9.2.2), (9.51) implies $r_{v^{(\ell)}} \neq 0$. Also, if p was a function of x only, then all solutions of (9.63) should satisfy a relation $y(t) = p(x(t))$, which is absurd from Lemma 9.2.1. We may then apply Lemma 9.9.2 (page 238) and assert that $k \geq 1$, $\ell \geq 1$, $p_{u^{(k-1)}} \neq 0$, $p_{v^{(\ell-1)}} \neq 0$.

Since p does not depend on $v^{(\ell)}$, (9.60) implies that the right-hand side of (9.59) does not depend on $v^{(\ell)}$ either; since $p_{u^{(k-1)}} \neq 0$, r must be affine with respect to $v^{(\ell)}$, i.e.

$$r \ = \ \tau \ + \ \sigma\, v^{(\ell)} \ , \tag{9.61}$$

with σ and τ some functions of $u, \ldots, u^{(k-1)}, x, v, \ldots, v^{(\ell-1)}$. Since p, $q = p_x$, λ and $q_x = p_{xx}$ do not depend on $v^{(\ell)}$, (9.54) implies that $\mathcal{Q}$ does not depend on $v^{(\ell)}$ either; with $p_x = q$, and r given by (9.61), the expression of $\mathcal{Q}_{v^{(\ell)}}$ is $\sigma p_{xu^{(k-1)}} + p_{xv^{(\ell-1)}}$ while, from (9.59), the expression of $\mathcal{P}_{v^{(\ell)}}$. Collecting this, one gets

$$\sigma p_{u^{(k-1)}} + p_{v^{(\ell-1)}} = \sigma p_{xu^{(k-1)}} + p_{xv^{(\ell-1)}} = 0 \ . \tag{9.62}$$

Since $p_{u^{(k-1)}} \neq 0$ and $p_{v^{(\ell-1)}} \neq 0$, (9.62) implies $E\sigma = 0$, and also $\sigma_x = 0$, $\sigma \neq 0$. Then, since $r_x \neq 0$ (see (9.51)), (9.61) implies $\tau_x \neq 0$. With the above remarks, (9.59) yields $\mathcal{P} = \lambda = Fp + \tau p_{u^{(k-1)}}$ and hence, from (9.60), the first relation in (9.20). In a similar way, (9.53) yields $\mathcal{Q} = Fp_x + \tau p_{xu^{(k-1)}}$, and substituting in (9.54), one obtains (the terms involving $\dot{x}$ disappear according to (9.60)) $Fp_x - \delta(x, p, p_x, p_{xx}) + p_{xu^{(k-1)}}\tau = 0$ with δ defined by (9.7). This proves that p satisfies (9.20), equivalent to (9.16) according to Remark 9.3.4, and hence that p is a solution of $\mathcal{E}_{k,\ell}^{\gamma,\delta}$.

To prove by contradiction that it is K-regular for some $K \leq k+\ell+1$, assume that $ED^ip = 0$ for $0 \leq i \leq k+\ell$. Then $p_x, p, \ldots, D^{(k+\ell-1)}p, x, \ldots, x^{(k+\ell-1)}$ are $2k+2\ell+1$ functions in the $2k+2\ell$ variables $u, \ldots, u^{(k-1)}, v, \ldots, v^{(\ell-1)}, x, \ldots, x^{(k+\ell-1)}$. At points where the Jacobian matrix has constant rank, there is at least one nontrivial relation between them. From point 2 of Definition 9.2.2, this would imply that all solutions of system (9.3) satisfy this relation, say $R(z(t), y(t), \ldots, y^{(k+\ell-1)}(t), x(t), \ldots, x^{(k+\ell-1)}(t)) = 0$, which is absurd from Lemma 9.2.1. □

9.6.2 The case where S and T are zero

Here, the situation is slightly more complicated : we also establish that any parameterization "derives from" a solution of the system of PDEs (9.16), but this is correct only if J is not zero, and there are two distinct (non equivalent) choices for γ and δ. If $J \neq 0$, we saw, in section 9.4, that possibly after a change of coordinates, system (9.3) can be written as (9.44), which we re-write here without the tildes :

$$\dot{z} = \kappa(x,y,z)\,(\dot{y} - \alpha(x,y,z)\,\dot{x})\,(\dot{y} - \beta(x,y,z)\,\dot{x}) \;+\; a(x,y,z)\,\dot{x} + b(x,y,z)\,\dot{y} + c(x,y,z)\;, \quad (9.63)$$

where $\kappa, \alpha, \beta, a, b, c$ are real analytic functions of three variables and $\kappa \neq 0$, $\alpha - \beta \neq 0$, $\partial\alpha/\partial x \neq 0$, $\partial\beta/\partial x \neq 0$. We state the theorem for this class of systems, because it is simpler to describe the two possible choices for γ and δ than with (9.3), knowing that $S = T = 0$.

Lemma 9.6.4. *If system (9.63) admits a parameterization (φ, ψ, χ) of order (k, ℓ) at a point, then $\varphi_{u^{(k)}}$ is a nonzero real analytic function.*

Proof. After a change of coordinates (9.45), use Lemma 9.6.1. □

Theorem 9.6.5. *Let (x_0, y_0, z_0) be a point where κ, $\alpha - \beta$, α_3 and β_3 are nonzero, and k, ℓ, L three integers. If system (9.63) has a parameterization of order (k, ℓ) at $\mathcal{X} = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ldots, x_0^{(L)}, y_0^{(L)})$ with φ, ψ, χ defined on $U \subset \mathbb{R}^{k+\ell+2}$, then $k \geq 1$, $\ell \geq 1$ and, for any point $(u_0, \ldots, u_0^{(k)}, v_0, \ldots, v_0^{(\ell)}) \in U$ (not necessarily sent to $\mathcal{X}$ by the parameterization) such that*

$$\varphi_{u^{(k)}}(u_0, \ldots, u_0^{(k)}, v_0, \ldots, v_0^{(\ell)}) \neq 0,$$

there exist a neighborhood O of $(u_0, \ldots, u_0^{(k-1)}, \varphi(u_0 \cdots v_0^{(\ell)}), v_0, \ldots, v_0^{(\ell-1)})$ in $\mathbb{R}^{k+\ell+1}$ and a regular solution $p : O \to \mathbb{R}$ of one of the two systems $\mathcal{E}_{k,\ell}^{\gamma^1,\delta^1}$ or $\mathcal{E}_{k,\ell}^{\gamma^2,\delta^2}$ with γ^i, δ^i given by (9.46)-(9.47), such that p, φ, ψ, χ are related by (9.23), (9.24) and (9.25).

Remark 9.6.3 applies to this theorem in the same way as theorem 9.6.2.

Proof. Like in the beginning of the proof of Theorem 9.6.2, a parameterization (φ, ψ, χ) of order (k, ℓ) with $\varphi_{u^{(k)}} \neq 0$ yields an implicit form (9.52). Substituting in (9.63), one obtains an identity between two polynomials in $v^{(\ell+1)}$ and $\dot{x}$. The coefficient of $(v^{(\ell+1)})^2$ in the right-hand side must be zero and this yields that p cannot depend on $v^{(\ell)}$; the linear term in $v^{(\ell+1)}$ then implies that q does not depend on $v^{(\ell)}$ either. To go further, let us define, as in the proof of Theorem 9.6.2,

$$\mathcal{P} = Fp + rp_{u^{(k-1)}} + v^{(\ell)}p_{v^{(\ell-1)}}\;, \quad \mathcal{Q} = Fq + rq_{u^{(k-1)}} + v^{(\ell)}q_{v^{(\ell-1)}}\;, \quad (9.64)$$

with F as in (9.15). Still substituting in (9.63), the terms of degree 0, 1 and 2 with respect to $\dot{x}$ then yield

$$\begin{array}{l}\mathcal{Q}=\kappa(x,p,q)\mathcal{P}^2+b(x,p,q)\mathcal{P}+c(x,p,q)\,,\\ q_x=\kappa(x,p,q)\left(2p_x-\alpha(x,p,q)-\beta(x,p,q)\right)\mathcal{P}+a(x,p,q)+b(x,p,q)p_x\,,\\ 0=\left(p_x-\alpha(x,p,q)\right)\left(p_x-\beta(x,p,q)\right)\,.\end{array}\tag{9.65}$$

The factors in the third equation cannot both be zero because $\alpha-\beta\neq 0$. Let us assume

$$p_x-\alpha(x,p,q)=0\,,\quad p_x-\beta(x,p,q)\neq 0\tag{9.66}$$

(interchange the roles of α and β for the other alternative). Since $\alpha_3\neq 0$, the map A defined in (9.45) has locally an inverse A^{-1}, and the equation in (9.66) is equivalent to $(x,p,q)=A^{-1}(x,p,p_x)$; by differentiation an expression of q_x as a function of x,p,p_x,p_{xx} is obtained ; solving the second equation in (9.65) for $\mathcal{P}$ and substituting q and q_x, one obtains $\mathcal{P}=\gamma^1(x,p,p_x,p_{xx})$ with γ^1 defined by (9.46)-(9.47). If one had chosen the other alternative in (9.66), A and γ^1 would be replaced by B and γ^2.

Since $\mathcal{P}$ is also given by (9.64), the relation (9.109) holds with $f=\gamma^1$; also, for the same reasons as in the proof of Theorem 9.6.2 (two lines further than (9.60)), $r_{v^{(\ell)}}$ is nonzero and it would be absurd that p depends on x only. One may then apply Lemma 9.9.2 (page 238) and deduce that $k\geq 1$, $\ell\geq 1$, $p_{u^{(k-1)}}\neq 0$, $p_{v^{(\ell-1)}}\neq 0$.

Since neither p nor $\mathcal{P}=\gamma^1(x,p,p_x,p_{xx})$ depend on $v^{(\ell)}$ and $p_{u^{(k-1)}}\neq 0$, the first equation in (9.64) implies that r assumes the form (9.61) with σ and τ some functions of the $k+\ell+1$ variables $u,\dot{u},\ldots,u^{(k-1)},x,v,\dot{v},\ldots,v^{(\ell-1)}$, and that two relations hold : on the one hand $\sigma p_{u^{(k-1)}}+p_{v^{(\ell-1)}}=0$, i.e. one of the relations in (9.20), and on the other hand the first relation in (9.20) with $\gamma=\gamma^1$. Similarly, the second equation in (9.64) yields $\sigma q_{u^{(k-1)}}+q_{v^{(\ell-1)}}=0$ and $Fq+\tau q_{u^{(k-1)}}=\mathcal{Q}=\kappa\mathcal{P}^2+b\mathcal{P}+c$. Applying $F+\tau\partial/\partial u^{(k-1)}$ and E to the first relation in (9.66) and using the four relations we just established, one obtains on the one hand the second relation in (9.20), with $\delta=\delta^1$ (δ^1 defined in (9.46)-(9.47)) and on the other hand $\sigma p_{xu^{(k-1)}}+p_{xv^{(\ell-1)}}=0$. The relations $\sigma_x=0$, $\sigma\neq 0$ and $\tau_x\neq 0$ are then obtained exactly like at the end of the proof of theorem 9.6.2 ; hence p satisfies (9.20) with $\gamma=\gamma^1$ and $\delta=\delta^1$; this proves, thanks to Remark 9.3.4, that p is a solution of $\mathcal{E}_{k,\ell}^{\gamma^1,\delta^1}$ (it would be $\mathcal{E}_{k,\ell}^{\gamma^2,\delta^2}$ if one had chosen the other alternative in (9.66)). The last paragraph of the proof of Theorem 9.6.2 can be used to prove that this solution is K-regular with $K\leq k+\ell+1$. □

9.7 Flat outputs and differential flatness

Definition 9.7.1 (flatness, endogenous parameterization [37])**.** *A pair $A=(a,b)$ of real analytic functions on a neighborhood of $(x_0,y_0,z_0,\ldots,x_0^{(j)},y_0^{(j)})$ in $\widehat{\Omega}\times\mathbb{R}^{2j-2}$ is a* flat output of order j *at $\mathcal{X}=(x_0,y_0,z_0,\ldots,x_0^{(L)},y_0^{(L)})$ (with $L\geq j\geq 0$) for system (9.3) if there exists a Monge parameterization (9.10) of some order (k,ℓ) at $\mathcal{X}$ such that any germ $(x(.),y(.),z(.),u(.),v(.))\in\mathcal{V}\times\mathcal{U}$ (with U,V possibly smaller than in (9.10)) satisfies (9.67) if and only if it satisfies (9.68) :*

$$\left.\begin{array}{rcl}\varphi\big(u(t),\dot{u}(t),\ldots,u^{(k)}(t),v(t),\dot{v}(t),\ldots,v^{(\ell)}(t)\big)&=&x(t)\\ \psi\big(u(t),\dot{u}(t),\ldots,u^{(k)}(t),v(t),\dot{v}(t),\ldots,v^{(\ell)}(t)\big)&=&y(t)\\ \chi\big(u(t),\dot{u}(t),\ldots,u^{(k)}(t),v(t),\dot{v}(t),\ldots,v^{(\ell)}(t)\big)&=&z(t)\end{array}\right\}\,,\tag{9.67}$$

$$\left.\begin{array}{l}\dot{z}(t)=h\big(x(t),y(t),z(t),\,\dot{y}(t)-z(t)\dot{x}(t)\,\big)\;+\;g\big(x(t),y(t),z(t),\,\dot{y}(t)-z(t)\dot{x}(t)\,\big)\,\dot{x}(t)\\ u(t)=a\big(x(t),y(t),z(t),\dot{x}(t),\dot{y}(t),\ddot{x}(t),\ddot{y}(t),\ldots,x^{(j)}(t),y^{(j)}(t)\big)\\ v(t)=b\big((x(t),y(t),z(t),\dot{x}(t),\dot{y}(t),\ddot{x}(t),\ddot{y}(t),\ldots,x^{(j)}(t),y^{(j)}(t)\big)\end{array}\right\}\tag{9.68}$$

System (9.3) is called flat *if and only if it admits a flat output of order j for some $j \in \mathbb{N}$. A Monge parameterization is* endogenous[5] *if and only if there exists a flat output associated to this parameterization as above.*

In control theory, flatness is a better known notion than Monge parameterization. For general control systems, it implies existence of a parameterization (obvious in the above definition), and people conjecture [42] that the two notions are in fact equivalent, at least away from some singular points. In any case, our results are relevant to both : systems (9.3) that are proved to be parameterizable are also flat and our efforts toward proving that the other ones are not parameterizable would also prove that they are not flat.

Theorem 9.3.7 gave a procedure to derive a parameterization of (9.3) from a regular solution p of $\mathcal{E}_{k,\ell}^{\gamma,\delta}$, and we saw in Section 9.5 that, unless $S = T = J = 0$, these are the only possible parameterizations. One can tell when such a parameterization is endogenous :

Proposition 9.7.2. *Let $p : O \to \mathbb{R}$, with $O \subset \mathbb{R}^{k+\ell+1}$ open, be a regular solution of system $\mathcal{E}_{k,\ell}^{\gamma,\delta}$. The parameterization of order (k, ℓ) of system (9.3) associated to p according to Theorem 9.3.7 is endogenous if and only if p is exactly $(k + \ell - 2)$-regular ; then, the associated flat output is of order $j \leq k + \ell - 2$.*

Proof. In the end of the proof of Theorem 9.3.7, it was established that (9.67), written $\Gamma(u, v) = (x, y, z)$, is equivalent to (9.35) if (x, y, z) is a solution of (9.3). If either $i_0 < k$ or $j_0 < \ell$ in (9.35), then there are, for fixed $x(.), y(.), z(.))$, infinitely many solutions $u(.), v(.))$ of (9.35) while there is a unique one for (9.68). Hence $i_0 = k$ and $j_0 = \ell$ if (9.67) is equivalent to (9.68) ; then $K = i_0 + j_0 - 2 = k + \ell - 2$ so that p is $(k + \ell - 2)$-regular and (9.35) (where u and v do not appear in the right-hand side) is of the form (9.68) with $j = K = k + \ell - 2$. □

The main result in Chapter 8 is a necessary condition for "(x, u)-dynamic linearizability" ((x, u)-flatness might be more appropriate) of system (9.1). For system (9.1), it means existence of a flat output whose components are functions of $\xi^1, \xi^2, \xi^3, \xi^4, w^1, w^2$; for system (9.3), it translates as follows. The functions γ and δ in (9.1) are supposed to be related to g and h in (9.3) according to (9.6) and (9.7).

Definition 9.7.3. *System (9.1) is "(x, u)-dynamic linearizable" is and only if system (9.3) admits a flat output of order 2 of a special kind : $A(x, y, z, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}) = \mathfrak{a}(x, y, z, \lambda, \dot{x}, \dot{\lambda})$ for some smooth $\mathfrak{a}$.*

The following proposition is useful to recover the main result from Chapter 8, see Remark 9.5.6.

Proposition 9.7.4. *If system* (9.1) *is "(x, u)-dynamic linearizable" in the sense of* Chapter 8*, then* (9.3) *admits a parameterization of order (k, ℓ) with $k \leq 3$ and $\ell \leq 3$.*

Proof. Consider the map $(x, y, z, \lambda, \dot{x}, \dot{\lambda}, \ldots, x^{(4)}, \lambda^{(4)}) \mapsto \begin{pmatrix} \mathfrak{a}(x, y, z, \lambda, \dot{x}, \dot{\lambda}) \\ \dot{\mathfrak{a}}(x, y, z, \lambda, \dot{x}, \dot{\lambda}, \ddot{x}, \ddot{\lambda}) \\ \ddot{\mathfrak{a}}(x, y, z, \lambda, \dot{x}, \dot{\lambda}, \ldots, x^{(3)}, \lambda^{(3)}) \\ \mathfrak{a}^{(3)}(x, y, z, \lambda, \dot{x}, \dot{\lambda}, \ldots, x^{(4)}, \lambda^{(4)}) \end{pmatrix}$.
Its Jacobian is 8×12, and has rank 8, but the 8×8 sub-matrix corresponding to derivatives with respect to $\dot{x}, \dot{\lambda}, \ldots, x^{(4)}, \lambda^{(4)}$ has rank 4 only. Hence x, y, z, and λ can be expressed as functions of the components of $\mathfrak{a}, \dot{\mathfrak{a}}, \ddot{\mathfrak{a}}, \mathfrak{a}^{(3)}$, yielding a Monge parameterization of order at most $(3, 3)$. □

[5] This terminology (endogenous vs. exogenous) is borrowed from the authors of [37, 68] ; it usually qualifies feedbacks rather than parameterizations, but the notion is exactly the same.

9.8 Conclusion

Let us discuss both flatness (see Section 9.7) and Monge parameterization. For convenience, assume $k \leq \ell$ and call F-systems the systems (9.3) such that $S = T = J = 0$ and C-systems all the other ones.

F-systems are flat ; this was proved in Chapter 8. This paper adds that they admit a Monge parameterization of order (1,2), but does *not* prove differential flatness of any system not known to be flat up to now : C-systems are not believed to be flat. It does not either prove non-flatness of any system : it only *conjectures* that no C-system admits a parameterization, and hence none of them is flat. To the best of our knowledge, no one knows whether simple systems like (9.14-b) or (9.14-c) are flat of not.

The first contribution of the paper is to prove that a C-system admits a parameterization of order (k, ℓ) if and only if the PDEs $\mathcal{E}_{k,\ell}^{\gamma,\delta}$, for suitable γ, δ, admit a "regular solution" p. The second contribution is to prove that, for any γ, δ, there is no regular solution to $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ if either $k \leq 2$ or $k = \ell = 3$ (this does not contradict existence of parameterizations of order $(1, 2)$ for F-systems : these do not "derive from" a solution of these PDEs). We guess, in Conjecture 9.3.8, that even for higher values of the integers k, ℓ, *none* of these PDEs have any regular solution ; this would imply that C-systems are not flat.

Besides recovering the results from Chapter 8 with far more natural and elementary arguments, we believe that some insight was gained on Monge parameterizations of *any order* for "C-systems", by reducing non-parameterizability to non-existence of solutions to a systems of PDEs that can easily be written for any k, ℓ.

The main perspective raised by this paper is to **prove Conjecture 9.3.8**. The only *theoretical* difficulty is, in fact, that no a priori bound on the integers k, ℓ is known. Indeed, as explained in Section 9.3.3, for fixed k, ℓ, γ, δ, it amounts to a classical problem. To prove Proposition 9.3.9, we solved, in a synthetic manner, that problem for $k \leq 2$ or $k = \ell = 3$ and arbitrary γ and δ. We lack a non-finite argument, or a better understanding of the structure, to go to arbitrary k, ℓ. Let us comment more on the (non trivial) case where γ and δ are *polynomials*, for instance the very simple ones in (9.49). For fixed k, ℓ, the question can be formulated in terms of differential polynomial rings : does the differential ideal generated by left-hand sides of the equations (9.49) contain the polynomials ED^ip ? Differential elimination (see [86] or the recent survey [49]) is relevant here ; finite algorithms have been already implemented in computer algebra. Although we have not yet succeeded (because of complexity) in carrying out these computations, even on example (9.49) for $(k, \ell) = (3, 4)$, and although it will certainly not *provide* a bound on k, ℓ, we do believe that computer algebra is a considerable potential help.

Another perspective is to enlarge the present approach to **higher dimensional control systems**. For instance, what would play the role of our system of PDEs $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ when, instead of (9.3), one considers a single relation between more than three scalar functions of time (this captures, instead of (9.1), control affine systems with n states and 2 controls, $n > 4$) ? We have very little insight on this question : the present paper strongly takes advantage of the special structure inherent to our small dimension ; the situation could be far more complex.

9.9 Proofs

Proof of Lemma 9.3.6

For this proof only, the notation $\mathcal{F}_{i,j}$ $(0 \leq i \leq k,\ 0 \leq j \leq \ell)$ stands either for the following family of $i+j$ vectors in $\mathbb{R}^{K+2}$ or for the corresponding $(K+2) \times (i+j)$ matrix :

$$\mathcal{F}_{i,j} \ = \ \left(\frac{\partial \pi}{\partial u^{(k-i)}}, \ldots, \frac{\partial \pi}{\partial u^{(k-1)}}, \frac{\partial \pi}{\partial v^{(\ell-j)}}, \ldots, \frac{\partial \pi}{\partial v^{(\ell-1)}} \right)$$

with the convention that if i or j is zero the corresponding list is empty ; $\mathcal{F}_{i,j}$ depends on $u, \ldots, u^{(k-1)}, v, \ldots, v^{(\ell-1)}, x, \ldots, x^{(K)}$. Let us first prove that, at least outside a closed subset of empty interior,

$$\mathrm{rank}\mathcal{F}_{k,\ell} \ = \ K+2\ . \tag{9.69}$$

Indeed, if it is smaller at all points of $O \times \mathbb{R}^K$, then, around points (they form an open dense set) where it is locally constant, there is at least one function R such that a non-trivial identity $R(p_x, p, \ldots, D^K p, x, \ldots, x^{(K)}) = 0$ holds and the partial derivative of R with respect to at least one of its $K+2$ first arguments is nonzero. Since p is K-regular, applying E to this relation, shows that R does not depend on $D^K p$, and hence does not depend on $x^{(K)}$ either. Then, applying ED, ED^2 and so on, and using the fact that, according to (9.21), Dp_x is a function of $p_x, p, Dp, x, \dot{x}$, we get finally a relation $R(p_x, p, x) = 0$ with $(R_{p_x}, R_p) \neq (0,0)$. Differentiating with respect to $u^{(k-1)}$, one obtains $R_{p_x} p_{x u^{(k-1)}} + R_p p_{u^{(k-1)}} = 0$; hence, from the first relation in (9.16-c), $R_{p_x} \neq 0$, and the relation $R(p_x, p, x) = 0$ implies, in a neighborhood of almost any point, $p_x = f(p, x)$ for some smooth function f. From Lemma 9.3.5, this would contradict the fact that the solution p is K-regular. This proves (9.69).

Let now W_s $(1 \leq s \leq K+2)$ be the set of pairs (i,j) such that $i+j = s$ and the rank of $\mathcal{F}_{i,j}$ is s at least at one point in $O \times \mathbb{R}^K$, i.e. one of the $s \times s$ minors of $\mathcal{F}_{i,j}$ is a nonzero real analytic function on $O \times \mathbb{R}^K$. The lemma states that W_{K+2} is nonempty ; in order to prove it by contradiction, suppose that $W_{K+2} = \varnothing$ and let $\overline{s}$ be the smallest s such that $W_s = \varnothing$. From (9.16-c), W_1 contains $(1,0)$, hence $2 \leq \overline{s} \leq K+2 < k+\ell+1$. Take (i', j') in $W_{\overline{s}-1}$; $\mathcal{F}_{i',j'}$ has rank $i'+j'$ (i.e. is made of $i'+j'$ linearly independent vectors) on an open dense set $A \subset O \times \mathbb{R}^K$. Let the $i_1 \leq k$ and $j_1 \leq \ell$ be the largest such that $\mathcal{F}_{i_1,j'}$ and $\mathcal{F}_{i',j_1}$ have rank $\overline{s}-1$ on A. On the one hand, since $i'+j' = \overline{s}-1 < k+\ell$, one has either $i' < k$ or $j' < \ell$. On the other hand since $W_{\overline{s}}$ is empty, it contains neither $(i'+1, j')$ nor $(i', j'+1)$; hence the rank of $\mathcal{F}_{i'+1,j'}$ is less than $i'+j'+1$ if $i' < k$, and so is the rank of $\mathcal{F}_{i',j'+1}$ if $j' < \ell$.

To sum up, the following implications hold : $i' < k \Rightarrow i_1 \geq i'+1$ and $j' < \ell \Rightarrow j_1 \geq j'+1$. From (9.69), one has either $i_1 < k$ or $j_1 < \ell$. Possibly exchanging u and v, assume $i_1 < k$; all the vectors $\partial\pi/\partial u^{(k-i_1)}$, ..., $\partial\pi/\partial u^{(k-i'+1)}$, $\partial\pi/\partial u^{(\ell-j_1)}$, ..., $\partial\pi/\partial u^{(\ell-j'+1)}$ are then linear combinations of the vectors in $\mathcal{F}_{i',j'}$, while $\partial\pi/\partial u^{(k-i_1-1)}$ is not :

$$\mathrm{rank}\mathcal{F}_{i',j'} = i'+j'\ , \quad \mathrm{rank}\mathcal{F}_{i_1,j_1} = i'+j'\ , \quad \mathrm{rank}\left(\frac{\partial \pi}{\partial u^{(k-i_1-1)}}\ , \mathcal{F}_{i',j'} \right) = i'+j'+1 \tag{9.70}$$

on an open dense subset of $O \times \mathbb{R}^K$, that we still call A although it could be smaller. In a neighborhood of any point in this set, one can, from the third relation, apply the inverse function theorem and obtain, for an open $\Omega \subset \mathbb{R}^{k+\ell+K+1}$, a map $\Omega \to \mathbb{R}^{i'+j'+1}$ that expresses $u^{(k-i')}, \ldots, u^{(k-1)}$, $v^{(\ell-j')}, \ldots, v^{(\ell-1)}$ and $u^{(k-i_1-1)}$ as functions of

$$u, \ldots, u^{(k-i_1-2)}, u^{(k-i_1)}, \ldots, u^{(k-i'-1)}, v, \ldots, v^{(\ell-j'-1)}, x, \ldots, x^{(K)}$$

and $i'+j'+1$ functions chosen among $p_x, p, Dp, \ldots, D^K p$ ($i'+j'+1$ columns defining an invertible minor in $\left(\partial\pi/\partial u^{(k-i_1-1)}\,,\, \mathcal{F}_{i',j'}\right)$). Focusing on $u^{(k-i_1-1)}$, one has

$$u^{(k-i_1-1)} = B\left(u, \ldots, u^{(k-i_1-2)}, u^{(k-i_1)}, \ldots, u^{(k-i'-1)}, v, \ldots, v^{(\ell-j'-1)}, x, \ldots, x^{(K)}, p_x, p, \ldots, D^K p\right) \tag{9.71}$$

where B is some smooth function of $k+\ell+2K+2-i'-j'$ variables and we have written all the functions $p_x, p, Dp, \ldots, D^{K-1}p$ although B really depends only on $i'+j'+1$ of them.

Differentiating (9.71) with respect to $u^{(k-i')}, \ldots, u^{(k-1)}, v^{(\ell-j')}, \ldots, v^{(\ell-1)}$, one has, with obvious matrix notation, $\left(\frac{\partial B}{\partial p_x}\ \frac{\partial B}{\partial p}\ \cdots\ \frac{\partial B}{\partial D^{K-1}p}\right)\mathcal{F}_{i',j'} = 0$, where the right-hand side is a line-vector of dimension $i'+j'$; from (9.70), this implies

$$\left(\frac{\partial B}{\partial p_x}\ \frac{\partial B}{\partial p}\ \cdots\ \frac{\partial B}{\partial D^{K-1}p}\right)\mathcal{F}_{i_1,j_1} \;=\; 0\,, \tag{9.72}$$

where the right-hand side is a now a bigger line-vector of dimension i_1+j_1. Differentiating (9.71) with respect to $u^{(k-i_1)}, \ldots, u^{(k-i'-1)}, v^{(\ell-j_1)}, \ldots, v^{(\ell-j'-1)}$ and using (9.72) yields that B does not depend on its arguments $u^{(k-i_1)}, \ldots, u^{(k-i'-1)}$ and $v^{(\ell-j_1)}, \ldots, v^{(\ell-j'-1)}$. B cannot depend on $D^K p$ either because $ED^K p \neq 0$ and all the other arguments of B are constant along E ; then it cannot depend on $x^{(K)}$ either because $x^{(K)}$ appears in no other argument ; (9.71) becomes

$$u^{(k-i_1-1)} = B\left(u, \ldots, u^{(k-i_1-2)}, v, \ldots, v^{(\ell-j_1-1)}, x, \ldots, x^{(K-1)}, p_x, p, \ldots, D^{K-1}p\right)\ .$$

Applying D, using (9.21) and substituting $u^{(k-i_1-1)}$ from above, one gets, from some smooth C,

$$u^{(k-i_1)} = C\big(\, u, \ldots, u^{(k-i_1-2)},\ \underbrace{v, \ldots, v^{(\ell-j_1)}}_{\text{rm empty if } j_1=\ell},x, \ldots, x^{(K)}, p_x, p, \ldots, D^K p\big)\ . \tag{9.73}$$

Differentiating with respect to $u^{(k-i')}, \ldots, u^{(k-1)}, v^{(\ell-j')}, \ldots, v^{(\ell-1)}$ yields

$$\left(\frac{\partial C}{\partial p_x}\ \frac{\partial C}{\partial p}\ \cdots\ \frac{\partial C}{\partial D^{K-1}p}\right)\mathcal{F}_{i',j'} = 0\ ,$$

the right-hand side being a line-vector of dimension $i'+j'$. From the first two relations in (9.70), $\partial\pi/\partial u^{(k-i_1-1)}$ is a linear combination of the columns of $\mathcal{F}_{i',j'}$, hence one also has $\left(\frac{\partial C}{\partial p_x}\ \frac{\partial C}{\partial p}\ \cdots\ \frac{\partial C}{\partial D^{K-1}p}\right)\frac{\partial\pi}{\partial u^{(k-i_1-1)}} \;=\; 0\ .$
This implies that the derivative of the right-hand side of (9.73) with respect to $u^{(k-i_1-1)}$ is zero. This is absurd.

Proof of Lemmas 9.3.10, 9.3.11 and 9.3.12

We need some notations and preliminaries. With F, E and τ defined in (9.15) and (9.17), define the vector fields

$$X = \frac{\partial}{\partial x}\,,\quad Y = F + \tau\frac{\partial}{\partial u^{(k-1)}} \tag{9.74}$$

$$X_1 = [X,Y],\ X_2 = [X_1,Y]\,,\qquad E_2 = [E,Y],\ E_3 = [E_2,Y]\ . \tag{9.75}$$

Then (9.20) obviously implies

$$Yp = \gamma(x,p,p_x,p_{x,x})\,,\qquad Yp_x = \delta(x,p,p_x,p_{x,x})\,,\qquad X\sigma = 0\,, \tag{9.76}$$

and a simple computation yields (we recall E from (9.17)) :

$$\begin{gathered} X_1=\tau_x\,\frac{\partial}{\partial u^{(k-1)}}\,,\quad X_2=\tau_x\,\frac{\partial}{\partial u^{(k-2)}}+(\cdots)\frac{\partial}{\partial u^{(k-1)}}\,,\\ E=\frac{\partial}{\partial v^{(\ell-1)}}+\sigma\,\frac{\partial}{\partial u^{(k-1)}}\,,\quad E_2=\frac{\partial}{\partial v^{(\ell-2)}}+\sigma\,\frac{\partial}{\partial u^{(k-2)}}+(\cdots)\frac{\partial}{\partial u^{(k-1)}}\,,\\ E_3=\frac{\partial}{\partial v^{(\ell-3)}}+\sigma\,\frac{\partial}{\partial u^{(k-3)}}+(\cdots)\frac{\partial}{\partial u^{(k-2)}}+(\cdots)\frac{\partial}{\partial u^{(k-1)}}\,. \end{gathered} \tag{9.77}$$

The vector field X_1 and X_2 are linearly independent because $\tau_x\neq 0$, see (9.20). Computing the following brackets and decomposing on X_1 and X_2, one gets

$$[X,X_1]=\lambda X_1\,,\quad [X_1,X_2]=\lambda' X_1+\lambda'' X_2\,, \tag{9.78}$$

$$[X,E]=0\,,\quad [X,E_2]=\mu X_1\,,\quad [X,E_3]=\mu' X_1+\mu'' X_2\,,\quad [E_2,X_2]=\nu' X_1+\nu'' X_2\,. \tag{9.79}$$

for some functions $\lambda,\lambda',\lambda'',\mu,\mu',\mu'',\nu',\nu''$.

Proof of Lemma 9.3.10. From (9.16-c), $y=p(u,\dots,u^{(k-1)},x,v,\dots,v^{(\ell-1)})$ defines local coordinates $u,\dots u^{(k-2)}$, y,x, $v,\dots,v^{(\ell-1)}$. Composing p_x by the inverse of this change of coordinates, there is a function α of $k+\ell+1$ variables such that $p_x=\alpha(u,\dots,u^{(k-2)},\,p\,,x,\dots,v^{(\ell-1)})$ identically. Since $Ep=Ep_x=0$ (see (9.20)), applying E to both sides of this identity yields that α does not depend on its argument $v^{(\ell-1)}$. Similarly, if $k\geq 2$, differentiating both sides of the same identity with respect to $u^{(k-1)}$ and $u^{(k-2)}$, the fact that the determinant in the lemma is zero implies that α does not depend on its argument $u^{(k-2)}$. To sum up, p and p_x satisfy an identity

$$p_x=\alpha(u,\dots,u^{(k-3)},\,p\,,x,v,\dots,v^{(\ell-2)}),$$

where the first list is empty if $k=1$ or $k=2$. Now define two integers $m\leq k-3$ and $n\leq \ell-2$ as the smallest such that α depends on $u,\dots,u^{(m)},x,y,v,\dots,v^{(n)}$, with the convention that $m<0$ if $k=1$, $k=2$, or α depends on none of the variables $u,\dots,u^{(k-3)}$ and $n<0$ if α depends on none of the variables $v,\dots,v^{(\ell-2)}$.

Applying Y to both sides of the above identity yields

$$Yp_x=\alpha_y\,Yp+\sum_{i=0}^{m}u^{(i+1)}\alpha_{u^{(i)}}+\sum_{i=0}^{n}v^{(i+1)}\alpha_{v^{(i)}}\,,$$

where, if $m<0$ or $n<0$, the corresponding sum is empty. Using (9.76), since $p_{xx}=\alpha_x+\alpha\alpha_y$, one can replace Yp with $\gamma(x,y,\alpha,\alpha_x+\alpha\alpha_y)$ and Yp_x with $\delta(x,y,\alpha,\alpha_x+\alpha\alpha_y)$ in the above equation, where all terms except the last one of each non-empty sum therefore depend on $u,\dots,u^{(m)},x,y,v,\dots,v^{(n)}$ only. Differentiating with respect to $u^{(m+1)}$ and $v^{(n+1)}$ yields $\alpha_{u^{(m)}}=\alpha_{v^{(n)}}=0$, which is possible only if $m<0$ and $n<0$, hence the lemma. □

Proof of Lemma 9.3.11. From (9.38), setting

$$y=p(u,\dots,u^{(k-1)},x,v,\dots,v^{(\ell-1)}),\;\; z=p_x(u,\dots,u^{(k-1)},x,v,\dots,v^{(\ell-1)}), \tag{9.80}$$

one gets some local coordinates $(u,\dots,u^{(k-3)},x,y,z,v,\dots,v^{(\ell-1)})$. In these coordinates, the vector fields X and Y defined by (9.74) have the following expressions, where χ and α are some functions, to be studied further :

$$X \;=\; \frac{\partial}{\partial x}+z\frac{\partial}{\partial y}+\alpha\frac{\partial}{\partial z}\,, \tag{9.81}$$

$$Y \;=\; \gamma\frac{\partial}{\partial y}+\delta\frac{\partial}{\partial z}+\chi\frac{\partial}{\partial u^{(k-3)}}+\sum_{i=0}^{k-4}u^{(i+1)}\frac{\partial}{\partial u^{(i)}}+\sum_{i=0}^{\ell-1}v^{(i+1)}\frac{\partial}{\partial v^{(i)}}\,. \tag{9.82}$$

In the expression of Y, the third term is zero if $k=2$, the fourth term ($\sum_{i=0}^{k-4}\cdots$) is zero if $k=2$ or $k=3$, and the notations γ and δ are slightly abusive : γ stands for the function

$$(u,\ldots,u^{(k-3)},x,y,z,v,\ldots,v^{(\ell-1)})\mapsto\gamma(x,y,z,\alpha(u,\ldots,u^{(k-3)},x,y,z,v,\ldots,v^{(\ell-1)}))\ ,$$

and the same for δ. With the same abuse of notations, (9.16-e) reads

$$X\gamma-\delta\neq 0. \tag{9.83}$$

The equalities $(\sigma\frac{\partial}{\partial u^{(k-1)}}+\frac{\partial}{\partial v^{(\ell-1)}})u^{(k-2)}=\frac{\partial}{\partial x}u^{(k-2)}=\frac{\partial}{\partial u^{(k-1)}}u^{(k-2)}=0$ are obvious in the original coordinates. Since the inverse of the change of coordinates (9.80) is given by

$$u^{(k-2)}=\chi(u,\ldots,u^{(k-3)},x,y,z,v,\ldots,v^{(\ell-1)}),\quad u^{(k-1)}=Y\chi(u,\ldots,u^{(k-3)},x,y,z,v,\ldots,v^{(\ell-1)}),$$

and E, X and X_1 are given by (9.77), those equalities imply

$$E\chi=X\chi=X_1\,\chi=0\ . \tag{9.84}$$

Then, from (9.75), (9.81) and (9.82),

$$\begin{aligned} X_1 &= (X\gamma-\delta)\,\frac{\partial}{\partial y}\ +\ (X\delta-Y\alpha)\,\frac{\partial}{\partial z}\ , && (9.85)\\ [X,X_1] &= \big(X^2\gamma-2X\delta-Y\alpha\big)\,\frac{\partial}{\partial y}\ +\ \big(X^2\delta-XY\alpha-X_1\alpha\big)\,\frac{\partial}{\partial z}\ . && (9.86)\end{aligned}$$

With these expressions of X and X_1, the first relation in (9.78) implies :

$$\begin{vmatrix} X\gamma-\delta & X^2\gamma-2X\delta+Y\alpha\\ X\delta-Y\alpha & X^2\delta-XY\alpha-X_1\alpha\end{vmatrix}=0\ . \tag{9.87}$$

The definition of α implies $Xz=\alpha$. In the original coordinates, this translates into the identity $p_{xx}=\alpha(u,\ldots,u^{(k-3)},x,p,p_x,v,\ldots,v^{(\ell-1)})$. Since $Ep=Ep_x=Ep_{x,x}=0$ (see (9.20)), applying E to both sides of this identity yields that α does not depend on its argument $v^{(\ell-1)}$. Also, if $k\geq 3$, differentiating both sides with respect to $u^{(k-1)}$, $u^{(k-2)}$ and $u^{(k-3)}$, we obtain that the determinant (9.37) is zero if and only if α does not depend on its argument $u^{(k-3)}$. To sum up, under the assumptions of the lemma,

$$\alpha \text{ depends on }\ u,\ldots,u^{(k-4)},x,y,z,v,\ldots,v^{(\ell-2)}\ \text{ only} \tag{9.88}$$

with the convention that the first list is empty if $k=2$ or $k=3$. Now define two integers $m\leq k-4$ and $n\leq\ell-2$ as the smallest such that α depends on $u,\ldots,u^{(m)},x,y,v,\ldots,v^{(n)}$, with the convention that $m<0$ if $k=2$, $k=3$, or α depends on none of the variables $u,\ldots,u^{(k-4)}$, and $n<0$ if α depends on none of the variables $v,\ldots,v^{(\ell-2)}$. We have

$$m\geq 0\ \Rightarrow\ \alpha_{u^{(m)}}\neq 0\ ,\qquad n\geq 0\ \Rightarrow\ \alpha_{v^{(n)}}\neq 0\ . \tag{9.89}$$

Since m is no larger that $k-4$, χ does not appear in the expression of $Y\alpha$:

$$Y\alpha\ =\ \gamma\alpha_y+\delta\alpha_z+\sum_{i=0}^{m}u^{(i+1)}\alpha_{u^{(i)}}+\sum_{i=0}^{n}v^{(i+1)}\alpha_{v^{(i)}} \tag{9.90}$$

where the first (or second) sum is empty if m (or n) is negative.

In the left-hand side of (9.87), all the terms depend only on $u,\ldots,u^{(m)},x,y,z,v,\ldots,v^{(n)}$, except $Y\alpha$, $XY\alpha$ and $X_1\alpha$ that depend on $u^{(m+1)}$ if $m\geq 0$ or on $v^{(n+1)}$ if $n\geq 0$ (see above) ; the determinant is a polynomial of degree two with respect to $u^{(m+1)}$ and $v^{(n+1)}$ with coefficients depending on $u,\dots,u^{(m)},x,y,z,v,\dots,v^{(n)}$ only, and the term of degree two, coming from $(Y\alpha)^2$, is

$$\left(\alpha_{u^{(m)}}u^{(m+1)}+\alpha_{v^{(n)}}v^{(n+1)}\right)^2 \ .$$

Hence (9.87) implies $\alpha_{u^{(m)}}=\alpha_{v^{(n)}}=0$ and, from (9.89), negativity of m and n are negative. By definition of these integers, this implies that α depends on (x,y,z) only : in the original coordinates, one has $p_{xx}=\alpha(x,p,p_x)$. □

Before proving Lemma 9.3.12, we need to extract more information from the previous proof :

Lemma 9.9.1. *Assume, as in Lemma 9.3.11, that p is a solution of $\mathcal{E}^{\gamma,\delta}_{k,\ell}$ satisfying (9.38), but assume also that $\ell\geq k\geq 3$ and the determinant (9.37) is nonzero. Then $[X,E_2]=[X,E_3]=0$.*

Proof. Starting as in the proof of Lemma 9.3.11, one does not obtain (9.88) but, since (9.37) is nonzero,

$$\alpha \text{ depends on } \ u,\dots,u^{(k-4)},x,y,z,v,\dots,v^{(\ell-2)} \ \text{ and } \ \alpha_{u^{(k-3)}}\neq 0 \ . \tag{9.91}$$

Since $Ep=Ep_x=0$, one has $E=\partial/\partial v^{(\ell-1)}$ in these coordinates. The first equation in (9.84) then reads $\chi_{v^{(\ell-1)}}=0$, and (9.75) and (9.82) yield

$$E_2=\frac{\partial}{\partial v^{(\ell-2)}}\ , \quad [X,E_2]=-\,\alpha_{v^{(\ell-2)}}\frac{\partial}{\partial z}\ .$$

Since $[X,E_2]=\mu X_1$ (see (9.79)), relations (9.85) and (9.83) imply that $\alpha_{v^{(\ell-2)}}$, μ, and the bracket $[X,E_2]$ are zero, and prove the first part of the lemma. Let us turn to $[X,E_3]$: from (9.75) and (9.82), one gets, since E_2 and X commute, and $X\chi=0$,

$$E_3=\chi_{v^{(\ell-2)}}\frac{\partial}{\partial u^{(k-3)}}+\frac{\partial}{\partial v^{(\ell-3)}}\ , \quad [X,E_3]=-(E_3\alpha)\frac{\partial}{\partial z}\ . \tag{9.92}$$

In order to prove that $E_3\alpha=0$, let us examine equation (9.87). For short, we use the symbol $\mathcal{O}$ to denote *any function* that depends on $u,\dots,u^{(k-3)},x,y,z,v,\dots,v^{(\ell-3)}$ only. For instance, $X\gamma-\delta=\mathcal{O}$, and all terms in the determinant are of this nature, except the following three :

$$\begin{aligned} Y\alpha &= \chi\,\alpha_{u^{(k-3)}}+v^{(\ell-2)}\alpha_{v^{(\ell-3)}}+\mathcal{O},\\ XY\alpha &= \chi\,X\alpha_{u^{(k-3)}}+v^{(\ell-2)}X\alpha_{v^{(\ell-3)}}+\mathcal{O},\\ X_1\alpha &= -\alpha_z\left(\chi\,\alpha_{u^{(k-3)}}+v^{(\ell-2)}\alpha_{v^{(\ell-3)}}\right)+\mathcal{O}\end{aligned}$$

(we used $X\chi=0$). Setting $\zeta=\chi\,\alpha_{u^{(k-3)}}+v^{(\ell-2)}\alpha_{v^{(\ell-3)}}$, one has

$$X\zeta=\frac{X\alpha_{u^{(k-3)}}}{\alpha_{u^{(k-3)}}}\zeta+\mathbf{b}\,v^{(\ell-2)} \quad \text{with} \quad \mathbf{b}=X\alpha_{v^{(\ell-3)}}-\alpha_{v^{(\ell-3)}}\frac{X\alpha_{u^{(k-3)}}}{\alpha_{u^{(k-3)}}}\ , \tag{9.93}$$

and equation (9.87) reads

$$\zeta^2+\mathcal{O}\,\zeta-(X\gamma-\delta)\,\mathbf{b}\,v^{(\ell-2)}+\mathcal{O} \ = \ 0\ . \tag{9.94}$$

Differentiating with respect to X and using (9.93) yields

$$2\,\frac{X\alpha_{u^{(k-3)}}}{\alpha_{u^{(k-3)}}}\,\zeta^2+\left(2\mathbf{b}\,v^{(\ell-2)}+\mathcal{O}\right)\zeta+\mathcal{O}\,v^{(\ell-2)}+\mathcal{O} \ = \ 0\ .$$

Then, eliminating ζ between these two polynomials yields the resultant

$$\begin{vmatrix} 1 & \mathcal{O} & -(X\gamma-\delta)\mathbf{b}\,v^{(\ell-2)}+\mathcal{O} & 0 \\ 0 & 1 & \mathcal{O} & -(X\gamma-\delta)\mathbf{b}\,v^{(\ell-2)}+\mathcal{O} \\ 2\frac{X\alpha_{u^{(k-3)}}}{\alpha_{u^{(k-3)}}} & 2\mathbf{b}\,v^{(\ell-2)}+\mathcal{O} & \mathcal{O}\,v^{(\ell-2)}+\mathcal{O} & 0 \\ 0 & 2\frac{X\alpha_{u^{(k-3)}}}{\alpha_{u^{(k-3)}}} & 2\mathbf{b}\,v^{(\ell-2)}+\mathcal{O} & \mathcal{O}\,v^{(\ell-2)}+\mathcal{O} \end{vmatrix} = 0\ .$$

This is a polynomial of degree at most three with respect to $v^{(\ell-2)}$, the coefficient of $(v^{(\ell-2)})^3$ being $-4\mathbf{b}^3(X\gamma-\delta)$. Hence $\mathbf{b}=0$ and, from (9.94), ζ does not depend on $v^{(\ell-2)}$. This implies $E_3\alpha=0$ because, from (9.92) and the definition of ζ, one has $\zeta_{v^{(\ell-2)}}=E_3\alpha$. □

Proof of Lemma 9.3.12. The independent variables in $\mathcal{E}_{3,3}^{\gamma,\delta}$ are $u,\dot u,\ddot u,x,v,\dot v,\ddot v$. Since the determinant (9.37) is nonzero, one defines local coordinates $(x,y,z,w,v,\dot v,\ddot v)$ by

$$y=p(u,\dot u,\ddot u,x,v,\dot v,\ddot v),\quad z=p_x(u,\dot u,\ddot u,x,v,\dot v,\ddot v),\quad w=p_{xx}(u,\dot u,\ddot u,x,v,\dot v,\ddot v). \tag{9.95}$$

In these coordinates, X and Y, defined in (9.74), have the following expressions, with ψ and α some functions to be studied further :

$$X = \frac{\partial}{\partial x}+z\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\alpha\frac{\partial}{\partial w}\ , \tag{9.96}$$

$$Y = \gamma\frac{\partial}{\partial y}+\delta\frac{\partial}{\partial z}+\psi\frac{\partial}{\partial w}+\dot v\frac{\partial}{\partial v}+\ddot v\frac{\partial}{\partial \dot v}\ . \tag{9.97}$$

Then, using, for short, the following notation Γ :

$$\Gamma = X\gamma-\delta \neq 0\ , \tag{9.98}$$

one has

$$X_1 = \Gamma\frac{\partial}{\partial y}+(X\delta-\psi)\frac{\partial}{\partial z}+(X\psi-Y\alpha)\frac{\partial}{\partial w}, \tag{9.99}$$

$$[X,X_1] = (X\Gamma-X\delta+\psi)\frac{\partial}{\partial y}+\left(X^2\delta-2X\psi+Y\alpha\right)\frac{\partial}{\partial z}+\left(X^2\psi-XY\alpha-X_1\alpha\right)\frac{\partial}{\partial w} \tag{9.100}$$

Also,

$$E=\frac{\partial}{\partial \ddot v}\ ,\qquad E_2=[E_1,Y]=\psi_{\ddot v}\frac{\partial}{\partial w}+\frac{\partial}{\partial \dot v}\ ,\qquad [X,E_2]=\psi_{\ddot v}\frac{\partial}{\partial z}+(X\psi_{u^{(k-1)}}-E_2\alpha)\frac{\partial}{\partial w}$$

but, from Lemma 9.9.1, one has $[X,E_2]=0$, hence $\psi_{\ddot v}=0$, $E_2=\partial/\partial\dot v$ and $\alpha_{\dot v}=0$. Then

$$E_3=[\frac{\partial}{\partial \dot v},Y]=\psi_{\dot v}\frac{\partial}{\partial w}+\frac{\partial}{\partial v}\ ,\qquad [X,E_3]=\psi_{\dot v}\frac{\partial}{\partial z}+(X\psi_{\dot v}-E_3\alpha)\frac{\partial}{\partial w}\ ,$$

but, from Lemma 9.9.1, one has $[X,E_3]=0$, hence $\psi_{\dot v}=0$, $E_3=\partial/\partial v$ and $\alpha_v=0$. To sum up,

$$E=\frac{\partial}{\partial \ddot v}\ ,\qquad E_2=\frac{\partial}{\partial \dot v}\ ,\quad E_3=\frac{\partial}{\partial v}\ , \tag{9.101}$$

α depends at most on (x,y,z,w) only and ψ on (x,y,z,w,v).
Notation : until the end of this proof, $\mathcal{O}$ stands for *any* function of x,y,z,w only. For instance, $\alpha=\mathcal{O}$, $\gamma=\mathcal{O}$, $\delta=\mathcal{O}$, $\Gamma=\mathcal{O}$, $X\Gamma=\mathcal{O}$, $X\delta=\mathcal{O}$ and $X^2\delta=\mathcal{O}$.

From (9.78), (9.99) and (9.100), one has $\begin{vmatrix} \Gamma & X\delta - \psi \\ X\Gamma - X\delta + \psi & X^2\delta - 2X\psi + Y\alpha \end{vmatrix} = 0$. Hence

$$X\psi \;=\; \frac{1}{2\Gamma}\psi^2 + \mathcal{O}\psi + \mathcal{O}\ . \tag{9.102}$$

We now write the expression (9.99) of X_1 as

$$X_1 = X_1^0 + \psi X_1^1 + \psi^2 X_1^2 \tag{9.103}$$

$$\text{with}\quad X_1^0 = \Gamma\frac{\partial}{\partial y} + \mathcal{O}\,\frac{\partial}{\partial z} + \mathcal{O}\,\frac{\partial}{\partial w}\ ,\quad X_1^1 = -\,\frac{\partial}{\partial z} + \mathcal{O}\,\frac{\partial}{\partial w}\ ,\quad X_1^2 = \frac{1}{2\Gamma}\frac{\partial}{\partial w}\ . \tag{9.104}$$

Note that X_1^0, X_1^1 and X_1^2 are vector fields in the variables x, y, z, w only. Now define :

$$U \;=\; -X_1^1 - \frac{\psi}{\Gamma}\frac{\partial}{\partial w} \;=\; \frac{\partial}{\partial z} + \left(\mathcal{O} - \frac{\psi}{\Gamma}\right)\frac{\partial}{\partial w}\ , \tag{9.105}$$

$$V \;=\; X_1^0 - \psi^2 X_1^2 \;=\; \Gamma\frac{\partial}{\partial y} + \mathcal{O}\,\frac{\partial}{\partial z} + \left(\mathcal{O} - \frac{\psi^2}{2\Gamma}\right)\frac{\partial}{\partial w}\ , \tag{9.106}$$

so that

$$X_1 = V - \psi U \tag{9.107}$$

and, from (9.97) and (9.103) one deduces the following expression of $X_2 = [X_1, Y]$:

$$X_2 \;=\; \big(Y\psi\big)\,U + \big(X_1\psi\big)\,\frac{\partial}{\partial w} + \psi^3\frac{\Gamma_w}{2\Gamma^2}\,\frac{\partial}{\partial w} + \psi^2\left(\frac{\gamma_w}{2\Gamma}\frac{\partial}{\partial y} + \mathcal{O}\frac{\partial}{\partial z} + \mathcal{O}\frac{\partial}{\partial w}\right) + \psi X_2^1 + X_2^0 \tag{9.108}$$

where X_2^1 and X_2^0 are two vector fields in the variables x, y, z, w only.

This formula and (9.101) imply $[E_2, X_2] = \big(Y\psi\big)_{\dot v}\,U = \psi_v\,U$; hence, from the last relation in (9.79), either ψ_v is identically zero or U is a linear combination of X_1 and X_2. We assume, until the end of the proof, that U is a linear combination of X_1 and X_2. This implies, using (9.107), that X_2 and X_1 are linear combinations of U and V ; hence U, V is another basis for X_1, X_2. Also, from (9.78) $[U, V]$ must be a linear combination of U and V. From (9.105) and (9.106),

$$[U, V] = \frac{X_1\psi}{\Gamma}\frac{\partial}{\partial w} - \psi^2\,\mathcal{O}\,\frac{\partial}{\partial w} + \psi\,W^1 + W^0$$

where W^1 and W^0 are two vector fields in the variables x, y, z, w only, and, finally, with Z^1 and Z^0 two other vector fields in the variables x, y, z, w only, one has, from (9.108)

$$X_2 - (Y\psi)\,U - \Gamma\,[U, V] \;=\; \psi^3\frac{\Gamma_w}{2\Gamma^2}\,\frac{\partial}{\partial w} + \psi^2\left(\frac{\gamma_w}{2\Gamma}\frac{\partial}{\partial y} + \mathcal{O}\frac{\partial}{\partial z} + \mathcal{O}\frac{\partial}{\partial w}\right) + \psi\,Z^1 + Z^0\ .$$

This vector field is also a linear combination of U and V. Computing the determinant in the basis $\partial/\partial y$, $\partial/\partial z$, $\partial/\partial w$, one has, using (9.105) and (9.106),

$$\det\big(U, V,\, X_2 - (Y\psi)U - \Gamma[U, V]\,\big) = \frac{\gamma_w}{\Gamma^3}\psi^4 + \mathcal{O}\psi^3 + \mathcal{O}\psi^2 + \mathcal{O}\psi + \mathcal{O} = 0\ .$$

It is assumed from the definition of $\mathcal{E}_{k,\ell}^{\gamma,\delta}$ that the partial derivative of γ with respect to its fourth argument is nonzero; hence $\gamma_w \neq 0$ and the above polynomial of degree 4 with respect to ψ is nontrivial; its coefficients depend on x, y, z, w only, hence ψ cannot depend on v.

We have proved that, in any case, both α and ψ depend on x, y, z, w only, and this yields the desired identities in the lemma. □

A technical lemma

Lemma 9.9.2. *Let p be a smooth function of $u,\ldots,u^{(k-1)},x,v,\ldots,v^{(\ell-1)}$, r a smooth function of $u,\ldots,u^{(k-1)},x,\ v,\ldots,v^{(\ell)}$, with $r_{v^{(\ell)}}\neq 0$, and f a smooth function of four variables such that*

$$\sum_{i=0}^{k-2} u^{(i+1)}p_{u^{(i)}} \ + \ rp_{u^{(k-1)}} \ + \ \sum_{i=0}^{\ell-1} v^{(i+1)}p_{v^{(i)}} \ = \ f(x,p,p_x,p_{xx}) \tag{9.109}$$

where, by convention, $rp_{u^{(k-1)}}$ is zero if $k=0$ and the first (resp. last) sum is zero if $k\leq 1$ (resp. $\ell=0$). Then either p depends on x only or

$$k\geq 1\,,\quad \ell\geq 1\,,\quad p_{u^{(k-1)}}\neq 0\,,\quad p_{v^{(\ell-1)}}\neq 0\,. \tag{9.110}$$

Proof. Let $m\leq k-1$ and $n\leq \ell-1$ be the smallest integers such that p depends on $u,\ldots,u^{(m)}$, $x,v,\ldots,v^{(n)}$; if p depends on none of the variables $u,\ldots,u^{(k-1)}$ (or $v,\ldots,v^{(\ell-1)}$), take $m<0$ (or $n<0$). Then $p_{u^{(m)}}\neq 0$ if $m\geq 0$ and $p_{v^{(n)}}\neq 0$ if $n\geq 0$.

The lemma states that either $m<0$ and $n<0$ or $k\geq 1$, $\ell\geq 1$ and $(m,n)=(k-1,\ell-1)$. This is indeed true :
- if $m=k-1$ and $k\geq 1$ then $n=\ell-1$ and $\ell\geq 1$ because if not, differentiating both sides in (9.109) with respect to $v^{(\ell)}$ would yield $r_{v^{(\ell)}}p_{u^{(k-1)}}=0$, but the lemma assumes that $r_{v^{(\ell)}}\neq 0$,
- if $m<k-1$ or $m=0$, (9.109) becomes : $\sum_{i=0}^{m} u^{(i+1)}p_{u^{(i)}}+\sum_{i=0}^{n} v^{(i+1)}p_{v^{(i)}}=f(x,p,p_x,p_{xx})$; if $m\geq 0$, differentiating with respect to $u^{(m+1)}$ yields $p_{u^{(m)}}=0$ and if $n\geq 0$, differentiating with respect to $u^{(m+1)}$ yields $p_{v^{(n)}}=0$; hence m and n must both be negative. □

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