Performance Evaluation

Lecture 2: Epidemics

Giovanni Neglia
INRIA – EPI Maestro
10 January 2013

There is more: Independence

- Theorem 2
  - Under the assumptions of Theorem 1, and that the collection of objects at time 0 is exchangeable $(X_1^N(0), X_2^N(0), \ldots X_N^N(0))$,

  then for any fixed n and t:

  $$\lim_{N\to\infty} \text{Prob}(\underline{X}_1^N(t)=i_1, \underline{X}_2^N(t)=i_2, \ldots \underline{X}_n^N(t)=i_n)=$$

  $$=\mu_{i1}(t)\mu_{i2}(t)\ldots\mu_{in}(t)$$

- MF Independence Property, a.k.a. Decoupling Property, Propagation of Chaos

Remarks

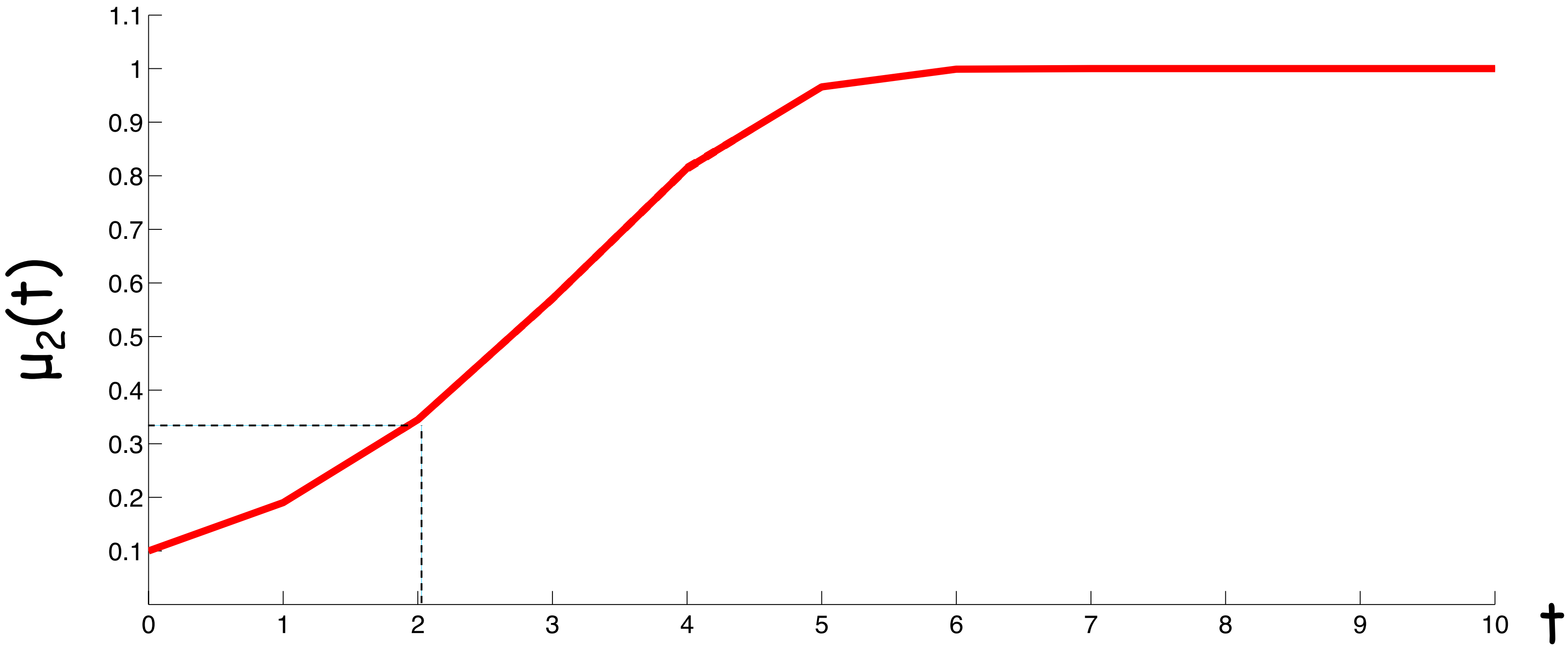
- $(X_1^N(0), X_2^N(0), \ldots X_N^N(0))$ exchangeable
  - Means that all the states that have the same occupancy measure $\mathbf{m}_0$ have the same probability
- $\lim_{N\to\infty} \text{Prob}(\underline{X}_1^N(t)=i_1, \underline{X}_2^N(t)=i_2, \ldots \underline{X}_n^N(t)=i_n)=$

  $=\mu_{i1}(t)\mu_{i2}(t)\ldots\mu_{in}(t)$
  - Application

    $\text{Prob}(X_1^N(k)=i_1, X_2^N(k)=i_2, \ldots X_k^N(k)=i_k) \approx$

    $\approx \mu_{i1}(k\varepsilon(N))\mu_{i2}(k\varepsilon(N))\ldots\mu_{ik}(k\varepsilon(N))$

Probabilistic interpretation of the occupancy measure
(SI model with $p=10^{-4}$, N=100)

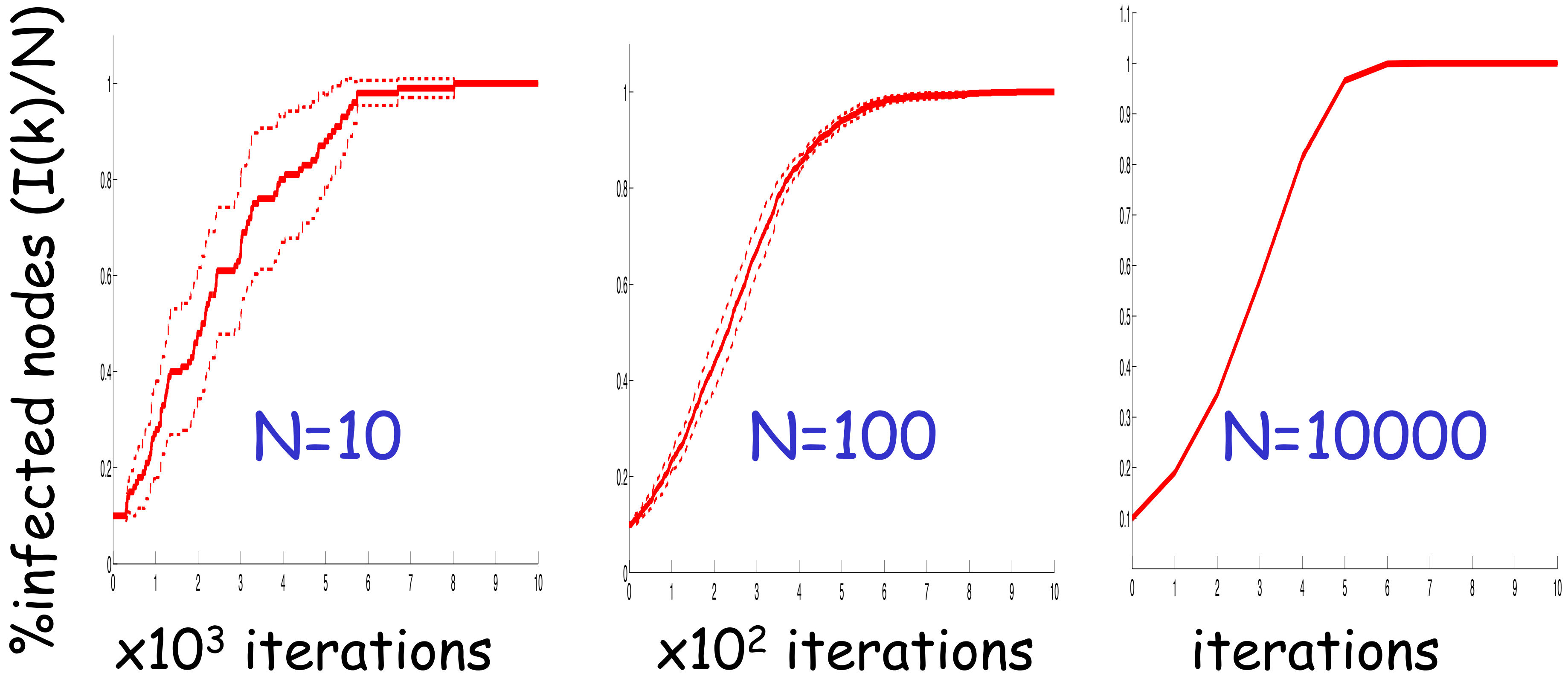


Prob(nodes 1,17,21 and 44 infected at k=200)=

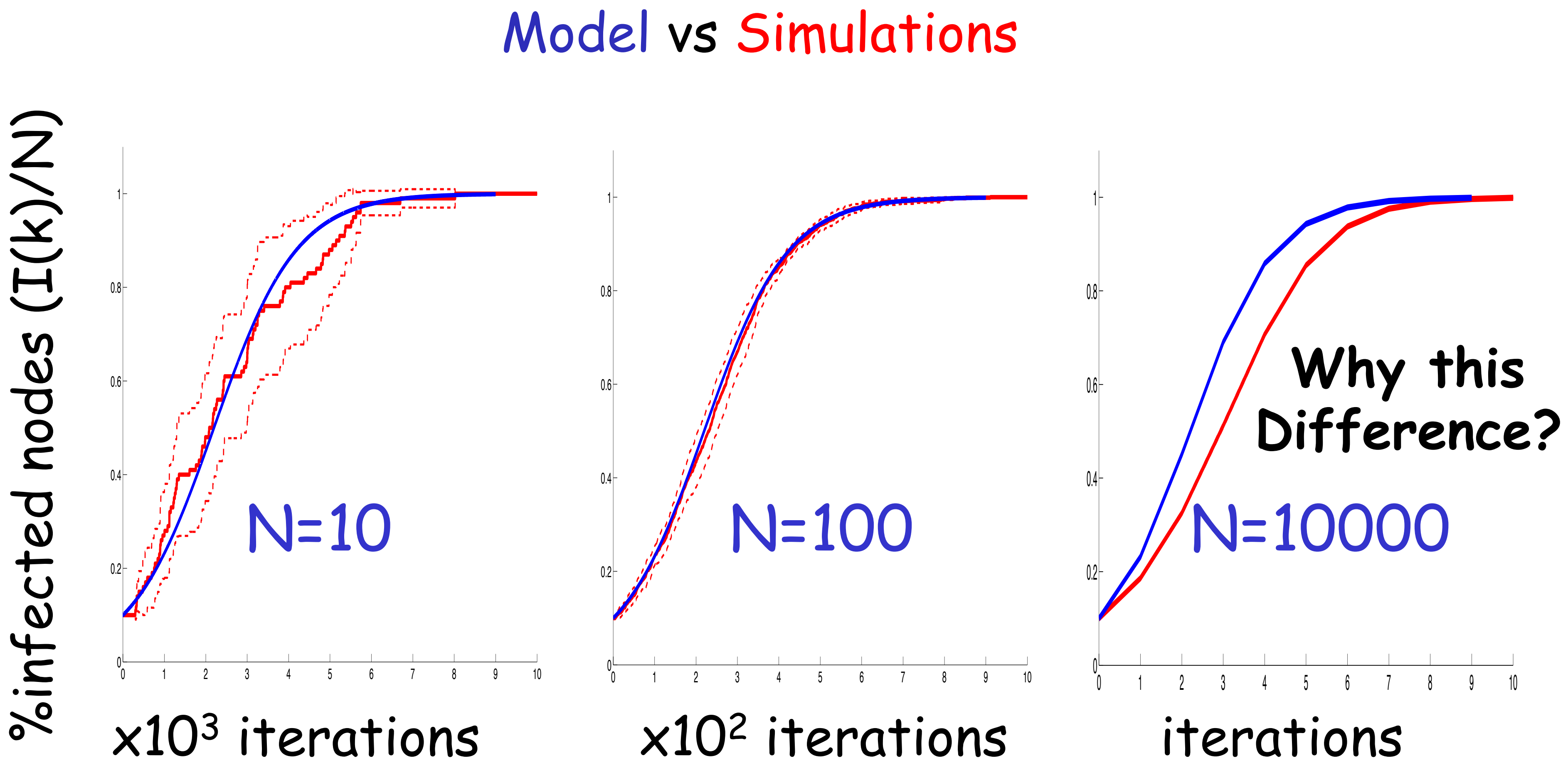
$=\mu_2(k\ p\ N)^4=\mu_2(2)^4 \approx (1/3)^4$

What if 1,17,21 and 44 are surely infected at k=0

On approximation quality
$p=10^{-4}$, $I(0)=N/10$, 10 runs



On approximation quality
$p=10^{-4}$, $I(0)=N/10$, 10 runs

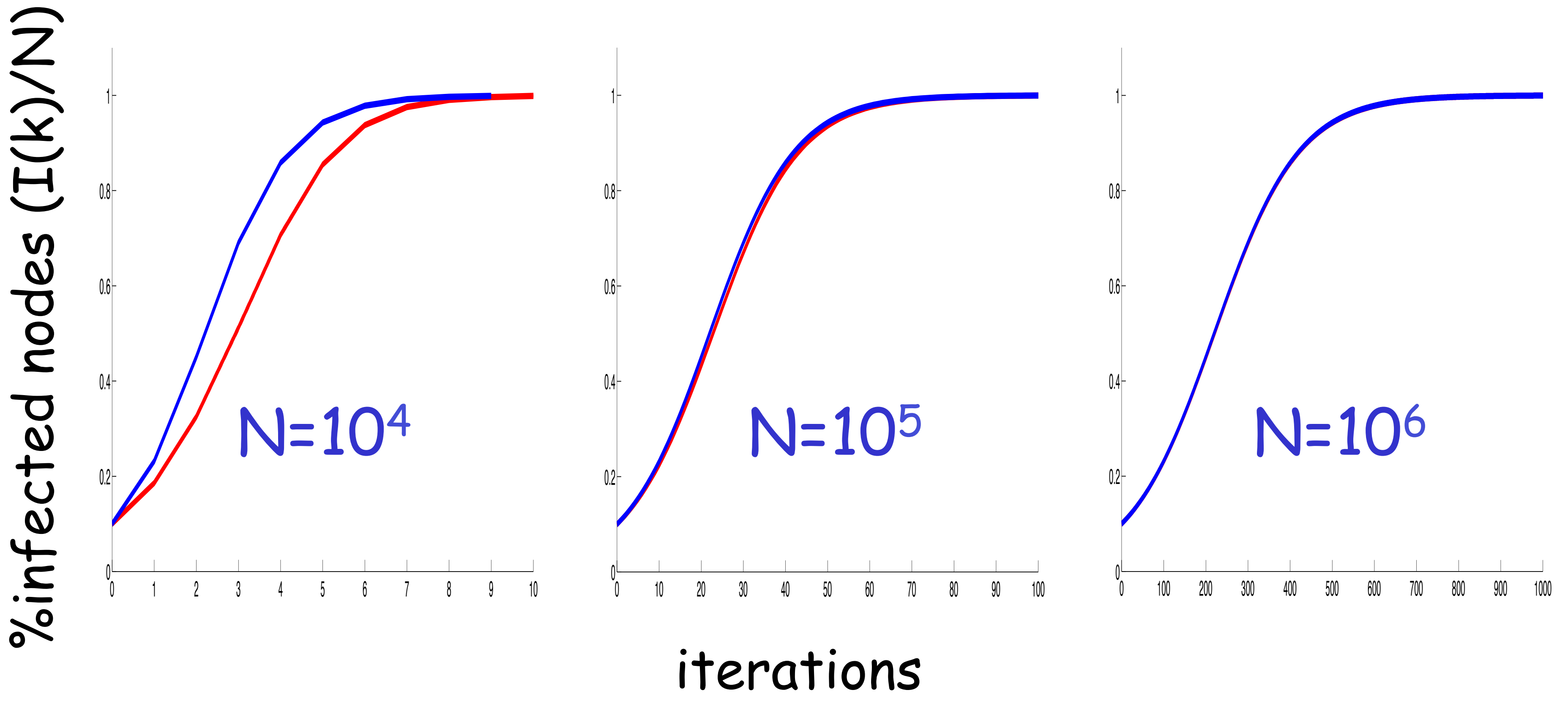


Why the difference?

- N should be large (the larger the better)
- p should be small
  - $p^{(N)}=p_0/N^2$
- For $N=10^4$ $p=10^{-4}$ is not small enough!
- What if we do the correct scaling?

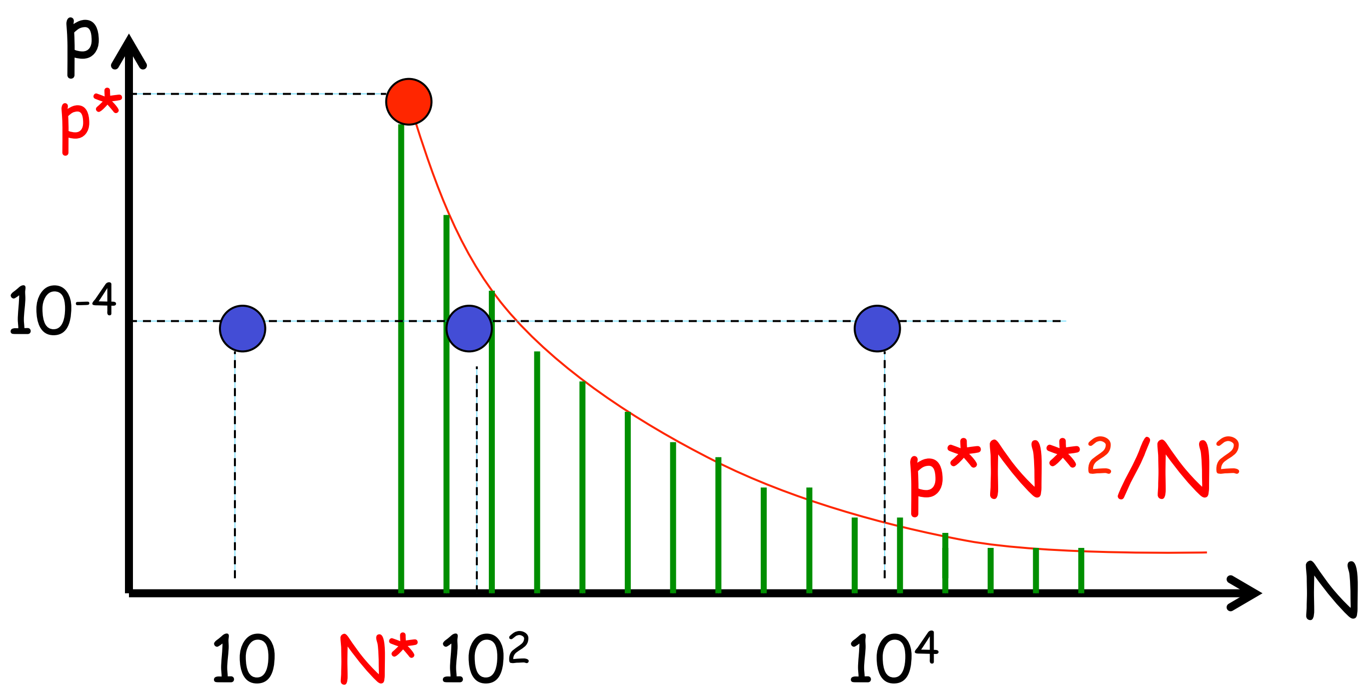
On approximation quality
$p=10^4/N^2$, $I(0)=N/10$, 10 runs

Model vs Simulations

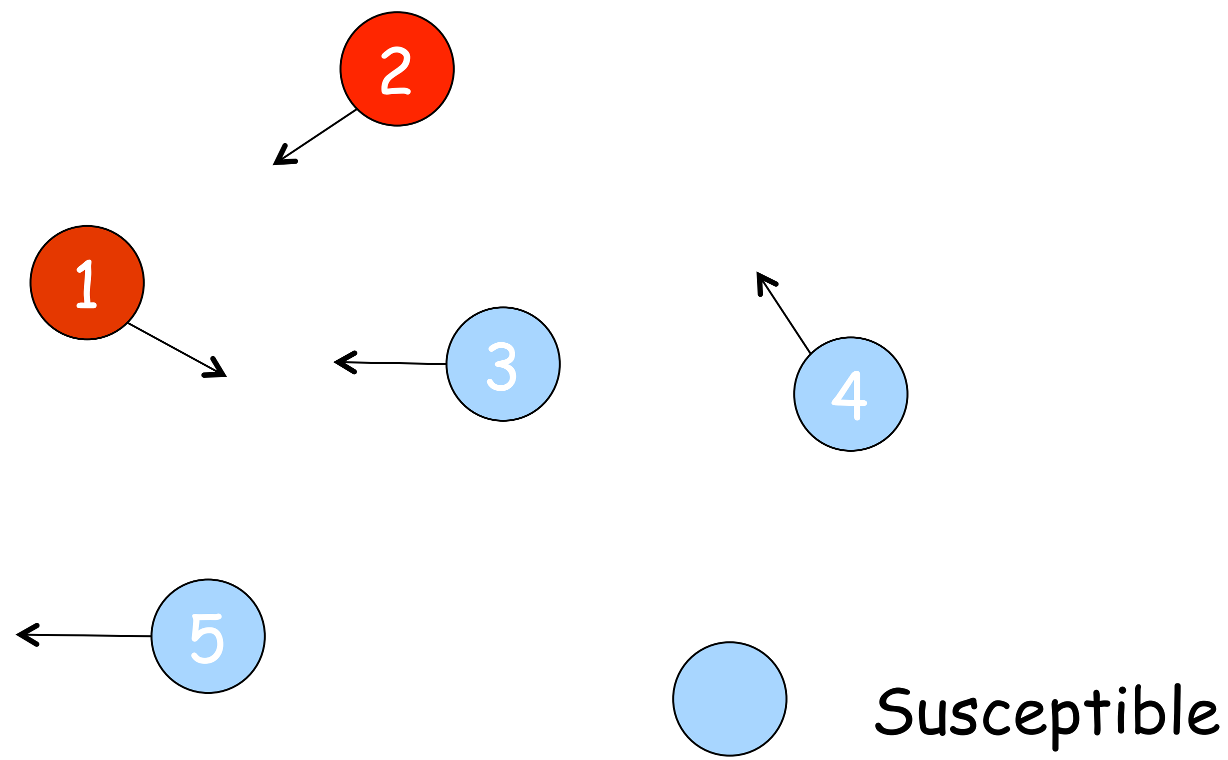


Lesson

- You need to check (usually by simulation) in which parameter region the fluid model is a good approximation.
  - e.g. $N > N^*$ $p < p^*/N^2$

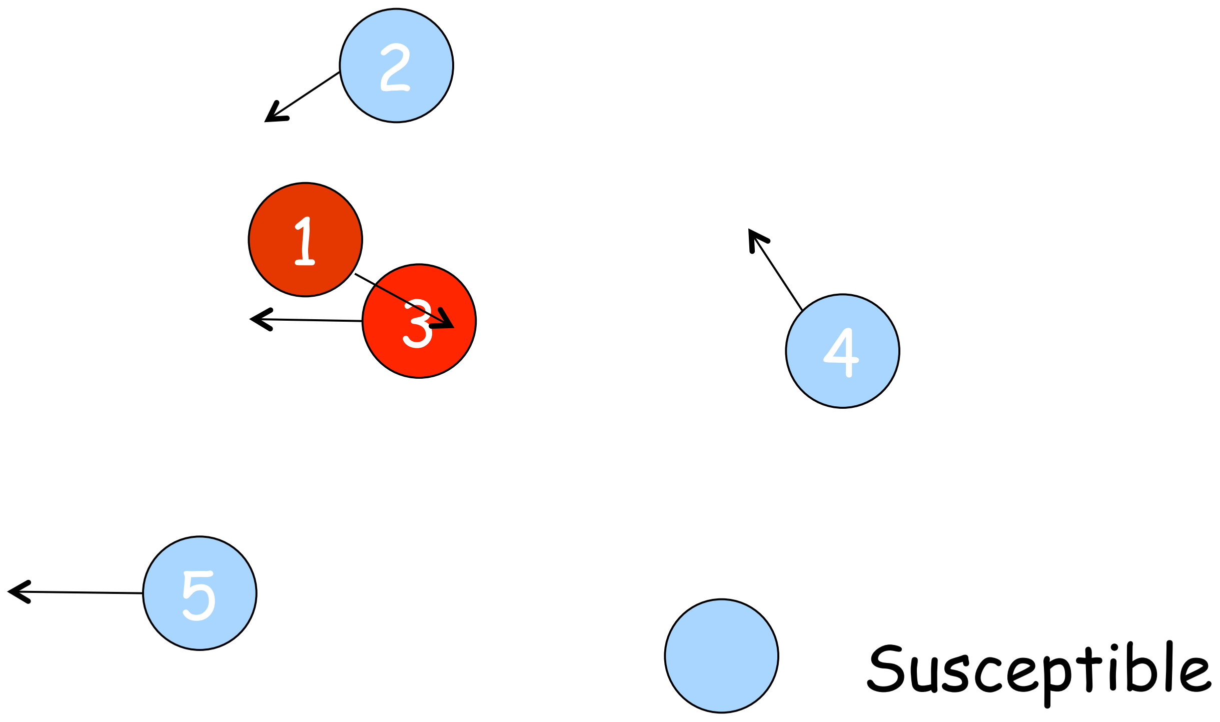


SIS model



At each slot there is a probability p that two given nodes meet, a probability r that a node recovers.

SIS model



At each slot there is a probability p that two given nodes meet,
a probability r that a node recovers.

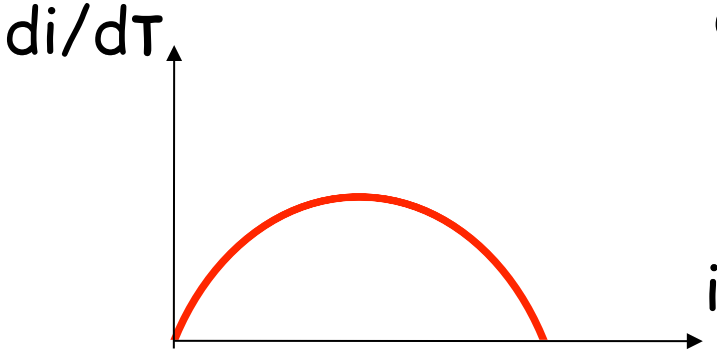
Let's practise

- Can we propose a Markov Model for SIS?
  - No need to calculate the transition matrix
- If it is possible, derive a Mean Field model for SIS
  - Do we need some scaling?

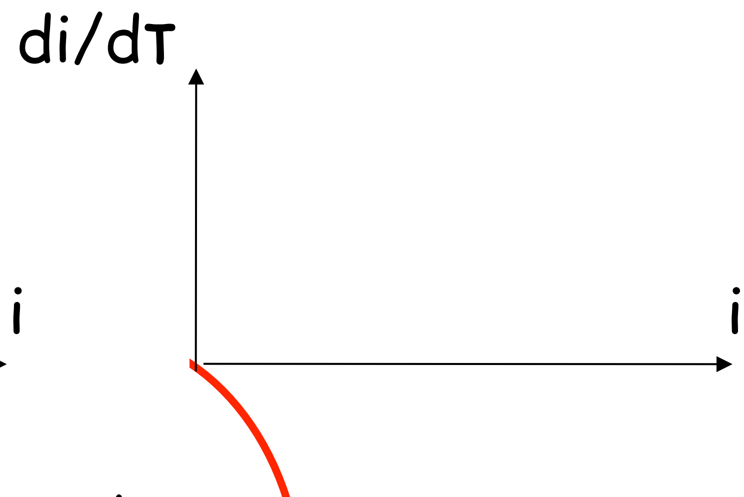
Study of the SIS model

- We need $p^{(N)}=p_0/N^2$ and $r^{(N)}=r_0/N$
- If we choose $\varepsilon(N)=1/N$, we get
  - $di(t)/dt = p_0\, i(t)(1-i(t)) - r_0\, i(t)$

$p_0 > r_0$



$p_0 < r_0$

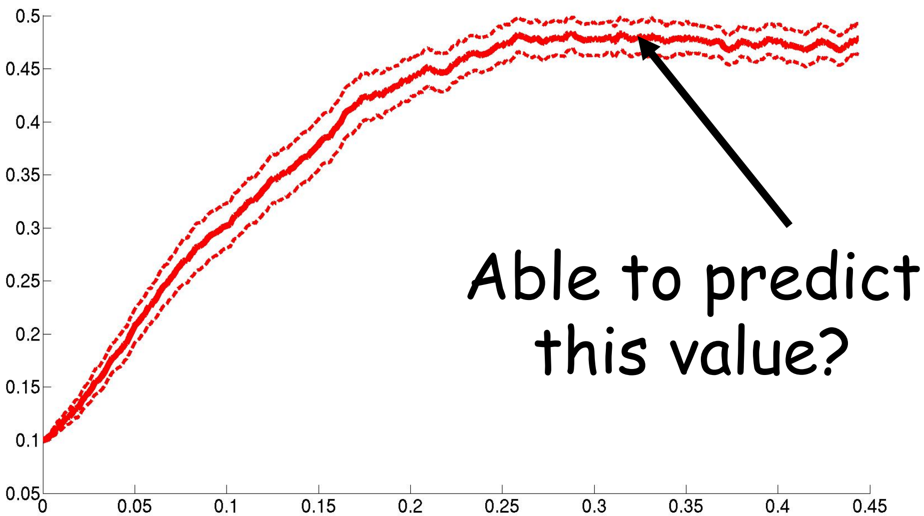


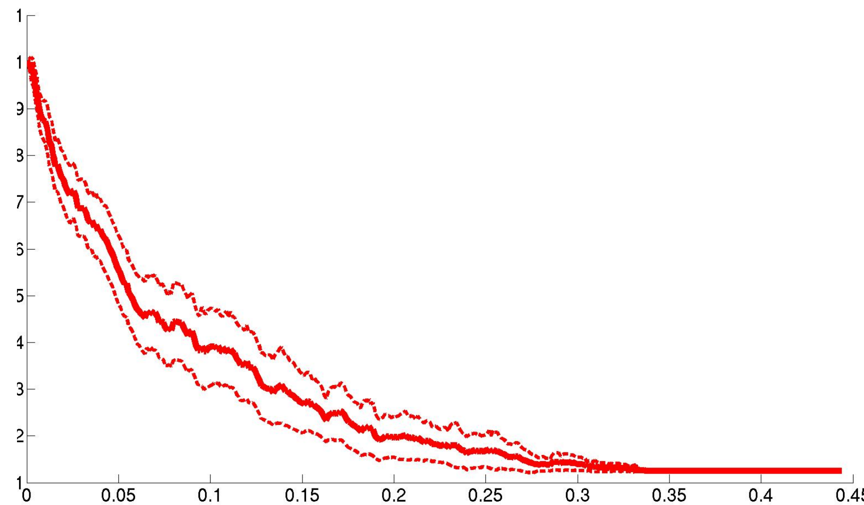
Epidemic Threshold: p_0/r_0

N=80, p_0=0.1

$r_0 = 0.05$

$r_0 = 0.125$





Study of the SIS model

- $\mu_2(t)=i(t)$
- $di(t)/dt=p_0\, i(t)(1-i(t)) - r_0\, i(t)$
- Equilibria, $di(t)/dt=0$
  - $i(\infty)=1-r_0/p_0$ or $i(\infty)=0$
  - If $i(0)>0$ and $p_0>r_0$ => $\mu_2(\infty)=1-r_0/p_0$

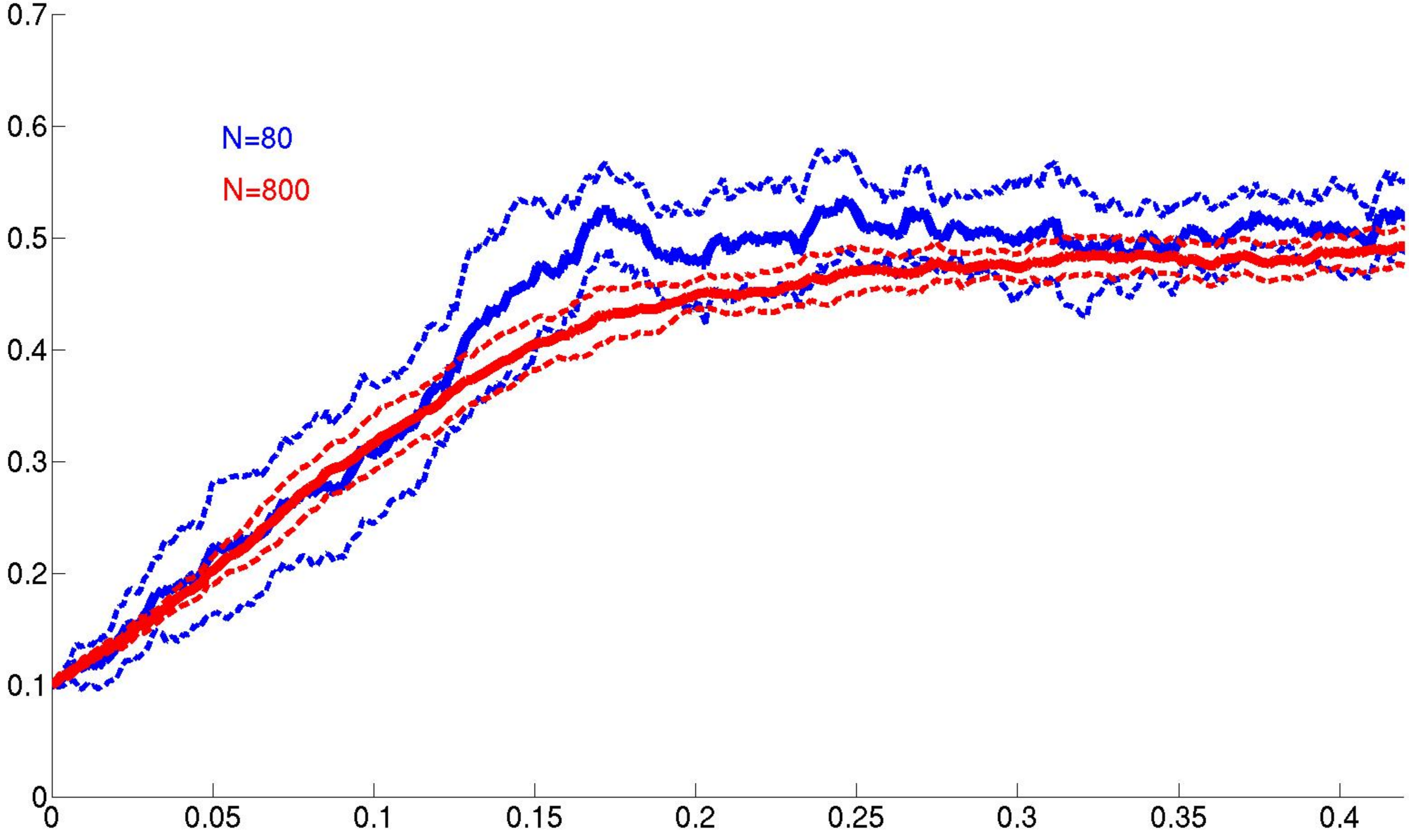
Study of the SIS model

- If $i(0)>0$ $p_0>r_{0,}$ $\mu_2(\infty)=1-r_0/p_0$
- $Prob(X_1^{(N)}(k)=1) \approx i(k\varepsilon(N))$
  - $Prob(X_1^{(N)}(\infty)=1) \approx \mu_2(\infty) = i(\infty) = 1-r_0/p_0$
- What is the steady state distribution of the MC?
  - (0,0,0,…0) is the unique absorbing state and it is reachable from any other state
  - Who is lying here?

Back to the Convergence Result

- Define $\underline{\mathbf{M}}^{(N)}(t)$ with t real, such that
  - $\underline{\mathbf{M}}^{(N)}(k\varepsilon(N))=\mathbf{M}^{(N)}(k)$ for k integer
  - $\underline{\mathbf{M}}^{(N)}(t)$ is affine on $[k\varepsilon(N),(k+1)\varepsilon(N)]$
- Consider the Differential Equation
  - $d\boldsymbol{\mu}(t)/dt=\mathbf{f}(\boldsymbol{\mu})$, with $\boldsymbol{\mu}(0)=\mathbf{m}_0$
- Theorem
  - For all T>0, if $\mathbf{M}^{(N)}(0) \to \mathbf{m}_0$ in probability (/mean square) as $N \to \infty$, then $\sup_{0\le t\le T}||\underline{\mathbf{M}}^{(N)}(t)-\boldsymbol{\mu}(t)|| \to 0$ in probability (/ mean square)

Some examples



Nothing to do with $t=\infty$?

- Theorem 3: The limits when N diverges of the stationary distributions of $\mathbf{M}^{(N)}$ are included in the Birkhoff center of the ODE
  - Birkhoff center: the closure of all the recurrent points of the ODE (independently from the initial conditions)
  - What is the Birkhoff center of $di(t)/dt=p_0\, i(t)(1-i(t)) - r_0\, i(t)$?

Nothing to do with t=∞?

- Theorem 3: The limits when N diverges of the stationary distributions of $\mathbf{M}^{(N)}$ are included in the Birkhoff center of the ODE
- Corollary: If the ODE has a unique stationary point $\mathbf{m}^*$, the sequence of stationary distributions $\mathbf{M}^{(N)}$ converges to $\mathbf{m}^*$

Outline

- Limit of Markovian models
- Mean Field (or Fluid) models
  - exact results
  - Extensions
    - Epidemics on graphs
      - Reference: ch. 9 of Barrat, Barthélemy, Vespignani "Dynamical Processes on Complex Networks", Cambridge press
  - Applications to networks