Lecture 4

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NOTE: The content of these notes has not been formally reviewed by the lecturer. It is recommended that they are read critically.

1 Introduction

In the previous classes we have seen how taking decisions locally, on the micro scale by each individual affect the general or the macroscopic state of the system. In the The fist example, the electrons on circuit by taking decision ~~they~~ optimize a global problem, which is minimization of the global ~~power~~ dissipated by Joule effect. But in the second example we have seen that ~~this~~ not always ~~correct~~, in fact the local policies may lead to inefficiency on the macro scale. In the last lesson, the goal was to allocate resources to get the maximum from the system capabilities.

WHAT IS NEW IN THIS LESSON? ELASTIC SOURCES WITH UNKNOWN UTILITY FUNCTIONS

2 Problem

Having a such graph with 3 sources: S1,S2 and S3 aiming to communicate with Destinations: D1, D2 and D3 respectively. The problem is how to allocate the shared links between different communications paths.

links are not really visible in this figure

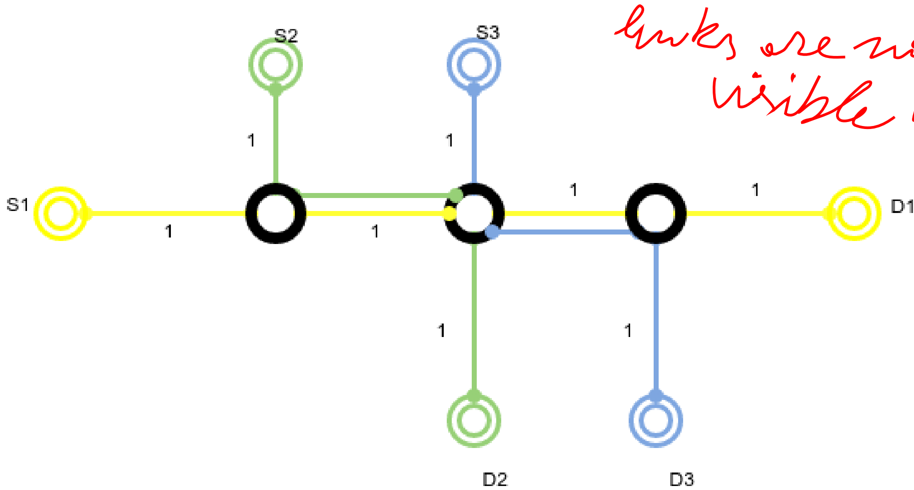


Figure 1: Communication network example

Considering T1,T2 and T3 respectively the throughput of each source S1,S2 and S3 looking at the shared links we have the following constraints: $T_1 + T_2 \leq 1$, $T_1 + T_3 \leq 1$

One may be "democratic" and do a Max-Min Fairness and give the following allocations:

$$T_1 = \frac{1}{2}, T_2 = \frac{1}{2}, T_3 = \frac{1}{2}$$

The global throughput will be :

$$T = \sum_{i=1}^{3} T_i = \frac{3}{2}$$

while another possible allocation one can ~~do~~ be:

$$T_1 = 0,\ T_2 = 1,\ T_3 = 1$$

And the global throughput will be:

$$T = \sum_{i=1}^{3} T_i = \cancel{\frac{3}{2}}\ 2 > 3/2$$

Doing ~~Maximum (minimum Ti)~~ max-min fairness may induce Low utilization of the system.

The question now is how to stay in the middle, between a fair allocation and a high utilisation of the system.

Here we can talk about Proportional Fairness.

The proportional fairness insure an optimal allocation to maximize T and in the same time (in some how) try to be fair.

DEFINITION **T*** is ~~Optimum, means~~ a max-min fair allocation if that for every other **T** :

$$\cancel{\mathbf{T}^* \geq \mathbf{T}}$$

NO PROPORTIONAL FAIRNESS DOES NOT GUARANTEE A HIGHER THROUGHPUT (E.G. $\frac{5}{3} < 2$)

~~**T*** is proportional fair if. (with: $T* = \sum T_i^*$)~~

$$\forall T_i : \sum_i \frac{T_i - T_i^*}{T_i^*} \leq 0$$

back to our example, doing proportional fairness we get the following results

$$T_1^* = \frac{1}{3},\ T_2^* = \frac{2}{3},\ T_3^* = \frac{2}{3}\ and\ \sum T_i^* = \frac{5}{3}$$

DEFINITION Weighted proportional fairness: More generally given a vector of weight $w_i > 0$, we talk about weighted proportional fairness T^* is proportional fair if:

$$\sum_i \frac{W_i(T_i - T_i^*)}{T_i^*} \leq 0 \quad , W_i > 0$$

It can be proved that $\underline{T}^*$ is weight proportional fair if only if it solves:

$$Maximize(\sum W_r ln(X_r))$$

$$A.\underline{X} \leq \underline{C}$$

$$\underline{X} \geq \underline{0}$$

C_l^* : *capacity of link "l"*
$A_{e,r} = 1\ if\ \ l \in r\ and\ 0\ if\ else$

The Utility function:

$$Maximize(\sum U_r(X_r))$$

$$subject\ to : \mathbf{A}.\underline{X} \leq \underline{C}$$

$$\underline{X} \geq \underline{0}$$

WHY DO WE INTRODUCE THIS PROBLEM?

The question is how to get the utility function

WHAT ARE THE PROPERTIES OF U?

we define: $\lambda_r = \frac{W_r}{X_r}$ WHAT IS THE INTERPRETATION OF λ?

NOTATION IS DIFFERENT FROM THE REST OF THE COURSE AND NOT ALWAYS COHERENT

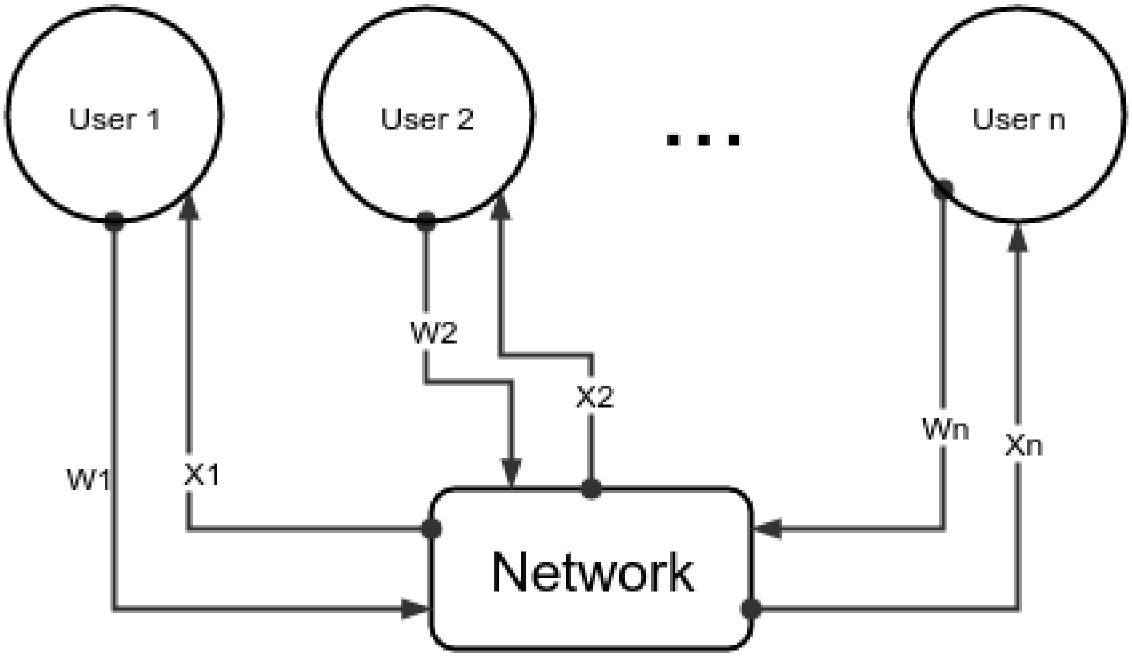


Figure 2: The users network relation

The User optimization system is the following, $User(U_r, \lambda_r)$:

$$Maximize\ (U_r(\frac{W_r}{\lambda_r}) - W_r)$$

INTERPRETATION?

$$Wr \leq 0$$

The Network optimization system is the following, Network($\underline{W}$,$\mathbf{A}$,$\underline{C}$):

$$maximizesigma(Wr.ln(Xr))$$

$$A.\underline{X} \leq \underline{C}$$

$$\underline{X} > 0$$

User + Network == System

Decomposition Theorem: 1. *If $\exists \underline{X}$, $\underline{W}$ and $\underline{\lambda}$ such that:*

1. $\forall r, W_r = X_r.\lambda_r$
2. *$\underline{X}$ is the solution of NETWORK($\underline{W}$,$\mathbf{A}$,$\underline{C}$)*
3. *$\forall r$, W_r is the solution of $USER(U_r,\ \lambda_r)$*

Then $\underline{X}$ is the solution of SYSTEM($\underline{U}$,A,$\underline{C}$).

3 Solving Problem

In order to solve the SYSTEM problem we will use theorem 2 seen in the previous courses.But the hypothesises of the theorem are not all satisfied.In fact the derivative of U is not undefiled in 0. So we will find another border for $\underline{X}$:

$$\exists \underline{X}^*\ and\ let\ b =\ min_r(\frac{X_r^*}{2})$$

• WHAT IS $\underline{X}^*$? (THE GLOBAL MAXIMUM) WHY DOES IT EXIST?
$x_r^* > 0$ $\forall r$. WHY?

OTHERWISE THERE IS NO ADVANTAGE TO CONSIDER PROBLEM

So $\forall(r): \ X_r \geq b > 0$

Coming back to the SYSTEM problem with the new constraint border "b":

$$Maximize(\sum_r (W_r . \ln(X_r))$$

$$A.X^* \leq C^*$$

$$X_r \geq b$$

Now we are in the framework of theorem 2.

Let's define the Lagrangian function:

$$\mathcal{L}_S = \sum_r U_r(X_r) + \sum_l \vartheta_l (C_l - \sum_{r/l \in r} X_r)$$

We have:

$$\frac{\partial Ls}{\partial X_r} = 0$$

WHY?

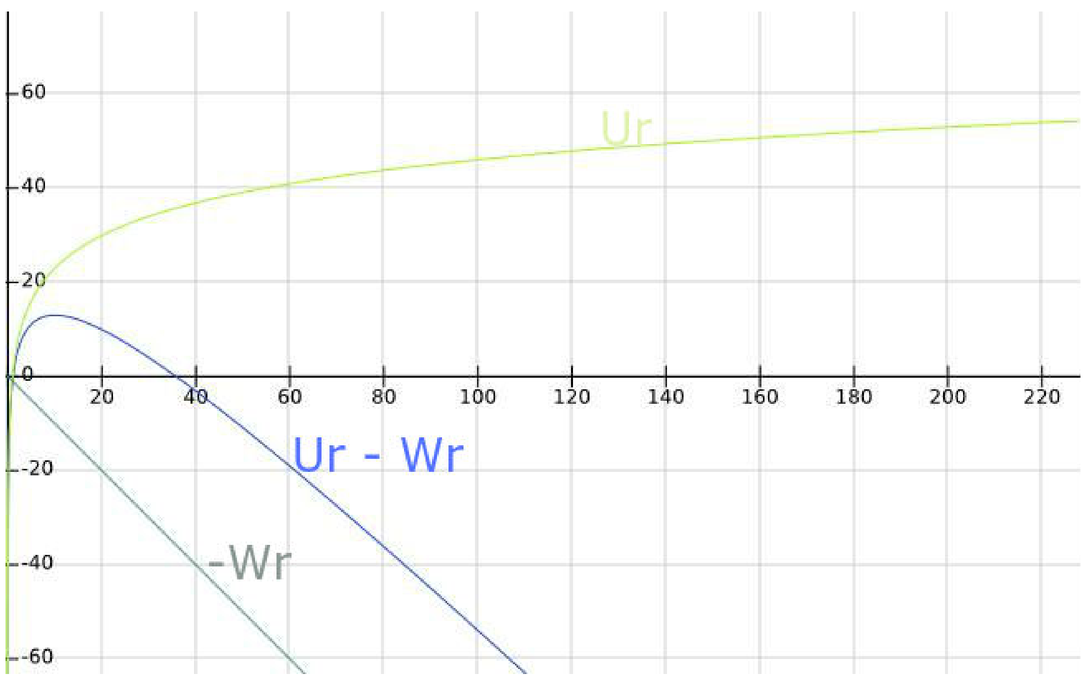
then

$$U_r(X_r)' + \sum_{l/l \in r} \vartheta_l = 0$$

Now let's try to solve the two problems, USER and NETWORK:

USER:

All the conditions of theorem 1 are now verified so:



W_r

Figure 3: The plot of Utility function of the User

$$\frac{\partial L_r}{\partial W_r} = \frac{\partial (U_r(\frac{W_r}{\lambda_r}) - W_r)}{\partial W_r} = 0$$

then

$$\frac{1}{\lambda_r}.U_r(\frac{W_r}{\lambda_r})' - 1 = 0$$

So

$$\lambda_r = U_r(\frac{W_r}{\lambda_r})'$$

Since $U_r(\frac{W_r}{\lambda_r})'$ is monotone then $(U_r(\frac{W_r}{\lambda_r})')^{-1}$ exists and we have:

WHY?

$$(U_r(\lambda_r)')^{-1} = \frac{W_r}{\lambda_r}$$

Thus

$$W_r = \lambda_r.(U_r(\lambda_r)')^{-1} \quad (equation : 0)$$

NETWORK: Once again we are in the form~~s~~ of theorem ~~1~~ ? Let's define

NO. THE PROBLEM IS DIFFERENT BUT WE CAN STILL STUDY IT WITH THE SAME APPROACH BECAUSE ...

$$\mathcal{L}_N = L(\underline{X}, \underline{\mu}) = \sum_r W_r. \ln(X_r) + \sum_l \mu_l.(C_l - \sum_{r/l \in r} X_r)$$

the Lagrangian of the network

We have (according to theorem ~~1~~ 2)

$$\frac{\partial L_N}{\partial X_r} = 0$$

So:

$$\frac{W_r^*}{X_r^*} - \sum_{l \in r} \mu_l^* = 0, \mu_l^* \geq 0 \quad (equation : 1)$$

But we have

$$W_r^* = X_r^*.\lambda_r^* \quad (equation : 2)$$

Thus (equation 1 + equation 2)

$$\lambda_r^* = \sum_{l \in r} \mu_l^* \quad (equation : 3)$$

THE UNDERLYING REASONING IS MISSING

Then we get the fourth equation:
equation 0 + equation 2 + equation 3 == equation 4

$$U_r{}'(X_r) = \sum_{l \in r} \mu_l^* \quad (equation : 4)$$

A practical way to solve the optimization problem is to do the penalty approach. The penalty method replaces a constrained optimization problem by a series of unconstrained problems whose solutions ideally converge to the solution of the original constrained problem