The Simplex Method or Solving Linear Program

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Motivation

- Most popular method to solve linear programs.
- Principle: smartly explore basic solutions (corner point solutions), improving the value of the solution.

The simplex

Start with a problem written under the standard form.

$$
\begin{array}{lrcrcrcr}
\text{Maximize} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{Subject to :} & & & & & & & \\
& 2x_1 & + & 3x_2 & - & x_3 & \leq & 5 \\
& 4x_1 & + & x_2 & - & 2x_3 & \leq & 11 \\
& 3x_1 & + & 4x_2 & - & 2x_3 & \leq & 8 \\
& & & & & x_1, x_2, x_3 & \geq & 0.
\end{array}
$$

The simplex

First step: introduce new variables, slack variables.

$$2x_1 + 3x_2 + x_3 \leq 5$$

We note x_4 the slack (difference) between the right member and 5, that is

$$x_4 = 5 - 2x_1 - 3x_2 - x_3.$$

The inequation can now be written as

$$x_4 \geq 0.$$

The simplex

Similarly, for the 2 others inequalities:

$$\begin{array}{rcrcrcr} 4x_1 & + & x_2 & - & 2x_3 & \leq & 11 \\ 3x_1 & + & 4x_2 & - & 2x_3 & \leq & 8 \end{array}$$

We define x_5 and x_6:

$$\begin{array}{rcrcrcrcr} x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \end{array}$$

And the inequalities can be written as

$$x_5 \geq 0, x_6 \geq 0.$$

The simplex

To summarize, we introduce three slack variables x_4, x_5, x_6:

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ z & = & & & 5x_1 & + & 4x_2 & + & 3x_3. \end{array}$$

The problem can be written as:

$$\text{Maximize } z \text{ subject to } x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

slack variables x_4, x_5, x_6 decision variables x_1, x_2, x_3. The two problems are equivalent.

The simplex

Second step: Find an initial solution.

In our example, $x_1 = 0, x_2 = 0, x_3 = 0$ is feasible.

We compute the value of x_4, x_5, x_6.

$$x_4 = 5 - 2x_1 - 3x_2 - x_3 = 5$$

Similarly, $x_5 = 11$ and $x_6 = 8$.

We get an initial solution

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 5, x_5 = 11, x_6 = 8$$

of value $z = 0$

The simplex

Dictionary:

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ \hline z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 . \end{array}$$

Basic variables: x_4, x_5, x_6, variables on the left.
Non-basic variable: x_1, x_2, x_3, variables on the right.

A dictionary is feasible if a feasible solution is obtained by setting all non-basic variables to 0.

The simplex

Dictionary:

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 .
\end{array}
$$

Basic variables: x_4, x_5, x_6, variables on the left.
Non-basic variable: x_1, x_2, x_3, variables on the right.

A dictionary is feasible if a feasible solution is obtained by setting all non-basic variables to 0.

The simplex

Simplex strategy: find an optimal solution by successive improvements.

Rule: we increase the value of the variable of largest positive coefficient in z.

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & \boxed{5}\,x_1 & + & 4x_2 & + & 3x_3.
\end{array}
$$

Here, we try to increase x_1.

The simplex

How much can we increase x_1?

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ \hline z & = & & & 5x_1 & + & 4x_2 & + & 3x_3. \end{array}$$

We have $\boxed{x_4 \geq 0}$.

It implies $5 - 2x_1 \geq 0$, that is $\boxed{x_1 \leq \frac{5}{2}}$.

The simplex

How much can we increase x_1?

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ \hline z & = & & & 5x_1 & + & 4x_2 & + & 3x_3. \end{array}$$

We have $\boxed{x_4 \geq 0}$.

It implies $5 - 2x_1 \geq 0$, that is $\boxed{x_1 \leq 5/2}$.

Similarly,

$x_5 \geq 0$ gives $x_1 \leq 11/4$.

$x_6 \geq 0$ gives $x_1 \leq 8/3$.

The simplex

How much can we increase x_1?

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ \hline z & = & & & 5x_1 & + & 4x_2 & + & 3x_3. \end{array}$$

We have $\boxed{x_4 \geq 0}$.

It implies $5 - 2x_1 \geq 0$, that is $\boxed{x_1 \leq 5/2}$ Strongest constraint

Similarly,

$x_5 \geq 0$ gives $x_1 \leq 11/4$.

$x_6 \geq 0$ gives $x_1 \leq 8/3$.

The simplex

How much can we increase x_1?

$$\begin{array}{rcrcrcrcr} \color{red}{x_4} & \color{red}{=} & \color{red}{5} & \color{red}{-} & \color{red}{2x_1} & \color{red}{-} & \color{red}{3x_2} & \color{red}{-} & \color{red}{x_3} \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ \hline z & = & & & 5x_1 & + & 4x_2 & + & 3x_3. \end{array}$$

We have $\boxed{x_4 \geq 0}$.

It implies $5 - 2x_1 \geq 0$, that is $\boxed{x_1 \leq 5/2}$ Strongest constraint

We get a new solution: $x_1 = 5/2$, $x_4 = 0$
with better value $z = 5 \cdot 5/2 = 25/2$.
We still have $x_2 = x_3 = 0$ and now $x_5 = 11 - 4 \cdot 5/2 = 1$,
$x_6 = 8 - 3 \cdot 5/2 = 1/2$

The simplex

We build a new feasible dictionary.

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ \hline z & = & & & 5x_1 & + & 4x_2 & + & 3x_3. \end{array}$$

x_1 enters the bases and x_4 leaves it:

$$x_1 = 5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4$$

The simplex

We replace x_1 by its expression in function of x_2, x_3, x_4.

$$\begin{array}{rclclclcl}
x_1 & = & 5/2 & - & 1/2x_4 & - & 3/2x_2 & - & 1/2x_3 \\
x_5 & = & 11 & - & 4(5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4) & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3(5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4) & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & 5(5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4) & + & 4x_2 & + & 3x_3.
\end{array}$$

The simplex

Finally, we get the new dictionary:

$$\begin{array}{ccccccccccc}
x_1 & = & \frac{5}{2} & - & \frac{3}{2} & x_2 & - & \frac{1}{2} & x_3 & - & \frac{1}{2} & x_4 \\
x_5 & = & 1 & + & 5 & x_2 & & & & + & 2 & x_4 \\
x_6 & = & \frac{1}{2} & + & \frac{1}{2} & x_2 & - & \frac{1}{2} & x_3 & + & \frac{3}{2} & x_4 \\
\hline
z & = & \frac{25}{2} & - & \frac{7}{2} & x_2 & + & \frac{1}{2} & x_3 & - & \frac{5}{2} & x_4 .
\end{array}$$

The simplex

Finally, we get the new dictionary:

$$
\begin{array}{rcrcrlcrlcrl}
x_1 & = & \boxed{5/2} & - & \frac{3}{2} & x_2 & - & \frac{1}{2} & x_3 & - & \frac{1}{2} & x_4 \\
x_5 & = & \boxed{1} & + & 5 & x_2 & & & & + & 2 & x_4 \\
x_6 & = & \boxed{1/2} & + & \frac{1}{2} & x_2 & - & \frac{1}{2} & x_3 & + & \frac{3}{2} & x_4 \\
\hline
z & = & \boxed{25/2} & - & 7/2 & x_2 & + & 1/2 & x_3 & - & 5/2 & x_4.
\end{array}
$$

We can read the solution directly from the dictionary:
Non basic variables: $x_2 = x_3 = x_4 = 0$.
Basic variables: $x_1 = 5/2$, $x_5 = 1$, $x_6 = 1/2$.
Value of the solution: $z = 25/2$.

The simplex

$$\begin{array}{ccccccccc} x_1 & = & \frac{5}{2} & - & \frac{3}{2} & x_2 & - & \frac{1}{2} & x_3 & - & \frac{1}{2} & x_4 \\ x_5 & = & 1 & + & 5 & x_2 & & & & + & 2 & x_4 \\ x_6 & = & \frac{1}{2} & + & \frac{1}{2} & x_2 & - & \frac{1}{2} & x_3 & + & \frac{3}{2} & x_4 \\ \hline z & = & \frac{25}{2} & - & \frac{7}{2} & x_2 & + & \frac{1}{2} & x_3 & - & \frac{5}{2} & x_4. \end{array}$$

New step of the simplex:

- x_3 enters the basis (variable with largest positive coefficient).
- 3^d equation is the strictest constaint $x_3 \leq 1$.
- x_6 leaves the basis.

The simplex

New feasible dictionary:

$$\begin{array}{rcrcrcrcr} x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\ x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\ x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\ \hline z & = & 13 & - & 3x_2 & - & x_4 & - & x_6. \end{array}$$

With new solution:

$$x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0$$

of value $z = 13$.

This solution is optimal.

All coefficients in z are negative and $x_2 \geq 0, x_4 \geq 0, x_6 \geq 0$, *so* $z \leq 13$.

The simplex

New feasible dictionary:

$$\begin{array}{rcrcrcrcr} x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\ x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\ x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\ \hline z & = & 13 & - & 3x_2 & - & x_4 & - & x_6. \end{array}$$

With new solution:

$$x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0$$

of value $z = 13$.

This solution is optimal.

All coefficients in z are negative and $x_2 \geq 0, x_4 \geq 0, x_6 \geq 0$, *so* $z \leq 13$.

Take Aways

- Most popular method to solve linear programs.
- Principle: smartly explore basic solutions (corner point solutions), improving the value of the solution.
- Complexity:
  - In theory, NP-complete (can explore a number of solutions exponentiel in the number of variables and constraints).
  - In practice, almost linear in the number of constraints.
- Polynomial methods exists: the ellipsoid method.