E. D. Sciences Fondamentales et Appliquées
Habilitation à Diriger des Recherches

Équivalence et linéarisation des systèmes de contrôle

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« **Équivalence et linéarisation des systèmes de contrôle** »

System: the real physical plant.
Model: a mathematical object, "representing" the system.

Equivalence, transformations, classification, linearization...
apply to **models**.

Class of models:

continuous-time finite-dimensional control systems ⇄ **Underdetermined ODEs**

$F(x, \dot{x}) = 0 \qquad x \in \boldsymbol{R}^n$ $\qquad$ F smooth, real analytic

Determined

$$\dot{x} = f(x)$$

Solution depends on d constants ($x(0)$).

Flow: $x(0) \mapsto x(t)$.

Under-determined [u: part of $\dot{x}$]

$$\dot{x} = f(x, u) \quad x \in \boldsymbol{R}^n, u \in \boldsymbol{R}^m \qquad (*)$$

Solution depends on $\qquad 0 < m \leq n$

- m functions of time $u(.)$
- and n constants $x(0)$.

"Set of solutions" for under-determined ODEs

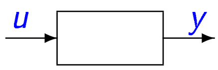
$\mathcal{B}$ = set of all (germs of) $t \mapsto (x(t), u(t))$ solution of $(*)$.

(behavior)

Trivial equation: no relation ($F = 0$).

What is a control system (model) ?

$\dot{x} = f(x, u)\,, \quad y = h(x)$



(In the sequel, $y = x$..)

- Input-output operator, transfer function (linear).
- State-space representation.
- Differential equations with control
  Calculus of variations, functional analysis.
- Dynamical systems $\longrightarrow$ dynamical poly-systems,
  families of vector fields, controllability.
- "Behavior" = collection of allowed signals
- Differentially algebraic extension of a purely transcendental differential field.

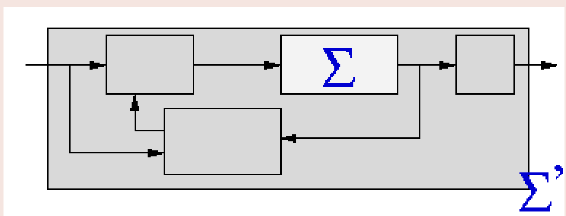
Our purpose

Goal

- Decide when two models "look alike",
  or can be transformed into one another.

Motivation

- Mathematical objects are born to be classified !
- **Modeling.** Help in choosing the right model to render observations.
- **Control.** A control for Σ may carry on to Σ'.



Equivalence

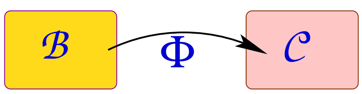
[É. Cartan, "sur l'équivalence absolue...", 1914]:

> La première idée qui vient à l'esprit, et qu'il s'agira de préciser, est la suivante : *deux systèmes seront dits « absolument équivalents » lorsqu'on pourra établir une correspondance univoque* (au moins dans un champ fonctionnel suffisamment petit) *entre les solutions de ces deux systèmes.*

$$
\begin{array}{llll}
(\Sigma) & \dot{x} = f(x,u)\,, & x \in \mathbf{R}^n,\ u \in \mathbf{R}^m, & \mathcal{B} = \{\text{solutions}\} \\
(\Sigma') & \dot{z} = g(z,v)\,, & z \in \mathbf{R}^{n'},\ v \in \mathbf{R}^{m'}, & \mathcal{C} = \{\text{solutions}\}
\end{array}
$$

Definition

Two systems are "equivalent" iff their (germs of) solutions are in one-to-one correspondence.



The nature of Φ matters a lot !!

Linear controllable systems

Linear system $(\Sigma): \ \dot{x} = Ax + Bu$. Transformation $\begin{cases} z = Px, \\ v = Kx + Qu \end{cases}$
yields $(\Sigma') : \ \dot{z} = P(A - BQ^{-1}K)P^{-1}z + PBQ^{-1}v$.

$$\Phi : \Big(t \mapsto (x(t), u(t))\Big) \mapsto \Big(t \mapsto (Px(t), Kx(t) + Qu(t))\Big)$$

Kronecker indices for Σ, or matrix pencil A, B.There are P, Q, K such that Σ' reads

$$\dot{\mathbf{Z}}_0 = \widetilde{\mathbf{A}}_0 \mathbf{Z}_0 \,, \ \ \dot{Z}_k = \begin{pmatrix} 0 & 1 & & (0) \\ & & \ddots & \\ & & (0) & 1 \\ & & & 0 \end{pmatrix} Z_k + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \mathbf{v_k}, \ \ k \geq 1$$

with $z = (\mathbf{Z_0}, Z_1 \ldots Z_m)$, $Z_k \in \boldsymbol{R}^{r_k}$, $r_0 + \cdots + r_m = n$.

- Σ is **controllable** iff $r_0 = 0$ (no $\mathbf{Z_0}$).

Linear contr^ble ∼ trivial. $\dot{z}_{k,1} = z_{k,2} \ \ldots \ z_{k,1}^{(r_k - 1)} = z_{k,r_k}, \ z_{k,1}^{(r_k)} = v_k$.
"Prolongation" of a trivial system.

Triviality is as important as linearity.

Nature of transformations

$\mathcal{B} \xrightarrow{\ \Phi\ } \mathcal{C}$

Two systems are equivalent iff *their solutions are in one-to-one correspondence.*

Merely bijection? This distinguishes control systems ($m \neq 0$) from systems with no control ($m = 0$) !
Functional transformations... continuity? smoothness?

Point-wise transformations:
Φ induced by a point transformation ϕ on state and input.

$$\pi_t(\,x(.).u(.)\,) = (x(t), u(t))$$

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Phi} & \mathcal{C} \\ \pi_t \downarrow & & \downarrow \pi_t \\ \boldsymbol{R}^{n+m} & \xrightarrow{\phi} & \boldsymbol{R}^{n'+m'} \end{array}$$

In between...
$\mathfrak{X}$ and $\mathfrak{Y}$ "smaller than" $\mathcal{B}$ and $\mathcal{C}$, and "larger than" $\boldsymbol{R}^{n+m}$ and $\boldsymbol{R}^{n'+m'}$.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Phi} & \mathcal{C} \\ \Pi_t \downarrow & & \downarrow \Pi_t \\ \mathfrak{X} & \xrightarrow{\phi} & \mathfrak{Y} \end{array}$$

Point-wise transformations

$$\begin{array}{lll} (\Sigma) & \dot{x} = f(x,u)\,, & x \in \boldsymbol{R}^n,\ u \in \boldsymbol{R}^m, \\ (\Sigma') & \dot{z} = g(z,v)\,, & z \in \boldsymbol{R}^{n'},\ v \in \boldsymbol{R}^{m'}, \end{array}$$

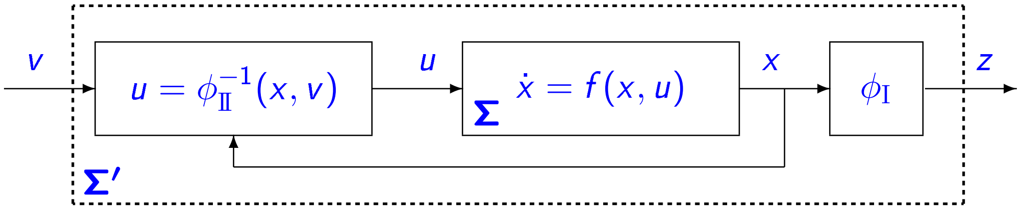
$$\left.\begin{array}{r} \phi \text{ conjugates} \\ (\Sigma) \text{ to } (\Sigma') \end{array}\right\} \overset{\Delta}{\Longleftrightarrow} \left\{ \begin{array}{c} (x(t),u(t)) \text{ is a solution of } (\Sigma) \\ \text{if and only if} \\ (z(t),v(t)) = \phi(x(t),u(t)) \\ \text{is a solution of } (\Sigma') \end{array}\right.$$

Proposition

If ϕ is a homeomorphism that conjugates Σ to Σ',

- ϕ must be **triangular**:
  $\phi(x,u) = (z,v) = (\,\phi_{\mathrm{I}}(x)\,,\,\phi_{\mathrm{II}}(x,u)\,)$,
- $n = n'$ and $m = m'$.
- if ϕ is a diffeomorphism, conjugacy is equivalent to
  $$\phi_{\mathrm{I}}{}'(x) f(x,u) \;=\; g(\,\phi_{\mathrm{I}}(x)\,,\,\phi_{\mathrm{II}}(x,u)\,)$$

- Control point of view:



- **Invariants for smooth static feedback:** a huge literature. [Brockett, Jakubczyk, Bonnard, Kupka, Tchon, Respondek, Zhitomirskii, Zelenko . . .]
  This is a very fine classification, usually no object is stable : equivalence classes have infinite co-dimension.

Topological vs. smooth linearization

$\dot{x} = f(x, u) = Ax + Bu + \varepsilon(x, u)$ ε of order 2 around $(0, 0)$.
Local behavior around $(0, 0)$ when A, B is controllable.

Engineering knowledge: If A, B controllable; one does not need ε.

Nonlinear modelling: Is a nonlinear model necessary, locally?

Natural question: Is the nonlinear system/model transformable into its linear approximation?

Smooth feedback linearization: A nonlinear system can very rarely be transformed to a linear one by smooth feedback.

Grobman-Hartman theorem: Generic systems without control are topologically linearizable... Control systems?

Topological equivalence for control systems: ϕ homeomorphism.

Theorem (Baratchart, JBP)

Topologically linearizable control systems are smoothly linearizable.

(almost smoothly, in fact)

Open questions

On topological vs. smooth equivalence. Does the result hold for equivalence between general control systems (*e.g.* whose linear approximation is controllable) ?

On "nonlinear" local phenomena. Does a nonlinear system locally "look like" its linear approximation when controllable ? Any qualitative phenomena ?

A Grobman-Hartman theorem for control systems

$$\Sigma : \dot{x} = f(x,u) = Ax + Bu + \varepsilon(x,u) \qquad \Sigma' : \dot{z} = Az + Bv$$

Remark: Grobman-Hartman theorem is about conjugating **flows**.

Then define, from a control system Σ,

- $\mathcal{U}$ some space of functions $\tau \mapsto u(\tau) \in \boldsymbol{R}^m$ (controls),
- $$\Pi_t : \begin{array}{ccc} \mathcal{B} & \to & \boldsymbol{R}^n \times \mathcal{U} \\ \big(x(.), u(.)\big) & \mapsto & \big(x(t), u^{+t}(.)\big) \end{array} \qquad \big(\; u^{+t}(\tau) = u(t+\tau) \; \big)$$

There is a flow $(\chi_T)_{T \in \boldsymbol{R}}$ on $\boldsymbol{R}^n \times \mathcal{U}$ such that $\chi_T \circ \Pi_t = \Pi_{t+T}$ for all T,t, i.e. [Colonius-Kliemann]
$\chi_T(\,x(t), u^{+t}(.)\,) = (\,x(t+T), u^{+t+T}(.)\,)$.

Theorem (Baratchart, Chyba, JBP)

There exists ϕ *that conjugates,* locally around zero, Σ *to* Σ', *and even to* $\dot{z} = Az$ $(+0v)$.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Phi} & \mathcal{C} \\ \Pi_t \downarrow & & \downarrow \Pi_t \\ \boldsymbol{R}^n \times \mathcal{U} & \xrightarrow{\phi} & \boldsymbol{R}^n \times \mathcal{V} \end{array}$$

- ▶ Does not yield a compensator.
- ▶ Meaning for modelling?

(Endogenous) dynamic (feedback) transformations

$$n,m \quad \boxed{\dot x = f(x,u)} \qquad \begin{array}{ccc} \mathcal{B} & \overset{\Phi}{\rightarrow} & \mathcal{C} \\ \Pi_t \downarrow & & \downarrow \Pi_t \\ \mathfrak{X}? & \overset{\phi}{\rightarrow} & \mathfrak{Y}? \end{array} \qquad \boxed{\dot z = g(z,u)} \quad n',m'$$

Σ $\qquad$ Σ'

◀ jet spaces

Define $\Pi_t^K : \mathcal{B} \longrightarrow \mathcal{J}^K$ by

$$\Pi_t^K(\, x(.). \, u(.)\,) = (x(t), \sout{\dot x(t), \ldots, x^{(K)}(t)}, u(t), \dot u(t), \ldots, u^{(K)}(t))$$

Dynamic transformations: Φ such that, for some K, K', ϕ and ψ,

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Phi} & \mathcal{C} \\ \Pi_t^K \downarrow & & \downarrow \pi_t \\ \mathcal{J}^K & \xrightarrow{\phi} & \mathbf{R}^{n'} \times \mathbf{R}^{m'} \end{array} \quad , \quad \begin{array}{ccc} \mathcal{B} & \xleftarrow{\Phi^{-1}} & \mathcal{C} \\ \pi_t \downarrow & & \downarrow \Pi_t^{K'} \\ \mathbf{R}^{n} \times \mathbf{R}^{m} & \xleftarrow{\psi} & \mathcal{J}'^{K'} \end{array}$$

or (infinite jets): $\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Phi} & \mathcal{C} \\ \Pi_t^\infty \downarrow & & \downarrow \Pi_t^\infty \\ \mathcal{J}^\infty & \xrightarrow{\phi^\infty} & \mathcal{J}'^\infty \end{array}$

ϕ^∞ is invertible, ϕ and ψ are not.

- Integers K, K' are called **the order of Φ.**

(Endogenous) dynamic (feedback) transformations II

Dynamic equivalence

$$\begin{array}{c}\Sigma \\ \dot x = f(x,u)\end{array} \quad \begin{pmatrix} x \\ u \\ \dot u \\ \ddot u \\ \vdots \end{pmatrix} \quad \begin{array}{c} (z,v) = \phi(x,u,\ldots,u^{(K)}) \\ \longrightarrow \\ \longleftarrow \\ (x,u) = \psi(z,v,\ldots,v^{(K')}) \end{array} \quad \begin{pmatrix} z \\ v \\ \dot v \\ \ddot v \\ \vdots \end{pmatrix} \quad \begin{array}{c}\Sigma' \\ \dot z = g(z,v)\end{array}$$

Flatness: Σ is flat if this holds with Σ' trivial:

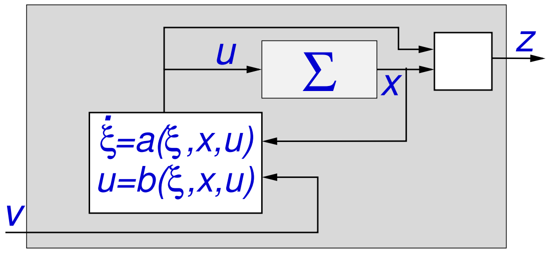
$$\begin{array}{c}\Sigma \\ \dot x = f(x,u)\end{array} \quad \begin{pmatrix} x \\ u \\ \dot u \\ \vdots \end{pmatrix} \quad \begin{array}{c} v = \phi(x,u,\ldots,u^{(K)}) \\ \longrightarrow \\ \longleftarrow \\ (x,u) = \psi(v,\ldots,v^{(K')}) \end{array} \quad \begin{pmatrix} v \\ \dot v \\ \ddot v \\ \vdots \end{pmatrix} \quad \begin{array}{c}\Sigma' \\ \text{no relation}\end{array}$$

[Fliess,Lévine,Martin,Rouchon]

(Endogenous) dynamic (feedback) transformations III

Dynamic equivalence

- x and z need not have the same dimension.
  Example (adding an integrator):
  $z = (x, u)$, $v = \dot{u}$, $g(z, v) = (f(z), v)$.
- Implementable via a compensator:



is Σ' (modulo "adding integrators").

Necessary conditions for dynamic equivalence

- Dynamic equivalence preserves m (= number of inputs, or of arbitrary functions of time defining the "general solution").
- A flat system must be ruled [Rouchon], [Sluis].

Definition (ruled system)

Σ is ruled iff each $\Sigma_x = \{f(x, u),\ u \in \boldsymbol{R}^m\}$ is a ruled submanifold of (the tangent space at x to) $\boldsymbol{R}^n$.

Theorem (JBP)

If Σ and Σ' are dynamic equivalent and $n = n'$, then, locally,

- *either they are static equivalent,*
- *or they are both* ***ruled****.*

If $n > n'$, Σ must be ruled.

Deciding dynamic equivalence

Beyond these necessary conditions, how to decide whether two systems are dynamic equivalent, or whether one system is flat ?

- No a priori bound on K is known.
- For **fixed** K, conditions on $\phi \rightarrow$ PDEs $\rightarrow$ in principle one may decide in finitely many operations on existence of ϕ, hence on "K-equivalence", or "K-flatness".

D. Avanessoff, JBP $(n, m) = (3, 2)$

Single relation $\dot{x}_3 = h(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2)$. Conditions for "3-flatness"

- Difficult question: how do the "$K+1$-conditions" project onto the "K-conditions" ?
- Another possibility: look for objects (transformations) depending on infinitely many variables and try afterwards to characterize finiteness.

[Baratchart, Avanessoff, JBP]: "very formal integrability".

Open questions

- How to bound *a priori* the number K of derivatives?
- **Conjecture:** with $(n, m) = (3, 2)$, systems that are not "2-flat" are in fact not flat.
  Note: they are not 3-flat.
- Is $\dot{x}_3 = x_2 + (\dot{x}_2 - x_3\dot{x}_1)^2\dot{x}_1$ flat?
  Control system: $\dot{x}_1 = u_1 \quad \dot{x}_2 = u_2 + x_3\, u_1 \quad \dot{x}_3 = x_2 + u_1\, {u_2}^2$.

1 Introduction

2 Equivalence

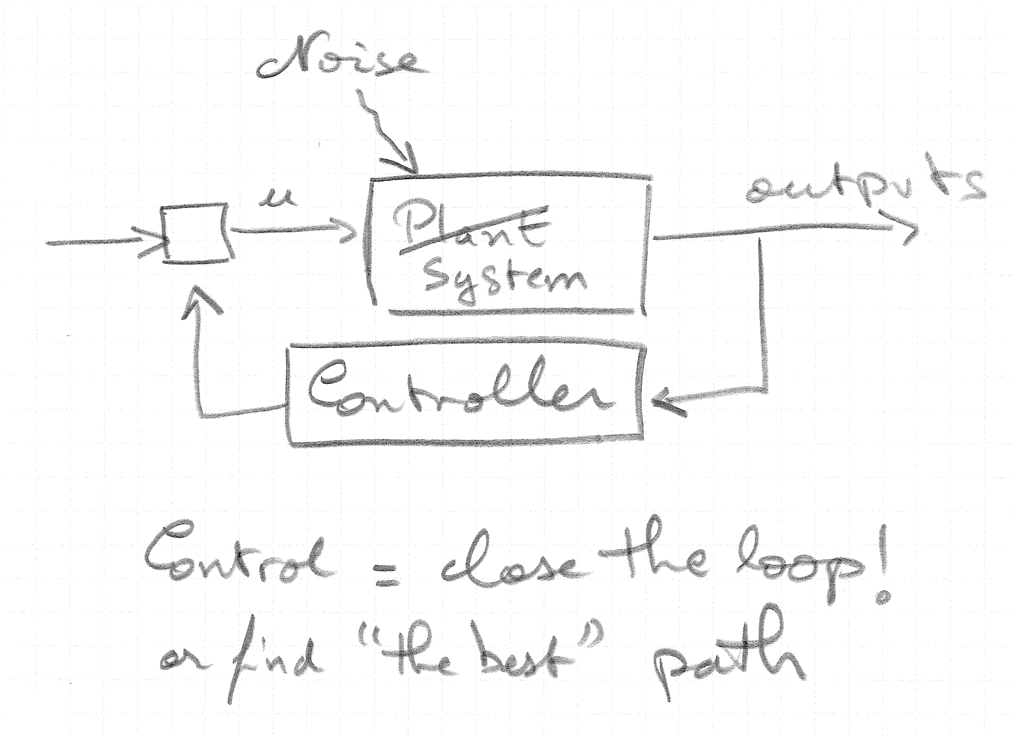
3 Static feedback

4 Local linearisation (smooth, topological..)

5 Dynamic equivalence

6 Conclusion, and other contributions

Real-life control problems



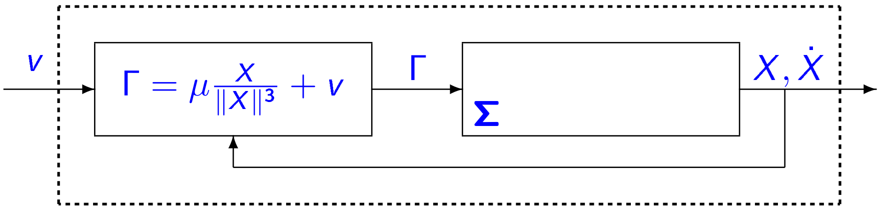
Low thrust satellite orbit transfer

$$\ddot{X} = \underbrace{-\mu \frac{X}{\|X\|^3}}_{\text{gravitation}} + \underbrace{\Gamma}_{}, \quad X \in \boldsymbol{R}^3, \qquad \boxed{\frac{\Gamma_{max}}{|\text{gravitat.}|} \approx 10^{-3}}$$

SMALL control, *or* perturbating acceleration

... but very efficient propulsion (high specific impulse).

- This system is TRIVIAL: linearizable, flat.... $\ddot{X} = v$



- This is (almost) irrelevant because Γ is small.
- Instead, averaging techniques from perturbation of Hamiltonian systems, adapted to control...
  plus control design techniques.

On equivalence and classification of control systems...

- This is conceptually central in control theory...
  ...although not the solution to all problems.
- A sound "classification" is almost unreachable
- We raised more questions than we solved !!

(i) Adaptive nonlinear control (Praly)
(ii) Output stabilization (Hirshorn, Cebuhar, Praly)
(iii) Time-varying stabilization (Morin, Samson)
(iv) Control Lyapunov functions and stabilisation (Faubourg)
(v) Control of a frequency converter in optic fibers, for Alcatel CIT, 1 patent (US). (Bombrun, Seyfert...)
(vi) Nonlinear model for a river flow (Litrico)
(vii) Small control and averaging $\longrightarrow$ Low-thrust satellite orbit transfer, for Thales Alenia Space (Bombrun)