Paradoxes in Game Theory and Networks

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OUTLINE

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Networking games

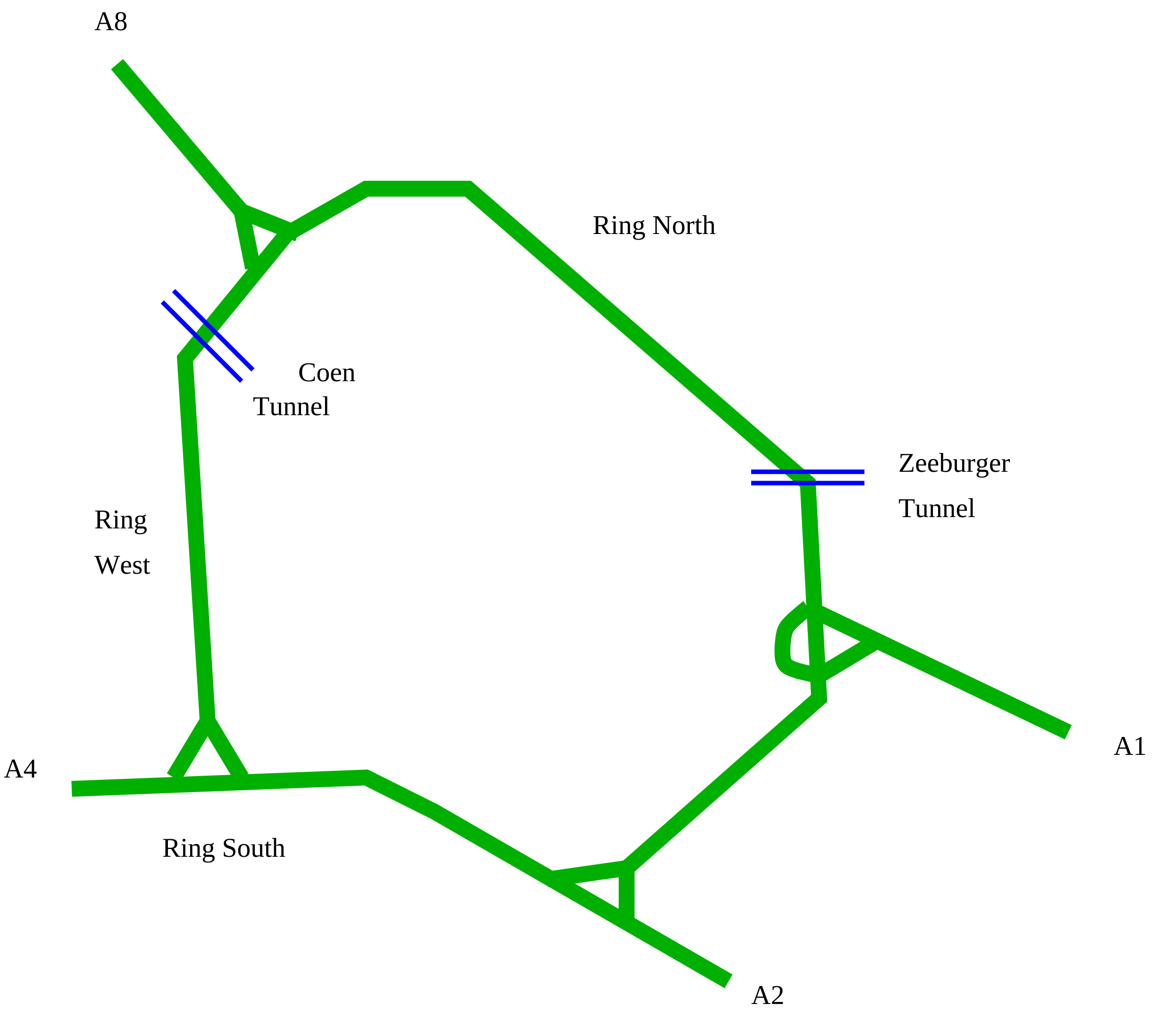
Road traffic

Individual choice of routes

- A large number of vehicles choose their route every day
- Each vehicle has a source and a destination.
- Typical objective of a vehicle: minimize delay
- Optimality concept: Wardrop Equilibrium

Example: The highway around Amsterdam

European project DRIVE II:
The vehicles have to decide the direction on the ring.
Objective: signalisation



Grouped choice of routes

- Example: transportation companies

- The influence of the choice of routes for a large number of vehicles of a company has a non-negligible influence on delays of other vehicles.

- Optimality concept: Nash Equilibrium.

Telecommunication networks

Individual choice of routes

- Ex: a large number of packets are transmitted over the network, and each packet makes its own routing choice.
- Ex: A large number of sessions share the network.
- A typical objective would be to minimise delays.

Grouped choice of routes

- Ex: A WEB application opens simultaneously several sessions (e.g. several images).

- Ex: Sessions cannot chooses their routes; the choice is done by the service provider for all its subscribers.

basic game theory notions: a reminder

Example: The Prisoner's Dilemma

- Emil and France are suspected of a crime.
- If both admit, they get 1 year in prison each.
- If both do not admit, they get 10 year in prison each.
- If one admits, he gets 15 years and the other is freed.

EMIL \ FRANCE	1	2
I	(1,1)	(15,0)
II	(0,15)	(10,10)

Figure 1: Prisoner's Dilemma

- The solution (1,1) is unstable: each prisoner may try to improve by deviating.
- Only stable solution is (10,10).
- **Nash Equilibrium:** A "stable" solution: no player has an incentive to deviate unilaterally.
- **Pareto Equilibrium:** A cooperative "dominating solution": there is no way to improve the performance of one player without harming another one.

Inefficiency of Nash equilibria

FRANCE

EMIL	1	2
I	(1,1)	(15,0)
II	(0,15)	(10,10)

Pareto Equilibrium

Nash Equilibrium

Nash Equilibria may be very inefficient!

Non-monotonicity of Nash equilibria

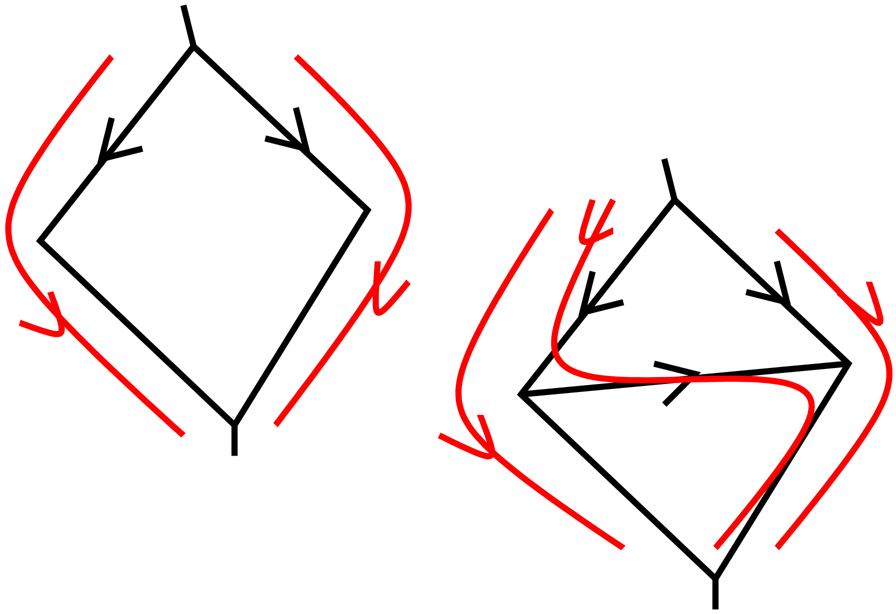
EMIL \ FRANCE	1	2
I	(1,1)	(15,5)
II	(5,15)	(10,10)

Assume we change the "0" to "5": a prisoner that does not admit is always punished (we may have prior information that the crime was committed together).

(1,1) is Nash equilibrium and unique Pareto solution!

Non-monotonicity in networks

Adding a link, or adding capacity to a link, increases delay for all users: **Braess Paradox**.



The paradox occurs in competitive routing (several users optimise). But also in the case of network optimisation using Bellman-Ford type algorithms (RIP).

References:

- D. Braess, “Uber ein paradoxen der werkehrsplannung”, *Unternehmenforschung*, **12**, 256-268, 1968.
- Y. A. Korilis, A. A. Lazar and A. Orda, “Avoiding the Braess Paradox in Noncooperative Networks”, *Proc. IEEE Conference on Decision & Control (San Diego)*, 864-878, 1997.

Nash Equilibrium in Networks

- N players
- Player i has X_i possible strategies,
- $x = (x_1, ..., x_N)$ is a multi-strategy, $X \stackrel{\text{def}}{=} \Pi_{i=1}^{N} X_i$.
- $\psi_i(x)$:= The cost for player i when x is used
- Given a multi-strategy x, define

(x^{-i}, y_i):= the multi-strategy obtained when player i switches to a strategy y_i.

- x is a Nash Equilibrium if for all i and $y \in X_i$,

$$\psi_i(x) \leq \psi_i((x^{[-i]}, y)).$$

The routing game

Network: a graph $G = (V, \mathcal{L})$ where

- V is a set of nodes
- L a class of directed links

Strategies of player i:

- player i sends a flow of Λ^i between the source $s(i)$ and the destination $d(i)$.
- The strategies are: $X_i = \{x_l^i\}, \quad x_l^i \in R_+$:
- Flow constraints:

$$r_v^i + \sum_{l \in In(v)} x_l^i = \sum_{l \in Out(v)} x_l^i,$$

$$\text{where} \qquad r_v^i = \begin{cases} \Lambda^i & \text{for } v = s(i), \\ -\Lambda^i & \text{for } v = d(i), \\ 0 & \text{otherwize} \end{cases}$$

Example of a network

- N players, all with the same source and destination.
- M links.
- $\lambda_l^{(k)} :=$ the flow of player k over link l.
- The cost functions:

$$\psi_i(\lambda) = \sum_{m=1}^{M} \lambda_m^i f_m(\lambda_m)$$

where $f_m(\lambda_m)$ is the delay (or cost) of link m, and

$$\lambda_m = \sum_{i=1}^{N} \lambda_m^i$$

Wardrop Equilibrium

Definition: x is a Wardrop Equilibrium if

$$\sum_{l \in k^*} f_l(x) \leq \sum_{l \in k} f_l(x)$$

où $k^* \in P_i^*$ et $k \in P_i$,

– P_i are all the routes between $s(i)$ and $d(i)$,

– P_i^* are all the routes between $s(i)$ and $d(i)$ over which the flow is strictly positive.

Ref: J.G. Wardrop,
"Some theoretical aspects of road traffic research", *Proc. Inst. Civ. Eng.* Part 2, 1, 325–378, 1952.

- First Principle of Wardrop: The durations of a trip on any road actually used are equal, and are greater than the duration of a trip of a single vehicle on a road which is not used.

- Second Principle of Wardrop:

The mean duration of the trip is minimal.

These principles have in fact appeared already in
A. C. Pigou, *The Economics of Welfare*, McMillan & Co., London, 1920.

Due to the fact that there are several individual decision makers, there are inefficiencies:

- The equilibrium is typically not socially optimal
- It has been observed that the increased capacity of a part of a system may lead to awkward behavior in terms of a system wide measure:

H. Kameda, J. Li, C. Kim and Y. Zhang, *Optimal Load Balancing in Distributed Computer Systems*, Springer, 1996.

- Braess paradox, in which increased capacity of a part of the system leads to the degradation in the benefits of *all* users

D. Braess, “Uber ein paradoxen der werkehrsplannung”, *Unternehmenforschung*, **12**, 256-268, 1968.

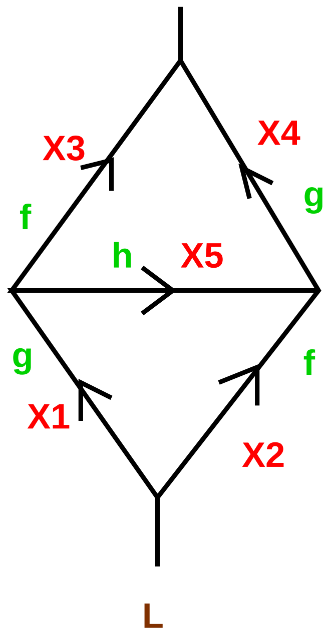
J. E. Cohen and F. P. Kelly, “A Paradox of Congestion in a Queueing Network,” *J. Appl. Prob. 27*, 730-734, 1990.

J. E. Cohen and C. Jeffries, “Congestion Resulting from Increased Capacity in Single-Server Queueing Networks,” *IEEE/ACM Trans. on Networking 5*, 2, 1220-1225, 1997.

B. Calvert, W. Solomon and I. Ziedins, “Braess's Paradox in a Queueing Network with State-Dependent Routing,” *J. Appl. Prob. 34*, 134-154, 1997.

Y. A. Korilis, A. A. Lazar and A. Orda, “Avoiding the Braess Paradox in Noncooperative Networks”, *Proc. IEEE Conference on Decision & Control (San Diego)*, 864-878, 1997.

Computing the equilibrium: Braess paradox



Assume: route 1-3, as well as 2-4 are used.

$$g(x_1) + f(x_3) = f(x_2) + g(x_4). \tag{1}$$

We now express x_5:

$$x_1 - x_3 = -x_2 + x_4.$$

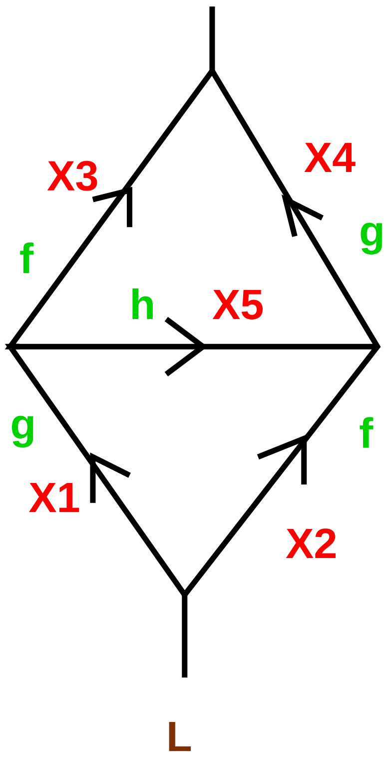
If f is linear then this implies

$$f(x_1) - f(x_3) = -f(x_2) + f(x_4).$$

Summing with (1), we get

$$f(x_1) + g(x_1) = f(x_4) + g(x_4).$$

If $f + g$ is strictly increasing then $x_1 = x_4$. Hence also $x_2 = x_3$.



Now, $x_2 = L - x_1$. Hence

$$x_5 = x_1 - x_3 = x_1 - x_2 = 2x_1 - L.$$

If route 1-5-4 is also used then

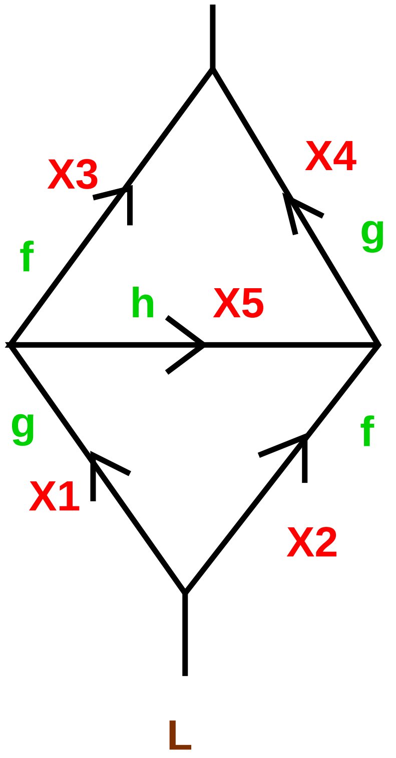
$$g(x_1) + h(x_5) = f(x_2).$$

We conclude that

$$g(x_1) + h(2x_1 - L) = f(L - x_1).$$

This gives x_1.

Original Example of Braess



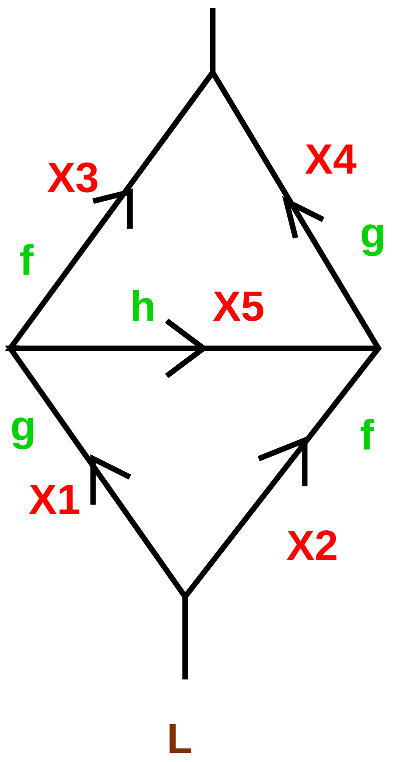
- Choose $L = 6$, $f(x) = 50+x$, $g(x) = 10x$, $h(x) = \infty$.

Then $x_1 = x_2 = 3$,

$$D_{13} = D_{24} = 83.$$

- Take $h = 10 + x$.

With $x_1 = x_3 = 3$, $D_{154} = 70 < D_{13}$. Not Wardrop Equilibrium!

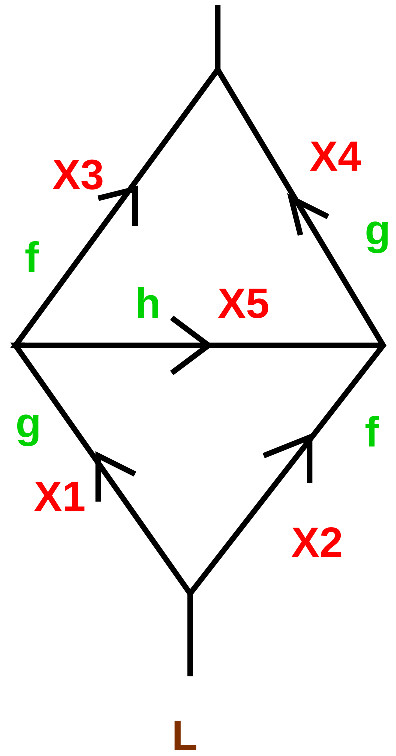


■ Suppose 1 unit moves from 2-4 to 1-5-4.

Then

$$D_{1-3} = 40 + 53 = 93, \quad D_{2-4} = 52 + 30 = 82,$$

$$D_{1-5-4} = 40 + 11 + 40 = 91.$$



■ Suppose 1 unit moves from 1-3 to 1-5-4. We have $x_1 = 4, x_2 = 2, x_5 = 2.$

Then

$$D_{1-3} = 40 + 52 = 92, \quad D_{2-4} = 52 + 40 = 92,$$

$$D_{1-5-4} = 40 + 12 + 40 = 92.$$

This satisfies Wardrop conditions!

We now check the equation:

$$g(x_1) + h(2x_1 - L) = f(L - x_1).$$

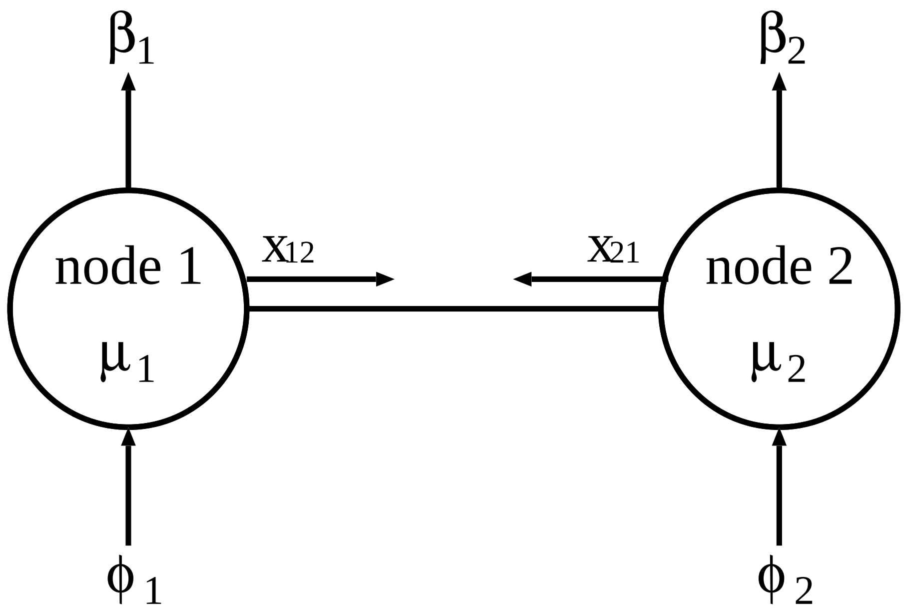
We get:

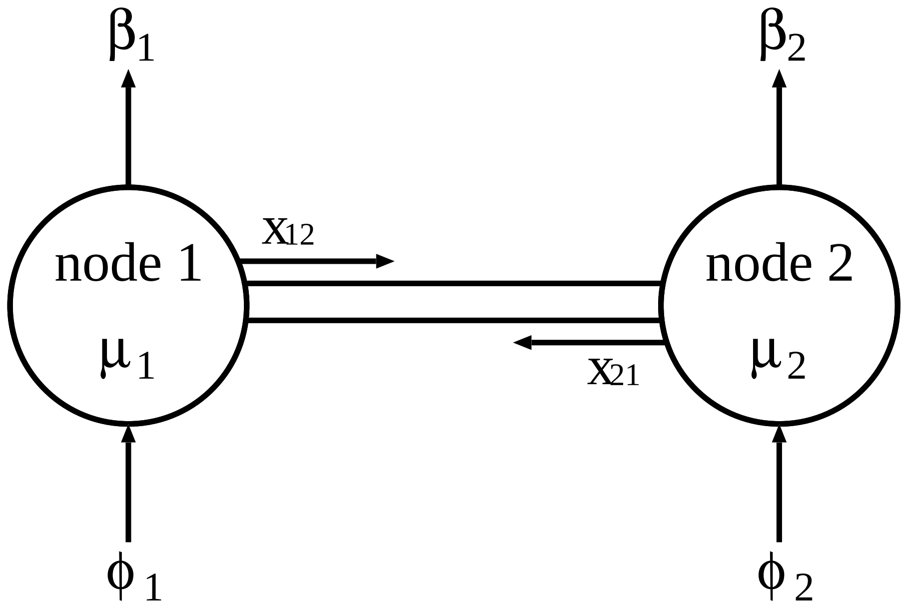
$$10x_1 + (10 + 2x_1 - 6) = (50 + 6 - x_1)$$

Hence $x_1 = 4$, $x_2 = L - x_1 = 2$, $x_5 = x_1 - x_2 = 2$.

Distributed computing

Joint work with Prof. H. Kameda and Mr. Y. Hosokowa, Univ. of Tsukuba, Japan.




β1
β2
x12
node 1
μ1
node 2
μ2
x21
φ1
φ2

The model

- Two nodes: 1, 2.
- Each node: a single exponential server, parameter μ_i.
- Node i has the external Poisson arrival with rate ϕ_i
- x_{ii} are processed at node i,
- x_{ij} $(i \neq j)$ are forwarded through the communication means to the other node j. The results of those jobs are returned back through the communication means to node i.

$$x_{ii} + x_{ij} = \phi_i, \qquad (i \neq j),$$

$$x_{ij}, \geq 0, \quad i, j = 1, 2.$$

x:=$(x_{11},\ x_{12},\ x_{21},\ x_{22})$.

- $\mathcal{C}$:= the x's that satisfy the constraints,
- $\Phi = \phi_1 + \phi_2$.
- Goal: optimize x_{ij} $(i, j = 1, 2)$
- The load on node i is $\beta_i := x_{ii} + x_{ij}$ $(i \neq j)$
- The expected processing (including queueing) time of a job that is processed at node i, is

$$1/(\mu_i - \beta_i)$$

for $\beta_i < \mu_i$ (otherwize it is infinite).

Communication means:

Two alternatives.

(A) a single-channel communication line

$$G(\lambda) = \frac{t}{1 - (x_{12} + x_{21})t}$$

if $x_{12} + x_{21} < 1$, and is otherwize infinite, where $\lambda = x_{12} + x_{21}$ is the network traffic.

The communication channel is modelled by a processor sharing server with service rate $1/t$; *i.e.*, the mean communication (without queueing) time is t, and, thus, the capacity of the communication line is t.

(B) Two unidirection communication lines 1 and 2.

The expected communication (with queueing) time of a job arriving at node i and being processed at node j ($\neq i$):

$$G_i(x_{ij}) = \frac{t}{1 - x_{ij}t}$$

if $x_{ij}t < 1$ (and is otherwize infinite).

For case (A), $(1/t) - x_{ij}$ $(i \neq j)$ is the amount of the line capacity left to be used by node i in forwarding the jobs arriving at node i.

The communication capacity for case (B) is greater than or equal to that for case (A).

The expected response time of a job that arrives at node i is

$$T_i(x) = \frac{1}{\phi_i} \sum_k x_{ik} T_{ik}(x),$$

where

$$T_{ii}(x) = \frac{1}{\mu_i - \beta_i}$$

if $\beta_i < \mu_i$ (and it is otherwize infinite), and for $j \neq i$,

$$\begin{aligned} T_{ij}(x) &= \frac{1}{\mu_j - \beta_j} + \frac{t}{1-(x_{ij}+x_{ji})t}, \qquad \text{for case (A)}, \\ &= \frac{1}{\mu_j - \beta_j} + \frac{t}{1-x_{ij}t}, \qquad \text{for case (B)}. \end{aligned}$$

The overall expected response time of a job that arrives at the system is

$$T(x) = \frac{1}{\Phi} \sum_i \phi_i T_i.$$

The overall optimum: find $\bar{x}$ that achieves

$$T(\bar{x}) = \min T(x) \quad \text{s.t.} \quad x \in \mathcal{C}.$$

The node optimum (Nash E): find $\tilde{x}$ s.t. for all i

$$T_i(\tilde{x}) = \min T_i(x), \quad \text{s.t. } x \in \mathcal{C}.$$

The individual optimum (Wardrop E): find $\hat{x}$ s.t. for all i

$$T_i(\hat{x}) = \min_{i \neq j} \{T_{ii}(\hat{x}), T_{ij}(\hat{x})\} \quad \text{s.t.} \quad \hat{x} \in \mathcal{C}$$

Paradoxes

Mutual forwarding

There should be no mutual forwarding in overall and individual optima:

In overall and individual optima, either one of x_{ij} $(i \neq j)$ must be zero [Kameda et al].

Nash equilibrium: mutual forwarding paradox.

Monotonicity

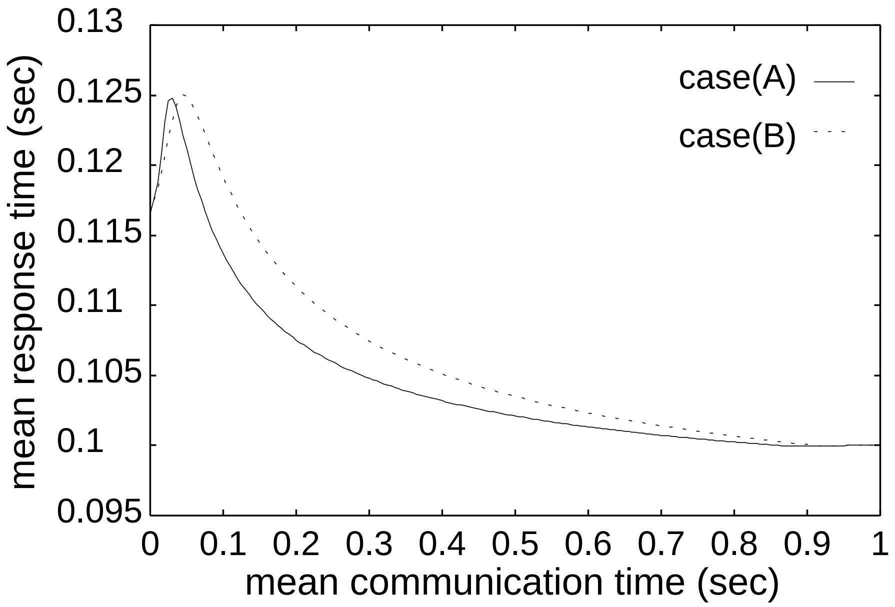
The overall response time may be decreasing in the capacity, both in the individual and the node optimal cases.

Braess Type Paradox

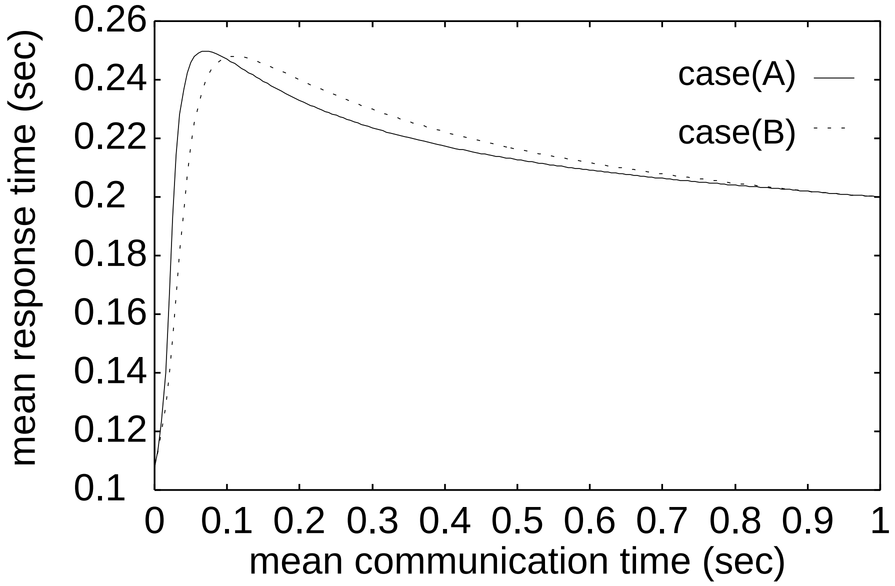
Three Braess-like paradoxes:

- For case (A) with the decrease of t,
- For case (B) with the decrease of t.
- By comparing cases (A) to case (B) where a single link has the same capacity in both cases.
- No paradox for the Wardrop case.

Class optimality, delays for node 1

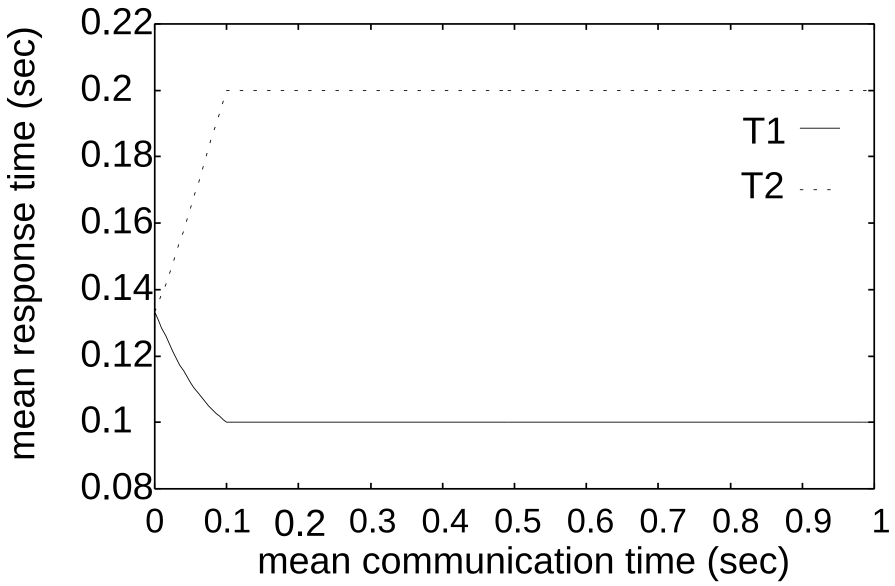


Class optimality, delays for node 2



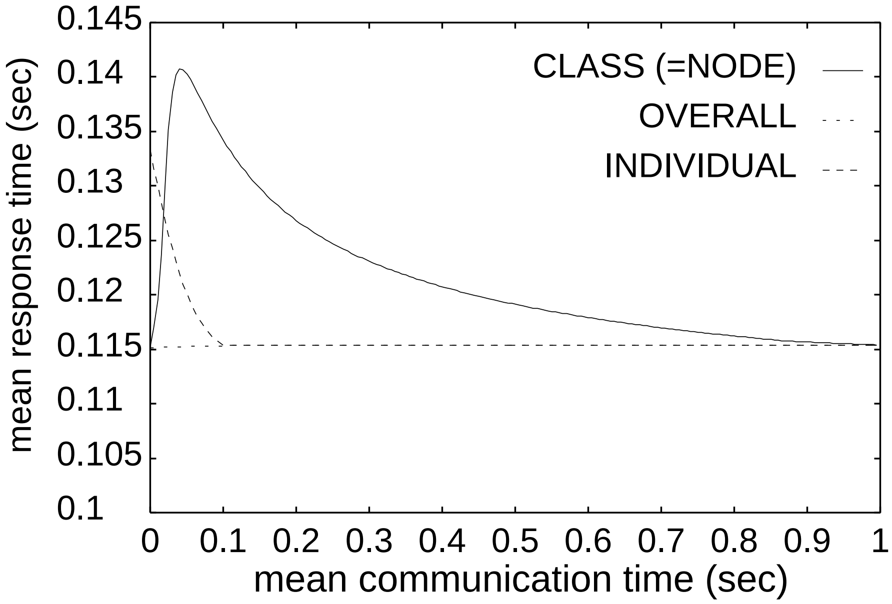
- Case (A) has less capacity. We would expect larger delay.
- In some region, both class 1 and class 2 increase delays when the communication means have larger capacity.

Individual optimality



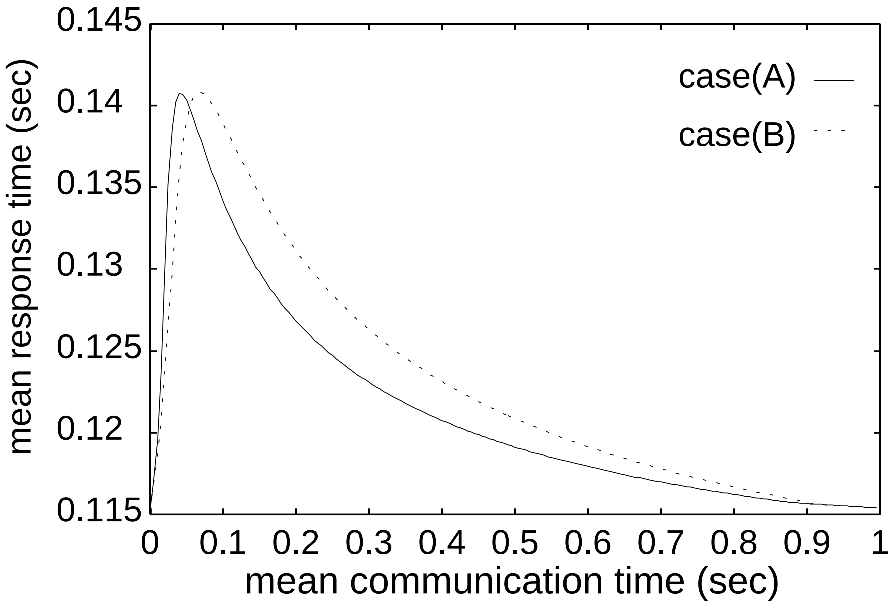
- Same values for cases A and B.
- No mutual forwarding.

Overall average delay, Case A



- Does not depend on θ for global optimization
- Decreases in θ for individual and class optimization.

Overall average delay, Case A versus B



Recall: case A less capacity (larger communication times).