Self-Stabilizing Repeated Balls-into-Bins

Emanuele Natale[†]

joint work with

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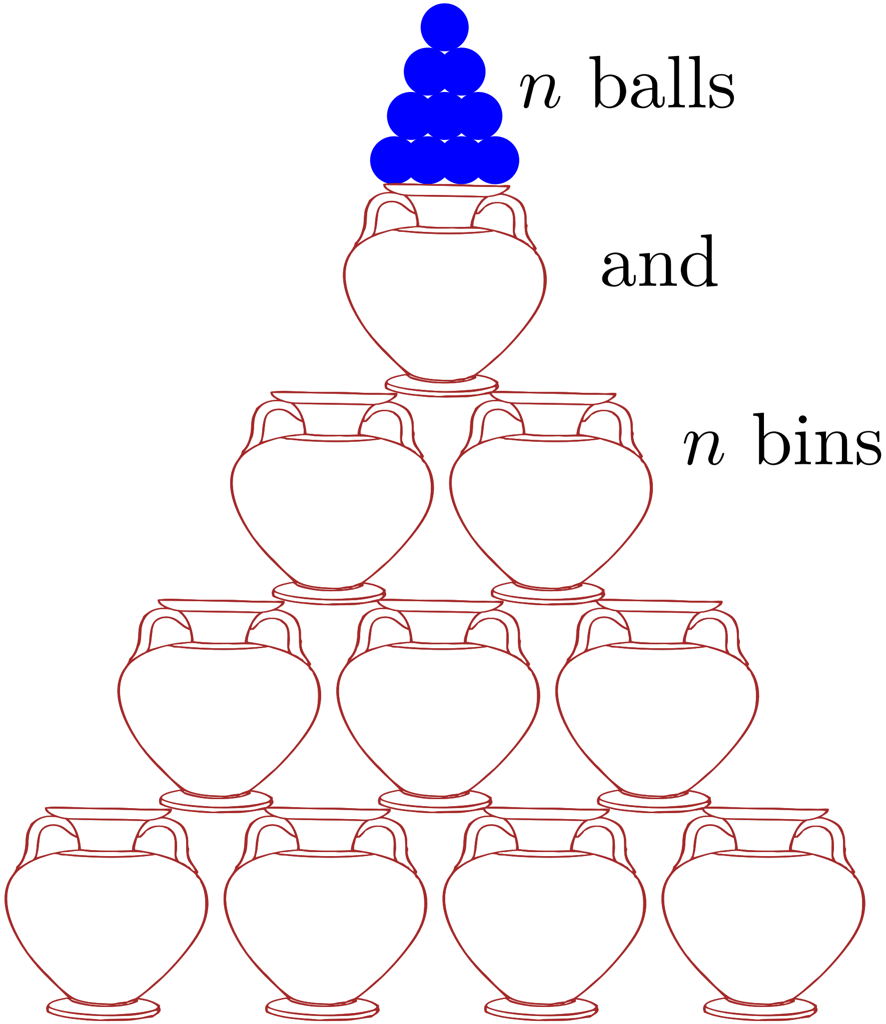
Balls-into-Bins



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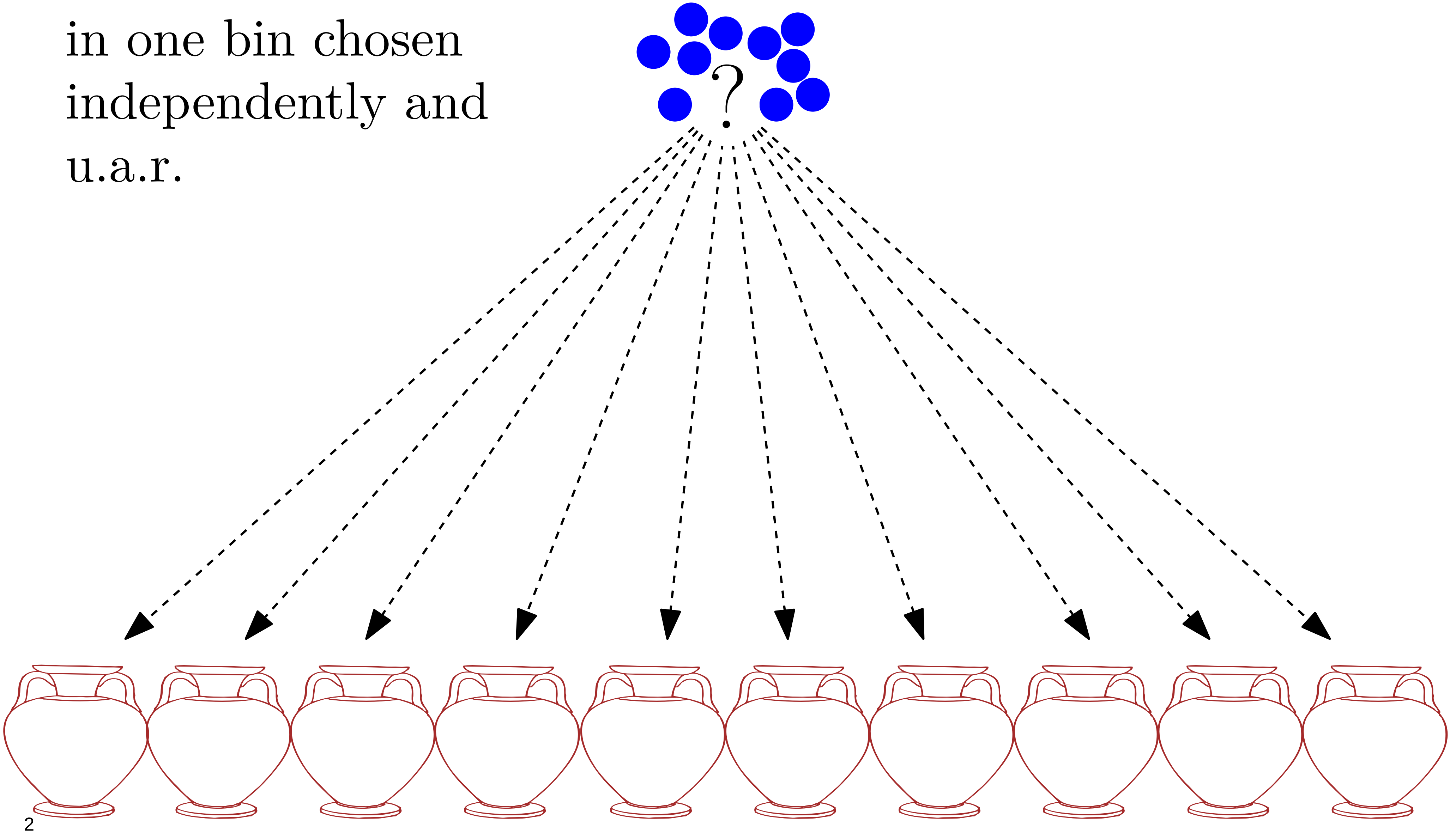

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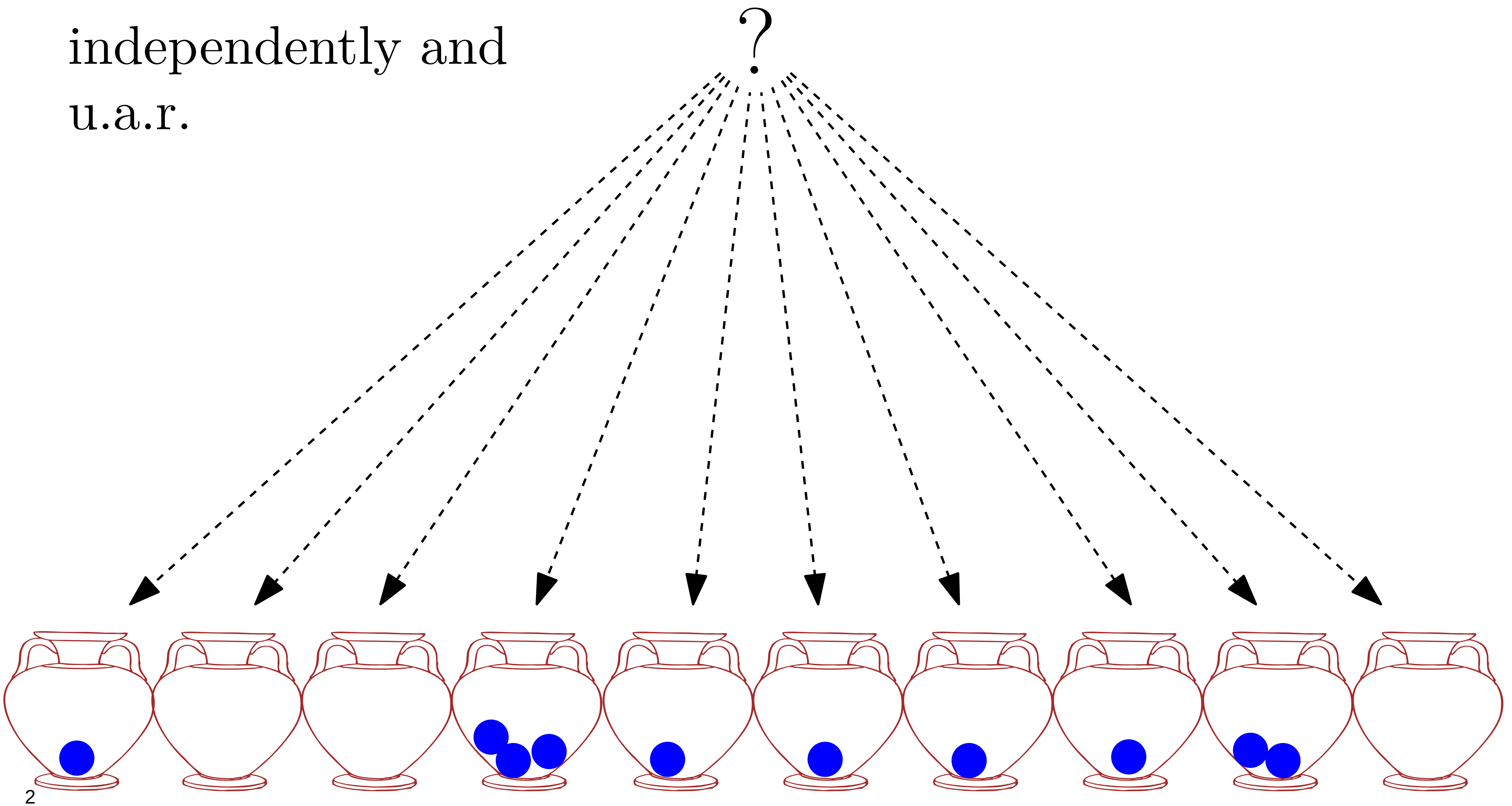

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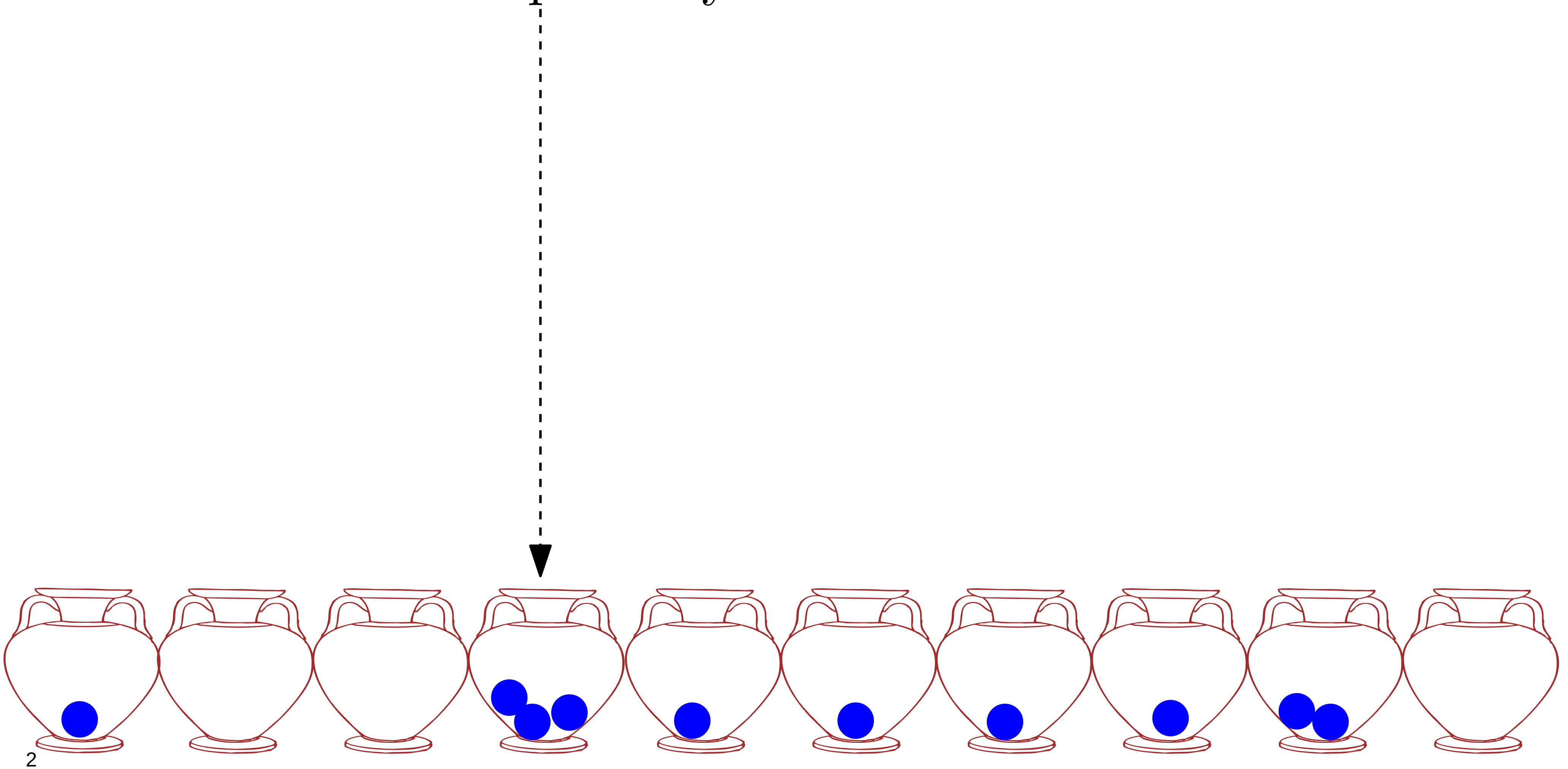
Maximum load: maximum number of balls that end up in any bin.



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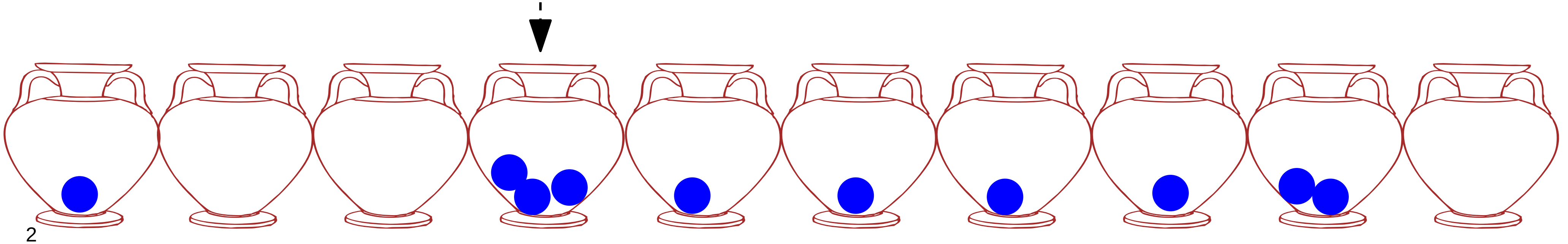
Applications: dynamic resource allocation, hashing, ...



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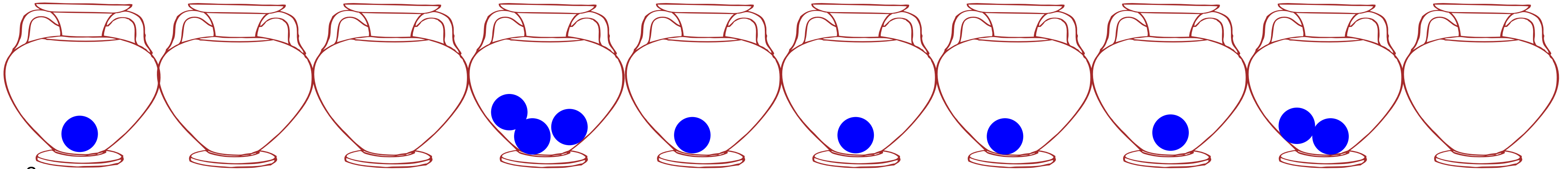

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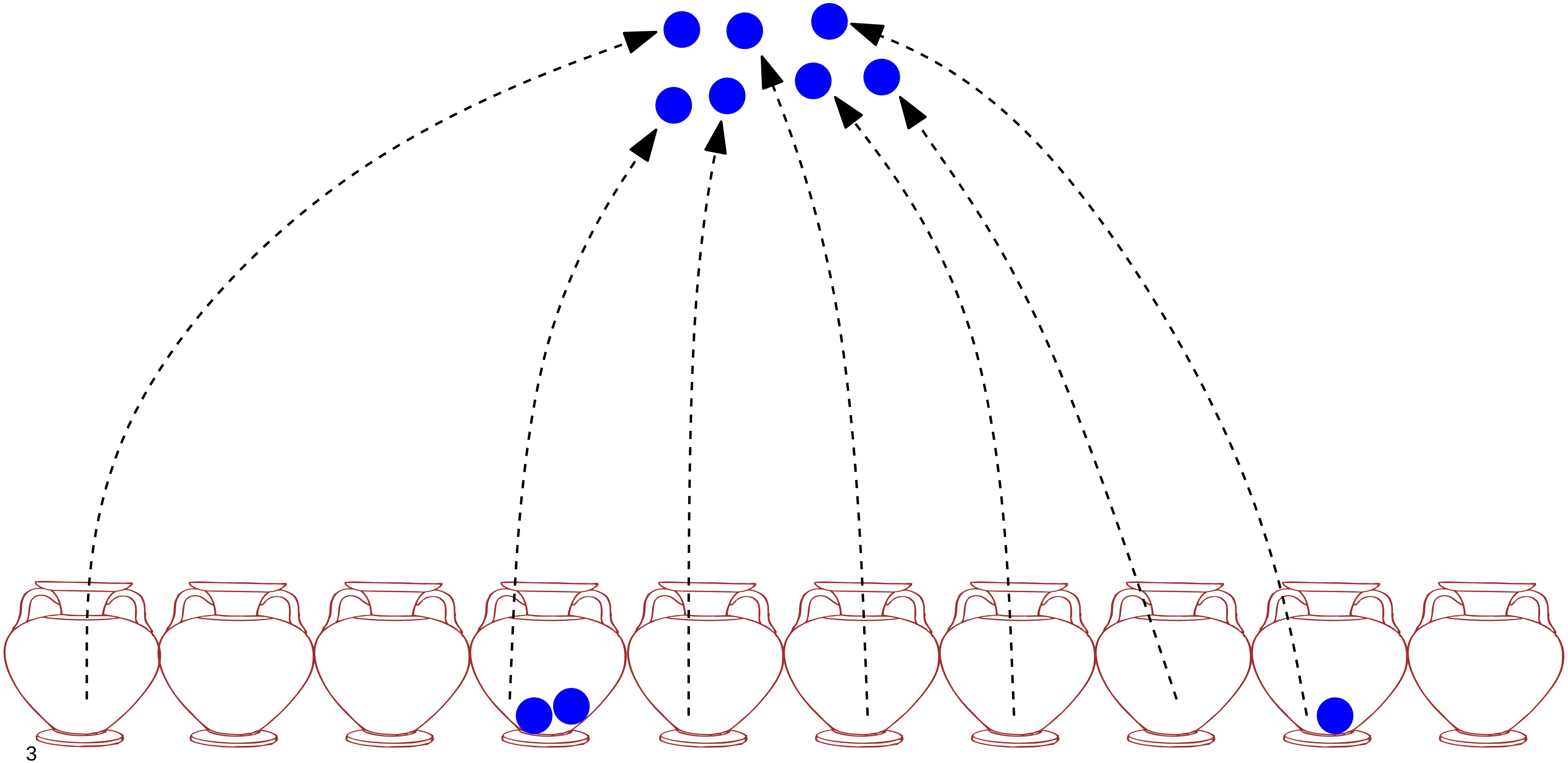

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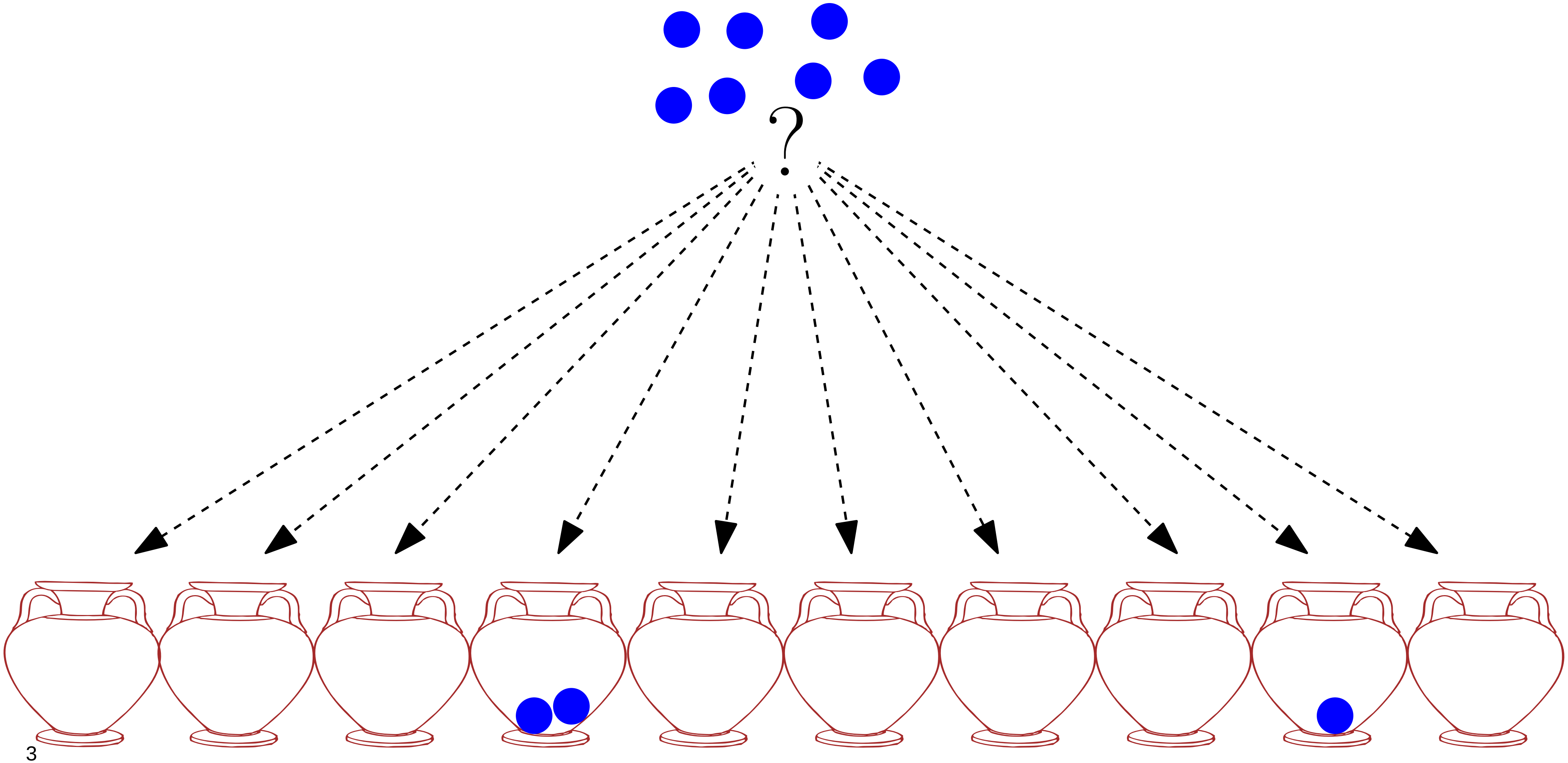

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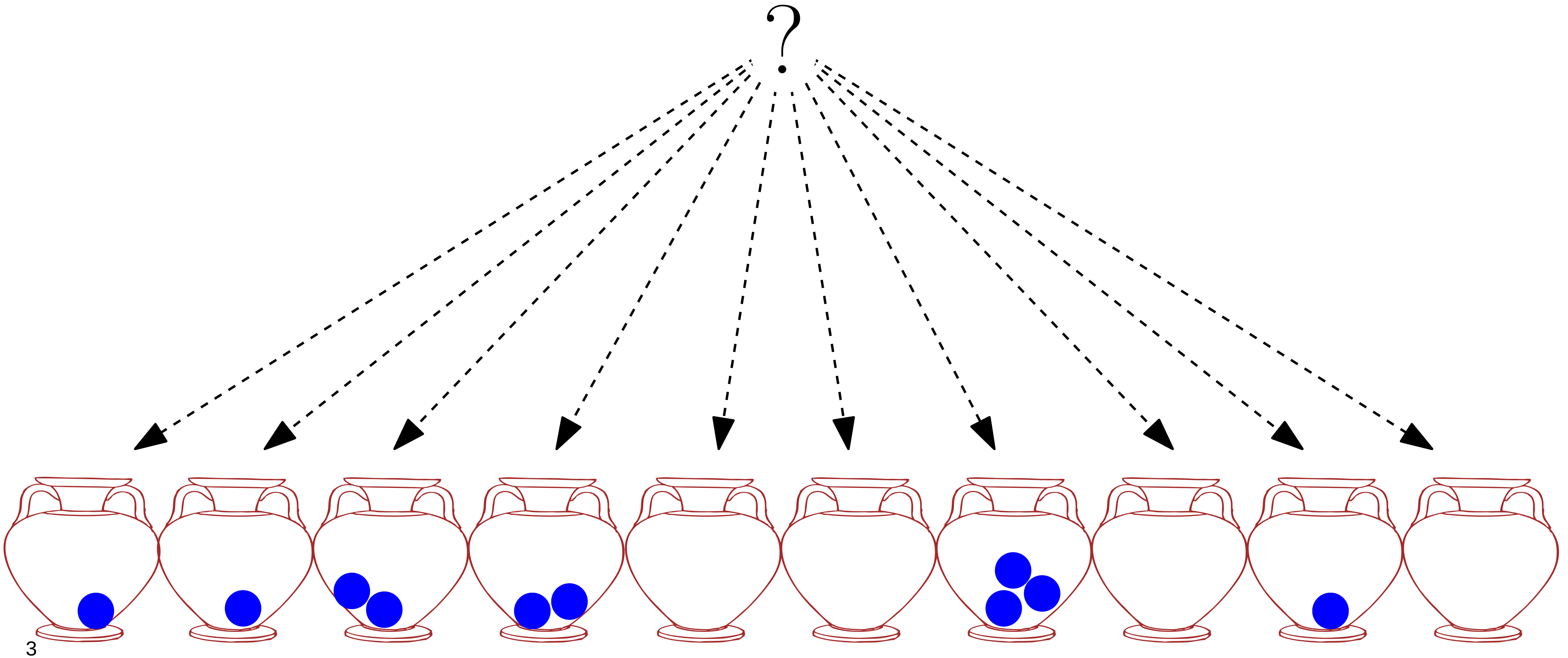

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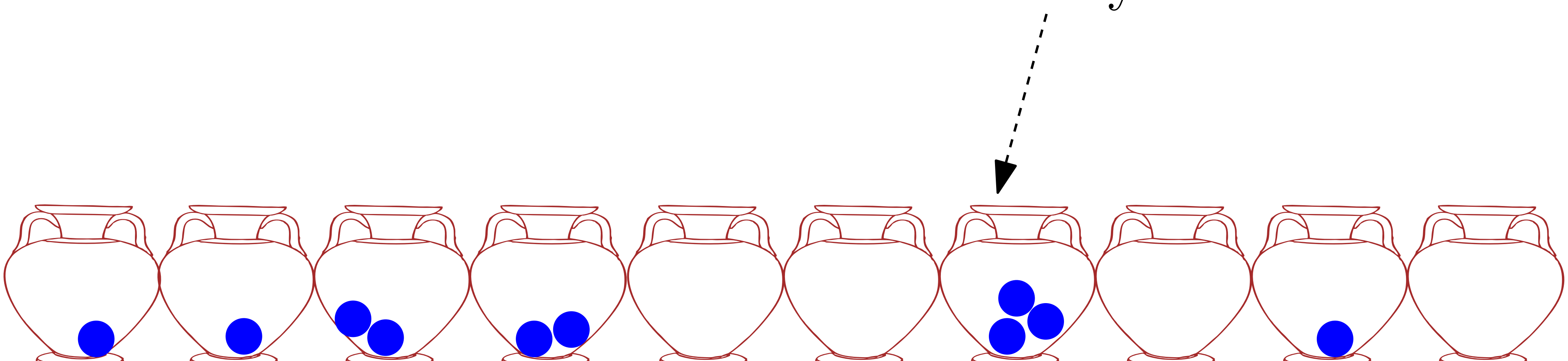

Our Contribution

From any configuration, in $O(n)$ rounds the process reaches a conf. with max. load $\mathcal{O}(\log n)$ w.h.p. and, from any conf. with max. load $\mathcal{O}(\log n)$, the max. load keeps $\mathcal{O}(\log n)$ for poly(n) rounds w.h.p.

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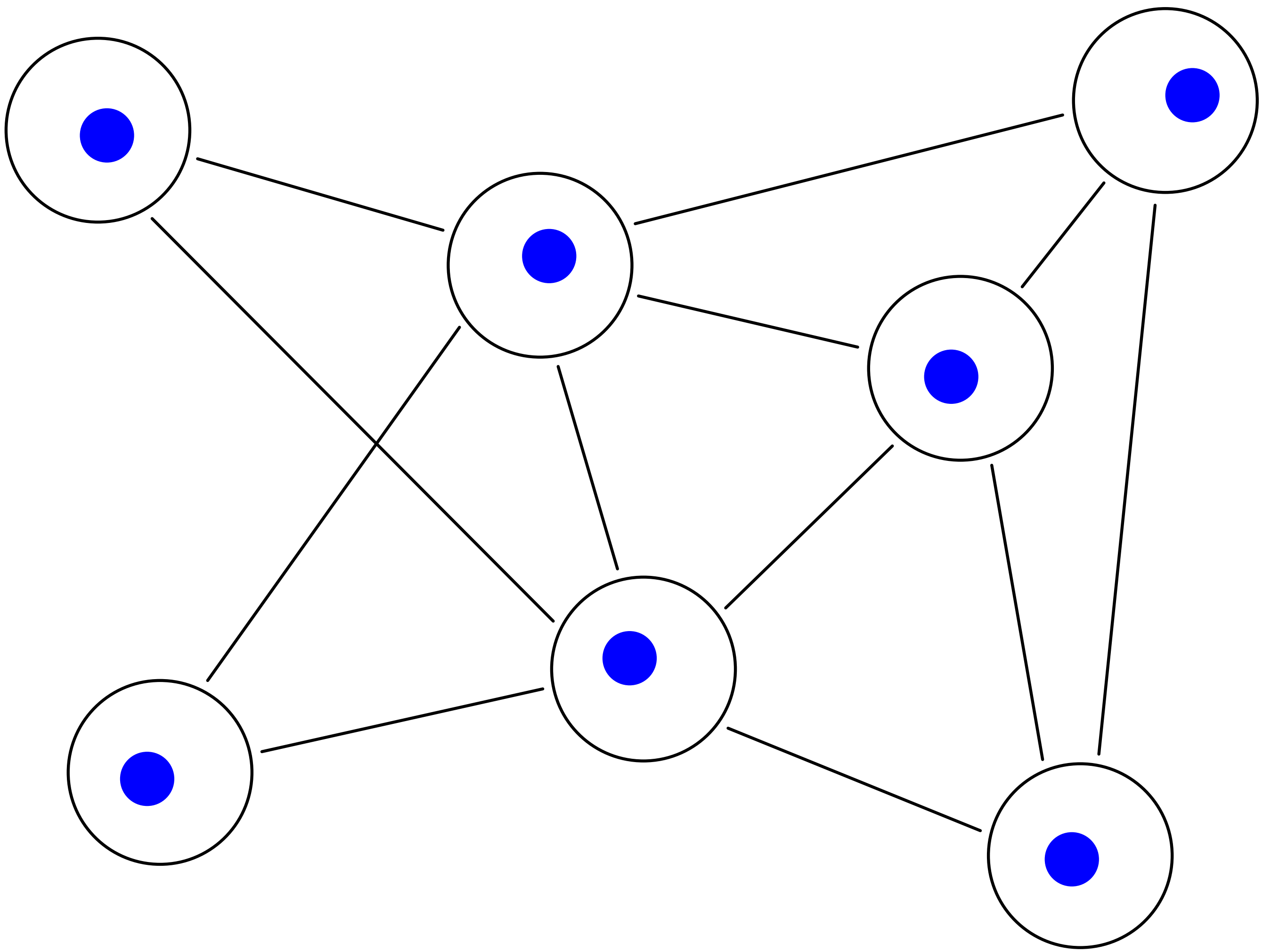
$\mathcal{GOSSIP}$ Model [Censor-Hillel et al., STOC '12]: nodes contact or call only one neighbor at each round.



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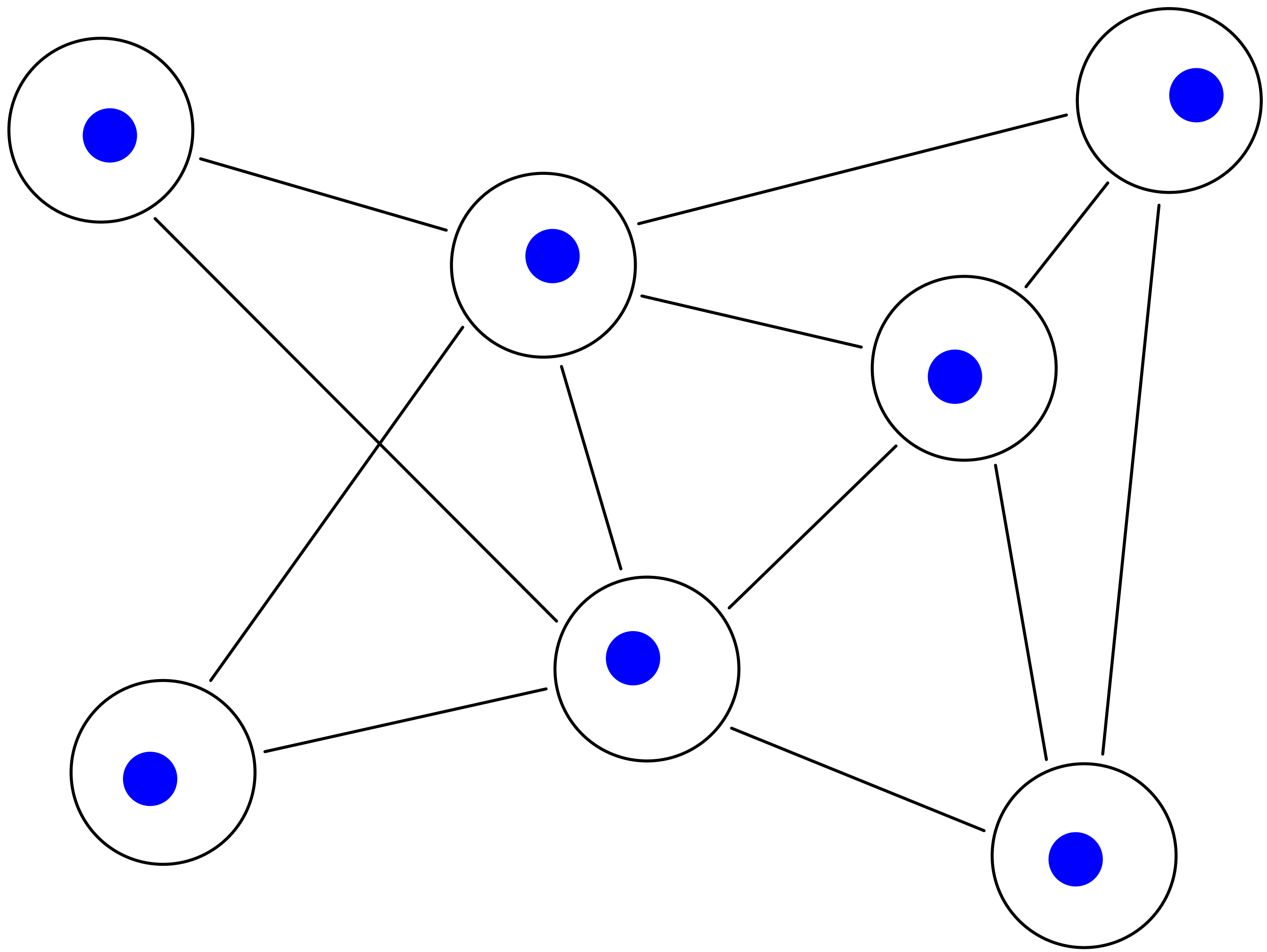

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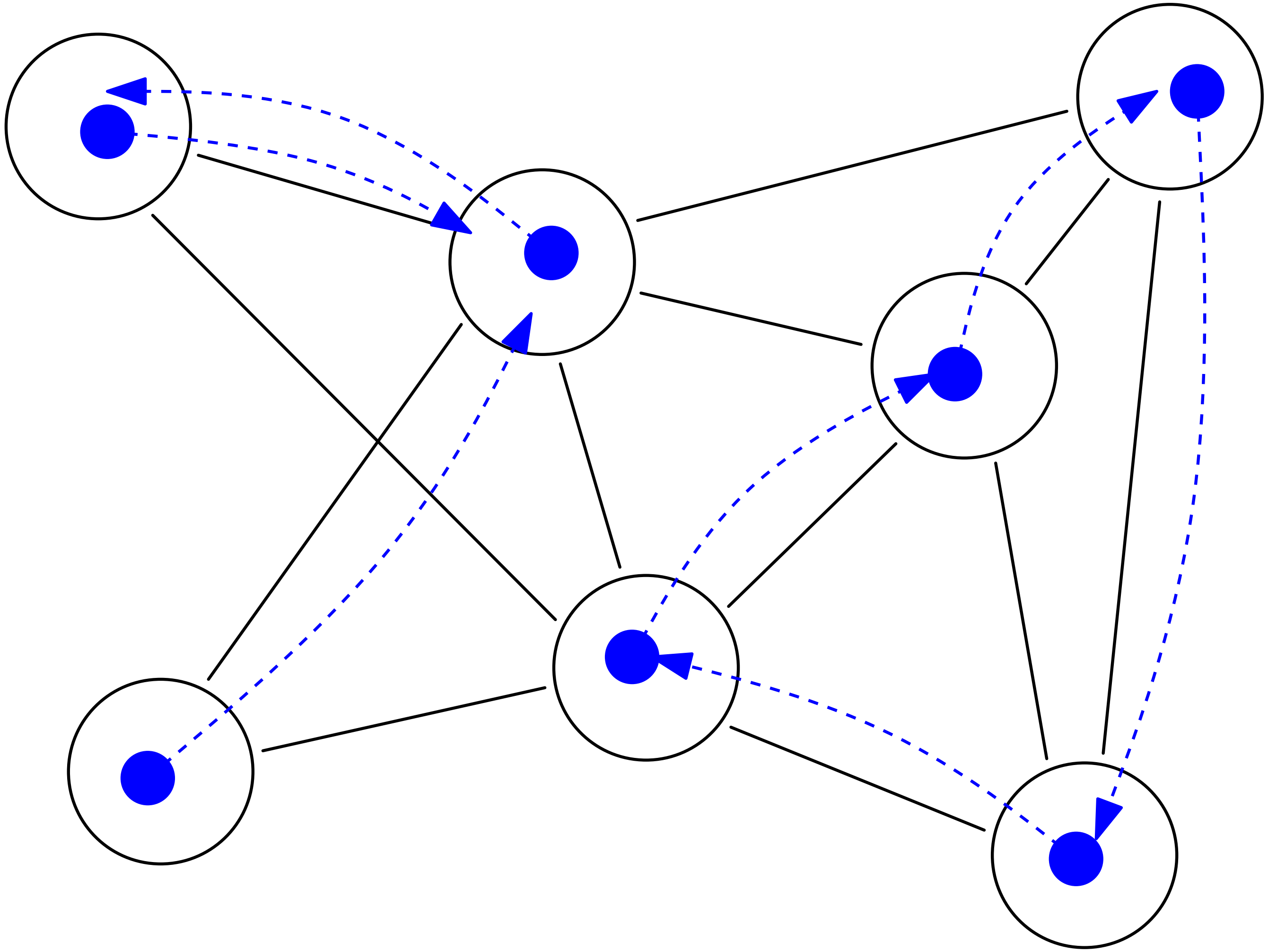

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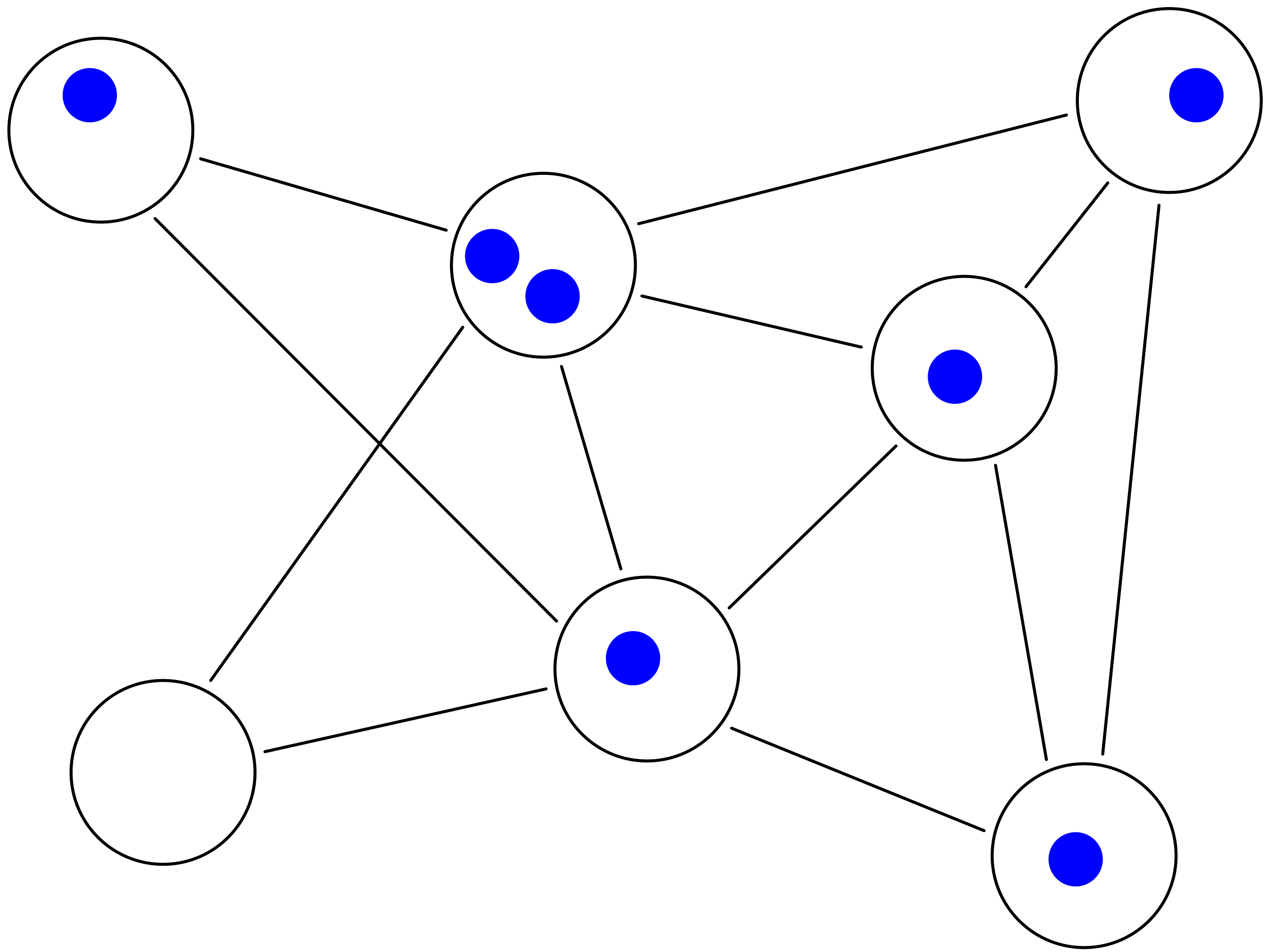

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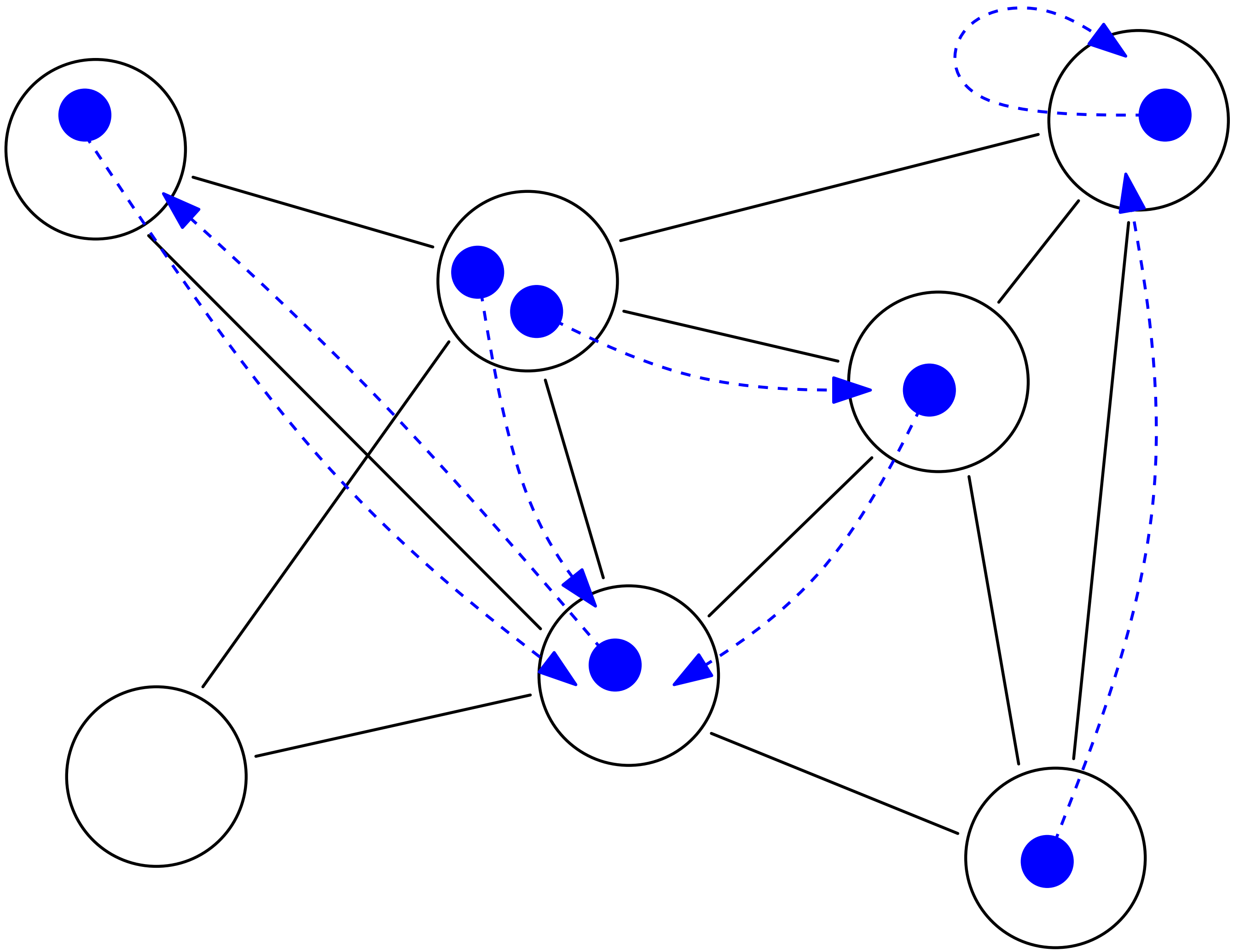

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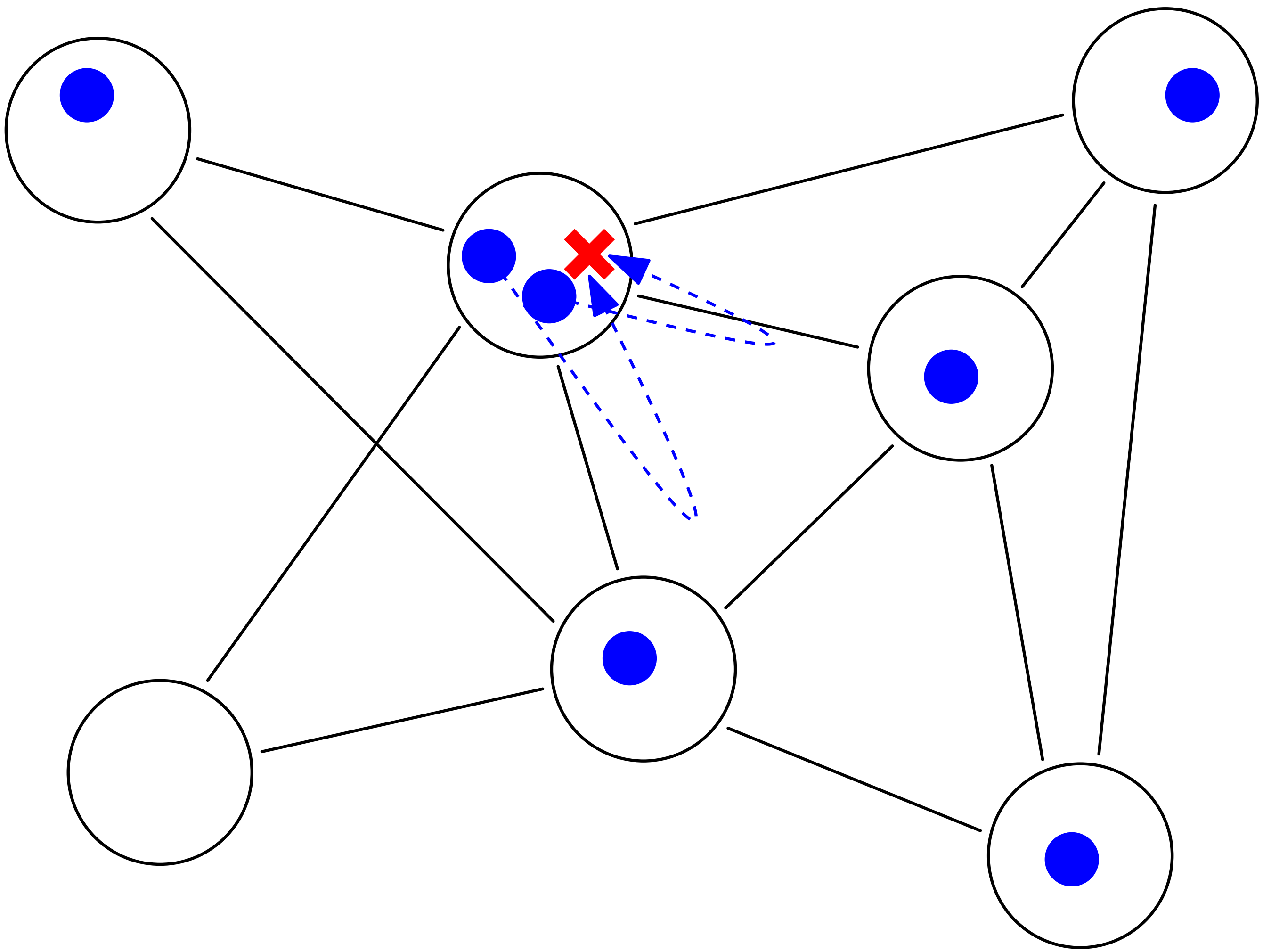

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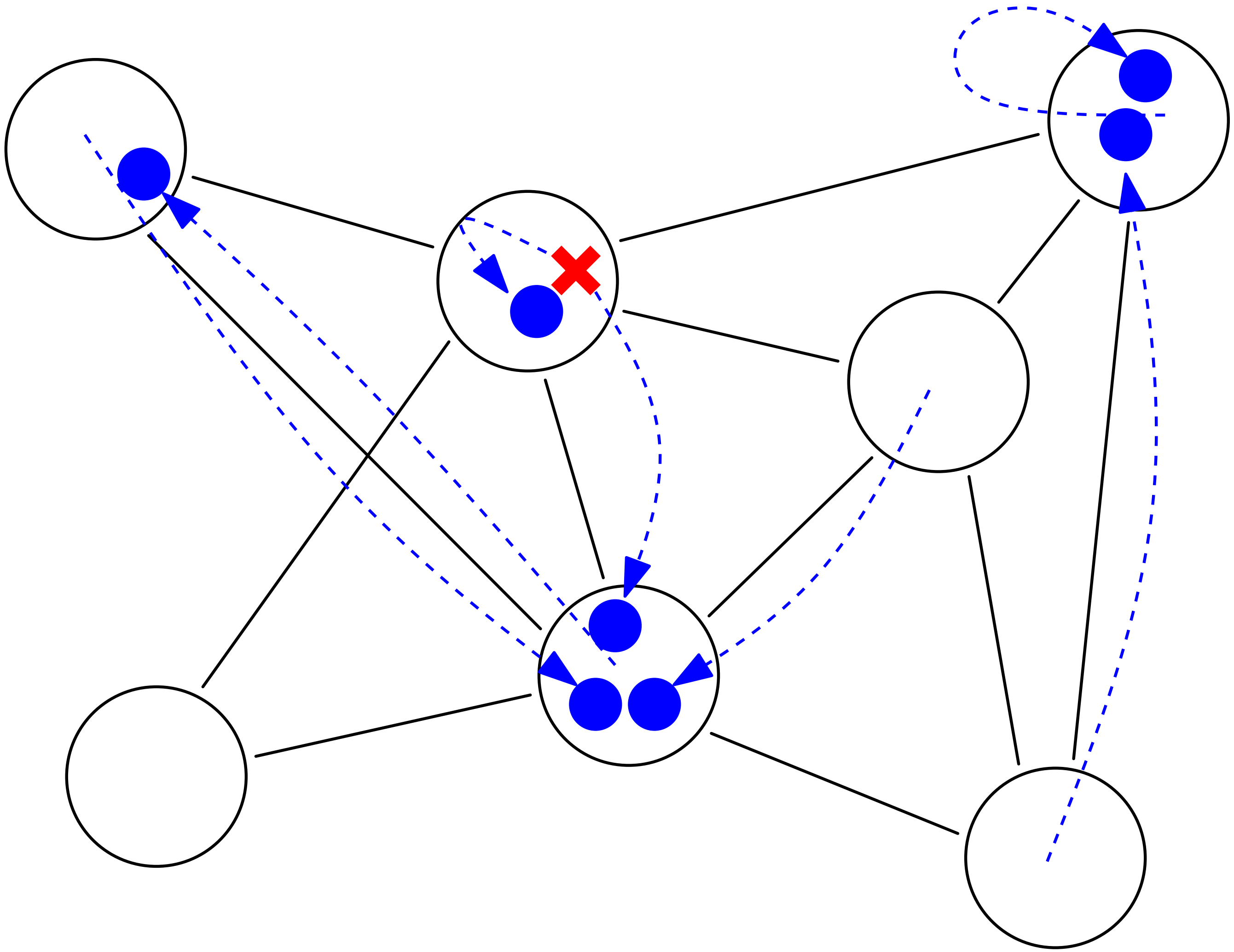

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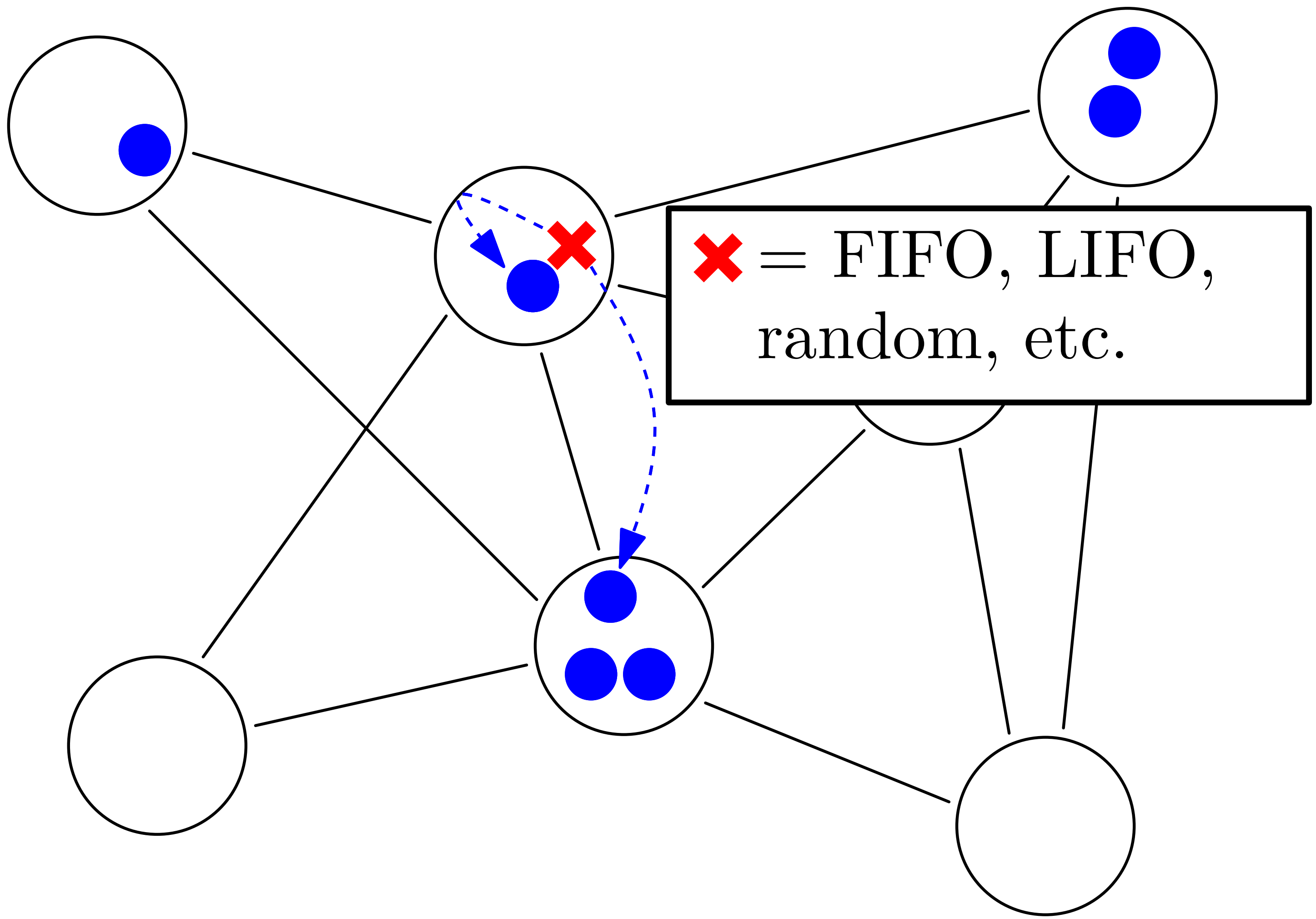

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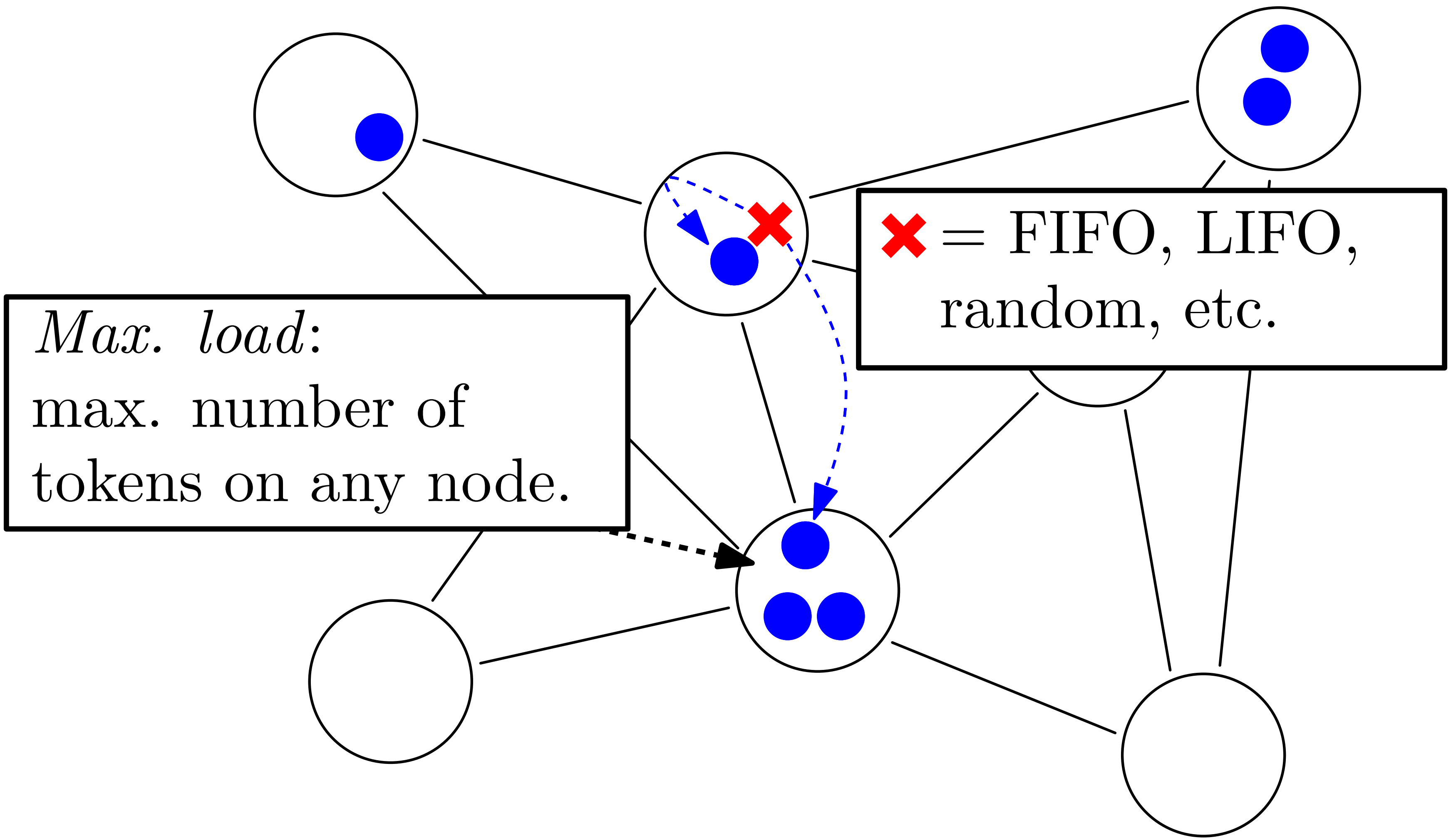

Some Related Work

Information exchange in phone-call model [Berenbrink et al. 2010, Elsässer et al. 2015]: analysis for polylog(n) rounds.

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Closed Jackson networks in queueing theory: asynchronous version of $\mathcal{GOSSIP}$ r.w.s (admits closed form solution).

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Repeated n balls in n bins =
n $\mathcal{GOSSIP}$ r.w.s on n-node complete graph
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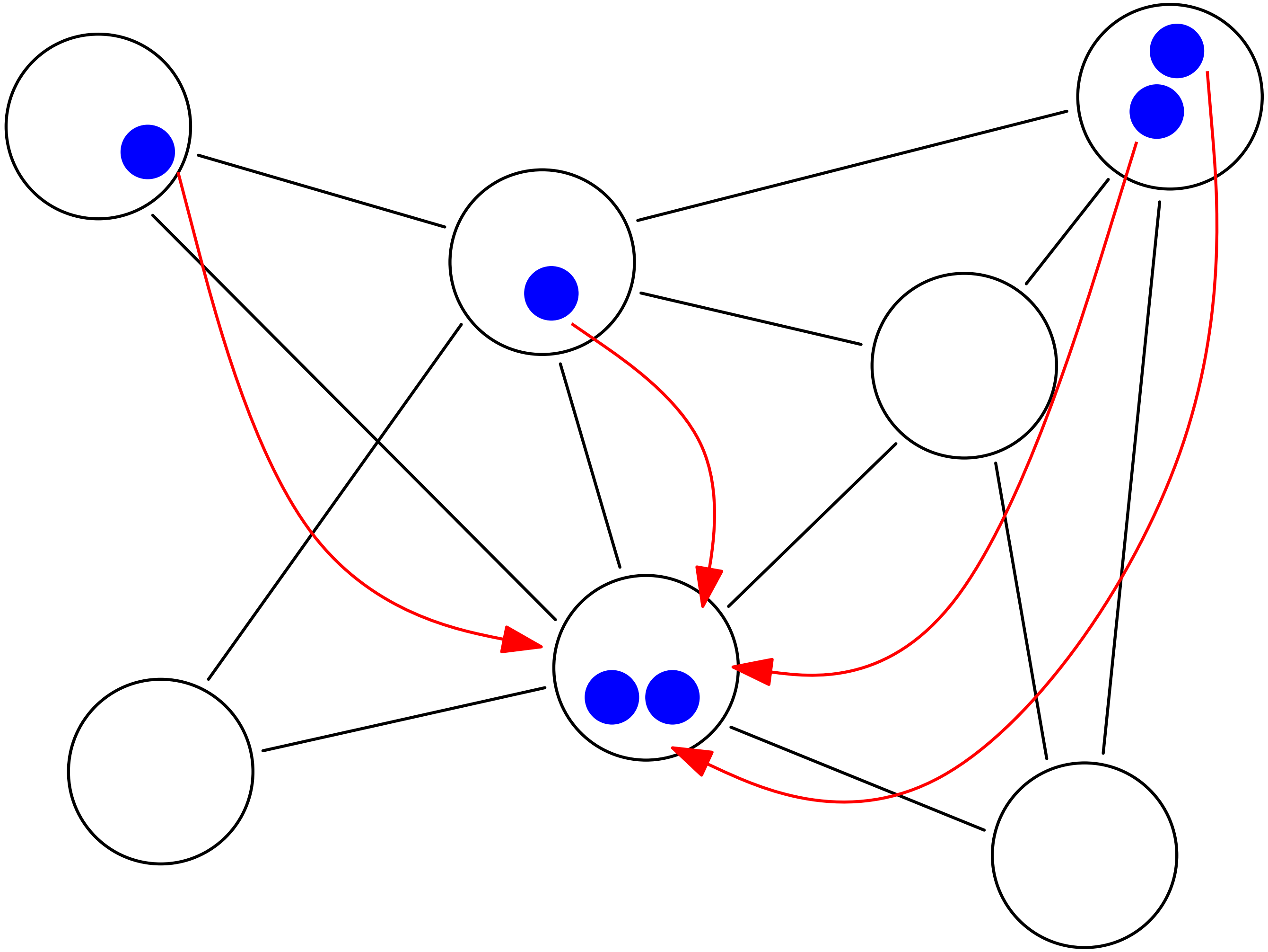
Corollary
After at most $\mathcal{O}(n)$ rounds the max. load of n $\mathcal{GOSSIP}$ r.w.s on n-node complete graph is $\mathcal{O}(\log n)$ w.h.p., and keeps $\mathcal{O}(\log n)$ for poly(n) rounds.

Adversarial Model

Every $\Omega(n)$ rounds: the adversary move the tokens
(cfr Adversarial Queuing Theory [Borodin et al., '01])



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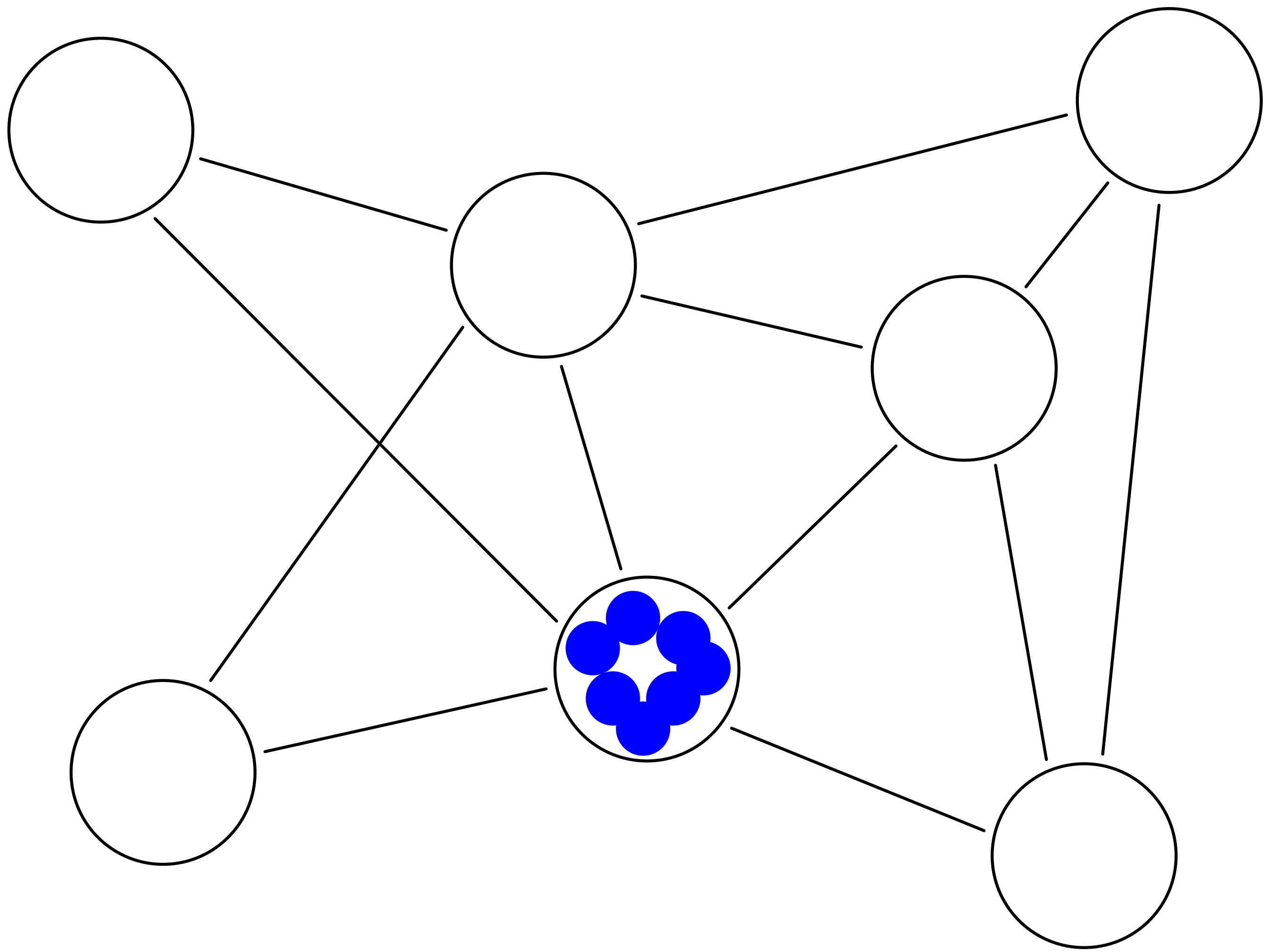

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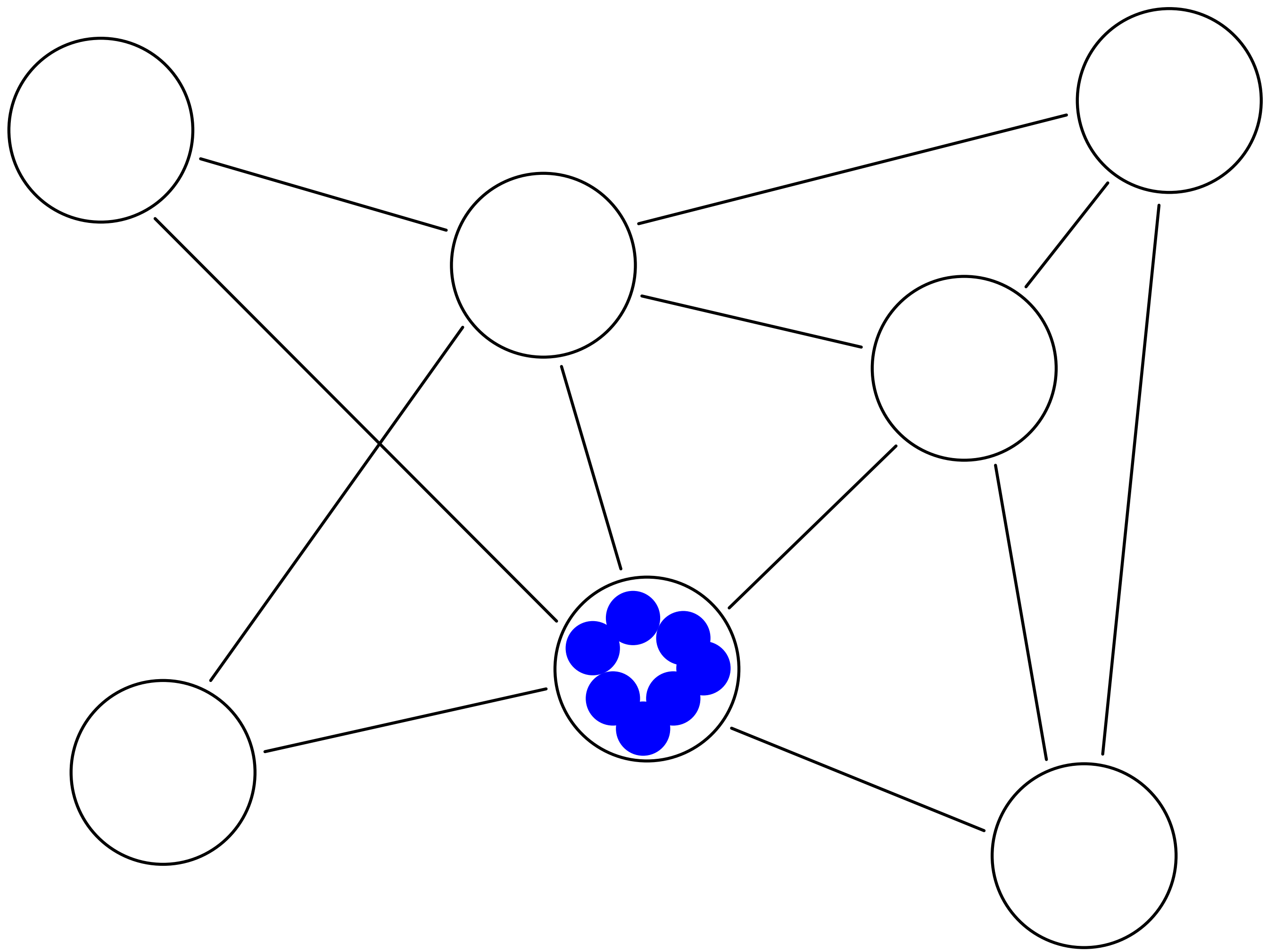

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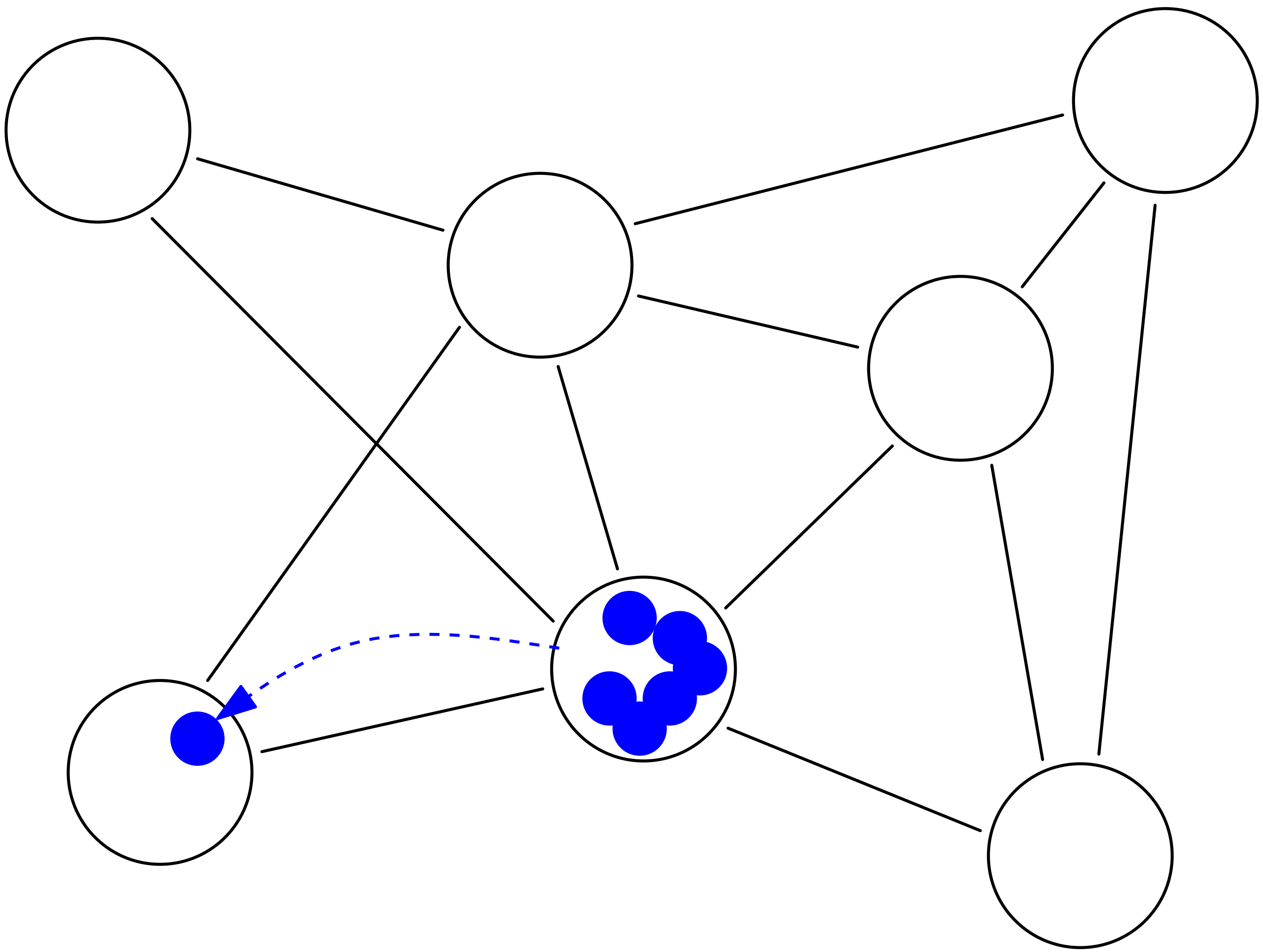

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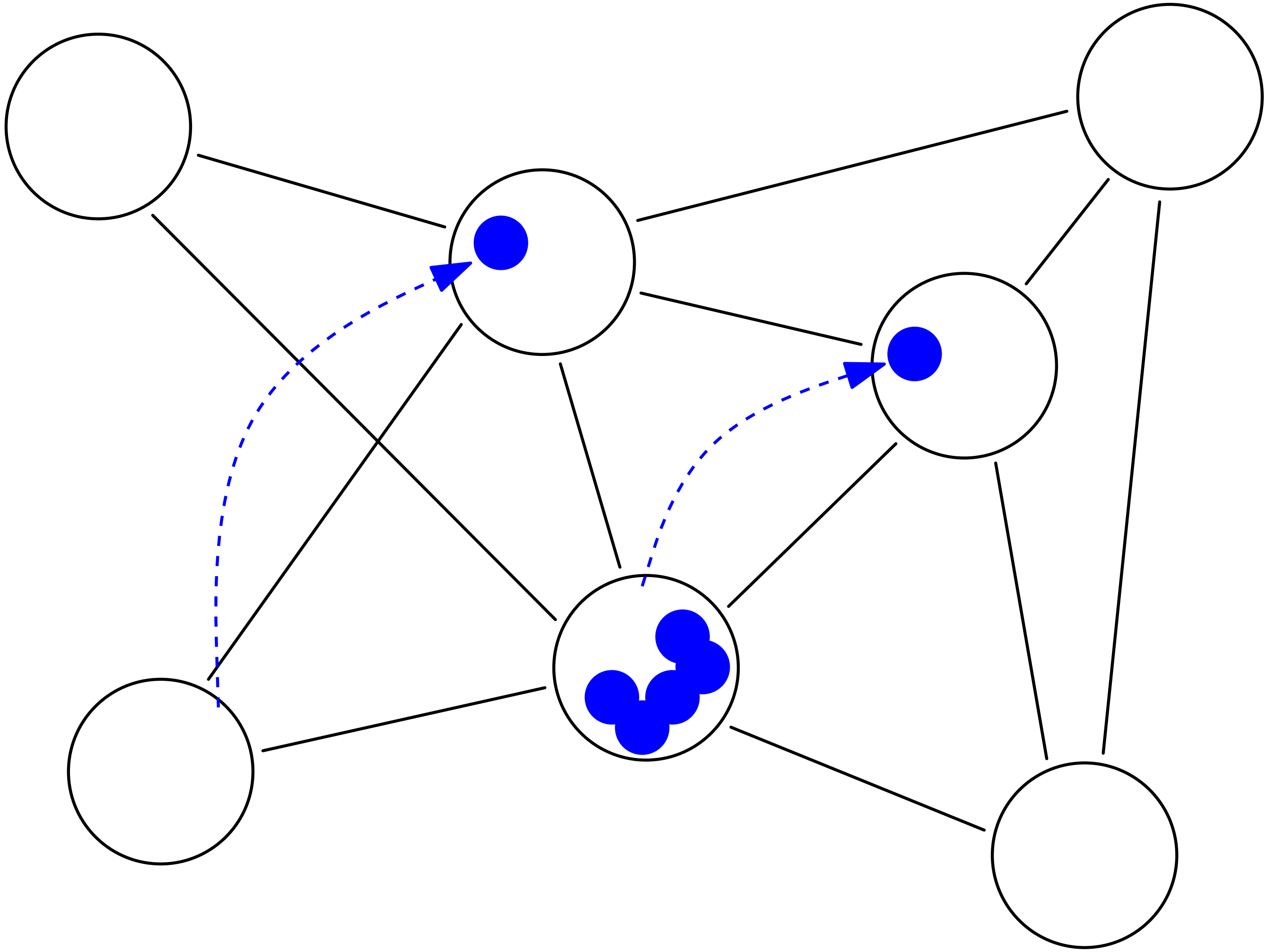

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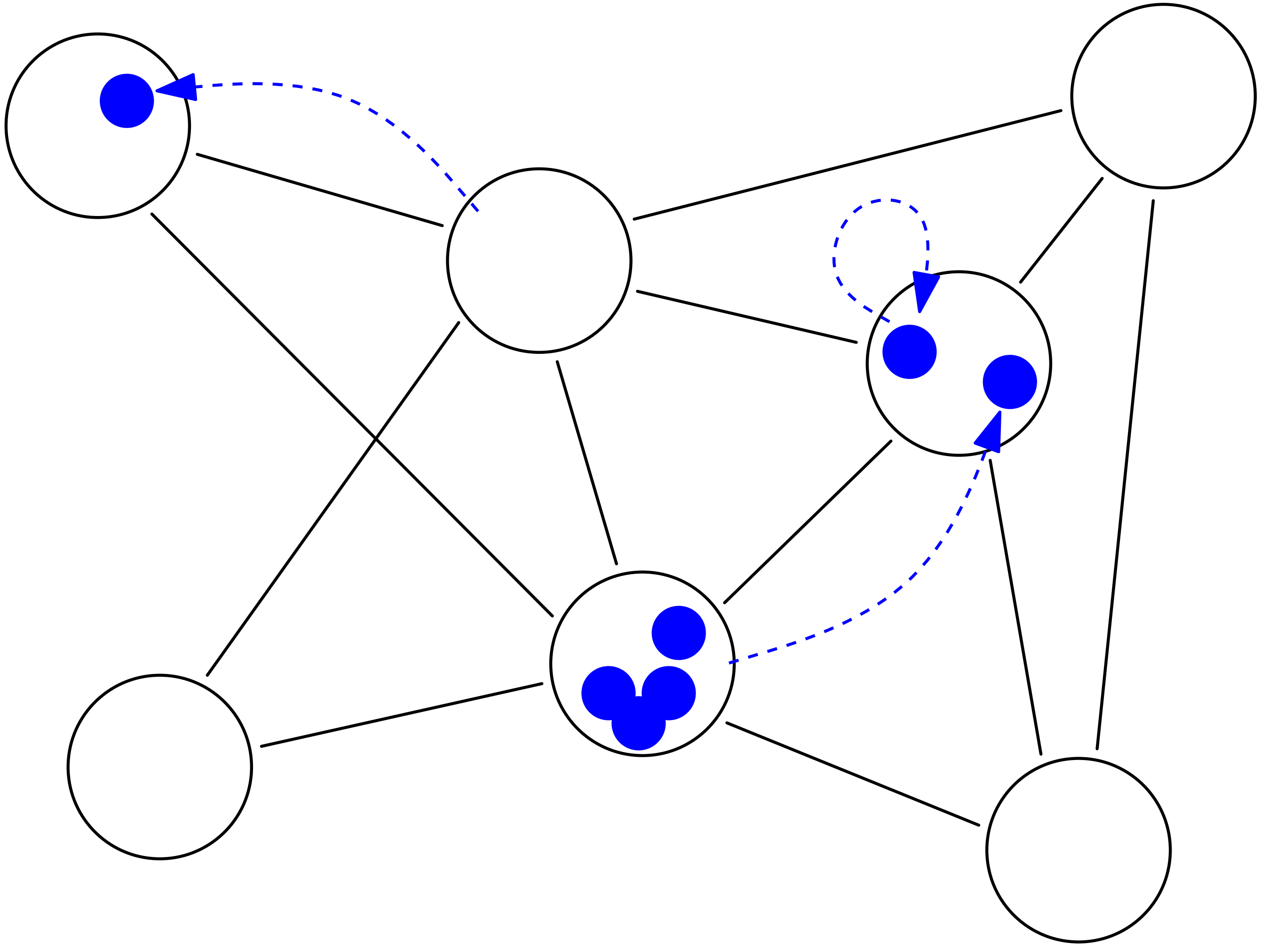

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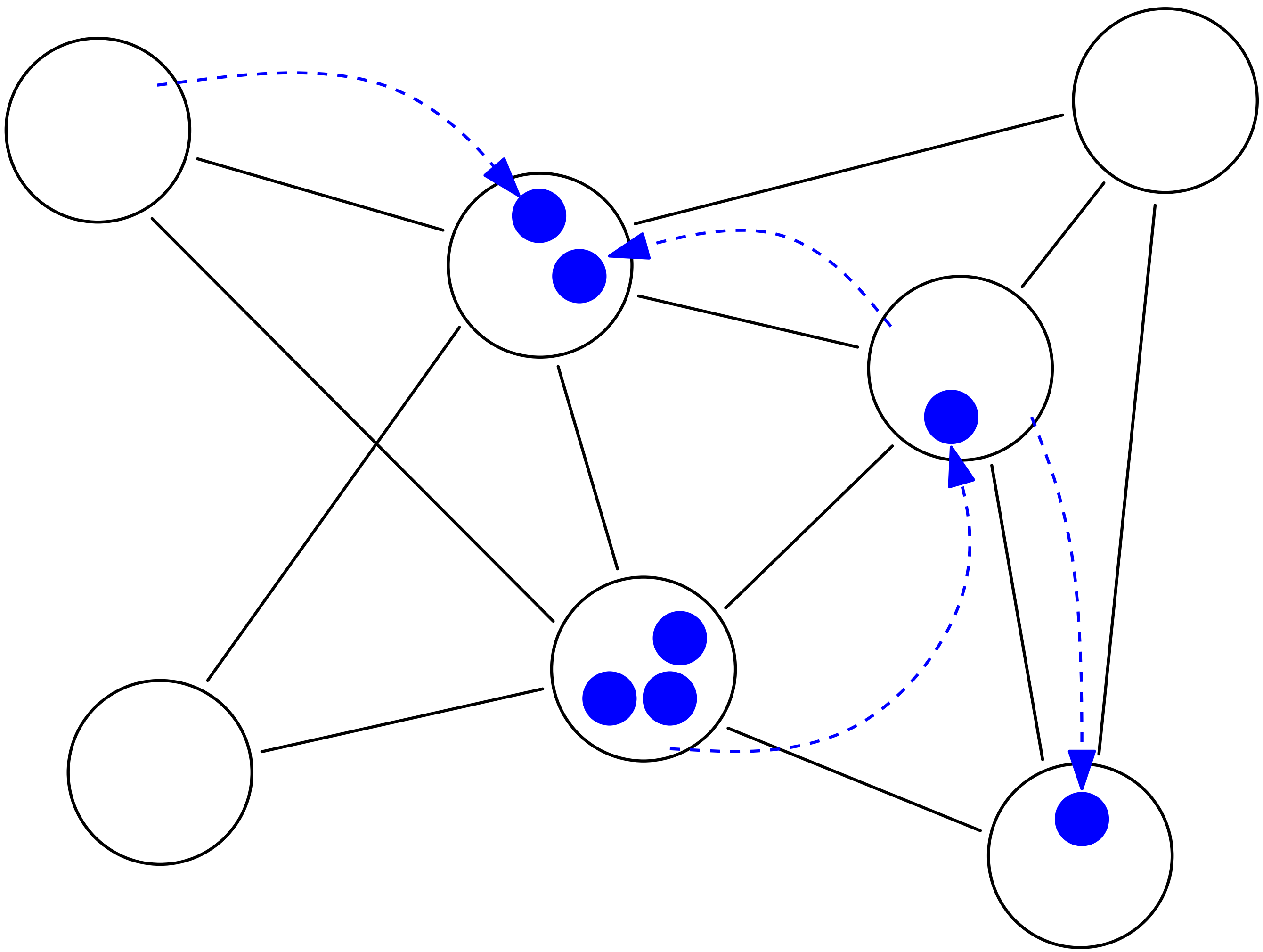

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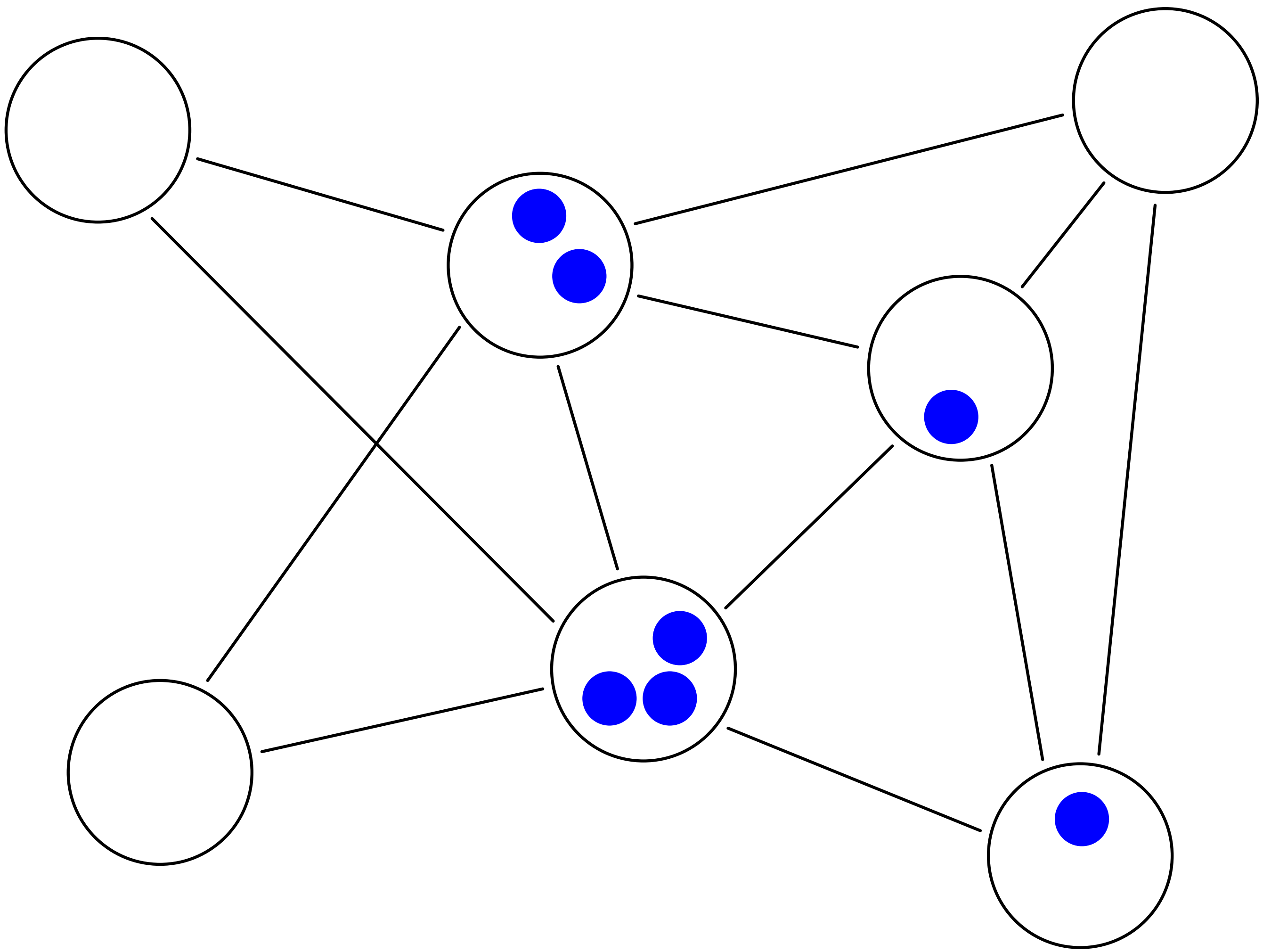

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Task assignment in mutual exclusion:
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Corollary
Cover time of n $\mathcal{GOSSIP}$ r.w.s on n-node complete graph is $\mathcal{O}(n \log^2 n)$ w.h.p.

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Stochastic dependence in balls-into-bins:
negative association,
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A coupling "w.h.p.": the tetris process

$M_t^{(RBB)}$:= time t max. load in repeated b.i.b.
$M_t^{(T)}$:= time t max. load in tetris proc.

$$\Pr(M_t^{(RBB)} \geq k) \leq \Pr(M_t^{(T)} \geq k) + t \cdot e^{-\Theta(n)}$$

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Proof

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3. Chernoff bound (negative association)

$\square$

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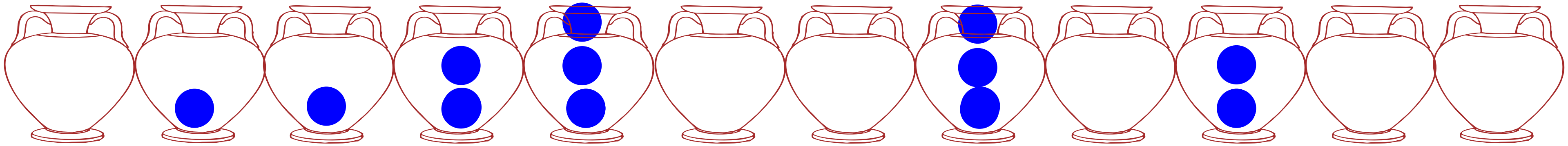
2- Throw $3n/4$ balls in the bins u.a.r.



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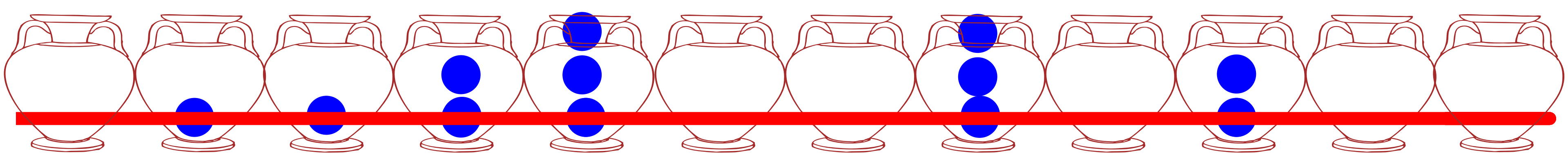

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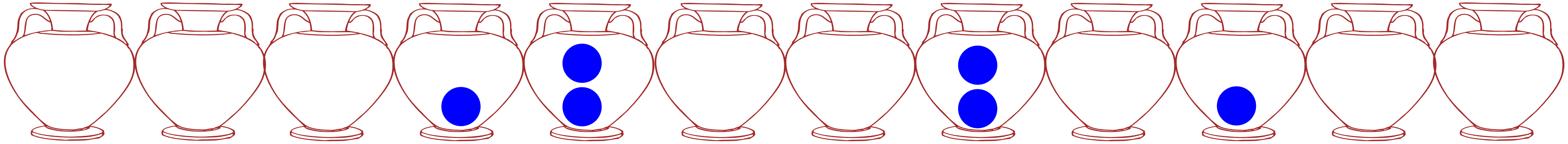

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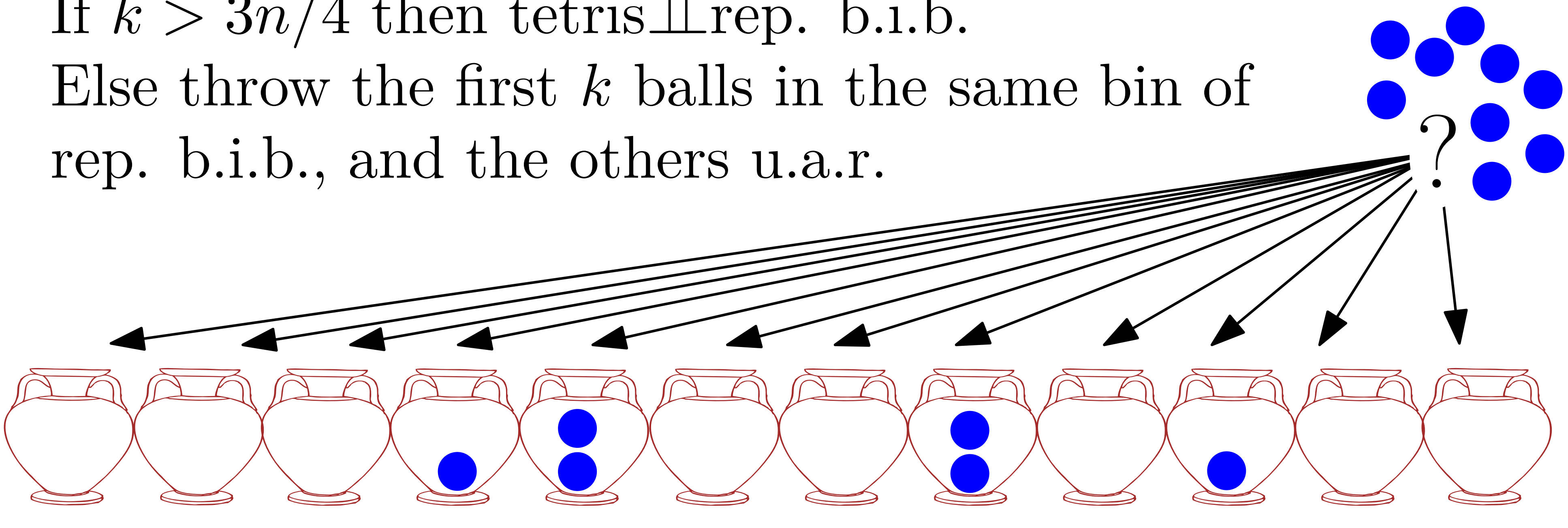

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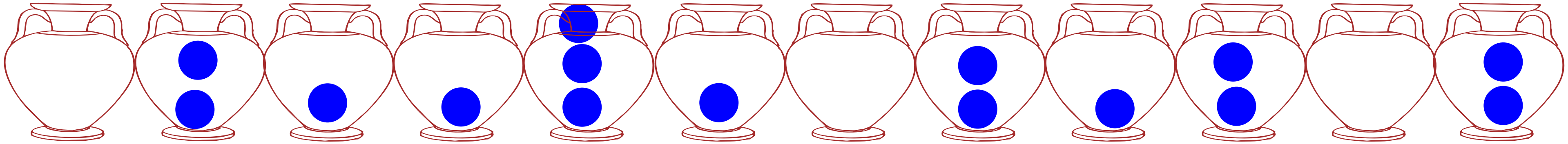

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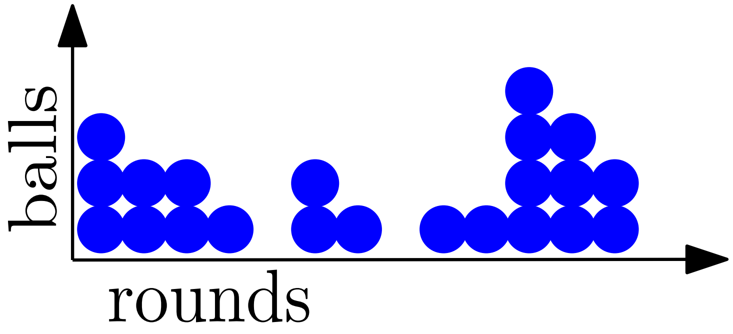

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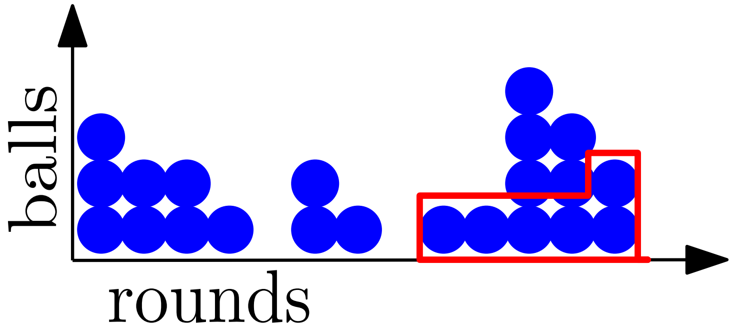

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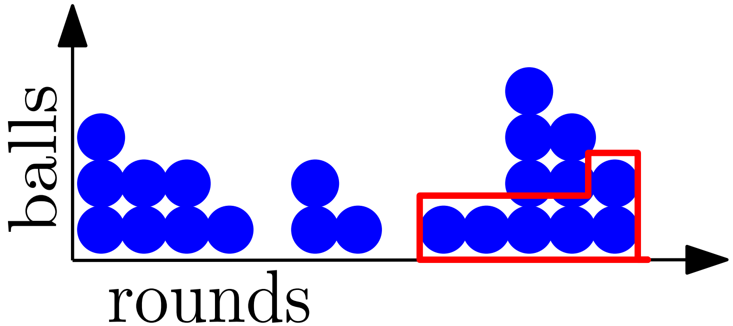

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□

Lemma

From any configuration, every bin in the tetris proc. is empty at least once every $5n$ rounds w.h.p.

Open Questions

$\mathcal{GOSSIP}$ **random walks**

Maximum load on other topologies?

On regular graphs?

On the ring?

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Repeated balls-into-bins

Maximum load of repeated

balls-into-bins with $\omega(n)$ balls?

$\Theta(n \log n)$ balls?

Thank You!

Self-stabilization, with high probability

$$\{\textit{legitimate}\text{ states}\} \subseteq \{\text{states of the system}\}$$

A system is self-stabilizing if:
- Starting from any state, reaches a *legitimate* state.
- If in a *legitimate* state, visits only *legitimate* states.

Self-stabilization, with high probability

$\{\textit{legitimate}\text{ states}\} \subseteq \{\text{states of the system}\}$

A system is self-stabilizing w.h.p. if:
- Starting from any state, reaches a *legitimate* state w.h.p.
- If in a *legitimate* state, visits only *legitimate* states for poly(n) rounds w.h.p.

Adversary-resilient

Here: legitimate = maximum load $\mathcal{O}(\log n)$