Performance Evaluation

Lecture 2: Epidemics

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Outline

- Limit of Markovian models
- Mean Field (or Fluid) models
  - exact results
  - extensions
  - Applications
    - Bianchi's model
    - Epidemic routing

Mean fluid for Epidemic routing (and similar)

1. Approximation: pairwise intermeeting times modeled as independent exponential random variables
2. Markov models for epidemic routing
3. Mean Fluid Models

Inter-meeting times under random mobility (from Lucile Sassatelli's course)

Inter-meeting times mobile/mobile have been shown to follow an exponential distribution

[Groenevelt et al.: The message delay in mobile ad hoc networks. Performance Evalation, 2005]

$\Pr\{ X = x \} = \mu \exp(- \mu x)$

CDF: $\Pr\{ X \leq x \} = 1 - \exp(- \mu x)$,   CCDF: $\Pr\{ X > x \} = \exp(- \mu x)$

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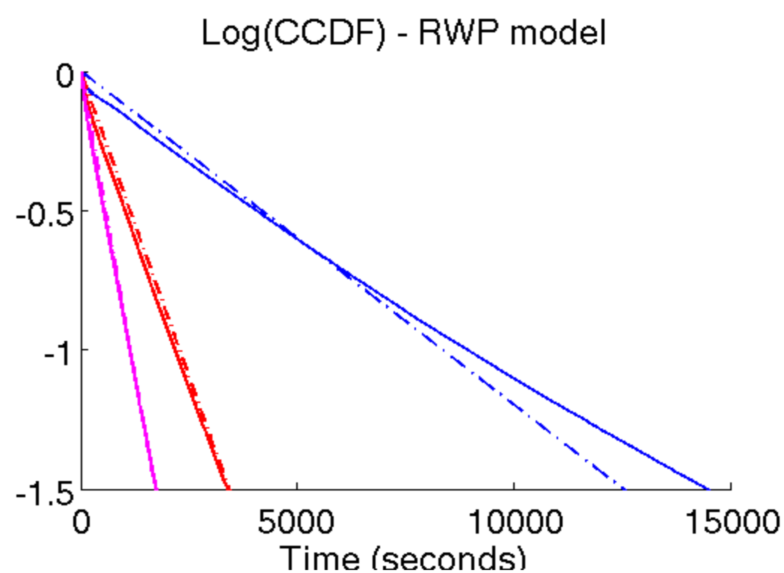
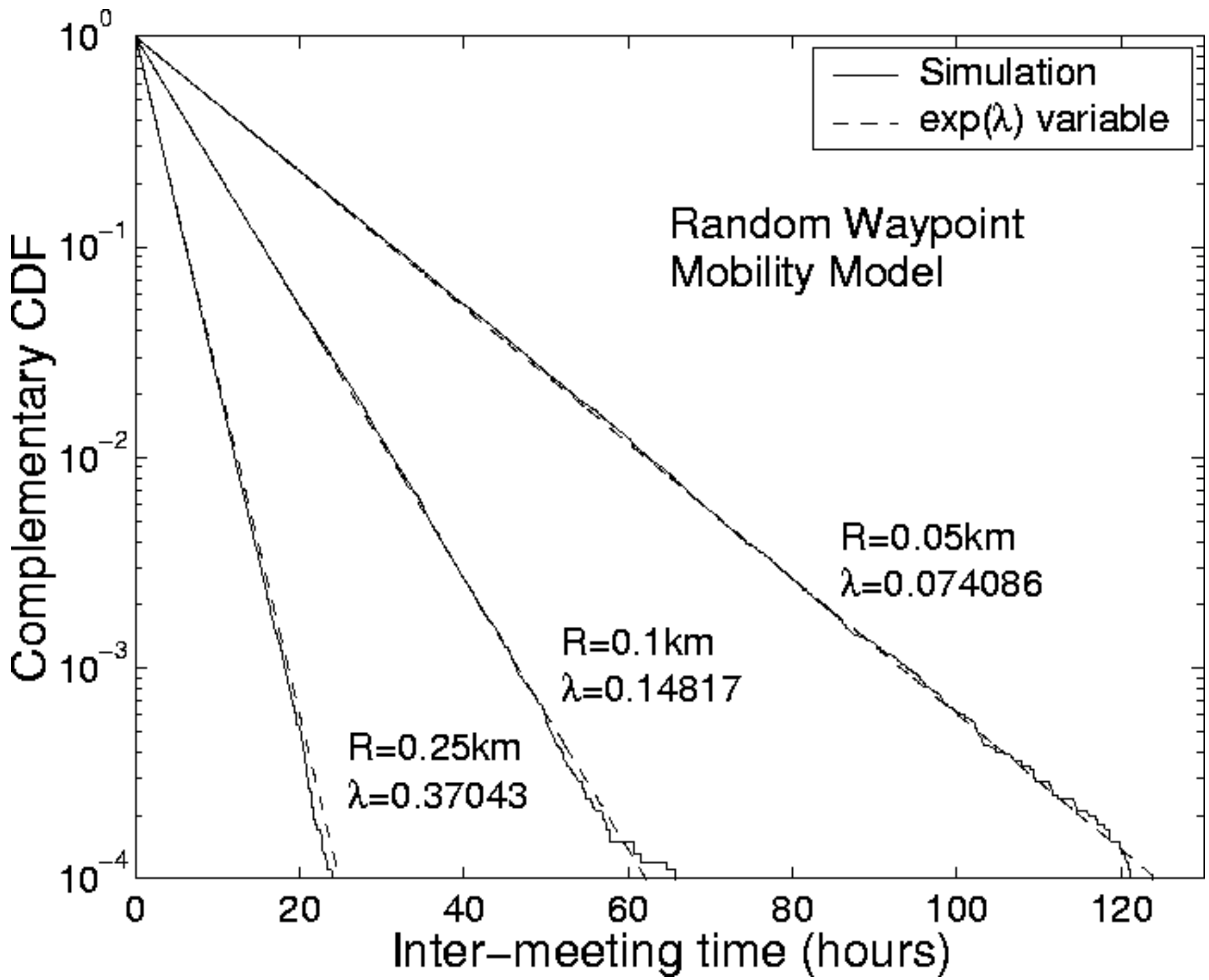


M. Ibrahim, *Routing and Performance Evaluation of Disruption Tolerant Networks*, PhD defense, UNS 2008

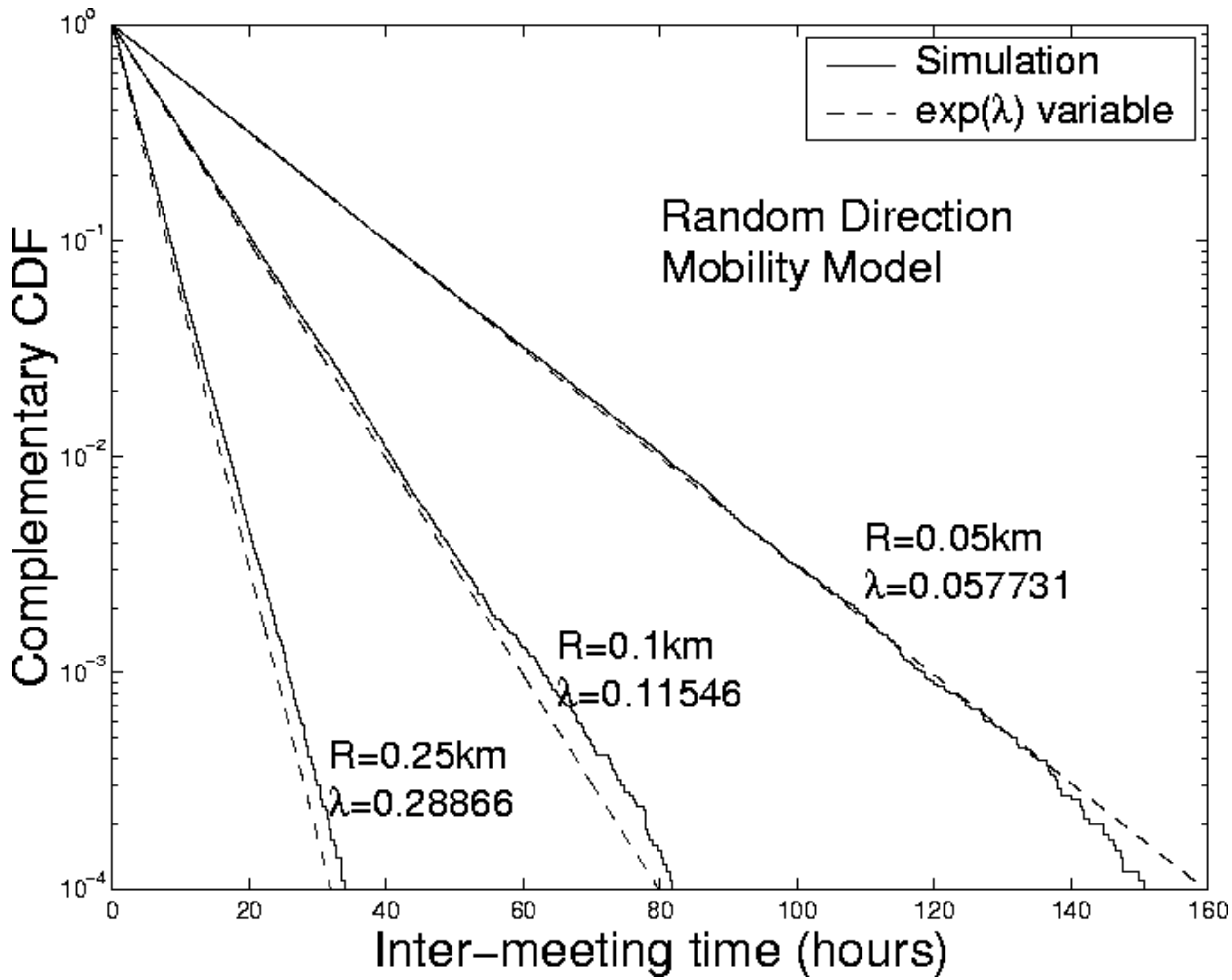
Pairwise Inter-meeting time



$$P(T > t) = e^{-\lambda t}$$

$$\log P(T > t) = -\lambda t$$

Pairwise Inter-meeting time



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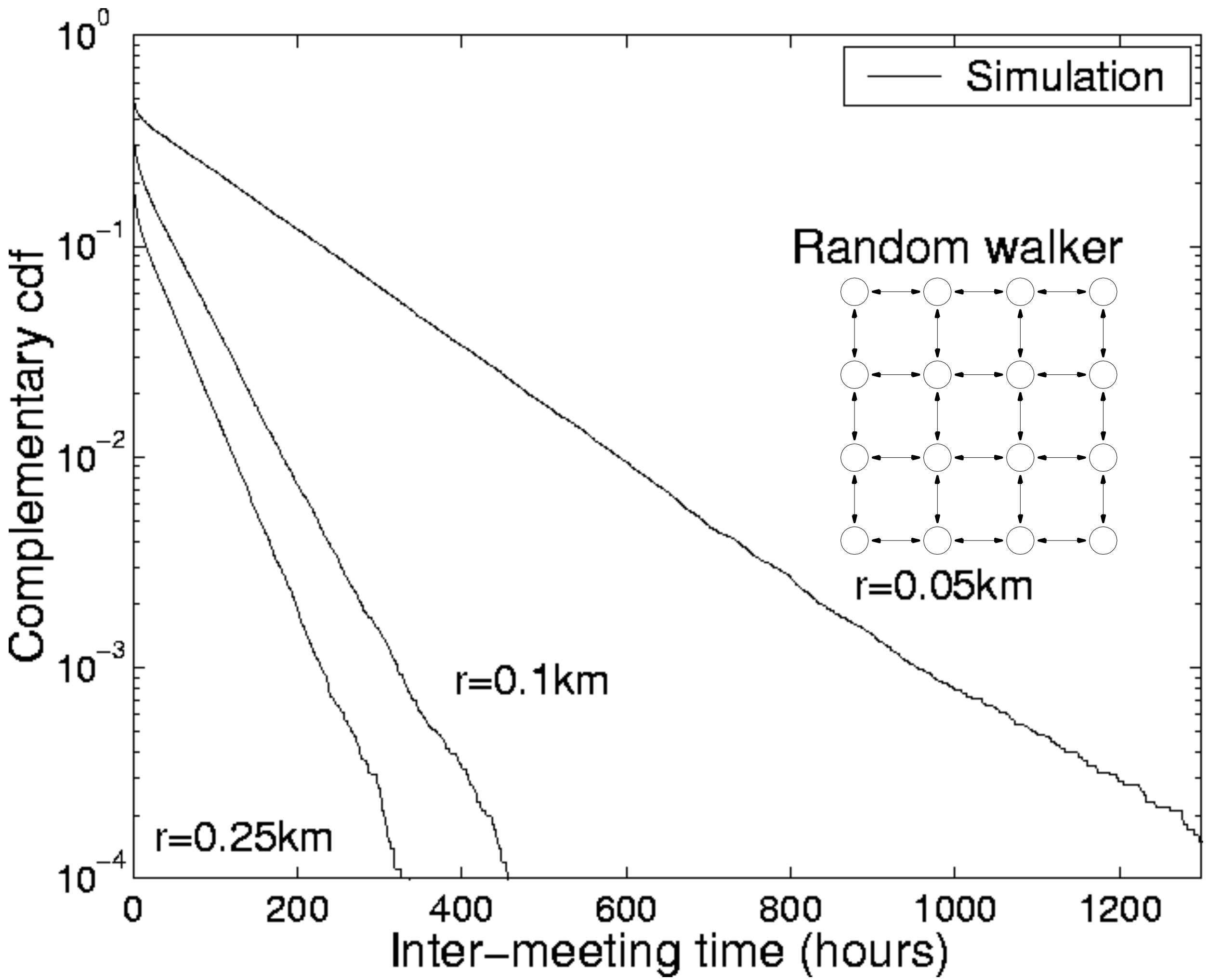
Pairwise Inter-meeting time

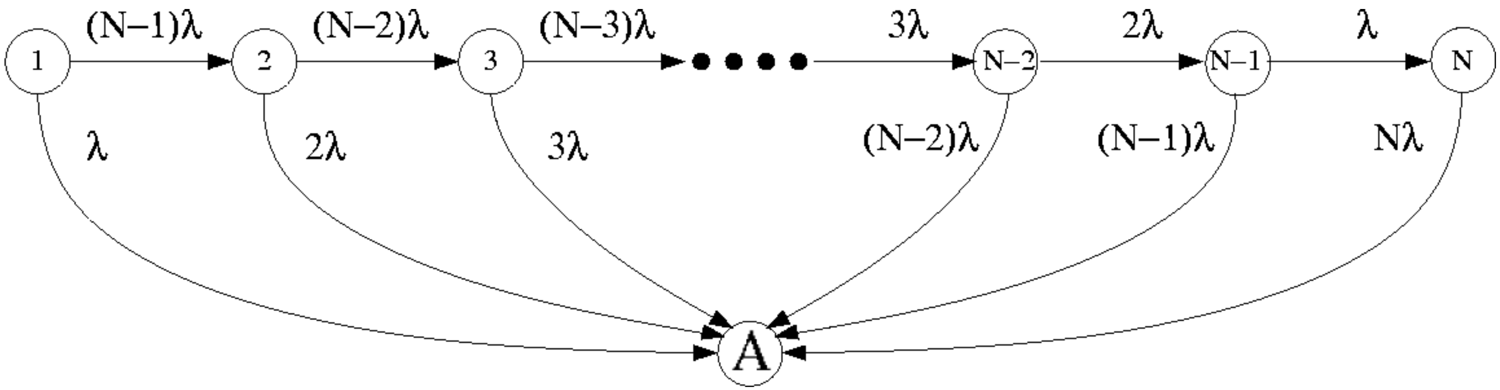


$$P(T > t) = e^{-\lambda t}$$

$$\log P(T > t) = -\lambda t$$

2-hop routing

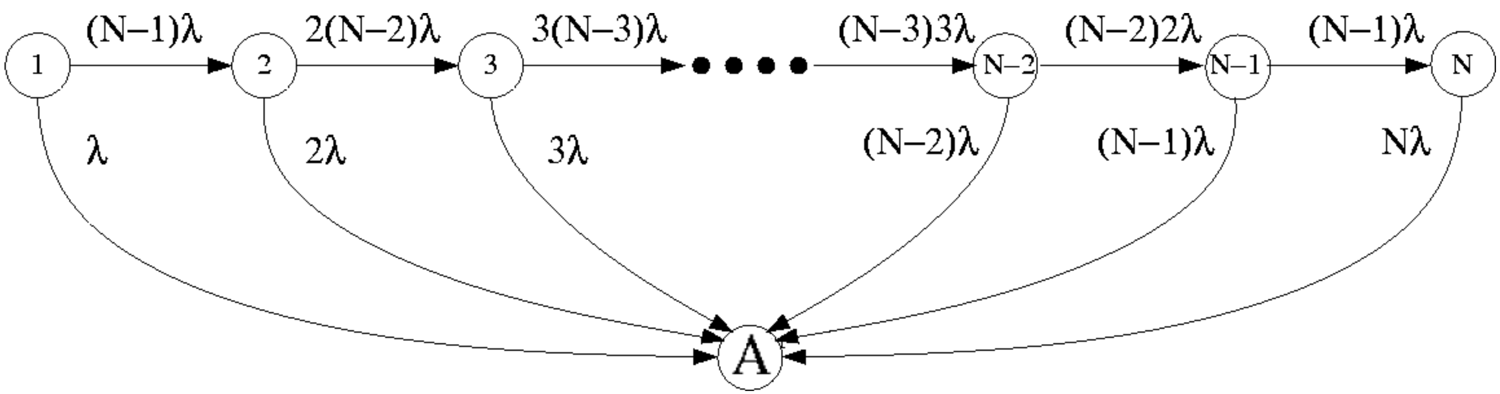
Model the number of occurrences of the message as an absorbing Continuous Time Markov Chain (C-MC):



- State i∈{1,…,N} represents the number of occurrences of the message in the network.
- State *A* represents the destination node receiving (a copy of) the message.

Epidemic routing

Model the number of occurrences of the message as an absorbing C-MC:



- State $i \in \{1,\ldots,N\}$ represents the number of occurrences of the message in the network.
- State *A* represents the destination node receiving (a copy of) the message.

A mean-field interaction model for modeling dissemination

(from Lucile Sassatelli's course)

- Time $t \in \mathbb{N}$ is discrete. There are N objects.
- Object n has state $Z_n^{(N)}(t)$ in $S = \{0, 1\}$.
- We assume that $\mathbf{Y}^{(N)}(t) = (Z_1^{(N)}(t), \ldots, Z_N^{(N)}(t))$ is a homogeneous Markov chain on S^N.
- We assume that we can observe the state of an object but not its label, i.e.,

$$\mathcal{K}^N(i_1, \ldots, i_N; i'_1, \ldots, i'_N) =$$

$$Pr\{Z_1^{(N)}(t+1) = i_1, \ldots, Z_N^{(N)}(t+1) = i_N | Z_1^{(N)}(t) = i'_1, \ldots, Z_N^{(N)}(t) = i'_N\}$$

is stable under any permutation.

$\rightarrow$ The process $\mathbf{Y}^{(N)}(t)$ is called a mean-field interaction model with N objects.

T. G. Kurtz, *Solutions of Ordinary Differential Equations as Limits of Pure Jump Markov Processes*, Journal of Applied Probability, vol. 7, no. 1, pp. 49-58, 1970.
M. Benaïm and J.-Y. Le Boudec, *A class of mean field interaction models for computer and communication systems*, Performance Evaluation, vol. 65, no. 11-12, pp. 823-838, 2008.

A mean-field interaction model for modeling dissemination

(from Lucile Sassatelli's course)

- Define the occupancy measure $\mathbf{M}^{(N)}(t)$ as the vector of frequencies of states $i \in S$ at t:
$M_i^{(N)}(t) = \frac{1}{N}\sum_{n=1}^{N} 1_{\{Z_i^{(N)}(t)=i\}}$. $\mathbf{M}^{(N)}(t)$ that takes vales in Δ.
$\mathbf{M}^{(N)}(t)$ is a homogeneous Markov chain.
- Let us define the drift $\mathbf{f}(\mathbf{m})$ for $\mathbf{m} \in \Delta$ as the expected change to $\mathbf{M}^{(N)}(t)$ in one time-slot:

$$
\begin{aligned}
\mathbf{f}^{(N)}(\mathbf{m}) &= \mathbb{E}[\mathbf{M}^{(N)}(t+1) - \mathbf{M}^{(N)}(t) | \mathbf{M}^{(N)}(t) = \mathbf{m}] \\
&= \sum_{\{i,i'\} \in S, i \neq i'} m_i P_{i,i'}^{(N)}(\mathbf{m})(\mathbf{e}_{i'} - \mathbf{e}_i)
\end{aligned}
$$

where $P_{i,i'}^{(N)}$ is the marginal transition probability:
$P_{i,i'}^{(N)}(\mathbf{m}) = Pr\{Z_n^{(N)}(t+1) = i' | Z_n^{(N)}(t) = i, \mathbf{M}^{(N)}(t) = \mathbf{m}\}$.

T. G. Kurtz, *Solutions of Ordinary Differential Equations as Limits of Pure Jump Markov Processes*, Journal of Applied Probability, vol. 7, no. 1, pp. 49-58, 1970.
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Convergence to the mean-field limit

(from Lucile Sassatelli's course)

If $\lim_{N\infty} \mathbf{f}^{(N)}(\mathbf{m}) = \mathbf{f}(\mathbf{m})$ exists for all $\mathbf{m} \in \Delta$,
Then $\mathbf{M}^{(N)}(t)$ converges to a deterministic process $\mu(t)$ that satisfies:

$$\begin{cases} \frac{d\mu(t)}{dt} = \mathbf{f}(\mu(t)) \\ \mu(0) = \mu_0 \text{ constant in } N \end{cases}$$

More exactly (Kurtz Th 3.1), $\forall \delta$:

$$\lim_{N\infty} Pr\{\sup_{s \leq t} ||\mathbf{M}^{(N)}(t) - \mu(t)|| > \delta\} = 0$$

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Performance modeling of dissemination under two-hop routing or epidemic routing

(from Lucile Sassatelli's course)

$$\mathbf{M}^{(N)}(t) = \begin{bmatrix} M_0^{(N)}(t) \\ M_1^{(N)}(t) \end{bmatrix} = \begin{bmatrix} 1 - M_1^{(N)}(t) \\ M_1^{(N)}(t) \end{bmatrix}$$

- Two-hop routing: $f_1(m_1) = \lambda s(1 - m_1)$, where s is the fraction of sources (constant in N)
- Epidemic routing: $f_1(m_1) = \lambda m_1(1 - m_1)$
- Let us rename $\mu_1(t)$ as $x(t)$, standing for the fraction of infected nodes.
- Let $X^{(N)}(t)$ be the number of infected nodes: $X^{(N)}(t)$ can be approximated by $Nx(t)$.

T. G. Kurtz, *Solutions of Ordinary Differential Equations as Limits of Pure Jump Markov Processes*, Journal of Applied Probability, vol. 7, no. 1, pp. 49-58, 1970.
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Performance modeling of dissemination under two-hop routing or epidemic routing

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From that we approximate $X^{(N)}(t)$ by the solution of:

Epidemic

$$\frac{dX^{(N)}(t)}{dt} = \beta X^{(N)}(t)(N - X^{(N)}(t)), \quad X^{(N)}(0) = 1$$

Two-hop

$$\frac{dX^{(N)}(t)}{dt} = \beta 1(N - X^{(N)}(t)), \quad X^{(N)}(0) = 0$$

- Defining T_d as the packet delivery delay, we can derive $P(t) = Pr\{Td < t\}$:

$$\frac{dP(t)}{dt} = \lambda x(t)(1 - P(t))$$

Proof: Exercise class

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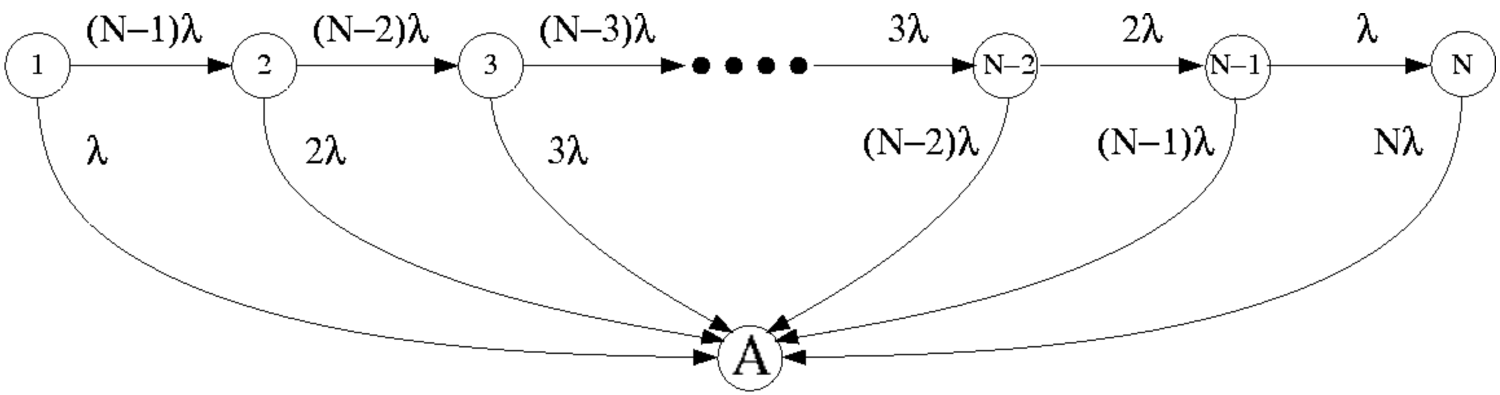
Zhang, X., Neglia, G., Kurose, J., Towsley, D.: Performance Modeling of Epidemic Routing. Computer Networks 51, 2867–2891 (2007)

A further issue

Model the number of occurrences of the message as an absorbing Continuous Time Markov Chain (C-MC):



- We need a different convergence result

[Kurtz70] Solution of ordinary differential equations as limits of pure jump markov processes, T. G. Kurtz, Journal of Applied Probabilities, pages 49-58, 1970

[Kurtz1970]

$\{X_N(t)$, N natural$\}$

a family of Markov process in Z^m

with rates $r_N(k,k+h)$,   k,h in Z^m

It is called density dependent if it exists a continuous function f() in $R^m x Z^m$ such that

$r_N(k,k+h) = N\, f(1/N\, k, h)$, $h<>0$

Define $F(x)=\Sigma_h\, h\, f(x,h)$

Kurtz's theorem determines when $\{X_N(t)\}$ are ***close*** to the solution of the differential equation:

$$\frac{\partial x(s)}{\partial s} = F(x(s)),$$

The formal result [Kurtz1970]

Theorem. Suppose there is an open set E in R^m and a constant M such that

$|F(x)-F(y)|<M|x-y|$, x,y in E

$\sup_{x \text{ in } E} \Sigma_h |h|\ f(x,h) < \infty$,

$\lim_{d \to \infty} \sup_{x \text{ in } E} \Sigma_{|h|>d} |h|\ f(x,h) = 0$

Consider the set of processes in $\{X_N(t)\}$ such that

$\lim_{N \to \infty} 1/N\ X_N(0) = x_0$ in E

and a solution of the differential equation

$$\frac{\partial x(s)}{\partial s} = F(x(s)), \quad x(0) = x_0$$

such that x(s) is in E for 0<=s<=t, then for each δ>0

$$\lim_{N\to\infty} \Pr\left\{ \sup_{0\le s\le t} \left| \frac{1}{N} X_N(s) - x(s) \right| > \delta \right\} = 0$$

Application to epidemic routing

$r_N(n_I) = \lambda\, n_I\, (N - n_I) = N\, (\lambda N)\, (n_I/N)\, (1 - n_I/N)$

assuming $\beta = \lambda N$ keeps constant (e.g. node density is constant)

$f(x,h) = f(x) = x(1-x)$, $F(x) = f(x)$

as $N \rightarrow \infty$, $n_I/N \rightarrow i(t)$, s.t.

$$i'(t) = \beta i(t)(1 - i(t))$$

with initial condition

$$i(0) = \lim_{N \to \infty} n_I(0)/N$$

Application to epidemic routing

$$i'(t) = \beta i(t)(1 - i(t)), \quad i(0) = \lim_{N\to\infty} n_I(0)/N$$

The solution is $$i(t) = \frac{1}{1 + \left(\frac{1}{i(0)} - 1\right)e^{-\beta t}}$$

And for the Markov system we expect

$$n_I(t) \approx N i(t) = \frac{N}{1 + \left(\frac{N}{n_I(0)} - 1\right)e^{-N\lambda t}}$$