Design of Minimal Fault Tolerant Networks: Asymptotic Bounds

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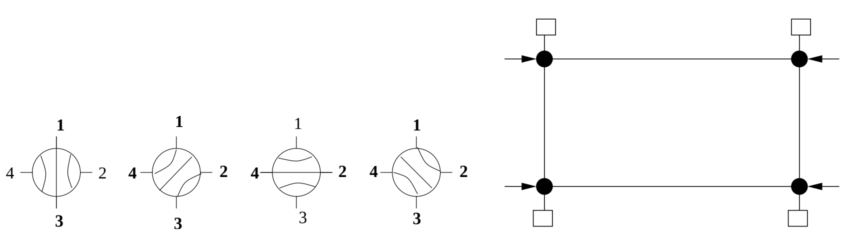
This paper deals with the design of on board networks in satellites (also called Traveling wave tube Amplifiers (TWTA)). These networks should connect signals arriving on some ports of the satellite to amplifiers, even in case of failures of some amplifiers. They are made of links and expensive switches each with 4 links. So, the aim is to design networks having as few switches as possible and satisfying the following property: *there exist p edge-disjoint paths from the p signals arriving on $p+\lambda$ ports (inputs) to any set of p amplifiers (outputs) chosen from the $p+k$ total number of outputs*. We call such networks *valid* (p,λ,k)*-networks* and want to determine the minimum number of switches $\mathcal{N}(p,\lambda,k)$ of such networks. By symmetry we suppose $\lambda \leq k$ and we note $n := p+k$. We give tight results for small values of k and asymptotic results when $k = O(\log n)$ which are tight when $k = \Theta(\lambda)$ and when $\lambda = 0$.

Keywords: key words : fault tolerant networks, switching networks, routing, TWTA redundancy, expanders, connectivity, disjoint paths.

1 Introduction

Problem and Motivation. The problem we consider here was asked by Alcatel Space Industry. Signals incoming in a telecommunication satellite have to be routed through an on-board network to amplifiers. The satellites under consideration are for example used for TV and video transmission (like the Eutelsat or Astra series) as well as for private applications. This network consists of switches with 4 links and which can realize the connections displayed in figure 1A. The signals enter the network through ports and exit through amplifiers. In the following ports and amplifiers will be refered as inputs and outputs. They are respectively represented in figures by arrows ($\rightarrow$) and boxes ($\square$) as shown in Figure 1B. Each input and output are connected to one of the switches of the network.

The difficulties to design such networks come from two symmetric facts. On one hand the amplifiers may fail during satellite lifetime and cannot be repaired. So more amplifiers are needed than the number of signals which have to be routed. On the other hand as the satellite is rotating on itself, all the inputs are not well oriented to capt the incoming signals. So, at each moment, a lot of inputs are unused. We want to



A: A switch can use 4 different states. B: A valid (3,1,1)-network.

Fig. 1: Introduction to networks.

be able to route the incoming signals from the used inputs to valid amplifiers, that is to find disjoint paths between the used inputs and valid outputs in the interconnection network. All the amplifiers are identical so a signal can be routed to any of them. If a network is able to route p signals to amplifiers in presence of λ useless inputs and of k faulty amplifiers, we will say that this network is a valid (p,λ,k)-fault tolerant network. An example of a $(3,1,1)$-fault tolerant network is given in Figure 1B. As a matter of fact all the possible configurations for the unused input and the non valid output, the three used inputs can be connected to the valid outputs. Other examples are presented in Figure 2. Realizing such a network is easy, but it is difficult to optimise it. To decrease launch costs, it is crucial to minimise the network physical weight, i.e. for us, to minimize the number of switches. As launch costs are dramatically high, it is worth saving even one switch. So our aim will be to construst valid (p,λ,k)-networks with the minimum number of switches denoted by $\mathcal{N}(p,\lambda,k)$. On one side we consider asymptotic cases. The networks are large in the sense that n and k go to infinity. For symmetry reasons, we can assume $k \geq \lambda$. We give results for $k = O(\log n)$. On the other side we have also studied exact values for small k.

Related Work. The problem in which all the inputs are used, that is $\lambda = 0$, has been introduced in [BDD02]. In [BPT] a general theory is introduced for $\lambda = 0$ and several results are obtained for small values of k. For example it is proven that $\mathcal{N}(p,0,4) = \lceil \frac{5p}{4} \rceil$. In [BDH$^+$03] and [DHMP05] the case of switches with $2k > 4$ links is considered for $\lambda = 0$. In [BHT06] the authors consider a variant of (p,0,k)-networks where some signals are priorities and should be sent to amplifiers offering the best quality of service. Finally in [BD02] the authors study the case were all the amplifiers are different. (In this work we consider that all the amplifiers are identical so we won't have to care how the signals go through the switches (indeed if we are in a forbidden state we can exchange the exit of the signals, see [BDD02])). More details could be found in extended versions of this paper.

2 Formalization

Notations. Given a function f, we define $f(A) := \sum_{a\in A} f(a)$ for any finite set A. For a subset W of vertices of a graph $G = (V,E)$, let us denote $\delta(W)$ the number of edges connecting W and $V \backslash W$.
$(p,\lambda,k)-$**networks and valid** $(p,\lambda,k)-$**networks.** A $(p,\lambda,k)-$*network* is a triple $N = \{(V,E),i,o\}$ where $G = (V,E)$ is a graph and i,o are integral functions defined on V called input and output functions, such that for any $v \in V$, $i(v)+o(v)+deg(v) = 4$. The total number of inputs is $i(V) = \Sigma_{v\in V} i(v) = p+\lambda$, and the total number of outputs is $o(V) = \Sigma_{v\in V} o(v) = p+k$. We note $n := p+k$. A *non-faulty output function* is a function o' defined on V such that $o'(v) \leq o(v)$ for any $v \in V$ and $o'(V) = p$. A *used input function* is a function i' defined on V such that $i'(v) \leq i(v)$ for any $v \in V$ and $i'(V) = p$. A (p,λ,k)-network is said *valid* if for any faulty output function o' and any used input function i', there are p edge-disjoint paths in G such that each vertex $v \in V$ is the initial vertex of $i'(v)$ paths and the terminal vertex of $o'(v)$ paths.
Design Problems. Let $\mathcal{N}(p,\lambda,k)$ denotes the minimum number of switches of a valid (p,λ,k)-network. The *Design Problem* consists in determining $\mathcal{N}(p,\lambda,k)$ and in constructing a minimum (p,λ,k)-network, or at least a valid (p,λ,k)-network with a number of vertices close to the optimal value. We introduce a second problem. We also consider only networks with a maximum number, $p+\lambda$, of switches with an input and an output in one to one correspondance on them and with $k-\lambda$ switches with only one output. To find minimum valid network like these is what we call the *Simplified Design Problem.* Networks of this kind are especially good for practical applications, as they simplify the routing process, minimize path lengths and lower interferences between signals.
Excess, Validity and Cut-criterion. We show that, to verify if a network is valid, instead of solving a flow/supply problem for each possible configuration of output failures and of used inputs, it is sufficient to look at an invariant measure of subsets of the network, the *excess*, as expressed in the following proposition.

Proposition 1 (Cut Criterion) *A $(p,\lambda,k)-$network is* valid *if and only if, for any subset of vertices $W \subset V$ the* excess of *W, defined by,*

$$\varepsilon(W) := \delta(W) + o(W) - \min(k, o(W)) - \min(i(W), p),$$

satisfies $\varepsilon(W) \geq 0$.

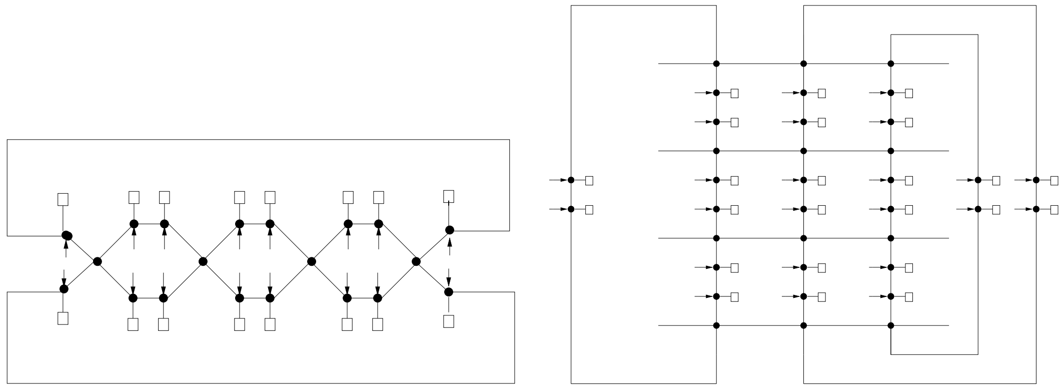


Fig. 2: A valid (12,4,4)-network and a valid (18,6,6)-network.

The intuition is that the signals arriving in W (in number at most $\min(i(W),p)$) should be routed either to the valid outputs of W (in number at least $o(W)-\min(k,o(W))$) or to the links going outside (in number $\delta(W)$). The omitted formal proof reduces to a supply/demand flow problem.

3 Results

Recall that $\lambda \leq k$ is assumed. We present *methodologies* to find and prove *lower bounds* for the minimal number of switches of valid networks, $\mathcal{N}(p,\lambda,k)$, and to find *constructions* close to minimal $(p,\lambda,k)-$networks. To prove lower bounds we use the *cut criterion* (Proposition 1) to exhibit forbidden patterns for small subgraphs, leading to linear equations linking different kinds of switches. To use these informations, we quasi-partition the graph into small connected components of similar size which doesn't overlap too much. This gives bounds for the whole network. For upper bounds, the idea is first to find *robust networks* and then to make a smart choice of the location of inputs and outputs. The last step is to prove the validity using the cut criterion. We show that robust networks can be obtained using known graphs with good expansion and good girth as the Ramanujan graphs. We study here networks for specific values of k and λ and *large networks*, where n and k go to infinity. Using the introduced methodologies, we prove *linear bounds* for these $(p,\lambda,k)-$networks. For small networks, we have obtained the following theorems.

Theorem 1 $\mathcal{N}(p,2,1) = \mathcal{N}(p,1,2) = \mathcal{N}(p,2,2) = p+2$

Theorem 2 *For $k \in \{3,4\}$ and $\lambda \geq 1$, we have*

$$\mathcal{N}(p,\lambda,k) \geq n + \frac{n}{4} - c''_4 \qquad \mathcal{N}(p,\lambda,k) \leq n + \frac{n}{4} + c'_4 - c,$$

with $c'_4 = \lceil \frac{n \mod 4}{4} \rceil$, $c = \lfloor \frac{k-\lambda}{2} \rfloor$ and $c''_4 = \frac{k-\lambda}{2} + \frac{k-\lambda}{8}$. Remark that the difference between the bounds is at most 1.

Theorem 3 *For $k \in \{5,6\}$ and $\lambda \geq 1$, we have*

$$\mathcal{N}'(p,\lambda,k) \geq n + \frac{n}{2} - c''_6 \qquad \mathcal{N}(p,\lambda,k) \leq n + \frac{n}{2} + c'_6 - c$$

with $c'_6 = 3\lceil \frac{n \mod 6}{6} \rceil$, $c = \lfloor \frac{k-\lambda}{2} \rfloor$ and $c''_6 = \frac{k-\lambda}{2} + \frac{k-\lambda}{4}$. Remark that the difference between the bounds is at most 4

Examples of $(p,4,4)$ and $(p,6,6)$-networks are given in Figure 2.

For large networks, *different cases* appear. We show that the nature of the problem differs with the order of k and λ. We distinguish 3 cases and give lower bounds (LB) for them: $\lambda = 0$ ($LB = (1-c_k)(n+\frac{n}{2})$), $\lambda = \Theta(k)$ ($LB = (1-c_k)(n+\frac{2}{3}n)$) and $\lambda = O(1)$ ($LB = (1-c_k)(n+\frac{n}{2})$), where c_k tends to zero when k tends to infinity. For $k \leq c \log n$ (where c is a constant depending on the expansion factors of regular graphs),

Cases ($k \leq c \log n$)	Lower Bound	Upper Bound
$\lambda = 0$	$(1-c_k)(n+\frac{1}{2}n)$	$n+\frac{1}{2}n$
Simplified Case	$(1-c_k)\ 2n$	$2n$
General Case $\lambda = \theta(k)$	$(1-c_k)(n+\frac{2}{3}n)$	$n+\frac{3}{4}n$
General Case	$(1-c_k)(n+\frac{1}{2}n)$	$n+\frac{3}{4}n$

Fig. 3: Summary ($n = p+k$ and c_k tends to 0, when k goes to infinity).

we give a construction (and so an upper bound) with $n+\frac{3}{4}n$ switches valid for any lambda and a specific construction with $n+\frac{n}{2}$ switches for $\lambda = 0$. In the last case ($\lambda = O(1)$), we succeed to tighten the bounds by introducing some restrictions on the switches that may be used. One of these restriction is the *Simplified Design Problem*. We solve it proving $LB = (1-c_k)\ 2n$ and giving a construction with $2n$ switches. The intuition is the following. An (n,r,c)-E-expander is a finite r-regular graph $G = (V,E)$ such that, for any set A of vertices of G with $|A| \leq |V|/2$, we have $\delta(A, V \setminus A) \geq c|A|$. Take a $(n,4,c = 1/4)-$E-expander, $G = (V,E)$, of girth g, $g \geq \frac{2}{3}\log n$ (explicit constructions of such graphs are given in [Mor94]). As G is $4-$regular, there exist a family F of vertex disjoint cycles covering all vertices of G. We first add n new vertices by splitting each edge of F into two edges. On each new vertex, we put an output and input. We now have a $(p,k,k)-$network, $\mathcal{R}$, with $2n$ switches. The proof ends by proving the validity of this network for $k \leq \frac{1}{6}\log n$. All these results are summarized in Figure 3. We also find a construction of valid $(p,\lambda,k)-$networks with $3n$ switches for k, no more of the order of $\log n$, but linear. The proof is based on the study of the expansion of small sets of graphs.

4 Conclusion

It remains a lot of work to do on this general problem for example on small values for specific demands of Alcatel but also we hope to find a tight asymptotic bound at least when $\lambda = O(1)$ and $k = O(\log n)$. We already have some results on networks without switches with two outputs or inputs as well as for networks with as many as possible (ie $p+\lambda$) switches with one input and one output and $k-\lambda$ switches with one output (extension of the simplified case). For values of λ and k of $O(\log n)$ it remains to tighten the bounds. It will be also interesting to study the case where $\lambda = \Theta(n)$ and $k = \Theta(n)$. We have yet only partial results.

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