Chapter 6

Matching in Graphs

Let G be a graph. Two edges are *independent* if they have no common endvertex. A set M of independent edges of G is called a *matching*. The *matching number*, denoted $\mu(G)$, is the maximum size of a matching in G.

In this chapter, we consider the problem of finding a *maximum matching*, i.e. with maximum size. In particular, we will try to characterise the graphs G that admit a *perfect matching*, i.e. a matching covering all vertices of G.

Let M be a matching. The vertices that are incident to an edge of M are *matched* or *covered* by M. If U is a set of vertices covered by M, then we say that M *saturates* U. The vertices which are not covered are said to be *exposed*.

Let $G = (V,E)$ be a graph and M a matching. An M-*alternating* path in G is a path whose edges are alternatively in $E \setminus M$ and in M. An M-alternating path whose two endvertices are exposed is M-*augmenting*. We can use an M-augmenting path P to transform M into a greater matching (see Figure 6.1). Indeed, if P is M-alternating, then the symmetric difference between M and $E(P)$

$$M' = M \triangle E(P) = (M \setminus (E(P) \cap M) \cup (E(P) \setminus M)$$

is also a matching. Its size $|M'|$ equals $|M| - 1 + x$ where x is the number of exposed ends of P.

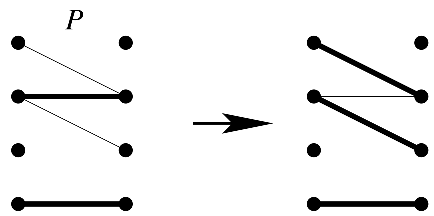


Figure 6.1: Getting of a greater matching from an augmenting path P. Bold edges are the one of the matching.

Hence if M is maximum, there are no augmenting paths. In fact, as shown by Berge, this necessary condition is also sufficient.

Theorem 6.1 (Berge 1957)**.** *Let M be a matching in a graph G. Then M is maximum if and only if there are no M-augmenting paths.*

Proof. Necessity was shown above so we just need to prove sufficiency. Let us assume that M is not maximum and let M' be a maximum matching. The symmetric difference $Q = M \triangle M'$ is a subgraph with maximum degree 2. Its connected components are cycles and paths where the edges of M and M' alternate. Hence, the cycles have even length and contain as many edges of M and of M'. Since M' is greater than M, Q contains at least one path P that contains more edges of M' than of M. Therefore, the first and the last edges of P belong to M', and so P is M-augmenting.

□

6.1 Matching in bipartite graphs

Let $G = ((A,B),E)$ be a bipartite graph. If $|A| \leq |B|$, the size of maximum matching is at most $|A|$. We want to decide whether it exists a matching saturating A. If there is such a matching M, then, for any subset S of A, the edges of M link the vertices of S to as many vertices of B.

Hence, we have a necessary condition, known as *Hall's c Condition*, for the existence of a matching saturating A:

$$|N(S)| \geq |S| \qquad \text{for all } S \subseteq A \tag{6.1}$$

where $N(S)$ is the set of vertices of $G \setminus S$ adjacent to at least one vertex of S. The set $N(S)$ is called the *neighbourhood* of S: $N(S) = \bigcup_{s \in S} N(s) \setminus S$.

Actually, this condition is sufficient.

Theorem 6.2 (Hall 1935)**.** *Let $G = ((A,B),E)$ be a bipartite graph. G has a matching saturating A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.*

We give two proofs of this theorem. The first one uses some basic arguments, while the second one is based on augmenting paths.

First proof: By induction on $|A|$, the result holding trivially for $|A| = 1$. Let us assume that $|A| \geq 2$ and that Condition (6.1) is sufficient for any matching saturating A' with $|A'| < |A|$.

i) Assume first that $|N(S)| \geq |S| + 1$ for every non-empty proper subset S of A. Let $ab \in E(G)$ and consider $G' = G - \{a,b\}$. Then, any subset $S \subseteq A \setminus \{a\}$ satisfies:

$$|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S|.$$

By the induction hypothesis, G' has a matching M' saturating $A \setminus \{a\}$. Hence, the matching $M' \cup \{ab\}$ is a matching of G saturating A.

ii) Assume now that there is a non-empty proper subset A' of A such that $|N(A')| = |A'|$. By the induction hypothesis, the subgraph G' induced by $A' \cup N(A')$ admits a matching M'

saturating A'.
Let $G'' = G - G'$. For any set $S \subseteq A \setminus A'$,

$$|N_{G''}(S)| \geq |N_G(S \cup A')| - |N_G(A')| \geq |S \cup A'| - |A'| \geq |S|.$$

Hence, by the induction hypothesis, G'' admits a matching M'' saturating $A \setminus A'$. The union $M' \cup M''$ is a matching of G saturating A. □

Second proof: The proof is algorithmic. Given a matching M with maximum size which does not cover a_0, it returns a set $S \subseteq A$ such that $|N(S)| < |S|$. Let $A_0 = \{a_0\}$ and $B_0 = N(a_0)$. Note that all vertices of B_0 are covered (if $b_0 \in B_0$ is not covered, the edge a_0b_0 can be added to the matching). If $B_0 = \emptyset$, $S = A_0$ is a set such that $|N(S)| < |S|$ and the algorithm terminates. Else, B_0 is matched with $|B_0|$ vertices of A distinct from a_0. We set $A_1 = N_M(B_0) \cup \{a_0\}$, where $N_M(B_0)$ is the set of vertices matched with vertices of B_0. We have $|A_1| = |B_0| + 1 \geq |A_0| + 1$. Let $B_1 = N(A_1)$. Again, no vertices in B_1 is exposed, otherwise there is an M-augmenting path. If $|B_1| < |A_1|$, the algorithm terminates with $|N(A_1)| < |A_1|$. If not, let $A_2 = N_M(B_1) \cup \{a_0\}$. Then$|A_2| \geq |B_1| + 1 \geq |A_1| + 1$. And so on, the following algorithm is executed until it terminates.

Algorithm 6.1 (Finding a set violating Hall's Condition)**.**

1. $A_0 := \{a_0\}$, $i := 0$;
2. If $|B_i| = |N(A_i)| < |A_i|$, terminate and return A_i;
3. Else, do $A_{i+1} := A_i \cup N_M(B_i)$, $i := i + 1$ and go to Step 2.

The algorithm eventually terminates because the sequence $|A_i|$ is strictly increasing. Hence, it returns a set $S \subseteq A$ such that $|N(S)| < |S|$. □

Hall's Condition (6.1) applies for matching saturating A but it can be generalised for matching of any size.

Theorem 6.3. *Let $G = ((A,B),E)$ be a bipartite graph and $k \in \mathbb{N}$. G has a matching of size k if and only if $|N(S)| \geq |S| - |A| + k$ for any $S \subseteq A$.*

Proof. Let us add $|A| - k$ new vertices to B, each of them being linked to all vertices of A. Then, in this new graph, $|N(S)| \geq |S|$ for any $S \subset A$. Thus, by Hall's Theorem (6.2), the new graph has a matching saturating A. But at most $|A| - k$ edges (the ones incident to a new vertex) are not in G. Hence G has a matching of size at least k. □

Corollary 6.4. *If $G = ((A,B),E)$ is a k-regular bipartite graph ($k \geq 1$), then G has a perfect matching.*

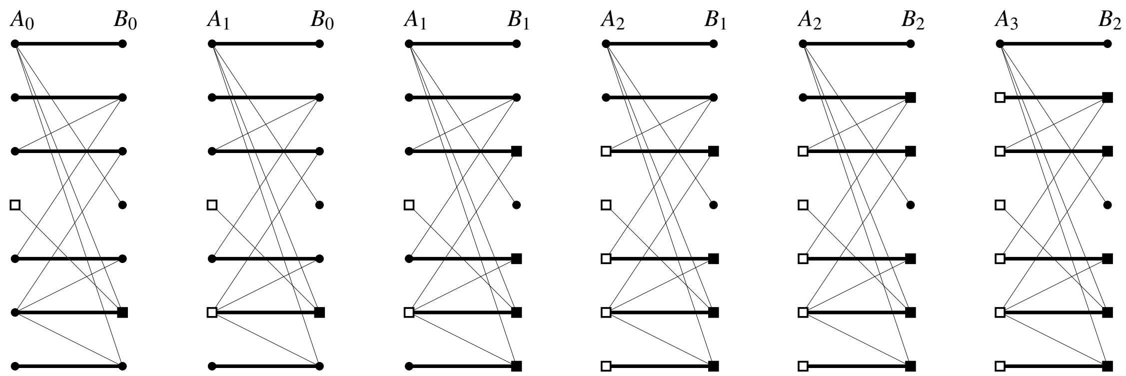


Figure 6.2: A run of Algorithm 6.1. The bold edges are those of the maximum matching. The vertices of A_i (resp. B_i) are represented by white (resp. black) squares.

Proof. If G is k-regular, then clearly $|A| = |B|$. So every matching saturating A is perfect.

Let $S \subseteq A$. The set S is incident to $k|S|$ edges that belong to the $k|N(S)|$ edges incident to $N(S)$. Hence, $k|S| \leq k|N(S)|$. Thus, G satisfies Hall's Condition (6.1) and so by Theorem 6.2, admits a matching saturating A, which is a perfect matching. □

Using the same method as in the second proof of Hall's Theorem, we give an algorithm which, given a bipartite graph $((A,B),E)$ computes either a matching saturating A or a set S such that $|N(S)| < |S|$. This algorithm, known as the *hungarian method*, is based on the sequence $a_o, b_1, a_1, b_2, a_2, \dots$ viewed as a tree T.

Algorithm 6.2 (Hungarian Method)**.**

0. Start with any matching M.

1. If M saturates A, then return M. Else, let $a_0 \in A$ be an exposed vertex. Let T be the tree consisting of the single vertex a_0. Let $A' = V(T) \cap A$ and $B' = V(T) \cap B$.

2. If $N(A') = B'$ then $|N(A')| < |A'|$ because $|A'| = |B'| + 1$. Return $S = A'$. Else, let $b \in N(A') \setminus B'$ and a' one of its neighbours in A'.

3. If b is covered by M, say with a, then add the edges $a'b$ and ba to T. (See Figure 6.3). Then A' becomes $A' \cup \{a\}$ and B', $B' \cup \{b\}$. Go to Step 2. Else P, the (a_0, b)-path in T, is an M-augmenting path. Replace M by $M \triangle E(P)$. Go to Step 1.

6.2 Matching and vertex-cover

We now show a duality theorem for the maximum matching in bipartite graphs.

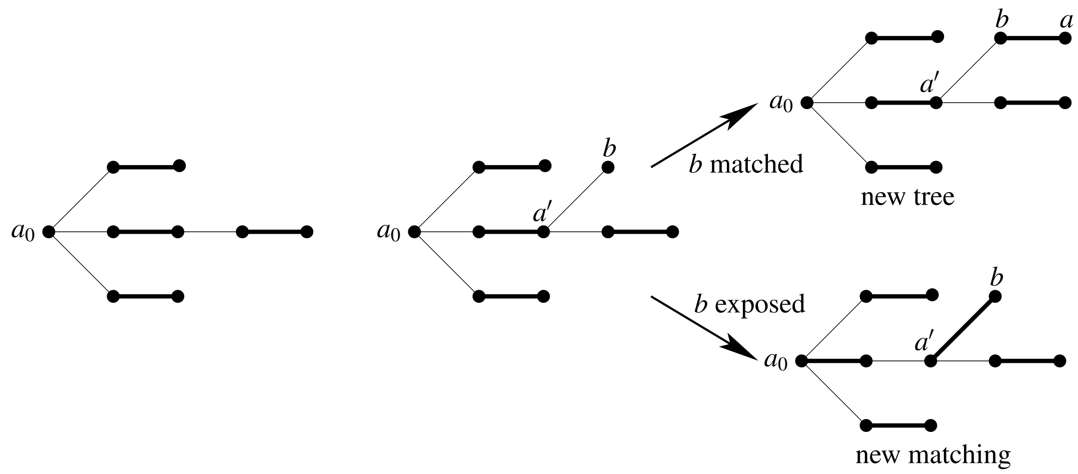


Figure 6.3: Step 3 of the Hungarian Method

Definition 6.5. Let $G = (V, E)$ be a graph. A set $K \subset V$ is a *vertex-cover* of E if any edge of G is incident to a vertex in K. The *vertex-cover number* of G, denoted $\tau(G)$, is the minimum size of a vertex-cover of G.

Let K be a vertex-cover of a graph. Then, for any matching M, K contains at least one endvertex of each edge of M. Hence, $|M| \le |K|$. So, the maximum size of a matching is at most the minimum size of a vertex-cover. For a bipartite graph, they are equal.

Theorem 6.6 (König 1931, Egerváry 1931)**.** *Let $G = ((A, B), E)$ be a bipartite graph. The size of a maximum matching equals the size of a minimum vertex-cover, that is*

$$\mu(G) = \tau(G).$$

Proof. Let M be a maximum matching, let U be the set of exposed vertices in A, and let V' be the set of vertices of G linked to U using M-alternating paths. Let $A' = A \cap V'$ and $B' = B \cap V'$. See Figure 6.4. The set B' is saturated by M. Indeed, if a vertex $b \in B'$ is not matched, then the M-alternating path that links b to a vertex in U is an M-augmenting path, contradicting the maximality of M (Theorem 6.1). Moreover, $N(A') = B'$ by definition of V'. Let $K = B' \cup (A \setminus A')$ Then, any edge of G has an endvertex in K. Therefore, K is a vertex-cover of G. But $|K| = |M|$ because $A \setminus A'$ is the set of vertices in A that are matched with some vertices in $B \setminus B'$. □

Hall's Theorem (6.2) can be deduced from this theorem.

Proof of Hall's Theorem (6.2): If G has no matching saturating A, then by Theorem 6.6, there is a vertex-cover K with less than $|A|$ vertices. Let $A' = A \cap K$ and $B' = B \cap K$. Then, $|A'| + |B'| = |K| < |A|$ and

$$|B'| < |A| - |A'| = |A \setminus A'|$$

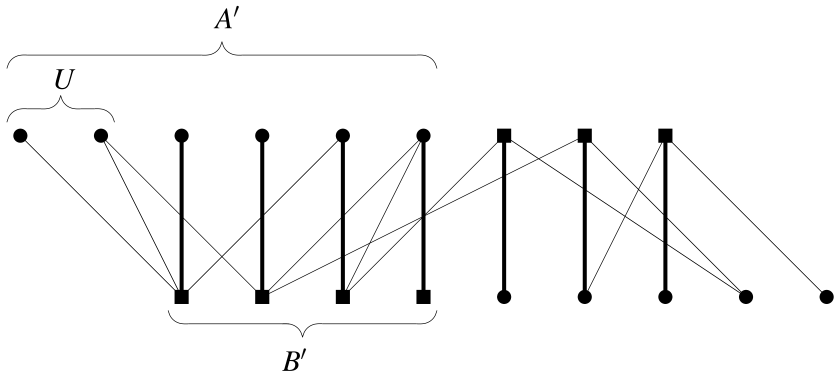


Figure 6.4: Finding a minimum vertex-cover (squares) from a maximum matching (bold edges).

By definition of a vertex-cover, there are no edges between $A \setminus A'$ and $B \setminus B'$, hence

$$|N(A \setminus A')| \leq |B'| < |A \setminus A'|$$

The set $A \setminus A'$ does not satisfies Hall's Condition (6.1). $\square$

An *edge-cover* of a graph G is a set of edge $F \subset E(G)$ such that every vertex $v \in V(G)$ is incident to an edge $e \in F$. Note that an edge-cover can exist only if G has no isolated vertices. The *edge-cover number* of G, denoted $\rho(G)$, is the minimum size of an edge-cover of G.

Theorem 6.7 (Gallai, 1959)**.** *Let $G = (V, E)$ be a graph without isolated vertices. Then*

$$\alpha(G) + \tau(G) = |V| = \rho(G) + \mu(G).$$

Proof. By definition, S is a stable set if and only if $V \setminus S$ is a vertex cover. Hence $\alpha(G) + \tau(G) = |V|$.

Let M be a matching of size $\mu(G)$. For each of the $|V| - 2|M|$ vertices v missed by M, add to M and edge incident to v. We obtain an edge-cover of size $|M| + (|V| - 2|M|) = |V| - |M|$. Hence $\rho(G) \leq |V| - \mu(G)$. Let F be an edge-cover of size $\rho(G)$. For each $v \in V$ delete from F, $d_F(v) - 1$ edges incident to v. We obtain a matching of size at least $|F| - \sum_{v \in V}(d_F(v) - 1) = |F| - (2|F| - |V|) = |V| - |F|$. Hence $\mu(G) \geq |V| - \rho(G)$. $\square$

Gallai's Theorem (6.7) and König-Egerváry Theorem (6.6) immediately imply the following.

Corollary 6.8 (König)**.** *Let G be a bipartite graph without isolated vertices. Then* $\alpha(G) = \rho(G)$.

6.3 Maximum-weight matching

Previous results on maximum matching in bipartite graphs can be generalized to maximum-weight matching in bipartite edge-weighted graphs. Note that, we may assume weights are non-negative.

The notion of vertex-cover can be generalized to edge-weighted graphs

Definition 6.9. Let G be an edge-weighted graph. A *fractional vertex-cover* is a function $c : V(G) \to \mathbb{R}^+$ such that, for any edge xy of G, $c(x)+c(y) \geq w(xy)$. The *weight* of a fractional vertex-cover c of G is $c(G) = \sum_{v \in V(G)} c(v)$.

Let M be a matching and c be a fractional vertex-cover in an edge-weighted graph (G,w). Then, the weight of c is at least the weight of M. Indeed, by summing all inequalities $c(x)+c(y) \geq w(xy)$ over all edges of M, we get $c(G) \geq w(M)$.

König-Egerváry Theorem is generalised to the weighted case:

Theorem 6.10. *In bipartite edge-weighted graph, the minimum weight of a fractional vertex-cover equals the maximum weight of a matching.*

Proof. Let (G,w) be a bipartite edge-weighted graph. Free to add edges of weight 0, we may assume that $G = K_{n,n}$.

We provide an algorithm to finding a matching M and a fractional vertex-cover c with same weight. Note that, for any edge xy of M, we have $c(x)+c(y) = w(xy)$.

Definition 6.11. Let c be a fractional vertex-cover of (G,w). The *excess* of an edge xy is $c(x)+c(y)-w(xy)$. The *equality graph* of c, denoted by G_c, is the graph induced by the edges of excess 0.

Algorithm 6.3.

0. Initialization: Take the fractional vertex-cover c defined by $c(a) := \max\{w(ab), ab \in E\}$ if $a \in A$ and $c(b) := 0$ if $b \in B$.
1. Find a maximum matching M in G_c.
2. If M is perfect in G, return "the maximum-weight matching is" M "and the minimum-weight fractional vertex-cover is" c.
3. Else, let K be a fractional vertex-cover with size $|M|$ in G_c. Let $R = A \cap K$ and $T = B \cap K$. Let $\varepsilon = \min\{c(a)+c(b)-w(ab) \mid a \in A \setminus R, b \in B \setminus T\}$. For any $a \in A \setminus R$, $c(a) := c(a)-\varepsilon$ and for any $b \in B \setminus T$, $c(b) := c(b)+\varepsilon$. Go to 1.

□

6.4 Matching in general graphs

Given a graph G, let $odd(G)$ denote the number of odd (i.e. with an odd number of vertices) connected components of G. Clearly, if G admits a perfect matching, then

$$odd(G-S) \leq |S| \qquad \text{for all} S \subset V(G).$$

Indeed, if G has a perfect matching, each odd component C of $G-S$ contains at least one vertex matched with a vertex not in C. Moreoever, this vertex must belong to S since there are no edges between distinct components of $G \setminus S$.

This necessary condition is actually sufficient.

Theorem 6.12 (Tutte 1947)**.** *A graph G admits a perfect matching if and only if $imp(G-S) \leq |S|$ for any $S \subset V(G)$.*

Proof. Let $G = (V,E)$ be a graph with no perfect matching. We will show that there is $S \subset V(G)$ such that $odd(G-S) > |S|$.

Let G' be the supergraph G with no perfect matching which has the maximum number of edges. For any S, a component of $G'-S$ is the union of components of $G-S$. Hence, an odd component of $G'-S$ contains an odd component of $G-S$. Hence, it is sufficient to find a *bad* set S, i.e., for which $odd(G'-S) > |S|$.

If G' contains a bad set, clearly it satisfies the Property ($\star$).

Property ($\star$): *S is complete, every component of $G'-S$ is complete, and every vertex of S is adjacent to all vertices of $G'-S$.*

Conversely, if a set satisfies Property ($\star$) then S or $\emptyset$ is bad. Indeed, if $\emptyset$ is not bad, then $|V(G)|$ is even. If, in addition, S is not bad, then it is possible to matched (independently) a vertex of each odd component with a vertex in S and complete the matching in a perfect matching of the graph.

A vertex is *universal* if it is adjacent to every vertex but itself. Let S be the set of universal vertices in G'. We shall prove that S satisfies ($\star$) and, so S is bad.

For sake of contradiction, assume that S does not satisfy ($\star$). Then, a component of $G-S$ contains two non-adjacent vertices a and a'. Let a, b and c be the first three vertices in a shortest (a,a')-path in this component. Then, $ab, bc \in E(G')$ and $ac \notin E(G')$. Since b is not in S, a vertex of G', say d, ins not adjacent to b. By maximality of G', $G' \cup ac$ contains a perfect matching M_1 and $G' \cup bd$ has a perfect matching M_2. Note that $ac \in M_1$ and $bd \in M_2$ otherwise M_1 or M_2 would be a perfect matching in G'.

The graph induced by $M_1 \cup M_2$ is the union of even cycles alternating edges of M_1 and M_2. Let C be the cycle containing bd. If C does not contain the edge ac, then by replacing in M_2 the edges of $E(C) \cap M_2$ by the edges in $E(C) \cap M_1$, we get a perfect matching of G', a contradiction. If C contains ac, then, let P be the path in $C \setminus bd$ with start d and last edge ac. Without loss of generality, we may assume that a is the terminus of P. Let C' be the cycle obtained from $P \setminus a$ by adding the edges bc and bd. Replacing in M_2 the edges of $E(C') \cap M_2$ by the edges of $E(C') \cap M_2$, we get a perfect matching of G', a contradiction. $\square$

6.5 Path-cover of digraphs

Consider a bipartite graph $G = ((A,B),E)$ and orient all its edges from A to B to get the digraph D. Then König-Egerváry's Theorem (Theorem 6.6) says how many disjoint directed paths are necessary to cover all vertices of D. Indeed, all directed paths have length 1 or 0, and the number of directed paths of such a "cover" is the smallest possible when it contains as many directed paths of length 1 as possible, i.e., a maximum matching.

In this section, we consider the following general question: given a digraph (non necessarily bipartite), how many paths are necessary to cover all its vertices?

Definition 6.13. A *path-cover* of a graph is a set of disjoint directed paths that cover all vertices. It can be viewed as a spanning forest every component of which is a directed path.

Recall that $\alpha(D)$ denotes the maximum size of a stable set in D.

Theorem 6.14 (Gallai and Milgram 1960)**.** *Every digraph D has a path-cover of at most $\alpha(D)$ directed paths.*

Proof. If P is a directed path, let us denote by $ter(P)$ its terminus. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two path-covers of D. We say that $\mathcal{P}_1 < \mathcal{P}_2$ if $\{ter(P) \mid P \in \mathcal{P}_1\} \subset \{ter(P) \mid P \in \mathcal{P}_2\}$ and $|\mathcal{P}_1| < |\mathcal{P}_2|$.

Let us show by induction that, if $\mathcal{P}$ is a path-cover which is minimum for $<$, then there is a stable set intersecting every directed path of $\mathcal{P}$.

Let $\mathcal{P} = \{P_1, P_2, \dots, P_m\}$ be a path-cover minimum for $<$ in D. Let $v_i = ter(P_i)$ for any $1 \leq i \leq m$. If $\{v_i \mid 1 \leq i \leq m\}$ is a stable, then, the result holds. W.l.o.g., we may assume that (v_2, v_1) is an arc. Since the concatenation of P_2 and (v_2, v_1) is a directed path, by minimality of $\mathcal{P}$, v_1 is not the unique vertex of P_1. Let v be the vertex preceding v_1 in P_1 and set $P_1' = P_1 - v_1$. Then $\mathcal{P}' = \{P_1', P_2, \dots, P_m\}$ is a path-cover of $D - v$.

Let us show that $\mathcal{P}'$ is minimum for $<$ in $D - v$. Assume that there is a path-cover $\mathcal{Q}' < \mathcal{P}'$. If $\mathcal{Q}$ contains a directed path P with terminus v or v_2, then replacing P by the concatenation of P and (v, v_1) or (v_2, v_1), we get a path-cover $\mathcal{Q} < \mathcal{P}$, a contradiction. If not, then $\mathcal{Q}'$ consists of at most $m-2$ directed paths and $\mathcal{Q}' \cup \{(v_1)\} < \mathcal{P}$, a contradiction.

Hence, $\mathcal{P}'$ is minimum for $<$ in $D - v$. So, by the induction hypothesis, $D - v$ admits a stable set intersecting every directed path of $\mathcal{P}'$. Clearly, this stable set also intersects every directed path of $\mathcal{P}$. $\square$

Remark 6.15. In 1990, Hahn and Jackson [2] conjectured that this theorem is best possible in the following strong sense. *For each positive integer k, there is a digraph D with stability number k such that deleting the vertices of any $k-1$ directed paths in D leaves a digraph with stability number k.* This was recently proved by Fox and Sudakov [1].

Definition 6.16. An *ordered set* is a pair $(P, \leq)$ such that P is a set and $\leq$ is a binary relation over P that is *reflexive* (for all $x \in P$, $x \leq x$), *antisymmetric* (if $x \leq y$ and $y \leq x$ then $x = y$) and *transitive* (if $x \leq y$ and $y \leq z$ then $x \leq z$). Two elements x and y of P are *comparable* if $x \leq y$ or $y \leq x$. A *chain* of $(P, \leq)$ is a set of pairwise comparable elements, and an *antichain* of $(P, \leq)$ is a set of pairwise non-comparable elements.

Corollary 6.17 (Dilworth 1950)**.** *For any ordered set $(P, \leq)$, the minimal number of chains covering P equals the maximum size of an antichain.*

Proof. If A is an antichain of maximum size in $(P, \leq)$, then at least $|A|$ chains are necessary to cover P because any chain contains at most one element of A. To show that $|A|$ chains are sufficient, consider the digraph $D = (P, E_<)$ with $E_< = \{(x,y) | x \leq y \text{ and } x \neq y\}$. Clearly, there is a one-to-one correspondence between the chains of $(P, \leq)$ and the directed paths of D, and there is a one-to-one correspondence between the antichains of $(P, \leq)$ and the stable sets of D. Hence Theorem 6.14 applied to D yields the result. □

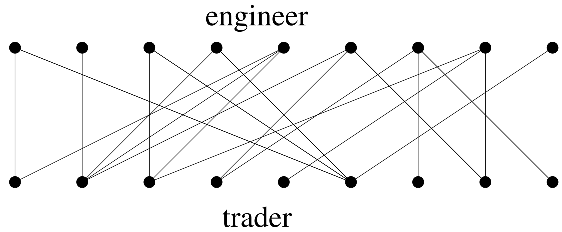
6.6 Exercises

Exercise 6.1. The french figure skating federation wants to form couples (one girl and one boy) for dance on ice in order to prepare the Olympic Winter Games. Six girls and six boys of high enough level volunteer. In view of temper incompatibility between some girls and some boys as well as aestethic criterias supposedly displeasing the judges (the federation does not have enough money to corrupt all judges), the following table has been designed. A cross in a square means that the two iceskaters cannot form a couple.

	Girl 1	Girl 2	Girl 3	Girl 4	Girl 5	Girl 6
Boy 1	×	×		×	×	
Boy 2						×
Boy 3	×			×	×	×
Boy 4	×				×	×
Boy 5		×	×			
Boy 6	×		×	×	×	

How many couples at most can the federation form? Justify your answer.

Exercise 6.2. A managing director has to launch the marketing of a new product. Several candidate products are at his disposal and he has to choose the best one. Hence, he let each of these products be analysed by a team made of an engineer and a trader who write a review together. The teams are made along the following graph; each edge corresponds to a product and its endvertices to the engineer and trader examining it.



How many people at least does the managing director gather in order to have the report on all the products? (The report can be given by either the engineer or the trader.)

Exercise 6.3.
1) Show that a tree has at most one perfect matching.
2) Show that a tree T has a perfect matching if and only if $odd(T-v)=1$ for all vertex v.

Exercise 6.4. Let $G=((A,B),E)$ be a bipartite graph.

1) Prove that the number of edges in a maximum matching is $|A|-\max_{A'\subset A}\{|A'|-|N(A')|\}$.

2) Deduce that if $|A|=|B|=n$ and $|E(G)|>(k-1)n$ then G has a matching of cardinality k.

Exercise 6.5. Let $G=((A,B),E)$ a bipartite graph such that $|N(S)|>|S|$ for all subset S of A distinct from $\emptyset$ and A. Show that every edge $e\in E(G)$ is in a matching saturating A.

Exercise 6.6. Let $G=((A,B),E)$ be a bipartite graph. Suppose that $S\subseteq A$, $T\subseteq B$ and G contains a matching M_1 saturating S and a matching M_2 saturating T. Prove that there exists a matching which saturates both S and T.

Exercise 6.7. Let $G=((A,B),E)$ be a bipartite graph such that $|A|=|B|=n$ and $\delta(G)\geq n/2$. Show that G has a perfect matching.

Exercise 6.8. Let $G=((A,B),E)$ be a bipartite graph and W be the set of vertices with minimum degree in G.
1) Prove that G contains a matching saturating $A\cap W$.
2) Deduce that there exists a matching saturating W. (*One could use Exercice 6.6*).

Exercise 6.9. A pack of $m\times n$ cards with m values and n colours is made of one card of each value and colour. The cards are arranged in an array with n lines and m columns. Show that there exists a set of m cards, one in each column, such that they all have distinct values.

Exercise 6.10. Prove that a bipartite graph G has a matching of size at least $|E(G)|/\Delta(G)$.

Exercise 6.11. Let $G=((A,B),E)$ be a connected bipartite graph such that $|A|=|B|=2k+3$. Show that if all the vertices have degree k or $k+1$ then G has a perfect matching unless it is a graph H which has an edge e such that $G\setminus e$ has two connected components which are subgraphs of $K_{k+2,k+1}$.

Exercise 6.12 (Alon)**.** Let $G=((A,B),E)$ be a bipartite graph such that $d(a)\geq 1$ for all $a\in A$ and $d(a)\geq d(b)$ for all $ab\in E$, where $a\in A$ and $b\in B$. Show that G has a matching saturating A.

Exercise 6.13 (Alon)**.** Let $G=((A,B),E)$ be a bipartite graph in which each vertex of A is of odd degree. Suppose that any two vertices of A have an even number of common neighbours. Show that G has a matching saturating A.

Exercise 6.14 (Petersen's Theorem)**.** Let G be a cubic graph.
1) Prove that if G is 2-edge-connected then it has a perfect matching.
2) Give an example of cubic graph with no perfect matching.
3)

a) Prove that if G is 2-edge-connected then, for all edge e, $G \setminus e$ has a perfect matching.

b) Deduce that if G has a unique bridge (separating edge) then G has a perfect matching.

Exercise 6.15. Let G be a connected graph.
1) Prove that if $|V(G)| \geq 4$ and every edge of G is in a perfect matching then G is 2-connected.
2) More generally, show that if $|V(G)| \geq 2k$ and every set of $k-1$ independent edges is included in a perfect matching then G is k-connected.

Exercise 6.16. Let G be a graph on at least $2k+2$ vertices which has a perfect matching. Show that if every set of k independent edges is included in a perfect matching then every set of $k-1$ independent edges is included in a perfect matching.

Exercise 6.17. Let D be a digraph. A *cycle-factor* of D is a spanning subdigraph F of D such that $d_F^+(v) = d_F^-(v) = 1$ for all $v \in V(D)$.

The *bipartite associated to D*, denoted $B(D)$, is defined by

- $V(B(D)) = V(G) \times \{1,2\}$ and
- $E(B(D)) = \{((u,1),(v,2)) \mid uv \in A(D)\}$

1) Show that D has a cycle-factor if and only if $B(D)$ has a perfect matching.

2) Assume that for every vertex v of D, $d^+(v) = d^-(v) = k$ for some fixed $k > 0$. Show that D has a cycle-factor.

3) Let G be a $2k$-regular graph. A *2-factor* in G is a spanning 2-regular graph. Deduce from the previous question that G has a 2-factor.

Exercise 6.18. Two persons are playing the following game on a graph. One after another the players choose vertices (one per turn) $v_1, v_2, v_3, \dots$ so that v_i is adjacent to v_{i-1} for all $i \geq 0$. The last player which is able to choose a vertex wins.

Prove that the first player has a winning strategy if and only if the graph has no perfect matching.

Exercise 6.19. The *claw* is the graph $K_{1,3} = (\{x, y_1, y_2, y_3\}, \{xy_1, xy_2, xy_3\})$. Show that an even connected graph with no claw as an induced subgraph has a perfect matching.

Exercise 6.20. Let G be a connected graph with an even number of edges. Prove that $E(G)$ may be partitionned into $|E(G)|/2$ sets of two adjacent edges. *One could show that a "line graph" with an even number of vertices has a perfect matching.*

Exercise 6.21 (Erdős ans Szekeres)**.** Prove that a sequence of $rs+1$ integers contains an increasing subsequence of $r+1$ integers or a decreasing subsequence of $s+1$ integers.

Exercise 6.22. Let G be a graph with vertices $v_1, \dots, v_n$. Give an algorithm that, given a sequence $d_1, \dots, d_n$, decides in polynomial time if G admits an orientation such that $d^+(v_i) = d_i$ for all $1 \leq i \leq n$.

Bibliography

[1] J. Fox and B. Sudakov. Paths and stability number in digraphs. *manuscript*.

[2] G. Hahn and B. Jackson. A note concerning paths and independence number in digraphs. *Discrete Math.* 82:327–329, 1990.