PhD course on Network Science

Module 2:

Albert-Barabasi's model

For power law networks

Power Law

- Where does it come from?
  - Albert-Barabasi's growth model
  - Highly Optimized Model
  - And other models
    - See Michael Mitzenmacher, A Brief History of Generative Models for Power Law and Lognormal Distributions

Albert-Barabasi's model

- Two elements
  - Growth
    - m_0 initial nodes, every time unit we add a new node with m links to existing nodes
  - Preferential attachment
    - The new node links to a node with degree k_i with probability

$$\Pi(k_i) = \frac{k_i}{\sum_{j=1,N} k_j}$$

The rich becomes richer

Albert-Barabasi's model

- Node i arrives at time t_i, its degree keeps increasing
- With a continuum approximation:

$$\frac{\partial k_i}{\partial t} = \frac{mk_i}{\sum_{j=1,N} k_j} = \frac{mk_i}{2tm} = \frac{k_i}{2t} \longrightarrow k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta}, \beta = \frac{1}{2}$$

- Then degree distribution at time t is:

$$P(k_i(t) < k) = P(t_i > t\,\frac{m^{1/\beta}}{k^{1/\beta}})$$

Albert-Barabasi's model

- At time t there are m_0+t nodes, if we consider that the t nodes are added uniformly at random in [0,t], then

$$P(t_i > x) = \frac{t - x}{t + m_0}$$

$$P(k_i(t) < k) = \frac{t}{t + m_0}\left(1 - \frac{m^{1/\beta}}{k^{1/\beta}}\right)$$

Albert-Barabasi's model

- The PDF is

$$P(k_i(t)=k)=\frac{\partial P(k_i(t)\le k)}{\partial k}=\frac{t}{t+m_0}\frac{1}{\beta}\frac{m^{1/\beta}}{k^{1/\beta+1}}$$

- For t->∞

$$P(k_i(t)=k)\xrightarrow[t\to\infty]{}\frac{1}{\beta}\frac{m^{1/\beta}}{k^{1/\beta+1}}\propto k^{-\gamma},\ \ \gamma=3$$

Albert-Barabasi's model

- If $\Pi(k_i) \propto a + k_i$, $P(k) \propto k^{-\gamma}$, $\gamma = 3 + \frac{a}{m}$

- Other variants:
  - With fitness $\Pi(k) = \frac{\eta_i k_i}{\sum_{j=1,N} \eta_j k_j}$
  - With rewiring (a prob. p to rewire an existing connection)
  - Uniform attaching with "aging": A vertex is deactivated with a prob. proportional to $(k_i+a)^{-1}$

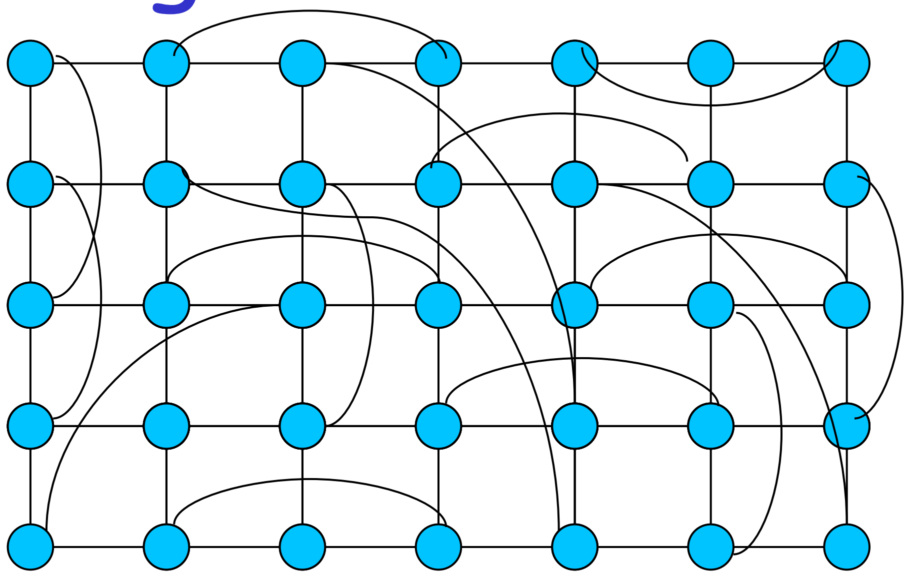
PhD course on Network Science

Module 3:
Navigation

Navigation

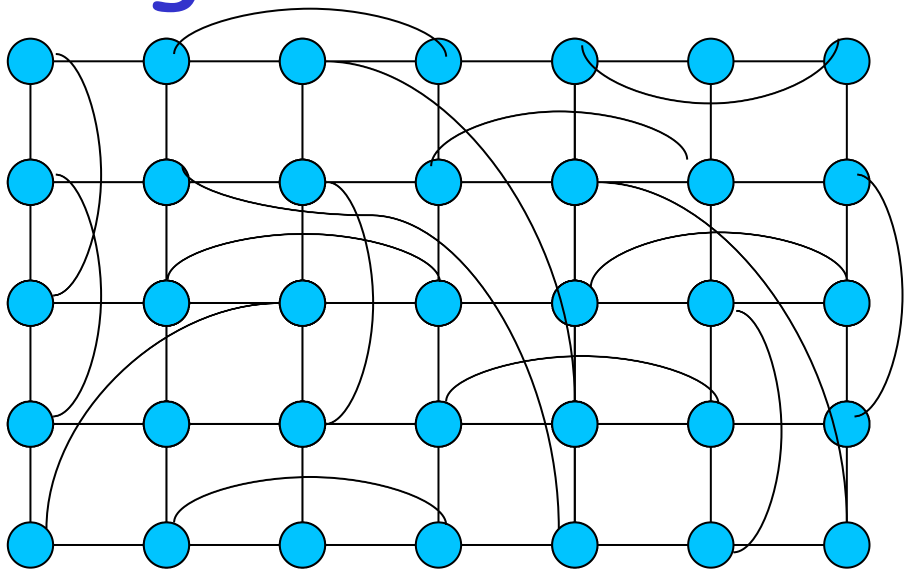
- In Small world nets there are short paths $O((\log(N))^{a})$
- But can we find them?
  - Milgram's experiment suggests nodes can find them using only local information
  - Standard routing algorithms require $O(N)$ information

Kleinberg's result



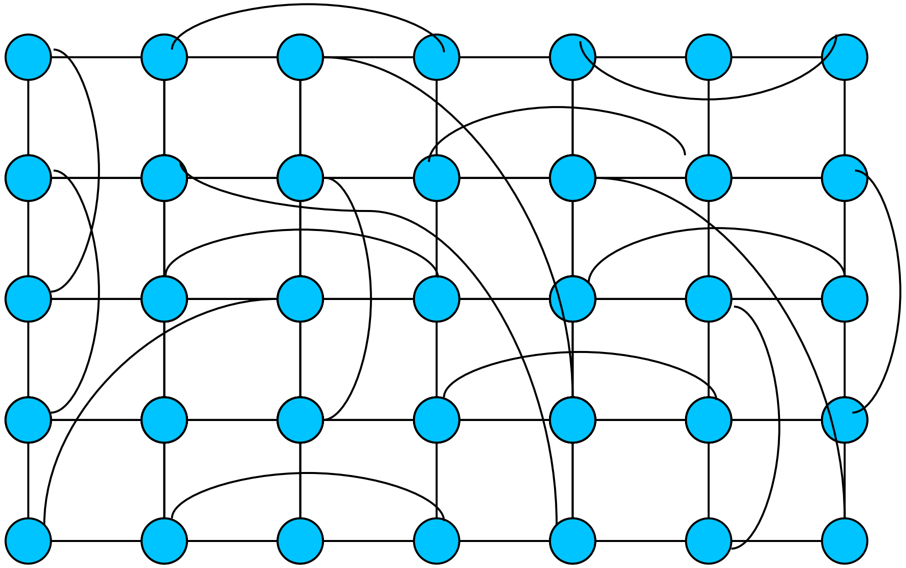
- Model: Each node has
  - Short-range connections
  - 1 long-range connection, up to distance r with probability prop. to $r^{-\alpha}$
  - For $\alpha=0$ it is similar to Watts-Strogatz model: there are short-paths

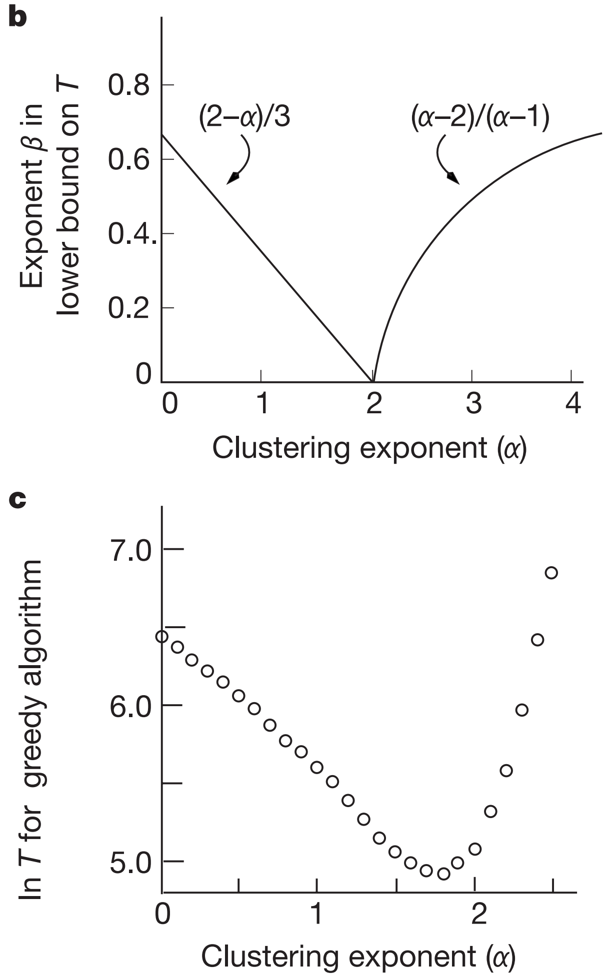
Kleinberg's result



- If α=2 the greedy algorithm (forward the packet to the neighbor with position closest to the destination) achieves avg path length $O((\log(N))^2)$

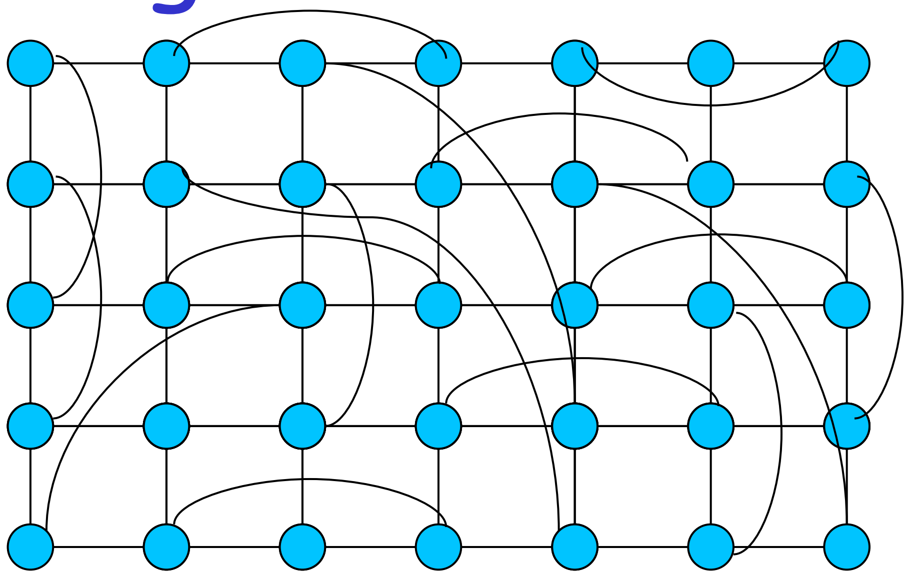
Kleinberg's result





- If α<>2 no local information algorithm can take advantage of small world properties
  - avg path length $\Omega(N^{\beta/2})$
    - where $\beta=(2-\alpha)/3$ for $0<=\alpha<=2$, $\beta=(\alpha-2)/(\alpha-1)$, for $\alpha>2$

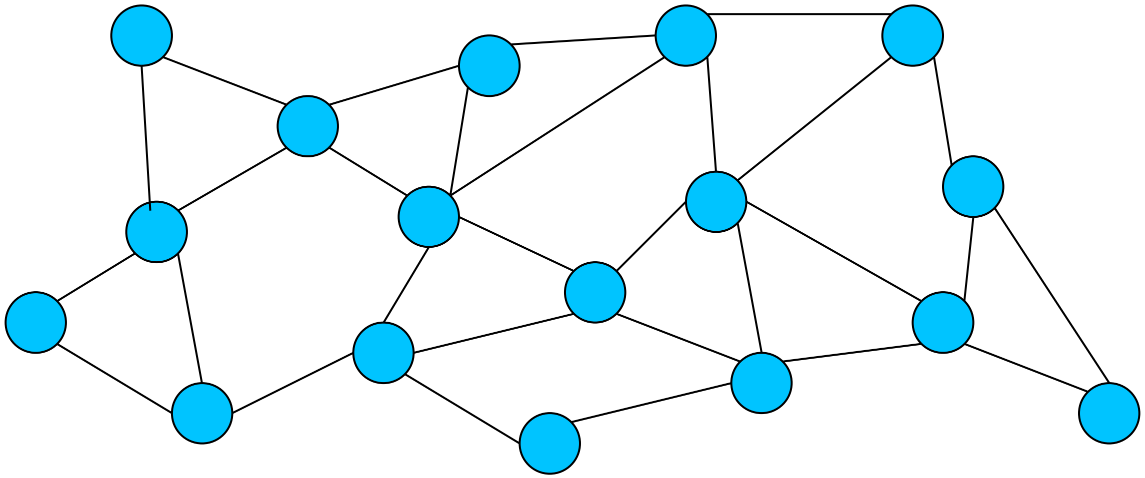
Kleinberg's result



- Conclusions
  - The larger α the less distant long-range contacts move the message, but the more nodes can take advantage of their "geographic structure"
  - α=2 achieved the best trade-off

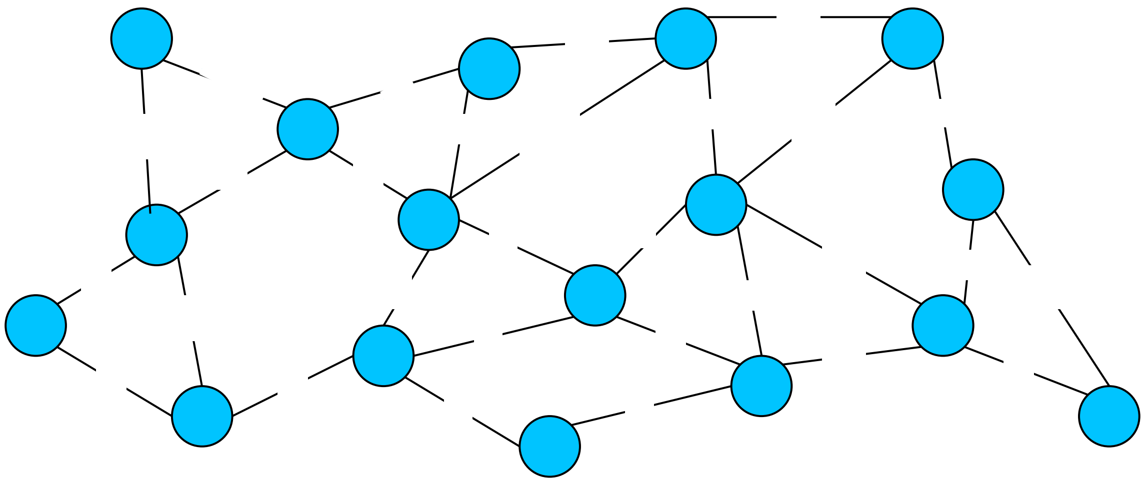
Configuration model

- A family of random graphs with given degree distribution



Configuration model

- A family of random graphs with given degree distribution
  - Uniform random matching of stubs



Configuration model

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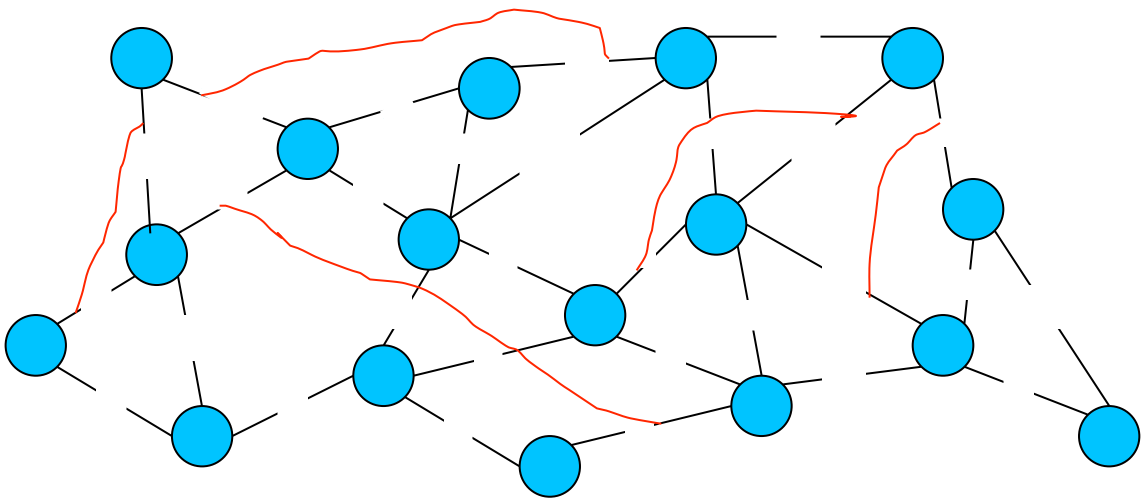
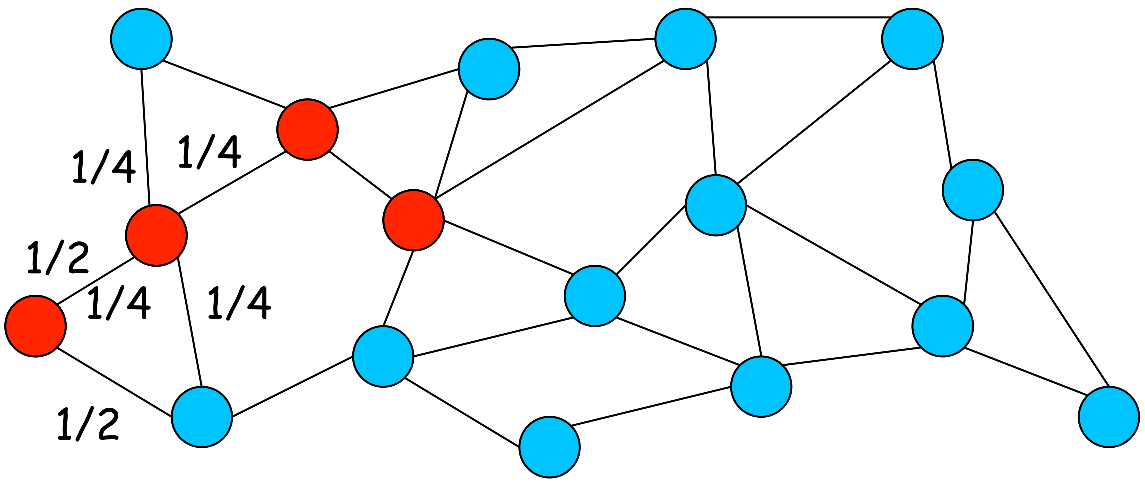


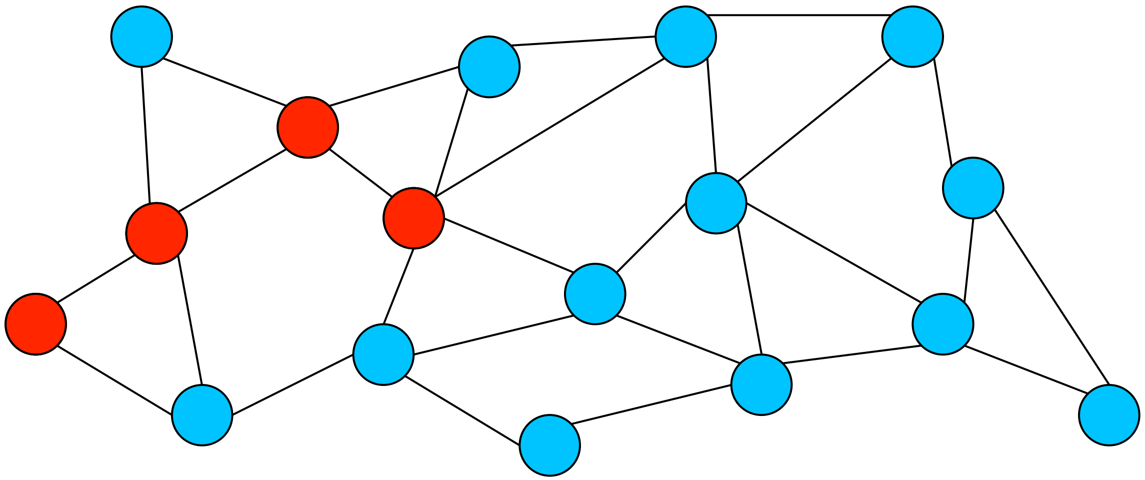
Back to Navigation: Random Walks

- What can we do in networks without a geographical structure?
  - Random walks



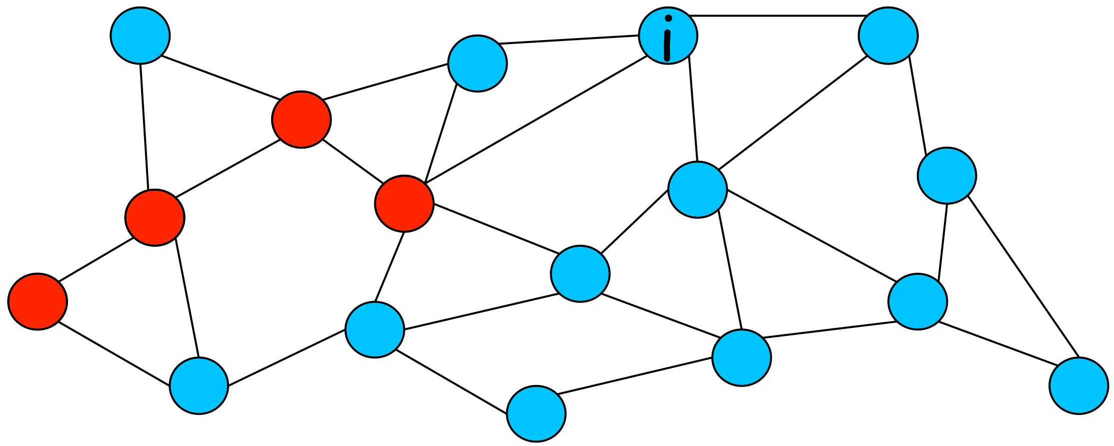
Back to Navigation: Random Walks

- How much time is needed in order to reach a given node?



Random Walks: stationary distribution

- $$\pi_i = \sum_{j \in N_i} \frac{1}{k_j} \pi_j$$
- $$\pi_i = \frac{k_i}{\sum_{i=1}^{N} k_j} = \frac{k_i}{2M}$$



- avg time to come back to node i starting from node i: $$\frac{1}{\pi_i} = \frac{2M}{k_i}$$
- Avg time to reach node i
  - intuitively $\approx \Theta(M/k_i)$

Another justification

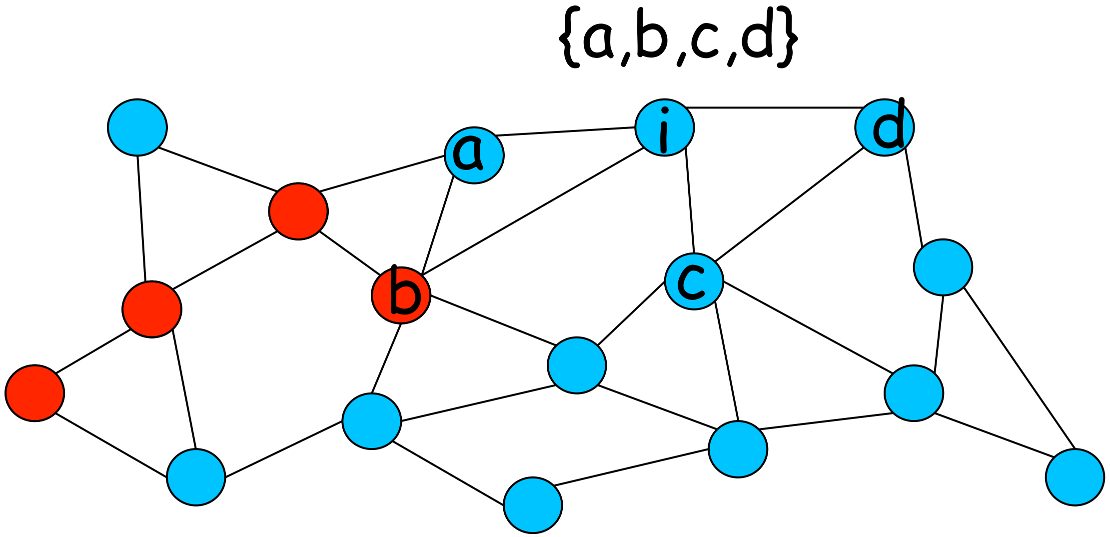
- Random walk as random edge sampling
  - Prob. to pick an edge (and a direction) leading to a node of degree k is $\frac{kp_k}{<k>}$
  - Prob. to arrive to a given node of degree k:

$$\frac{kp_k}{p_k N <k>} = \frac{k}{2M}$$

  - Avg. time to arrive to this node 2M/k
- …equivalent to a RW where at each step we sample a configuration model

Distributed navigation
(speed up random walks)

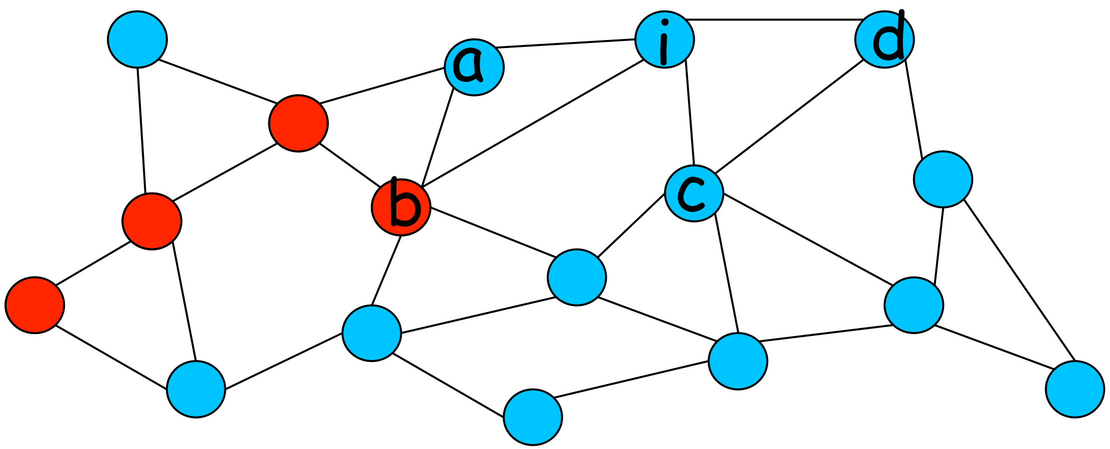
- Every node knows its neighbors



Distributed navigation
(speed up random walks)

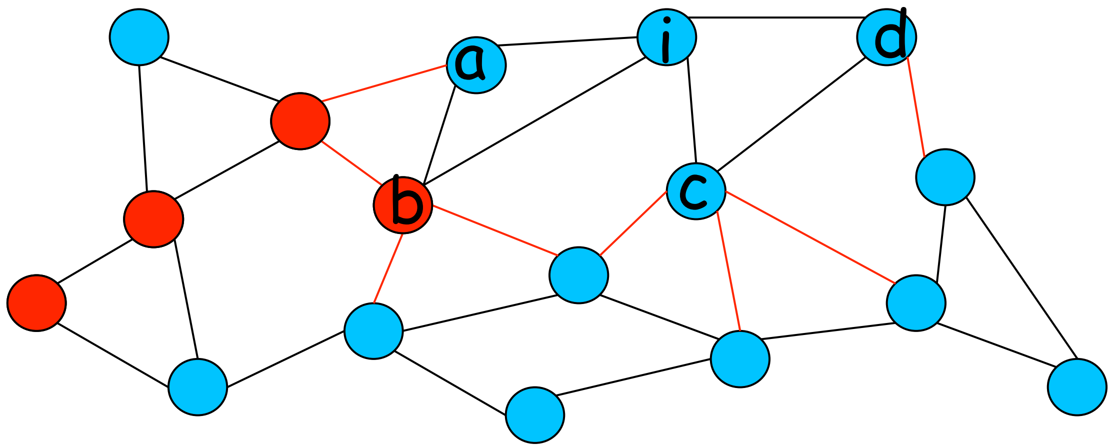
- Every node knows its neighbors
- If a random walk looking for *i* arrives in *a* the message is directly forwarded to *i*

{a,b,c,d}



Distributed navigation
reasoning 1

- We discover *i* when we sample one of the links of *i*'s neighbors
- Avg # of these links: $k_i \sum_k \left((k-1) \frac{k p_k}{<k>} \right) = k_i \left(\frac{<k^2>}{<k>} - 1 \right)$
- Prob. to arrive at one of them: $\frac{k_i}{2M} \left(\frac{<k^2>}{<k>} - 1 \right)$



Distributed navigation
reasoning 2

- Prob that a node of degree k is neighbor of node *i* given that RW arrives to this node from a node different from *i*

$$1-\left(1-\frac{k_i}{2M}\right)^{k-1} \approx \frac{k_i(k-1)}{2M}$$

- Prob that the next edge brings to a node that is neighbor of node i:

$$\sum_k \frac{k_i(k-1)}{2M}\frac{kp_k}{<k>} = \frac{k_i}{2M}\left(\frac{<k^2>}{<k>}-1\right)$$

Distributed navigation

- Avg. Hop# $\dfrac{2M}{k_i}\dfrac{<k>}{<k^2>-<k>}$
  - Regular graph with degree d: $\dfrac{2M}{d(d-1)}$
  - ER with <k>: $\dfrac{2M}{k_i(<k>-1)}$
  - Pareto distribution $\left(P(k)\approx\dfrac{\alpha x_m^{\alpha}}{x^{\alpha+1}}\right)$:

$$\approx\frac{2M}{k_i}\frac{(\alpha-2)(\alpha-1)}{x_m-(\alpha-2)(\alpha-1)}$$ If α->2…

Distributed navigation

- Application example:
  - File search in unstructured P2P networks through RWs