Plurality Consensus in the Gossip Model

Emanuele Natale

Sapienza Università di Roma

joint work with
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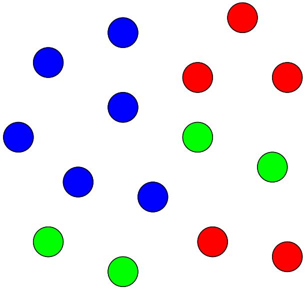
The Plurality Consensus Problem

- We have a set of nodes each having one color out of $\{1, \ldots, k\}$.



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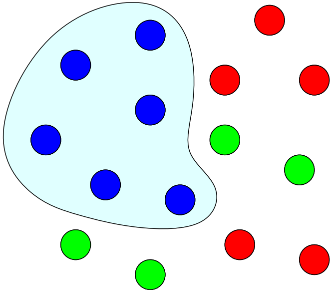

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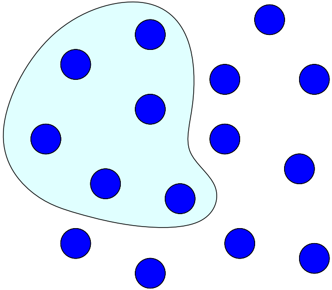
- We want to reach consensus on the plurality color.



Motivations and Applications

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- **Chemestry**: chemical reaction networks/population protocols (Angluin et al. '07).

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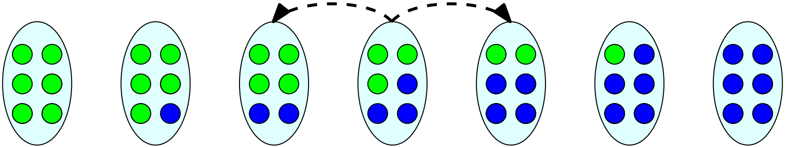
Probabilistic Polling (Peleg '99). Time divided in discrete rounds. All nodes *simultaneously* take the opinion of a random neighbor.
$\rightarrow$ Discrete time (parallel/synchronous) process. Initiated the study of Plurality Consensus in Computer Science.

Asynchronous vs Synchronous



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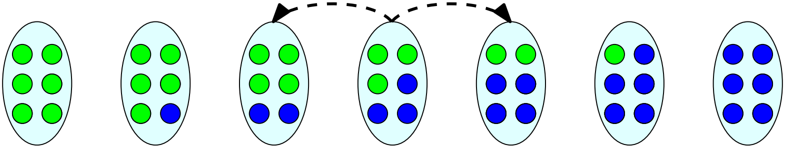
Asynchronous vs Synchronous



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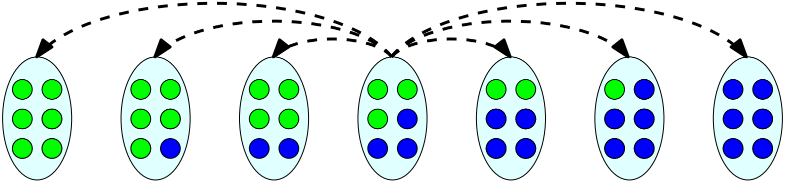

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- **Local memory and message size**: $O(\log n)$.

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Every task that can be solved in the $\mathcal{LOCAL}$ model in T rounds, can be solved in $O(T + \text{polylog} n)$ rounds in the $\mathcal{GOSSIP}$ model.
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But…using the preceding theorem, message size grows dramatically!

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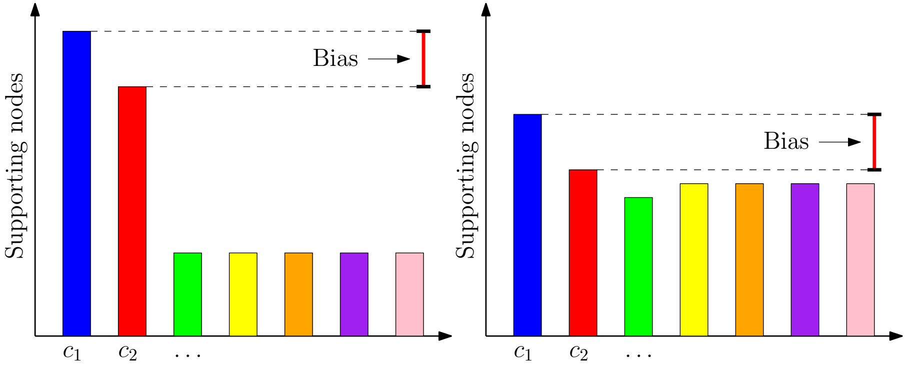
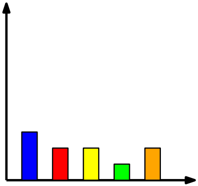
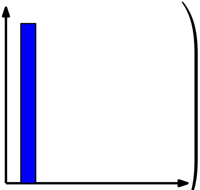


The Monochromatic Distance

$$\mathrm{md}(\mathbf{c}^{(\mathbf{0})}) := \sum_{i=1}^{k} \left(\frac{c_i^{(0)}}{c_1^{(0)}} \right)^2 = 1 + \mathcal{D}$$





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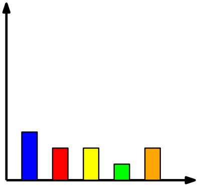
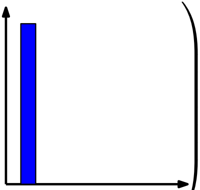
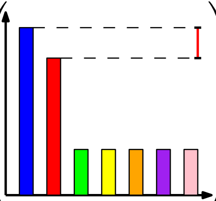
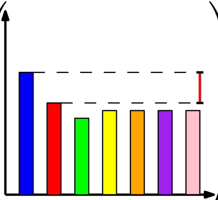
$$\mathrm{md}(\mathbf{c}^{(\mathbf{0})}) := \sum_{i=1}^{k} \left(\frac{c_i^{(0)}}{c_1^{(0)}} \right)^2 = 1 + \mathcal{D}\Bigg($$  ,  $$\Bigg)$$

$$1 \leq \mathrm{md}\Bigg($$  $$\Bigg) \ll \mathrm{md}\Bigg($$  $$\Bigg) \leq k$$

Our Results

First analysis for $k = \omega(1)$ of the Undecided-State Dynamics [Angluin et al., Perron et al., Babaee et al., Jung et al.]:

Upper Bound

If $k = O\left((n/\log n)^{1/3}\right)$ and $c_1 \geq (1+\epsilon) \cdot c_2$ with $\epsilon > 0$, then w.h.p. the Undecided-State Dynamics reaches plurality consensus in $O\left(\mathrm{md}(\mathbf{c}^{(0)}) \cdot \log n\right)$ rounds.

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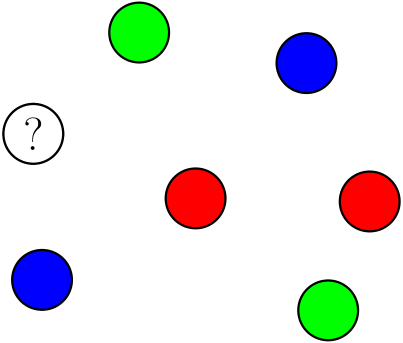
Lower Bound

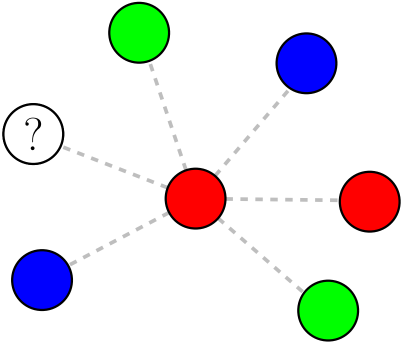
If $k = O\left((n/\log n)^{1/6}\right)$ then w.h.p. the Undecided-State Dynamics converges after at least $\Omega(\mathrm{md}(\mathbf{c}^{(0)}))$ rounds.

The Undecided-State Dynamics



Some nodes can be “undecided”.

The Undecided-State Dynamics



At the beginning of each round, each node observes a neighbor picked uniformly at random.

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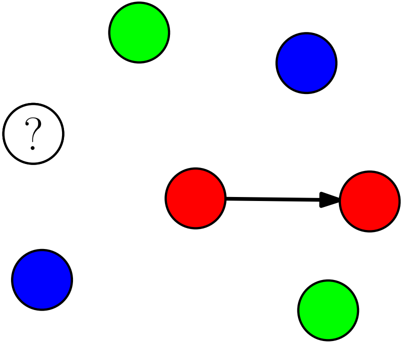
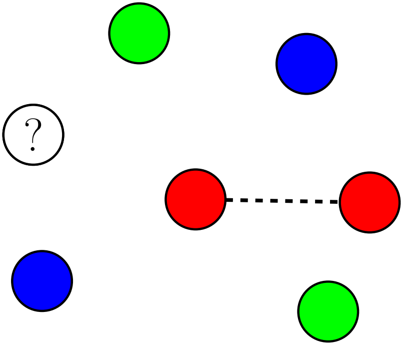


If the observed node shares the same color...

The Undecided-State Dynamics



. . . nothing happens;

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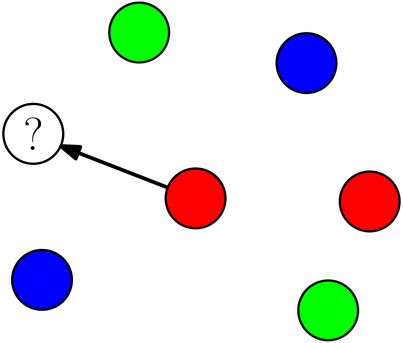

if the node observes an undecided one...

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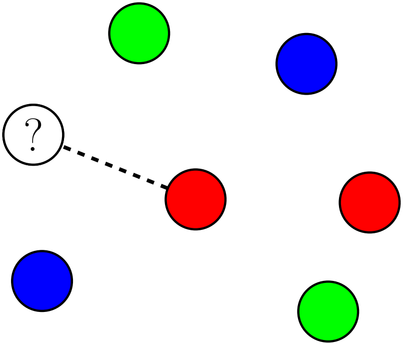
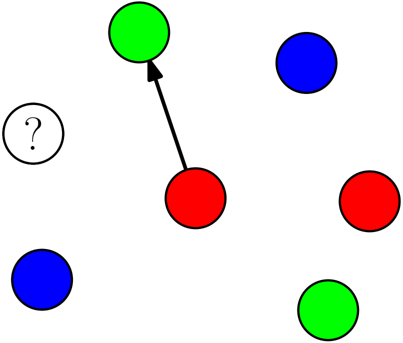


. . . nothing happens too;

The Undecided-State Dynamics



but, if the observed node has a different color...

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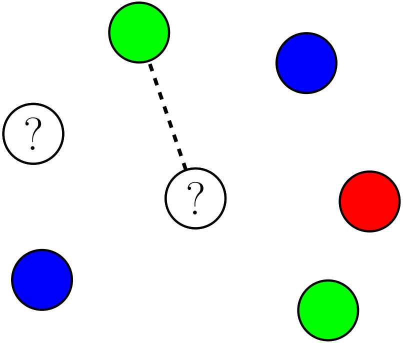
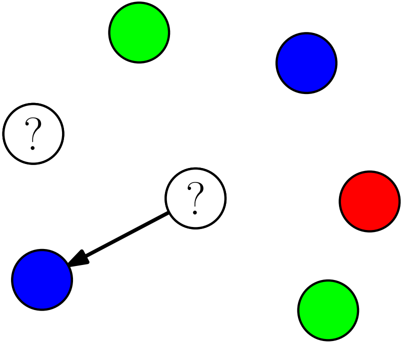


. . . then the node becomes undecided.

The Undecided-State Dynamics



Once undecided. . .

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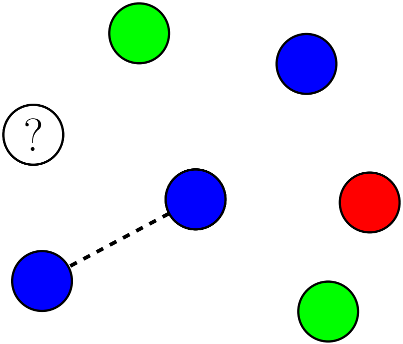

...the node copies the first color it sees.

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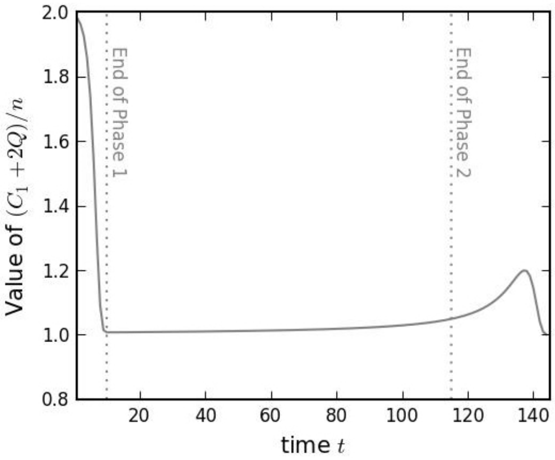
Remarks

W.h.p.:

- Plurality does not change.
- Growth factor of plurality is > 1.

Simulation of the growth factor:



Expected Behaviour of the Process

$$\begin{cases} \mathbf{E}\left[q^{(t+1)} \;\middle|\; \mathbf{c}^{(t)}\right] = \frac{1}{n}\left[\left(q^{(t)}\right)^2 + \left(n - q^{(t)}\right)^2 - \sum_i \left(c_i^{(t)}\right)^2\right] \\ \mathbf{E}\left[c_1^{(t+1)} \;\middle|\; \mathbf{c}^{(t)}\right] = c_1^{(t)} \cdot \frac{c_1^{(t)} + 2q^{(t)}}{n} \\ \qquad \vdots \\ \mathbf{E}\left[c_k^{(t+1)} \;\middle|\; \mathbf{c}^{(t)}\right] = c_k^{(t)} \cdot \frac{c_k^{(t)} + 2q^{(t)}}{n} \end{cases}$$

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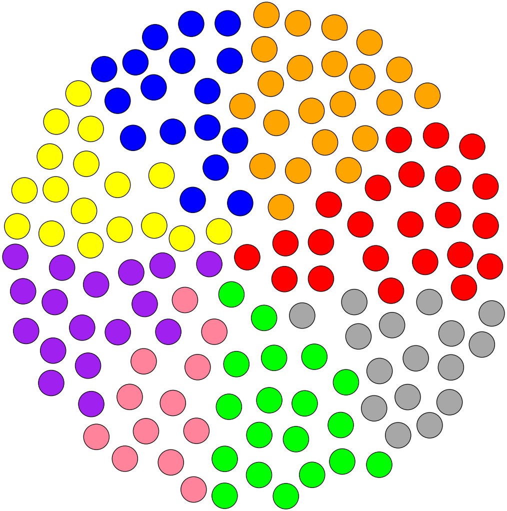
Lemma

$$\mathbf{E}\left[\frac{c_1^{(t+1)} + 2q^{(t+1)}}{n} \,\middle|\, \mathbf{c}^{(t)}\right] =$$

$$= 1 + \frac{\left(n - 2q^{(t)} - c_1^{(t)}\right)^2}{n^2} + \frac{2\left(R(\mathbf{c}^{(t)}) - \mathrm{md}(\mathbf{c}^{(t)})\right) \cdot (c_1)^2}{n^2}$$

First Round

Round 1: Each node observes another random one.



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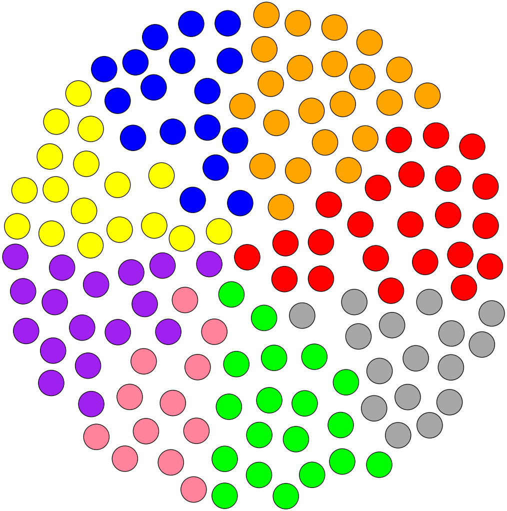
The larger the number of colors and the more uniform the initial distribution, the higher the expected number of undecided nodes.



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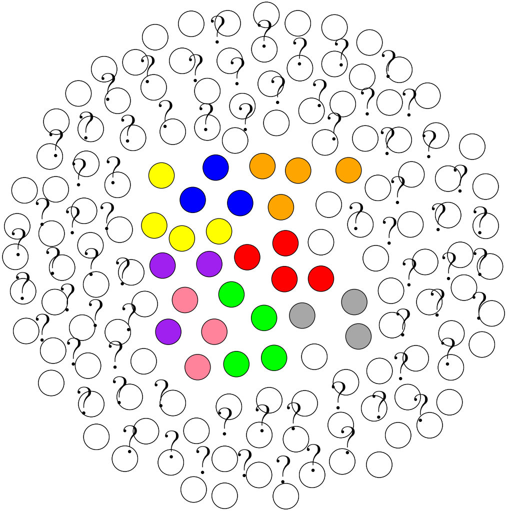

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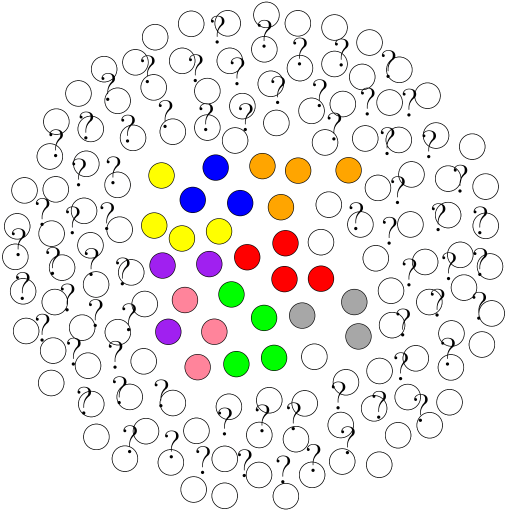
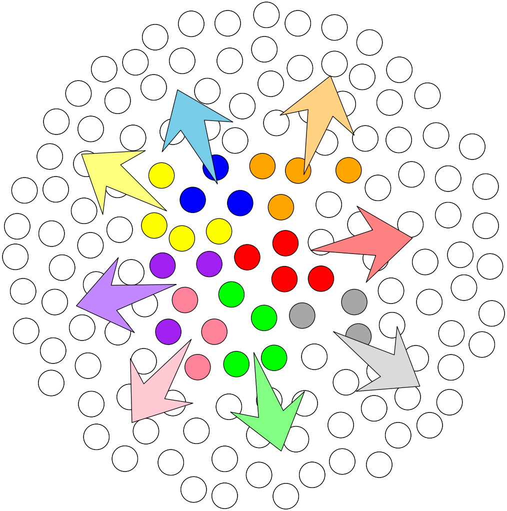
Colors with $c_i^{(0)} = O(\sqrt{n})$ nodes are likely to disappear.



Phase 1

If the initial distribution is quite uniform there are $\Omega(n)$ undecided nodes.

Undecided nodes take the first color they pull, causing colors to spread very fast.



Phase 1

Lemma

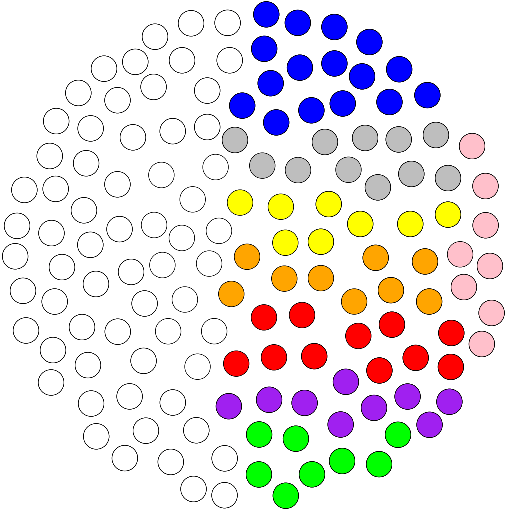
Within $T = O\left(\log \frac{R(\mathbf{c})^2}{\mathrm{md}(\mathbf{c})}\right)$ *rounds the system reaches a configuration such that w.h.p.*

$$c_1^{(T)} = \Theta\left(\frac{n}{\mathrm{md}(\mathbf{c})}\right)$$
$$q^{(T)} = \frac{n}{2}\left(1 \pm \Theta\left(\frac{1}{\mathrm{md}(\mathbf{c})}\right)\right)$$

and, for every i, $c_1^{(0)}/c_i^{(0)}$ *is approximately preserved.*

Phase 2

new colored
$\approx$
new undecided.



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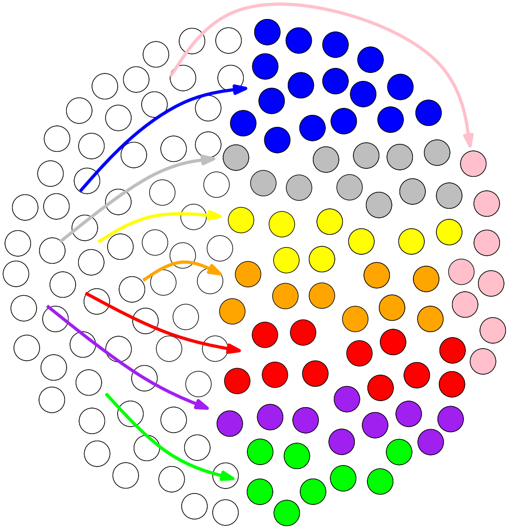

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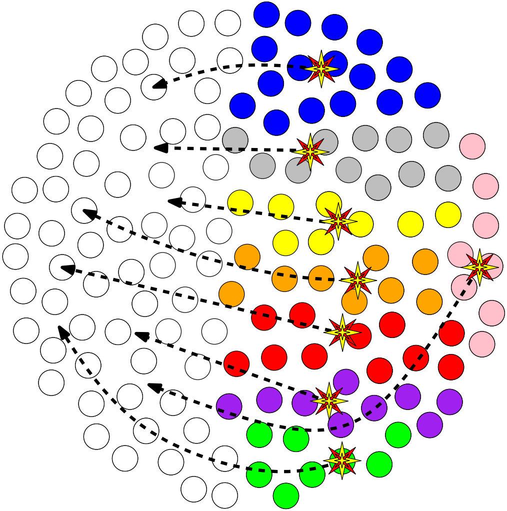

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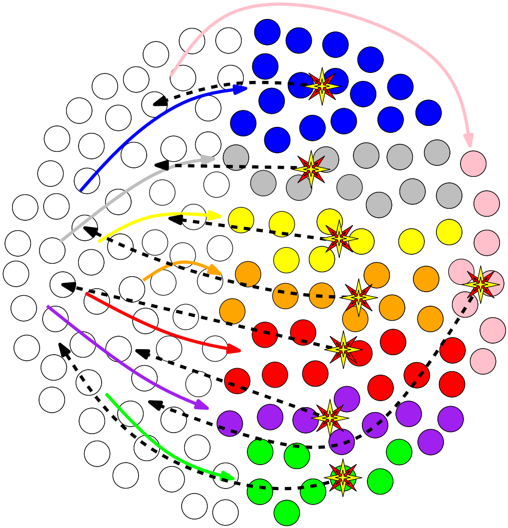

Phase 2

Average Growth:

$$\mathbf{E}\left[c_1^{(t+1)} \,\middle|\, \mathbf{c}^{(t)}\right] \approx c_1^{(t)} \left(1 + \Theta\left(\frac{1}{\mathrm{md}(\mathbf{c})}\right)\right)$$

$$\mathbf{E}\left[q^{(t+1)} \,\middle|\, \mathbf{c}^{(t)}\right] \approx \frac{n}{2} \left(1 - \Theta\left(\frac{1}{\mathrm{md}(\mathbf{c})}\right)\right)$$

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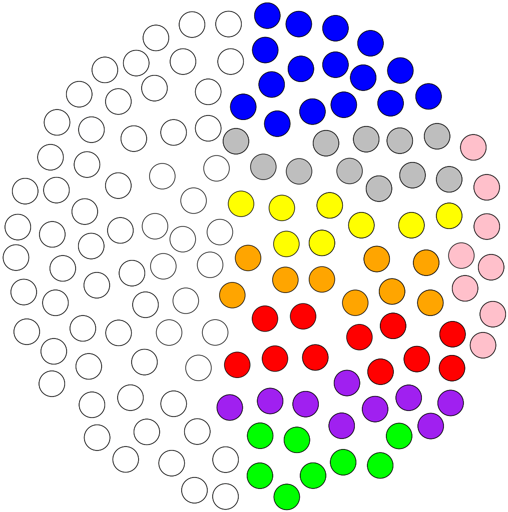
$\implies$ Lower bound of $\Omega\left(\mathrm{md}(\mathbf{c})\right)$.

Phase 2

The plurality has a small advantage $\Longrightarrow$ after long time the equilibrium breaks down.



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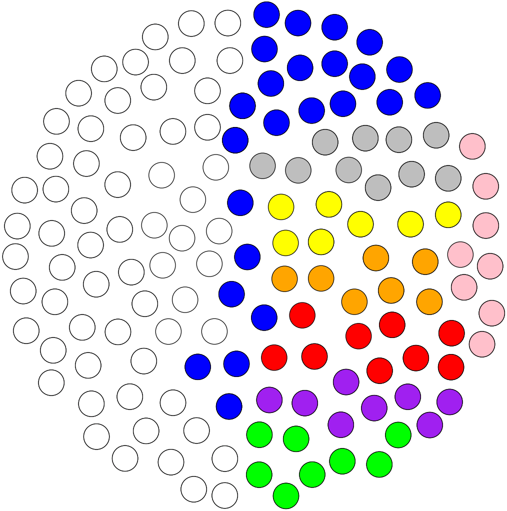

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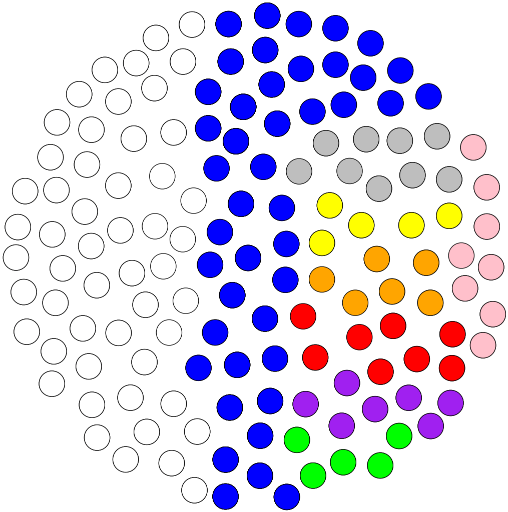

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$$\mathbf{E}\left[c_1^{(t+\text{md}(\mathbf{c}))} \,\middle|\, \mathbf{c}^{(t)}\right] \approx c_1^{(t)}\left(1+\Theta\left(\frac{1}{\text{md}(\mathbf{c})}\right)\right)^{\text{md}(\mathbf{c})}$$

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$\Longrightarrow$ After $O\left(\operatorname{md}(\mathbf{c})\log n\right)$ rounds, $R(\mathbf{c}^{(t)}) = 1 + o(1)$.

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$\implies$ Plurality Consensus is reached within $O(\log n)$ rounds.

Extension to d-Regular Expanders

Given a d-regular expander graph, $k = O\left((n/\log n)^{1/3}\right)$ and $c_1 \geq (1+\epsilon) \cdot c_2$ with $\epsilon > 0$, using polylogarithmic memory and message size the plurality consensus problem can be solved in w.h.p. $O(\mathrm{md}(\mathbf{c})\mathrm{polylog}(n))$ rounds.

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Issue. The $\mathcal{GOSSIP}$ model with $O(\mathrm{polylog} n)$ limit on message size: congestion when random walks meet.

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