Competitive routing in networks with polynomial cost

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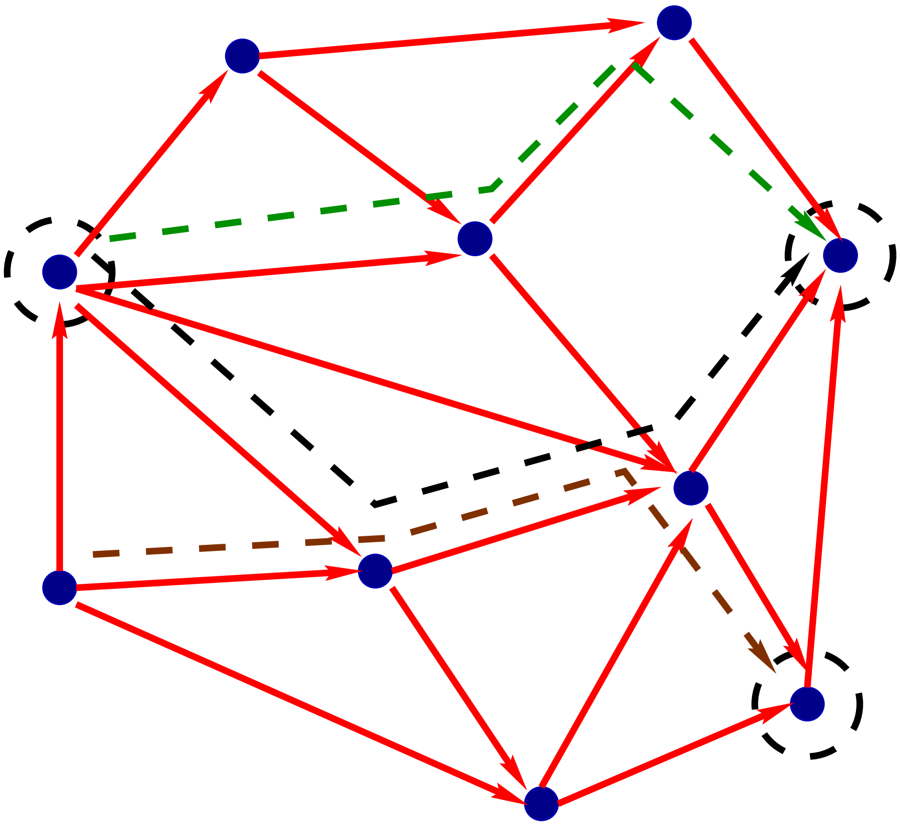


Figure 1: Competitive Routing

The Model

- N players (connections, or groups of connections) have to determine their routes.
- Each player minimises its own cost (e.g. its average delay).
- The optimisation concept is **Nash Equilibrium:** a routing strategy for each player such that no player can benefit from unilateral deviation.
- Problem: The Nash equilibrium is often inefficient.

Objectives:

- Design a pricing that yields an efficient solution.
- Study the uniqueness of the equilibrium.
- Computation of the equilibrium.

Main results

- We use a class of polynomial cost functions already used in the context of road traffic by the US Bureau of Public Roads (US Department of Commerce, 1964).

The cost for link l when there is a flow λ:

$$f_l(\lambda) = a(l)\lambda^{p(l)} + b_l, \qquad p(l) > 0.$$

Player i's cost is proportional to its average link costs:

$$J^i(\lambda_l^i, \lambda_l, l \in L) = \sum_{l \in L} \lambda_l^i f_l(\lambda_l).$$

We show for this cost (under some assumptions) Uniqueness, Efficiency and provide some Computations.

basic game theory notions: a reminder

Example: The Prisoner's Dilemma

- Emil and France are suspected of a crime.
- If both admit, they get 1 year in prison each.
- If both do not admit, they get 10 year in prison each.
- If one admits, he gets 15 years and the other is freed.

		FRANCE 1	FRANCE 2
EMIL	I	(1,1)	(15,0)
EMIL	II	(0,15)	(10,10)

Figure 2: Prisoner's Dilemma

- The solution (1,1) is unstable: each prisoner may try to improve by deviating.
- Only stable solution is (10,10).
- **Nash Equilibrium:** A "stable" solution: no player has an incentive to deviate unilaterally.
- **Pareto Equilibrium:** A cooperative "dominating solution": there is no way to improve the performance of one player without harming another one.

Inefficiency of Nash equilibria

FRANCE

EMIL	1	2
I	(1,1)	(15,0)
II	(0,15)	(10,10)

Pareto Equilibrium

Nash Equilibrium

Nash Equilibria may be very inefficient!

Non-monotonicity of Nash equilibria

FRANCE (columns) / **EMIL** (rows)

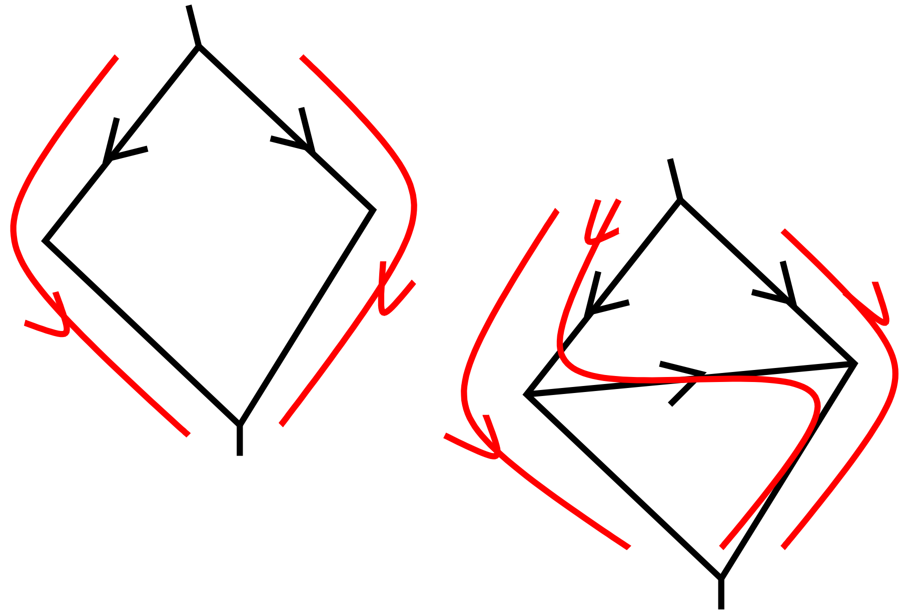
	1	2
I	(1,1)	(15,5)
II	(5,15)	(10,10)

Assume we change the "0" to "5": a prisoner that does not admit is always punished (we may have prior information that the crime was committed together).

(1,1) is Nash equilibrium and unique Pareto solution!

Non-monotonicity in networks

Adding a link, or adding capacity to a link, increases delay for all users: **Braess Paradox**.



The paradox occurs in competitive routing (several users optimise). But also in the case of network optimisation using Bellman-Ford type algorithms (RIP).

References:

- D. Braess, "Uber ein paradoxen der werkehrsplannung", *Unternehmenforschung*, **12**, 256-268, 1968.

- Y. A. Korilis, A. A. Lazar and A. Orda, "Avoiding the Braess Paradox in Noncooperative Networks", *Proc. IEEE Conference on Decision & Control (San Diego)*, 864-878, 1997.

The model

Player i has to

- ship a flow with rate Λ_i between a source $s(i)$ and a destination $d(i)$.
- determine his link rates $\{\lambda_l^i\}$, $l \in L$.

Flow constraints:

$$r_i(x) + \sum_{l \in In(x)} \lambda_l^i = \sum_{l \in Out(x)} \lambda_l^i, \tag{1}$$

$\forall x \in V, \lambda_l \geq 0$, where

$$r_i(x) = \begin{cases} \Lambda^i & \text{if } x = s(i), \\ -\Lambda^i & \text{if } x = d(i), \\ 0 & \text{otherwise.} \end{cases}$$

Uniqueness of equilibrium

Recall: cost for link l when there is a flow λ:

$$f_l(\lambda) = a(l)\lambda^{p(l)} + b_l, \qquad p(l) > 0.$$

Define

$$p^* \stackrel{\text{def}}{=} 1 + \frac{2 + \sqrt{2N^2 + 2N}}{N - 1}$$

(Note that $p^* > 1 + \sqrt{2}$ for any $N \geq 2$).

Theorem 1 *Consider the BPR cost with* $0 < p(l) < p^*$ *for all links, and a general topology. Then the Nash equilibrium is unique.*

Proof: Define

$$G_l(\lambda_l) \stackrel{\text{def}}{=} \left\{ \frac{\partial^2 J_l^i}{\partial \lambda_l^i \partial \lambda_l^j}(\lambda_l) \right\}_{i,j}$$

If $G + G^T > 0$ then the equilibrium is unique see [Orda, Rom, Shimkin, 1993] based on [Rosen, 1965].

For example, if $p(l) = 1$ then

$$G_l = a_l \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix}$$

It is easy to see that $G_l + G_l^T > 0$ (sum of I and a nonnegative definite matrix).

With some algebra we establish this for all $p(l)$.

Efficiency of the equilibrium

We consider the following cost:

$$f_l(\lambda_l) = a(l)\lambda_l^p, \qquad p > 0.$$

All users have the same sources and destinations.

Global optimisation: determine the link flows $\overline{\lambda}$ that minimise

$$J(\lambda) = \sum_l = (\sum_i \lambda_l^i) f_l(\sum_i \lambda_l^i).$$

We weigh the link costs $f_l(\lambda_l)$ by the flow λ_l that uses that link.

Global optimisation problem: find $\overline{\lambda}$ that achieves

$$\min_\lambda J(\lambda) \quad s.t. \quad (1) \quad \text{holds.} \tag{2}$$

Theorem 2 *Let $\overline{\lambda}$ be the globally optimal solution and define $\alpha_i = \Lambda_i / \sum_{j=1}^N \Lambda_j$. Then $\hat{\lambda}_l^i = \alpha_i \overline{\lambda}$ is a Nash equilibrium, with the corresponding cost for user i being $J^i(\overline{\lambda}) = \alpha_i J(\overline{\lambda})$. In particular, $\hat{\lambda}_l^i$ is an efficient equilibrium.*

Sketch of Proof:

- $\hat{\lambda}$ is a Nash equilibrium if and only if it is feasible and the following holds for all i:

$$\sum_{l\in L} \frac{\partial J_l^i(\hat{\lambda})}{\partial \lambda_l^i}(\hat{\lambda}_l^i - \lambda_l^i) \leq 0 \tag{3}$$

for any feasible λ^i (e.g. Haurie and Marcotte, 85).

- In particular, $\overline{\lambda} = (\overline{\lambda}^1, ..., \overline{\lambda}^N)$ is globally optimal if and only if it is feasible (i.e. it satisfies the traffic constraints (1)) and the following holds:

$$\sum_{l\in L} \frac{\partial J_l(\overline{\lambda}_l)}{\partial \lambda_l}(\overline{\lambda}_l - \lambda_l) \leq 0 \tag{4}$$

for any feasible λ.

- We define $\hat{\lambda}_l^i = \alpha_i \overline{\lambda}_l$ where $\overline{\lambda}_l$ is the unique globally optimal solution,

- substitute in (3) and obtain the required inequality, using the fact that (4) holds!

Computation of the Nash equilibrium: Affine costs

Consider M parallel links and with the affine cost function for all links:

$$f_l(\lambda_l) = a_l \lambda_l + b_l.$$

Possible interpretation: expected delay ED_l of a packet in a light traffic regime in an M/G/1 queue:

$$ED_l = \frac{\beta_l^{(2)}}{1 - \lambda_l \beta_l} \lambda_l + \beta_l \sim \beta_l^{(2)} \lambda_l + \beta_l$$

($\beta_l, \beta_l^{(2)}$ are the two first moments of the service time).

Kuhn-Tucker conditions

K.T. conditions are necessary and sufficient for a solution $\lambda = \{\lambda_l^i\}_{l,i}$ to constitute a Nash equilibrium:

$$K_l^i(\lambda) = a_l\lambda_l + b_l + a_l\lambda_l^i = \mu^i \qquad \text{if } \lambda_l^i > 0, \forall i, l, \quad (5)$$

$$K_l^i(\lambda) = a_l\lambda_l + b_l \geq \mu^i \qquad \text{if } \lambda_l^i = 0, \forall i, l, \quad (6)$$

$$\sum_{l=1}^{M} \lambda_l^i = \Lambda^i, \quad \forall i, \qquad \lambda_l^i \geq 0 \quad \forall i, l. \quad (7)$$

- (5)-(7) constitute a system of **linear equalities and inequalities**
- they can be solved by standard LP methods.

Lemma 1 *Assume that the following condition holds:*

$$\min_i R(i) > \frac{\max_l b_l}{N+1} \tag{8}$$

where

$$R(i) \stackrel{\text{def}}{=} \frac{1}{\sum_{j=1}^{M} \frac{1}{a_j}} \left(\Lambda^i - \frac{1}{N+1} \sum_{j=1}^{M} \frac{b_j}{a_j} \right).$$

Then, the unique Nash equilibrium is given by

$$\lambda_l^i = \frac{\frac{1}{a_l}}{\sum_{j=1}^{M} \frac{1}{a_j}} \left(\Lambda^i + \frac{1}{N+1} \sum_{j=1}^{M} \frac{b_j - b_l}{a_j} \right), \tag{9}$$

$\forall l, i$. *Condition (8) holds if and only if each user sends a positive amount of flow to each link.*

Ex: $b_l = 0$, $\forall l$. Then

$$\lambda_l^i = \frac{1/a_l}{\sum_{j=1}^{M} 1/a_j} \Lambda^i.$$

Sketch of Proof:

- Assume that in equilibrium all users send positive flows on all links.
- Then we use the 1st K.T. condition to compute the equilibrium.
- The assumption on $R(i)$ then guarantees that λ_l^i are strictly positive.
- Hence the solution is consistent with our assumption on positive flows. Due to the uniqueness, this is indeed the solution.

related results

There are very few results on uniqueness of equilibria in networks.

[Orda, Rom, Shimkin, 1993] showed uniqueness in the following cases:

- (1) Parallel links
- (2) symmetric players, general topology,
- (3) "All positive flows": over some set of links L, all users send positive flows, over others, they all send zero flow. If two equilibria satisfy these conditions, then they are the same. Requires same sources and destinations.

[Orda, Rom, Shimkin, 1993] also provided a simple counterexample for uniqueness for a network of 4 nodes!

- In the context of road-traffic, there is a huge literature on routing (concerning atomless games). Optimality concept: Wardrop Equilibrium (1952).

- Easy to compute and is "unique" in general topologies.

- More recent work:

R. J. La and V. Anantharam, “Optimal routing control: game theoretic approach”, CDC97.