Simplex Method and Reduced Costs, Duality and Marginal Costs

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**** Simplex Method and Reduced Costs, Strong Duality Theorem ****

Simplex - Reminder

Start with a problem written under the standard form.

$$
\begin{array}{lrcrcrcr}
\text{Maximize} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{Subject to :} & & & & & & & \\
 & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\
 & 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\
 & & & & & x_1, x_2, x_3 & \geq & 0.
\end{array}
$$

Simplex - Reminder

Write the Dictionary:

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 .
\end{array}
$$

Basic variables: x_4, x_5, x_6, variables on the left.
Non-basic variable: x_1, x_2, x_3, variables on the right.

A dictionary is feasible if a feasible solution is obtained by setting all non-basic variables to 0.

Simplex - Reduced Costs

Write the Dictionary:

$$
\begin{array}{rclcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
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x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & \boxed{5}\,x_1 & + & \boxed{4}\,x_2 & + & \boxed{3}\,x_3.
\end{array}
$$

We call Reduced Costs the coefficients of z. The reduced cost of x_1 is 5, of x_2 is 4 and of x_3 is 3.

Reminder: If all reduced cost are non-positive, the solution is optimal and the simplex algorithm stops.

Simplex - Reduced Costs

Relationship between reduced costs, $\overline{c} = (\overline{c}_1, ..., \overline{c}_n)$ and optimal solution of the dual problem $\pi = (\pi_1, ..., \pi_m)$.

If we consider a general LP:

$$\begin{array}{lrcll} \text{Maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, 2, \cdots, m) \\ & x_j & \geq & 0 & (j = 1, 2, \cdots, n) \end{array}$$

Lemma: When the simplex algorithm finishes, we have:

$$\boxed{\overline{c}_j = c_j - \sum_{i=1}^{m} \pi_i A_{ij}}$$

Simplex - Reduced Costs

We consider a general LP:

$$\begin{array}{lrcll} \text{Maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, 2, \cdots, m) \\ & x_j & \geq & 0 & (j = 1, 2, \cdots, n) \end{array} \tag{1}$$

We introduce the following notations, A and B.

$$\begin{array}{lc} \text{Maximize} & c^T x \\ \text{Subject to:} & Ax = b \\ & x \geq 0 \end{array}$$

The method of the simplex finishes with an optimal solution x and an associated basis. Let $B(1), \ldots, B(m)$ be the indices of basic variables.

We define $B = [A_{B(1)} \ldots A_{B(m)}]$ the matrix associated to the basis.

We have $x_B = B^{-1} b$

Simplex - Reduced Costs

$$\begin{array}{rl} \text{Maximize} & c^T x \\ \text{Subject to:} & Ax = b \\ & x \geq 0 \end{array}$$

By studying what happens during a step of the simplex method, we can get the following expression for the reduced cost of variable x_j

$$\overline{c}_j = c_j - c_B^T B^{-1} A_j.$$

Simplex - Reduced Costs

When the method of the simplex finishes, the reduced costs are non-positive.

$$c^T - c_B^T B^{-1} A \leq 0^T.$$

Let π be such that

$$\pi^T = c_B^T B^{-1}$$

Simplex - Reduced Costs

We get

$$c^T - c_B^T B^{-1} A \leq 0^T.$$

$$c^T - \pi^T A \leq 0^T.$$

$$\pi^T A \geq c^T.$$

$$A^T \pi \geq c.$$

$\Rightarrow$ π is a feasible solution of the dual problem:

$$\begin{array}{ll} \text{Minimize} & \pi^T b \\ \text{Subject to:} & A^T \pi \geq c \\ & \pi \geq 0 \end{array} \tag{2}$$

Simplex - Reduced Costs

Moreover, the value of p equals the value of the optimal value of the primal:

$$\pi^T b = c_B^T B^{-1} b = c_B^T x_B = c^T x$$

$\Rightarrow \pi$ is an optimal solution of the dual problem (by the weak duality theorem).

Theorem [Strong Duality]: If the primal problem has an optimal solution,

$$x^* = (x_1^*, ..., X_n^*),$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, ..., y_n^*),$$

and

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$

Simplex - Reduced Costs

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Simplex - Reduced Costs

The Reduced Cost is

- *the amount by which an objective function coefficient would have to improve before it would be possible for a corresponding variable to assume a positive value in the optimal solution.*

** Dual Variables and Marginal Costs **

Signification of Dual Variables

$$\begin{array}{llcll} \text{Max} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{S. t.:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, \cdots, m) \\ & x_j & \geq & 0 & (j = 1, \cdots, n) \end{array} \qquad \begin{array}{llcll} \text{Min} & \sum_{i=1}^{m} b_i y_i & & & \\ \text{S. t.:} & \sum_{i=1}^{m} a_{ij} y_i & \geq & c_j & (j = 1, \cdots, n) \\ & y_i & \geq & 0 & (i = 1, \cdots, m) \end{array}$$

Signification can be given to variables of the dual problem:

"The optimal values of the dual variables can be interpreted as the marginal costs of a small perturbation of the right member b."

Signification of Dual Variables

Maximize	$\sum_{j=1}^{n} c_j x_j$				Minimize	$\sum_{i=1}^{m} b_i y_i$			
Subject to:	$\sum_{j=1}^{n} a_{ij} x_j$	$\leq$	b_i	$(i = 1, 2, \cdots, m)$	Subject to:	$\sum_{i=1}^{m} a_{ij} y_i$	$\geq$	c_j	$(j = 1, 2, \cdots, n)$
	x_j	$\geq$	0	$(j = 1, 2, \cdots, n)$		y_i	$\geq$	0	$(i = 1, 2, \cdots, m)$

Dimension analysis for a factory problem:

- x_j: production of a product j (chair, ...)
- b_i: available quantity of resource i (wood, metal, ...)
- a_{ij}: unit of resource i per unit of product j
- c_j: net benefit of the production of a unit of product j

Signification of Dual Variables

$$\begin{array}{llccl} \text{Maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, 2, \cdots, m) \\ & x_j & \geq & 0 & (j = 1, 2, \cdots, n) \end{array}$$

$$\begin{array}{llccl} \text{Minimize} & \sum_{i=1}^{m} b_i y_i & & & \\ \text{Subject to:} & \sum_{i=1}^{m} a_{ij} y_i & \geq & c_j & (j = 1, 2, \cdots, n) \\ & y_i & \geq & 0 & (i = 1, 2, \cdots, m) \end{array}$$

Dimension analysis for a factory problem:

- x_j: production of a product j (chair, ...)
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unit of resource i/unit of product j **euros/unit of product j** **euros/unit of resource i**

$$a_{1j} y_1 + \ldots + a_{nj} y_n > c_j$$

$\rightarrow y_i$ euro by unit of resource i. Unit value of resource i.

Signification of Dual Variables

Theorem: If the LP admits at least one optimal solution, then there exists $\varepsilon > 0$, with the property: If $|t_i| \leq \varepsilon \ \forall i = 1, 2, \cdots, m$, then the LP

$$\begin{array}{rrcll} \text{Max} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i + t_i & (i = 1, 2, \cdots, m) \\ & x_j & \geq & 0 & (j = 1, 2, \cdots, n). \end{array} \tag{3}$$

has an optimal solution and the optimal value of the objective is

$$z^* + \sum_{i=1}^{m} y_i^* t_i$$

with z^* the optimal solution of the initial LP and $(y_1^*, y_2^*, \cdots, y_m^*)$ the optimal solution of its dual.

To be remembered

- Definition of the Reduced Costs

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ \hline z & = & & & \boxed{5}\,x_1 & + & \boxed{4}\,x_2 & + & \boxed{3}\,x_3. \end{array}$$

- If all reduced cost are non-positive, the solution is optimal and the simplex algorithm stops.
- Relationship between reduced costs, $\overline{c} = (\overline{c}_1, \ldots, \overline{c}_n)$ and optimal solution of the dual problem $\pi = (\pi_1, \ldots, \pi_m)$.
  When the simplex algorithm finishes, we have:

$$\overline{c} = c^T - \pi^T A$$