UBINET, Master 2 IFI Algorithms for telecommunications

**Final Exam, November 2015**

3 hours

*Course and manuscript notes are allowed. No computers, cellphones, books.*

*Instruction and comments: the points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. All proposed solutions must be proved.* **All the exercises are independent.** *The points are indicated so you may adapt your effort.*

# 1 Linear Programming

**Exercise 1 (Modeling as a linear program (2 points, 15 minutes))** *The production manager of a chemical plant is attempting to devise a shift pattern for his workforce. Each day of every working week is divided into three eight-hour shift periods (00:01-08:00, 08:01-16:00, 16:01-24:00) denoted by night, day and late respectively. The plant must be manned at all times and the minimum number of workers required for each of these shifts over any working week is as below:*

| | *Mon* | *Tues* | *Wed* | *Thur* | *Fri* | *Sat* | *Sun* |
|---|---|---|---|---|---|---|---|
| *Night* | 5 | 3 | 2 | 4 | 3 | 2 | 2 |
| *Day* | 7 | 8 | 9 | 5 | 7 | 2 | 5 |
| *Late* | 9 | 10 | 10 | 7 | 11 | 2 | 2 |

*The union agreement governing acceptable shifts for workers is as follows:*

- *Each worker is assigned to work either a night shift or a day shift or a late shift and once a worker has been assigned to a shift they must remain on the same shift every day that they work.*
- *Each worker works four consecutive days during any seven day period.*

*In total there are currently 60 workers.*

1. *Formulate the production manager's problem as a linear program.*

**Exercise 2 (Solving a linear program (2 points, 15 minutes))**

$$\begin{array}{lrcrcr}
\textit{Maximize} & 40x_1 & + & 50x_2 & & \\
\textit{Subject to:} & & & & & \\
& x_1 & + & 2x_2 & \leq & 40 \\
& 4x_1 & + & 3x_2 & \leq & 120 \\
& x_1 & + & x_2 & \geq & 10 \\
& & & x_1, x_2 & \geq & 0
\end{array}$$

1. *Solve the linear program using the graphical method.*

**Exercise 3 (Duality and Optimality certificates. (2 points, 15 minutes))** *Consider the following linear program.*

$$\begin{array}{lrcrcr}
\textit{Maximize} & 2x_1 & + & x_2 & & \\
\textit{Subject to:} & & & & & \\
& x_1 & + & 2x_2 & \leq & 14 \\
& 2x_1 & - & x_2 & \leq & 10 \\
& x_1 & - & x_2 & \leq & 3 \\
& & & x_1, x_2 & \geq & 0
\end{array}$$

1. *Write the dual program of this linear program.*
2. *Is $x_1 = 20/3$, $x_2 = 11/3$ an optimal solution of the linear programme above? To answer, use the method to provide optimality certificates seen during the class. Detail all steps.*

**Exercise 4 (4 points, 15 minutes)** Formulate an integer linear program to solve the problem of positioning eight queens on the chessboard so that no queen is under threat by any other queen.

**Exercise 5 (Travelling Salesman (hard) (5 points, 15 minutes))** A travelling salesman must visit 7 customers in 7 different locations whose (symmetric) distance matrix is:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | - | 86 | 49 | 57 | 31 | 69 | 50 |
| 2 | | - | 68 | 79 | 93 | 24 | 5 |
| 3 | | | - | 16 | 7 | 72 | 67 |
| 4 | | | | - | 90 | 69 | 1 |
| 5 | | | | | - | 86 | 59 |
| 6 | | | | | | - | 81 |

1. Formulate a linear program to determine a visit sequence starting and ending at location 1, which minimizes the travelled distance.

*Hint: Write a constraint for each subtour, a subtour being a subset $S \subsetneq V$.*

## 2 Approximation algorithms

**Exercise 6 (Maximum Cut Problem (10 points))** Recall that, for any $c \geq 1$, a $c$-approximation algorithm for a maximization problem is an algorithm that computes **in polynomial-time** a feasible solution such that

$$OPT/c \leq value(solution) \leq OPT$$

where $OPT$ is the optimal value.
**Notations:** In this section, $n$ will always denote the number of vertices of a graph and $m$ will denote the number of its edges.

Let $G = (V, E)$ be a graph. A *cut* in $G$ is a partition of $V$ into two sets. Let $S \subseteq V$ be a subset of vertices. The *cost of the cut* $(S, V \setminus S)$, denoted by $cost(S)$, equals the number of edges between $S$ and $V \setminus S$, i.e., the size of the set $\{\{x, y\} \in E \mid x \in S, y \in V \setminus S\}$.

The Maximum Cut Problem takes a graph $G = (V, E)$ as input and the objective is to find a cut with maximum cost.

**Question 1** *Let $G = (A \cup B, E)$ be a bipartite graph (i.e., $A$ and $B$ are stable sets). Give a maximum cut of $G$. What is its cost? (prove that the solution is optimal)*

**Question 2** *Give an exponential-time algorithm that computes a maximum cut in arbitrary graphs. Prove that its time-complexity is $O(m \cdot 2^n)$.*

The Maximum Cut Problem is NP-hard, meaning that it does not admit a polynomial-time algorithm unless $P = NP$. The goal of next questions is to analyze an approximation algorithm for it.
**Definition:** Let $(S, V \setminus S)$ be a cut. A vertex $v$ is *movable* if

- either $v \in S$ and $cost(S) < cost(S \setminus \{v\})$;
- or $v \in V \setminus S$ and $cost(S) < cost(S \cup \{v\})$.

That is, $v$ is movable if moving $v$ on the other side strictly increases the cost of the cut.

**Question 3** *Apply Algorithm 1 on the example depicted in Figure 1*

**Question 4** *What is the maximum number of iterations of the While loop of Algorithm 1?*

*Let us assume that checking if a vertex is movable can be done in constant time. What is the order of magnitude of the time-complexity of Algorithm 1?*

**Algorithm 1** 2-approximation algorithm for MAXIMUM CUT

**Require:** A graph $G = (V, E)$
**Ensure:** A cut $(S, V \setminus S)$

$S = \emptyset$
**while** There is a movable vertex $v$ **do**
    Move the vertex $v$ on the other side, that is:

- if $v \in S$ then replace $S$ by $S \setminus \{v\}$
- if $v \in V \setminus S$ then replace $S$ by $S \cup \{v\}$

**return** $S$

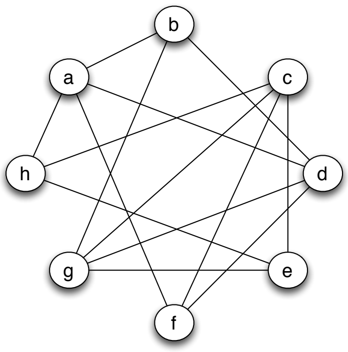


Figure 1: A graph with 8 nodes and 14 edges

**Notation:** Let $G = (V, E)$ be a graph, $v \in V$ and $X \subseteq V$. Let $deg_X(v)$ denote the degree of $v$ in $X$, that is the size of the set $\{w \in X \mid \{v, w\} \in E\}$. Let $d(v)$ denote the (classical) degree of $v$, i.e., $d(v) = deg_V(v)$.

**Question 5** *Let $(S, V \setminus S)$ be a solution computed by the algorithm.*

1. *Let $v \in S$. Show that $deg_S(v) \leq \lfloor d(v)/2 \rfloor$.*
2. *Similarly, show that $deg_{V \setminus S}(v) \leq \lfloor d(v)/2 \rfloor$ for any $v \in V \setminus S$.*

**Question 6** *Let $(S, V \setminus S)$ be a solution computed by the algorithm. Let $X = \{\{u, v\} \in E \mid u, v \in S\}$ be the set of edges between nodes in $S$. Let $Y = \{\{x, y\} \in E \mid x, y \notin S\}$ be the set of edges between nodes in $V \setminus S$. Let $Z = E \setminus (X \cup Y)$ be the set of edges between $S$ and $V \setminus S$.*

1. *By summing the degree of the nodes in $S$, and using previous question, show that $2|X| \leq |Z|$.*
2. *Similarly, show that $2|Y| \leq |Z|$.*
3. *Deduce that $|Z| \geq |E|/2$.*

**Question 7** *Prove that Algorithm 1 is a 2-approximation algorithm for the* MAXIMUM CUT *problem.*