Distributed Weight Selection in Consensus Protocols by Schatten Norm Minimization

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Abstract

This paper studies the weight optimization problem for average consensus protocols, by reformulating the problem to a Schatten norm minimization with parameter p. We show that as p approaches infinity, the optimal solution of the Schatten norm induced problem recovers the optimal solution of the original problem. Large p obtains matrices with faster consensus, but requires more information at each step of the algorithm. By tuning the parameter p in our proposed minimization, we can simply trade-off the quality of the solution (i.e., the speed of convergence) for communication/computation requirements (in terms of number of messages exchanged and volume of data processed). We then propose a distributed implementation for the Schatten norm minimization and we show that this algorithm outperforms the existing methods.

I. Introduction

A network is formed of nodes (or agents) and communication links that allow these nodes to share information and resources. We consider each node i in the network to have a scalar $x_i(0) \in \mathbb{R}$ called estimate that they can maintain in their local memory. Average consensus protocols are algorithms that by only neighbor to neighbor communication can allow nodes to reach consensus on the average of all initial estimates ($\frac{\sum_i x_i(0)}{n}$). Consensus algorithms are used

in many applications and distributed control problems for different and various systems [1], [2], and [3]. For a complete overview of consensus protocols we refer the reader to [4], [5], [6], [7] and the references therein.

An iterative algorithm for achieving the average consensus is the following: at iteration $k+1$, node i updates its state value x_i as follows:

$$x_i(k+1) = w_{ii}x_i(k) + \sum_{j \in N_i} w_{ij}x_j(k), \tag{1}$$

where N_i is the set neighbors of node i, w_{ij} is the weight selected by node i for the value sent by its neighbor j and w_{ii} is the weight selected by node i for it own value. We can put the weights in an n by n matrix W where n is the number of nodes in the network. Necessary and sufficient conditions for system (1) to converge starting from any initial condition [8] is to select $W\mathbf{1} = \mathbf{1}$, $\mathbf{1}^T W = \mathbf{1}^T$, and $\mu(W) < 1$ where $\mathbf{1}$ is the vector of all ones and $\mu(W)$ is the second largest eigenvalue of W in module.

Xiao and Boyd in [8] formulated a general Fastest Distributed Linear Averaging (FDLA) problem as a non-convex optimization to find W that guarantees fastest convergence. They showed that if a symmetric weight matrix is considered, then the symmetric FDLA problem can be formulated as a Semi-Definite Program (SDP) that can be solved by a centralized unit using interior point methods. Kim *et al.* in [9] approximate the general FDLA using the qth-order spectral norm (2-norm) minimization (q-SNM). For a symmetric weight matrix, the solution of the q-SNM is equivalent to that of the symmetric FDLA problem. Their algorithm is centralized and has a more expensive complexity than the SDP. Therefore, solving the problem of optimal weight selection in a distributed way is still an open problem.

In this paper we study distributed techniques to optimally select the weights of average consensus protocols. We address the problem of selecting the weights in a given network in order to have a fast speed of convergence for these protocols. We approximate the problem of optimal weight selection by the minimization of the Schatten p-norm of a matrix with some constraints related to the connectivity of the underlying network. We then provide a totally distributed gradient method to solve the Schatten norm optimization problem. By tuning the parameter p in our proposed minimization, we can simply trade-off the quality of the solution (i.e., the speed of convergence) for communication/computation requirements (in terms of number of messages exchanged and volume of data processed). The theoretical contribution of this paper is as follows:

we formulate a new optimization problem (the Schatten norm minimization) for weight selection problem of average consensus, then we show that the formulated problem is an approximation of the optimal weight selection problem, and finally we show that the Schatten norm problem can be implemented in a totally decentralized fashion. Simulation results on random graphs and on real networks show that our approach provides very good performance already for values of p that only needs limited information exchange.

The paper is organized as follows: In section II we give the notation used across the paper. In section III we propose Schatten p-norm minimization as an approximation of the original problem and in section IV we show how its solution can be computed in a distributed way and evaluate its computation and communication costs. Section V compares the performance of our algorithm and that of other known weight selection algorithms on different graph topologies. Section VI summarizes the paper.

II. NOTATION

The network is considered as an undirected graph $G = (V, E)$ where V is the set of nodes ($V = 1, \ldots, n$), the E is the set of edges ($E = 1, \ldots, m$, and $(i, j) \in E$ if nodes i and j are neighbors and can communicate). We denote by $\mathbf{w}$ the vector of dimensions $m \times 1$, whose l-th element w_l is the weight associated to link l, then if $l \sim (i, j)$ it holds $w_l = w_{ij} = w_{ji}$. A is the adjacency matrix of graph G, i.e., $a_{ij} = 1$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise. $\mathcal{C}_G$ is the set of all real $n \times n$ matrices M corresponding to graph G, i.e., $m_{ij} = 0$ if $(i, j) \notin E$. D is a diagonal matrix where d_{ii} (or simply d_i) is the degree of node i in the graph G. $\mathcal{I}$ is the $n \times m$ incidence matrix of the graph, such that for each $l \sim (i, j) \in E$ $\mathcal{I}_{il} = +1$ and $\mathcal{I}_{jl} = -1$ and the rest of the elements of the matrix are null. L is the laplacian matrix of the graph, so $L = D - A$. It can also be seen that $L = \mathcal{I}\mathcal{I}^T$. The $n \times n$ identity matrix is denoted by I_n. Given that W is real and symmetric, it has real eigenvalues (and then they can be ordered). We denote by λ_i the i-th largest eigenvalue of W, and by μ the largest eigenvalue in module non considering λ_1, i.e., $\mu = \max\{\lambda_2, -\lambda_n\}$. σ_i is the i-th largest singular value of a matrix. $\mathrm{Tr}(X)$ is the trace of the matrix X and $\rho(X)$ is its spectral radius. $||X||_{\sigma p}$ denotes the Schatten p-norm of matrix X, i.e., $||X||_{\sigma p} = (\sum_i \sigma_i^p)^{1/p}$. Finally we use the symbol $\frac{\mathrm{d}}{\mathrm{d}X} f(X)$, where f is a differentiable scalar-valued function $f(X)$ with matrix argument $X \in \mathbb{R}^{m \times n}$, to denote the $n \times m$ matrix whose (i, j) entry is $\frac{\partial f(X)}{\partial x_{ji}}$.

III. SCHATTEN NORM MINIMIZATION

The problem of finding the weight matrix that guarantees the fastest convergence can be formalized as follows (see [8]):

$$\begin{aligned} &\underset{W}{\text{Argmin}} \quad \mu(W) \\ &\text{subject to} \quad W = W^T,\ W\mathbf{1} = \mathbf{1},\ W \in \mathcal{C}_G, \end{aligned} \tag{2}$$

where the last constraint on the matrix W is derived from the assumption that nodes can only communicate with their neighbors and then necessarily $w_{ij} = 0$ if $(i, j) \notin E$. Problem 2 is called in [8] the "symmetric FDLA problem".

The above minimization problem is a convex one and the function $\mu(W)$ is non-smooth convex function. It is convex since when W is a symmetric matrix, we have $\mu(W) = \rho(W - G_1) = ||W - G_1||_2$ which is a composition between an affine function and the convex matrix L-2 norm function[10]. The function $\mu(W) = \rho(W - G_1)$ is non-smooth since the spectral radius of a matrix is not differentiable at points where the eigenvalues coalesce [11]. The process of minimization itself in (2) tends to make them coalesce at the solution.

We change the original minimization problem in (2) by considering a different cost function that is a monotonic function of the Schatten Norm. The minimization problem we propose is the following one:

$$\begin{aligned} &\underset{W}{\text{Argmin}} \quad f(W) = ||W||_{\sigma p}^p \\ &\text{subject to} \quad W = W^T,\ W\mathbf{1} = \mathbf{1},\ W \in \mathcal{C}_G, \end{aligned} \tag{3}$$

where p is an even positive integer. The following result establishes that (3) is a smooth convex optimization problem and also it provides an alternative expression of the cost function in terms of the trace of W^p. For this reason we refer to our problem also as *Trace Minimization* (TM).

Proposition 1. *$f(W) = ||W||_{\sigma p}^p = Tr(W^p)$ is a scalar-valued smooth convex function on its feasible domain when p is an even positive integer.*

Proof: We have $\text{Tr}(W^p) = \sum_{i=1}^n \lambda_i^p$. Since W is symmetric, its non-zero singular values are the absolute values of its non-zero eigenvalues [12]. If p is even, $\sum_{i=1}^n \lambda_i^p = \sum_{i=1}^n \sigma_i^p$. Therefore, $\text{Tr}(W^p) = ||W||_{\sigma p}^p$.

The Schatten norm $||W||_{\sigma p}$ is a nonnegative convex function. Hence f is convex because it is the composition of a non-decreasing convex function—function x^p where x is non-negative—and a convex function [10].

The function is also differentiable and we have

$$\frac{\mathrm{d}}{\mathrm{d}W}\mathrm{Tr}(W^p) = pW^{p-1}, \tag{4}$$

(see [13, p. 411]). ■

We now illustrate the relation between (3) and the optimization (2). The following lemmas will prepare the result:

Lemma 1. *For any symmetric weight matrix W whose rows (and columns) sum to 1 and with eigevalues $\lambda_1(W) \geq \lambda_2(W) \geq \cdots \geq \lambda_n(W)$, there exist two integers $K_1 \in \{1, 2, \ldots n-1\}, K_2 \in \{0, 1, 2, \ldots n-1\}$ and a positive constant $\alpha < 1$ such that for any positive integers p and q where $p = 2q$ we have:*

$$1 + \tau(W)^p K_1 \leq \mathit{Tr}(W^p) \leq 1 + \tau(W)^p(K_1 + K_2\alpha^p), \tag{5}$$

where

$$\tau(W) = \begin{cases} \rho(W) = \max\{\lambda_1(W), -\lambda_n(W)\} & \textit{if } \rho(W) > 1, \\ \mu(W) = \max\{\lambda_2(W), -\lambda_n(W)\} & \textit{if } \rho(W) \leq 1. \end{cases} \tag{6}$$

Proof: Due to space limits the proof is presented in [14]. ■

We now show that the the Schatten-p optimization problem (3) gives a stable matrix.

Lemma 2. *Let us denote by $W_{(p)}$ the solution of the minimization problem* (3)*. If the graph of the network is strongly connected then $\tau\left(W_{(p)}\right) < 1$ for p sufficiently large.*

Proof: Due to space limits the proof is presented in [14]. ■

We are now ready to state our main results by the following proposition:

Proposition 2. *If the underlying graph is connected, then the solution of the Schatten Norm minimization problem* (3) *satisfies the consensus protocol convergence conditions for p sufficiently large. Moreover as p approaches ∞, this minimization problem is equivalent to the minimization problem* (2) *(i.e., to minimize the second largest eigenvalue $\mu(W)$).*

Proof: Due to space limits the proof is presented in [14]. ■

Proposition 3. *The Schatten Norm minimization* (3) *is an approximation for the original problem* (2) *with a guaranteed error bound,*

$$|\mu(W_{(SDP)}) - \mu(W_{(p)})| \leq \mu(W_{(SDP)}) \times \epsilon(p),$$

where $\epsilon(p) = (n-1)^{1/p} - 1$ *and where* $W_{(SDP)}$ *and* $W_{(p)}$ *are the solutions of* (2) *and* (3) *respectively.*

Proof: Due to space limits the proof is presented in [14]. ■

Remark: Comparing the results of Schatten Norm minimization (3) with the original problem (2), we observe that on some graphs the solution of problem (3) already for $p = 2$ gives the optimal solution of the original problem (2); this is for example the case for complete graphs.[1] However, on some other graphs, it may give a weight matrix that does not guarantee the convergence of the consensus protocol because the second largest eigenvalue is larger than or equal to 1 (the other convergence conditions are intrinsically satisfied). Nevertheless, a suitable projection of the matrix on the set of stable matrices can be done in distributed way (see our technical report [14] for more details).

IV. A Distributed Algorithm for Schatten Norm minimization

Given that problem (3) is smooth and convex, it can be solved by interior point methods which would be a centralized solution. In this section we are going to show a distributed gradient-type algorithm to solve problem (3).

By distributed algorithm we mean an algorithm where each node only needs to retrieve information from a limited neighborhood (possibly larger than N_i) in order to calculate the weights on its incident links. The constraint $W = W^T$ in the optimization requires any two neighbors i and j to choose the same weight on their common link $l \sim (i,j)$ i.e., $w_{ij} = w_{ji} = w_l$. The last condition $W\mathbf{1} = \mathbf{1}$ means that at every node i the sum of all weights on its incident links plus its self-weight w_{ii} must be equal to one. This condition is satisfied if nodes choose first weights on links, and then adapt consequently their self-weights w_{ii}. Moreover these two

[1]This can be easily checked. In fact, for any matrix that guarantees convergence of average consensus protocol, it holds $\mu(W) \geq 0$ and $\text{Tr}(W^2) \geq 1$ (because 1 is an eigenvalue of W). The matrix $\hat{W} = 1/n\mathbf{1}\mathbf{1}^T$ (corresponding to each link having the same weight $1/n$) has eigenvalues 1 and 0 with multiplicity 1 and $n-1$ respectively. Then $\mu(\hat{W}) = 0$ and $\text{Tr}(\hat{W}^2) = 1$. It follows that $\hat{W}$ minimizes both the cost function of problem (2) and (3).

constraints lead to the possibility to write W as follows: $W = I - \mathcal{I} \times \text{diag}(\mathbf{w}) \times \mathcal{I}^T$, where $\mathbf{w} \in \mathbb{R}^m$ is the vector of all the weight links w_l, $l = 1...m$. It follows that Schatten Norm minimization (3) is equivalent to the following unconstrained problem:

$$\text{minimize } h(\mathbf{w}) = Tr\left((I - \mathcal{I} \times \text{diag}(\mathbf{w}) \times \mathcal{I}^T)^p\right). \tag{7}$$

We will give a distributed algorithm to solve the Schatten Norm minimization (3) by applying gradient techniques to problem (7). Since the cost function to optimize is smooth and convex as we proved in Proposition 1, if the gradient technique converges to a stationary point, then it converges to the global optimum. The gradient method uses the simple iteration:

$$w_l^{(k+1)} = w_l^{(k)} - \gamma^{(k)} g_l^{(k)} \ \forall l = 1...m \ ,$$

where $\gamma^{(k)}$ is the stepsize at iteration k and $g_l^{(k)}$ is the l-th component of the gradient $\mathbf{g}^{(k)}$ of the function $h(\mathbf{w})$. At every iteration k, starting with a feasible solution for link weights, $w_l^{(k)}$, we calculate the gradient $g_l^{(k)}$ for every link, and then we obtain a new weight value $w_l^{(k+1)}$. There are different conditions on the function $h(.)$ and on the stepsize sequence that can guarantee convergence (see for example [15]). In our case, as we are interested in distributed implementation of the gradient method, careful selection of convergence conditions is required because the function $h(.)$ does not satisfy some conditions (e.g., Lipschitz continuity or a bounded gradient), or because stepsize calculation would require some global knowledge (e.g., the value of the function $h(.)$ or the module of its gradient). We will then add a further constraint, looking for a solution in a set X, and we will consider the following projected gradient method:

$$\mathbf{w}^{(k+1)} = P_X\left(\mathbf{w}^{(k)} - \gamma^{(k)} \mathbf{g}^{(k)}\right),$$

where $P_X()$ is the projection on the set X. We will show that by a particular choice of X and $\gamma^{(k)}$ the method converges to the solution of the original problem. Moreover, all the calculations can be performed in a distributed way on the basis of local knowledge. In particular, we will show that:

- nodes incident to l are able to calculate $g_l^{(k)}$ using only information they can retrieve from their (possibly extended) neighborhood;
- the stepsize sequence $\gamma^{(k)}$ is determined a priori and then nodes do not need to evaluate the function h or any other global quantity to calculate it;

- the projection on set X can be performed component-wise, and locally at each node;
- the global convergence of the projected gradient method is guaranteed.

We will start by g_l and show that it only depends on information local to nodes i and j incident to the link $l \sim (i,j)$, then we will discuss the choice of the stepsize $\gamma^{(k)}$ and of the projection set X.

A. Locally Computed Gradient

Consider the link $l \sim (i,j)$, since $w_l = w_{ij} = w_{ji}$ and $w_{ii} = 1 - \sum_{s \in N_i} w_{is}$, we have:

$$\frac{\mathrm{d}w_{st}}{\mathrm{d}w_l} = \begin{cases} +1 & \text{if } s=i \text{ and } t=j \\ +1 & \text{if } s=j \text{ and } t=i \\ -1 & \text{if } s=i \text{ and } t=i \\ -1 & \text{if } s=j \text{ and } t=j \\ 0 & \text{else.} \end{cases} \tag{8}$$

The gradient g_l of the function $h(\mathbf{w})$ for $l \sim (i,j)$ can be calculated as follows:

$$\begin{aligned} g_l &= \frac{\mathrm{d}h(\mathbf{w})}{\mathrm{d}w_l} \\ &= \frac{\mathrm{d}f(W)|_{W=I-\mathcal{I}\times \mathrm{diag}(\mathbf{w})\times \mathcal{I}^T}}{\mathrm{d}w_l} \\ &= \sum_{s,t} \frac{\partial f}{\partial w_{st}} \frac{\mathrm{d}w_{st}}{\mathrm{d}w_l} \\ &= \frac{\partial f}{\partial w_{ij}} \frac{\mathrm{d}w_{ij}}{\mathrm{d}w_l} + \frac{\partial f}{\partial w_{ji}} \frac{\mathrm{d}w_{ji}}{\mathrm{d}w_l} + \frac{\partial f}{\partial w_{ii}} \frac{\mathrm{d}w_{ii}}{\mathrm{d}w_l} + \frac{\partial f}{\partial w_{jj}} \frac{\mathrm{d}w_{jj}}{\mathrm{d}w_l} \\ &= \frac{\partial f}{\partial w_{ij}} + \frac{\partial f}{\partial w_{ji}} - \frac{\partial f}{\partial w_{ii}} - \frac{\partial f}{\partial w_{jj}} \\ &= p\big((W^{p-1})_{ji} + (W^{p-1})_{ij} - (W^{p-1})_{ii} - (W^{p-1})_{jj}\big). \end{aligned} \tag{9}$$

In the last equality we used equation (4).

It is well know from graph theory that if we consider W to be the adjacency matrix of a weighted graph G, then $(W^s)_{ij}$ is a function of the weights on the edges of the $i-j$ walks (i.e., the walks from i to j) of length exactly s (in particular if the graph is unweighted $(W^s)_{ij}$ is the number of distinct $i-j$ s-walks [16]). Since for a given p the gradient g_l, $l \sim (i,j)$, depends on the $\{ii, jj, ij, ji\}$ terms of the matrix W^{p-1}, g_l can be calculated locally by using

only the weights of links and nodes at most $\frac{p}{2}$ hops away from i or j.[2] Practically speaking, at each step, nodes i and j need to contact all the nodes up to $p/2$ hops away in order to retrieve the current values of the weights on the links of these nodes and the values of weights on the nodes themselves. An advantage of our approach is that it provides a trade-off between locality and optimality. In fact, the larger the parameter p, the better the solution of problem (3) approximates the solution of problem (2), but at the same time the larger is the neighborhood from which each node needs to retrieve the information. When $p = 2$, then g_l where $l \sim (i, j)$ only depends on the weights of subgraph induced by the two nodes i and j. For $p = 4$, the gradient g_l depends only on the weights found on the subgraph induced by the set of vertices $N_i \cup N_j$, then it is sufficient that nodes i and j exchange the weights of all their incident links.

B. Choice of Stepsize and Projection set

The global convergence of gradient methods (i.e., for any initial condition) has been proved under a variety of different hypotheses on the function h to minimize and on the step size sequence $\gamma^{(k)}$. In many cases the step size has to be adaptively selected on the basis of the value of the function or of the module of its gradient at the current estimate, but this cannot be done in a distributed way for the function $h(\mathbf{w})$. This leads us to look for convergence results where the step size sequence can be fixed ahead of time. Moreover the usual conditions, like Lipschitzianity or boundness of the gradient, are not satisfied by the function $h(.)$ over all the feasible set. For this reason we add another constraint to our original problem (7) by considering that the solution has to belong to a given convex and compact set X. Before further specifying how we choose the set X, we state our convergence result.

Proposition 4. *Given the following problem*

$$\begin{aligned} &\textit{minimize} && h(\mathbf{w}) = Tr\left((I - \mathcal{I} \times \textit{diag}(\mathbf{w}) \times \mathcal{I}^T)^p\right), \\ &\textit{subject to} && \mathbf{w} \in X \end{aligned} \tag{10}$$

where $X \subseteq \mathbb{R}^m$ is a convex and compact set, if $\sum_k \gamma^{(k)} = \infty$ and $\sum_k \left(\gamma^{(k)}\right)^2 < \infty$, then the

[2]If a link or a node is more than $p/2$ hops away both from node i and node j, then it cannot belong to a $i-j$ walk of length p.

following iterative procedure converges to the minimum of h in X:

$$\mathbf{w}^{(k+1)} = P_X\left(\mathbf{w}^{(k)} - \gamma^{(k)}\mathbf{g}^{(k)}\right), \tag{11}$$

where $P_X(.)$ is the projection operator on the set X and $\mathbf{g}^{(k)}$ is the gradient of h evaluated in $\mathbf{w}^{(k)}$.

Proof: The function h is continuous on a compact set X, so it has a point of minimum. Moreover also the gradient $\mathbf{g}$ is continuous and then bounded on X. The result then follows from Proposition $8.2.6$ in [17, pp. 480]. ■

For example, $\gamma^{(k)} = a/(b+k)$ where $a > 0$ and $b \geq 0$ satisfies the step size condition in Proposition 4.

While the convergence is guaranteed for any set X convex and compact, we have two other requirements. First, it should be possible to calculate the projection P_X in a distributed way. Second, the set X should contain the solution of the optimization problem (7). About the first issue, we observe that if X is the cartesian product of real intervals, i.e., if $X = [a_1, b_1] \times [a_2, b_2] \times \ldots [a_m, b_m]$, then we have that the l-th component of the projection on X of a vector $\mathbf{y}$ is simply the projection of the l-th component of the vector on the interval $[a_l, b_l]$, i.e.,

$$[P_X(\mathbf{y})]_l = P_{[a_l,b_l]}(y_l) = \begin{cases} a_l & \text{if } y_l < a_l, \\ y_l & \text{if } a_l \leq y_l \leq b_l, \\ b_l & \text{if } b_l < y_l. \end{cases} \tag{12}$$

Then in this case Eq. (11) can be rewritten component-wise as

$$w_l^{(k+1)} = P_{[a_l,b_l]}(w_l^{(k)} - \gamma^{(k)} g_l^{(k)}).$$

We have shown in the previous section that g_l can be calculated in a distributed way, then the iterative procedure can be distributed. About the second issue, since we have from the bound of matrix norms (see [13])

$$||W||_\infty \leq \rho(W), \tag{13}$$

where $||W||_\infty = \max_{i,j} |w_{ij}|$. Then we can choose X in such a way that we include in the feasibility set all the weight matrices with spectral radius at most 1.

A consequence of inequality (13) is that if we choose $X = [-1, 1]^m$ the weight vector of the matrix solution of problem (2) necessarily belongs to X (the weight matrix satisfies the

convergence conditions). The same is true for the solution of problem (7) for p large enough because of Proposition 2. The following proposition summarizes our results.

Proposition 5. *If the underlying graph is connected, then the following distributed algorithm converges to the solution of the Schatten norm minimization problem for p large enough:*

$$w_l^{(k+1)} = P_{[-1,1]}(w_l^{(k)} - \gamma^{(k)} g_l^{(k)}), \quad \forall l = 1, \ldots, m, \tag{14}$$

where $\sum_k \gamma^{(k)} = \infty$ and $\sum_k \left(\gamma^{(k)}\right)^2 < \infty$.

Remark: If p is larger than twice the diameter of the graph, then each agent requires the information of all other agents, and this requirement is restrictive. But we would directly implement the centralized version of the Schatten norm minimization that is still more scalable than the (necessarily centralized) solution of the original SDP problem (only feasible for graphs of few thousands of links).

C. Complexity of the Algorithm

Our distributed algorithm for Schatten Norm minimization requires to calculate at every iteration, the stepsize $\gamma^{(k)}$, the gradient $g_l^{(k)}$ for every link, and a projection on the feasible set X. Its complexity is determined by the calculation of link gradient g_l, while the cost of the other operations is negligible. For $p = 2$, $g_l = 2 \times (2W_{ij} - W_{ii} - W_{jj})$, so the computational complexity for nodes is negligible, but the communication complexity is 1 message carrying a single real value (w_{ii}) per link, per node and per iteration. For $p = 4$, $g_l = 4\big((W^3)_{ij} + (W^3)_{ji} - (W^3)_{ii} - (W^3)_{jj}\big)$, and as discussed in the previous section, each of the 4 terms can be calculated only locally from the weights within 2-hops from i or j. The computational complexity for calculating g_l is in the worst case $O(\Delta^3)$ where Δ is the largest degree in the network. In order to calculate g_l locally, the communication complexity would be to send two messages per link per node and per iteration. The first message carries at most Δ values (the weight vector $\mathbf{W}_i$) and the second message carries one real value ($(W^3)_{ii}$). Due to space limit, we refer the reader to the technical report [14] for details on both computational and communication costs.

V. Performance Evaluation

In this section we evaluate the speed of convergence of consensus protocols when the weight matrix W is selected according to our algorithm. We compare with other distributed approaches

(see [18], [8]): max degree weights (MD) ($w_l = \frac{1}{\max_i\{d_i\}+1} \quad \forall l = 1, \ldots, m$), local degree (metropolis) weights (LD) ($w_l = \frac{1}{\max\{d_i,d_j\}+1} \; l \sim (i,j) \quad \forall l = 1, \ldots, m$), and optimal constant weights (OC) ($w_l = \frac{2}{\lambda_1(L)+\lambda_{n-1}(L)} \quad \forall l = 1, \ldots, m$).

As we have discussed in section III, this speed is asymptotically determined by the second largest eigenvalue in absolute value ($\mu(W)$). The simulations are done on random graphs: Erdös-Renyi (ER) graphs and Random Geometric Graphs (RGG), given that they are connected. The random graphs are generated as following :

- For the ER random graphs, we start from n nodes fully connected graph, and then every link is removed from the graph by a probability $1-q$ and is left there with a probability q. We have tested the performance for different probabilities q given that the graph is connected.
- For the RGG random graphs, n nodes are thrown uniformly at random on a unit square area, and any two nodes within a connectivity radius r are connected by a link. We have tested the performance for different connectivity radii given that the graph is connected. It is known that for a small connectivity radius, the nodes tend to form clusters.

A. Comparison with the optimal solution

We first compare $\mu\left(W_{(p)}\right)$ of the solution $W_{(p)}$ of the Schatten p-norm (or Trace) minimization problem (3) with its minimum value obtained solving the symmetric FDLA problem (2). To this purpose we used the CVX solver ([19]). This allows us also to evaluate how well problem (3) approximates problem (2) for finite values of the parameter p. The results in Fig. 1 have been averaged over 100 random graphs with 20 nodes generated according to the Erdos-Renyi (ER) model, where each link is included with probability $q \in \{0.2, 0.3, 0.4, 0.5\}$. We see from the results that as we solve the trace minimization for larger p, the asymptotic convergence speed of our approach converges to the optimal one as proven in Proposition 2.

B. Other distributed approaches: Asymptotic Convergence Rate

We compare now our algorithm for $p = 2$ and $p = 4$ with other distributed weight selection approaches. Fig. 2 shows the results on connected Random Geometric Graphs (RGG) with 100 nodes for different values of the connectivity radius r. We provide 95% confidence intervals by averaging each metric over 100 different samples. We see in Fig. 2 that on RGG for $p = 2$ and $p = 4$ the TM algorithm reaches faster convergence than the other known algorithms even when

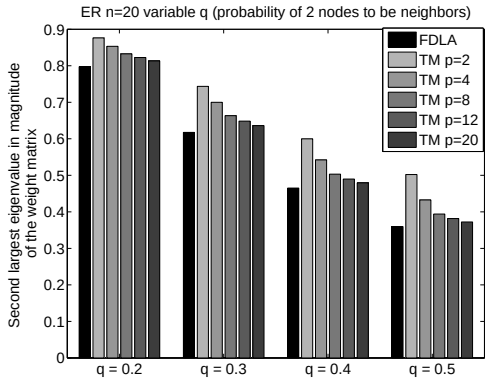

Fig. 1. Performance comparison between the optimal solution of the FDLA problem (labeled FDLA) and the approximated solutions obtained solving the Schatten Norm minimization for different values of p (labeled TM).

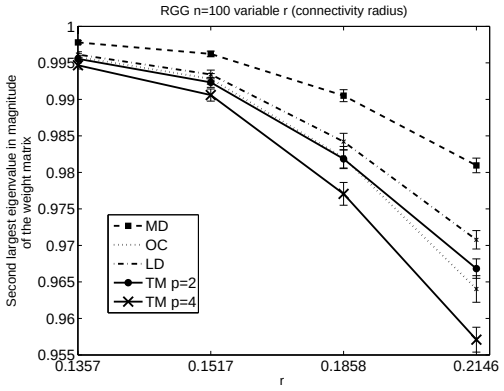

Fig. 2. Performance comparison between Schatten Norm minimization (TM) for $p = 2$ and $p = 4$ with other weight selection algorithms on RGG graphs.

the graph is well connected (large connectivity radius). However, the larger the degrees of nodes, the higher the complexity of our algorithm. Interestingly even performing trace minimization for the smallest value $p = 2$ nodes are able to achieve faster speed of convergence than a centralized solution like the OC algorithm.

C. Communication Complexity of Local Algorithms

As the Schatten norm minimization problem itself may take a long time to converge, whereas other heuristics can be obtained instantaneously, the complexity of the optimization algorithm can affect the overall procedure. We investigate in this section the communication complexity for

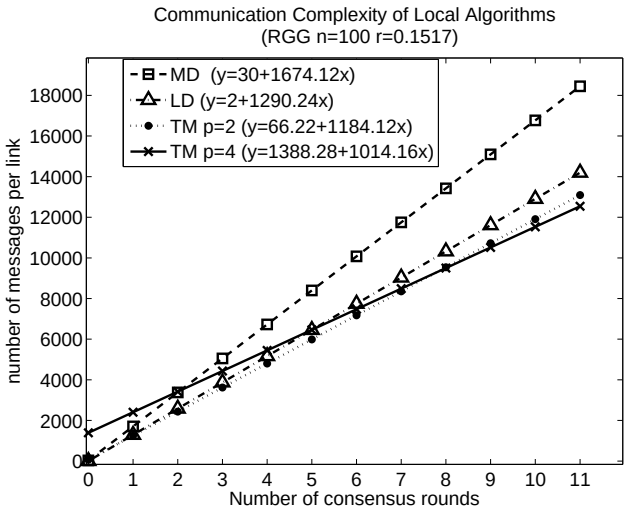

Fig. 3. Communication overhead of local algorithms. The figure shows that the Schatten norm algorithm ($TM\ p = 4$) provides better communication overhead than other local algorithms for networks whose topology is not changing very often, while $TM\ p = 2$ provides better communication overhead than other local algorithms for networks used less than 8 rounds.

optimizing the weights by the Schatten norm algorithm suggested in this paper and other local weight selection algorithms on RGG with 100 nodes and connectivity radius 0.1517. For each algorithm, the weights are calculated before starting the consensus rounds. The communication overhead of the local algorithms is plotted in Fig. 3. The figure shows the total number of messages transmitted on a link at each round, considering both those needed to calculate the weights and those needed to determine the average with a given precision. The TM algorithms have high initial communication overhead (due to the slow convergence of the gradient method for weight calculation), but then the more the consensus rounds we have the more the messages are saved in comparison to the simpler methods. Note that the asymptotic results are reflected in the slopes of the lines. As the figure shows, if the network is used for less than 8 times the $TM\ p = 2$ is recommended, while for networks that have a topology not changing very often the $TM\ p = 4$ should be selected.

VI. Conclusion

We have proposed in this paper an approximated solution for the FDLA problem by minimizing the Schatten p-norm of the weight matrix. Our approximated algorithm converges to the solution

of the FDLA problem as p approaches infinity, and in comparison to it, has the advantage to be suitable for a distributed implementation. Moreover, simulations on random and real networks show that the algorithm outperforms other common distributed algorithms for weight selection.

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