Avoiding Paradoxes in Routing Games *

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Strange behavior may occur in networks due to the non-cooperative nature of decision making, when the latter are taken by individual agents. In particular, the well known Braess paradox illustrates that when upgrading a network by adding a link, the resulting equilibrium may exhibit larger delays for all users. We present here some guidelines to avoid the Braess paradox when upgrading a network. We furthermore present conditions for the delays to be monotone increasing in the total demand.

1. Introduction

Service providers or the network administrator may often be faced with decisions related to upgrading of the network. For example, where should one add capacity? or where should one add new links? Decisions related to the network capacity and topology have direct influence on the equilibrium that would be attained.

A frequently used heuristic approach for upgrading a network is through *Bottleneck Analysis*. A system bottleneck is defined as "a resource or service facility whose capacity seriously limits the performance of the entire system" [10, p. 13]. Bottleneck analysis consists of adding capacity to identified bottlenecks until they cease to be bottlenecks. In a non-cooperative framework, however, this approach may have devastating effects; it may cause delays of all users to increase; in an economic context in which users pay the service provider, this may further cause a decrease in the revenues of the provider. The first problem has already been identified in road-traffic context by Braess [3] (see also [7,16]), and has further been studied in networking context in [2,4,6,5,11,13]. The focus of Braess paradox on the bottleneck link in a queueing context, as well as the paradoxical impact on the service provider have been studied in [14]. The Braess paradox has further been identified and studied in the context of distributed computing [8,9] where arrivals of jobs may be routed and performed on different processors.

Braess paradox illustrates that the network designer or service providers have to take into consideration the reaction of non-cooperative users to his decisions. This is in particular important when upgrading the network. Some upgrading guidelines have been proposed in [11–13] so as to avoid the Braess paradox or so as to obtain a better performance. Our first objective is to pursue that direction and to provide new guidelines for avoiding the Braess paradox when upgrading the network. Another related issue is that of monotonicity of the performance measures in the demand. Our second objective

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is to check under what conditions are delays as well as the marginal costs at equilibrium increasing in the demands.

The paper is organized as follows: In Section 2 we present the model, formulate the problem, and mention some related work. In Section 3 we present a sufficient condition for the monotonicity of performance measures when the demands increase. The proposed methods for capacity addition are studied in section 4 and 5.

2. Problem formulation

We consider a network $(\mathcal{N}, \mathcal{L})$ where $\mathcal{N}$ is a finite set of nodes and $\mathcal{L}$ is a set of directed links. For simplicity of notation and without loss of generality, we assume that at most one link exists between each pair of nodes (in each direction). Considering a node $v \in \mathcal{N}$, and let $In(v) = \{l \in \mathcal{L} | \exists w \in \mathcal{N}, l = (w, v)\}$ denote the set of its in-coming links, and $Out(v) = \{l \in \mathcal{L} | \exists w \in \mathcal{N}, l = (v, w)\}$ the set of its out-going links. Let c_l be the capacity of link l. $C = (c_l)_{l \in \mathcal{L}}$ is called the capacity configuration of the network.
We are given a set $\mathcal{I} = \{1, 2, .., I\}$ of users that share the network. We assume that all users ship flow from a common source s to a common destination d. Each user i has a throughput demand that is some process with average rate r^i, and let $r = \sum_{i \in \mathcal{I}} r^i$ the total throughput demand of users. User i splits its demand r^i among the paths connecting the source to the destination, so as to optimize some individual performance objective. Let x_l^i denote the expected flow that user i sends on link l. The user flow configuration $X^i = (x_l^i)_{l \in \mathcal{L}}$ is called a routing strategy of user i. The set of strategies of user i that satisfy the user's demand and preserve its flow at all nodes is called the strategy space of user i is given by $\mathcal{S}^i = \{X^i \in \mathbb{R}^{|\mathcal{L}|} : 0 \le x_l^i \le c_l, l \in \mathcal{L}; \sum_{l \in Out(v)} x_l^i = \sum_{l \in In(v)} x_l^i + r_v^i, v \in \mathcal{V}\}$, where $r_s^i = r^i$, $r_d^i = -r^i$ and $r_v^i = 0$ for $v \ne s, d$. The system flow configuration $X = (X^1, ..., X^I)$ is called a routing strategy taking values in the product space $\mathcal{S} = \bigotimes_{i \in \mathcal{I}} \mathcal{S}^i$.

The performance objective of user i is quantified by means of a cost function $J^i(X)$. The user aims to find a strategy $X^i \in \mathcal{S}^i$ that minimizes its cost. This optimization depends on the routing decisions of the other users, described by the strategy $X^{-i} = (X^1, .., X^{i-1}, X^{i+1}, .., X^I)$, since J^i is a function of the system flow configuration X.
Typically, the performance of a link l is manifested through some function $T_l(x_l)$, that measures the cost per unit of flow on the link, and depends upon the link's total flow $x_l = \sum_{i \in \mathcal{I}} x_l^i$. The users' cost then typically satisfy the following properties [15]:

A1 $J^i(X) = \sum_{l \in \mathcal{L}} x_l^i T_l(x_l)$.
A2 $T_l : [0, \infty) \to (0.\infty]$.
A3 $T_l(x_l)$ is positive, strictly increasing and convex.
A4 $T_l(x_l)$ is continuously differentiable.
Functions that comply with these assumptions are referred to as type-A functions. Note that $T_l(x_l)$ is the average delay on link l and depends only on the total flow $x_l = \sum_{i \in \mathcal{I}} x_l^i$ on that link. The average delay should be interpreted as a general congestion cost per unit of flow, that encapsulates the dependence of the quality of service provided by a finite capacity resource on the total load x_l offered to it.

Some of our general results will require that the costs are of type A, whereas other results will require the following more specific costs:
B1 J^i is a type-A cost function.
B2 T_l and T_l' are strictly decreasing with respect to capacity c_l of link l.
Functions that comply with these assumptions shall be referred to as type-B functions. Note that a type-B function is a special case of type-A.

A special kind of type-B cost functions is that which corresponds to an $M/M/1$ link model. In other words, suppose that:
C1 J^i is a type-B cost function.
C2 $T_l = \begin{cases} 1/(c_l - x_l) & x_l < c_l \\ \infty & x_l \geq c_l \end{cases}$, where c_l is the capacity of link l.
Functions that comply with these assumptions shall be referred to as type-C functions.

We note that the above different type of assumptions on the cost have already been introduced in the context of analysis of uniqueness of equilibria in [15].

Definition 2.1 *A Nash equilibrium of the routing game is a strategy from which no user finds it beneficial to unilaterally deviate. Hence, $X \in \mathcal{S}$ is a Nash equilibrium if: $X^i \in arg\min_{g^i \in \mathcal{S}^i} J^i(g^i, X^{-i}), i \in \mathcal{I}$.*

Consider the best reply X^i of user i to strategy X^{-i} of the other users. This is the unique solution to the (single-user) optimal routing problem for a network; the uniqueness follows since the cost function of type-A or type-B or type-C is a convex function of its strategy X^i and $\mathcal{S}^i$ is bounded for all $i \in \mathcal{I}$ (note that the uniqueness of best response strategies does not imply the uniqueness of the Nash equilibrium). The Kuhn-Tucker conditions imply that X^i is the optimal response of user i to X^{-i} if and only if there exist (Lagrange multipliers) $(\lambda_u^i)_{u \in \mathcal{V}}$ (that may depend on X^i and X^{-i}), such that [15,13]:

$$\begin{aligned} \lambda_u^i &= x_{uv}^i T_{uv}' + T_{uv} + \lambda_v^i, \text{ if } x_{uv}^i > 0, \quad (u,v) \in \mathcal{L}, \\ \lambda_u^i &\leq x_{uv}^i T_{uv}' + T_{uv} + \lambda_v^i, \text{ if } x_{uv}^i = 0, \quad (u,v) \in \mathcal{L}, \qquad \lambda_d^i = 0. \end{aligned} \tag{1}$$

Therefore, a strategy $X \in \mathcal{S}$ is a Nash equilibrium if and only if there exist λ_u^i, such that the conditions (1) are satisfied for all $i \in \mathcal{I}$. The Lagrange multiplier λ_s^i can be interpreted as the *marginal cost of user* i at the optimality point [13]. Due to this interpretation, the Lagrange multipliers, and in particular, λ_s^i (i.e. the marginal cost for at the source node) have been advocated in [13] as yet another important performance measure for the network. The latter was defined as the *price* for user i.

We shall study the monotonicity of the total price and the total cost at equilibrium in the demands $r = (r^i)_{i \in \mathcal{I}}$, and in the capacity allocation $(c_l)_{l \in \mathcal{L}}$. We recall that in the Braess paradox, the monotonicity in the capacity allocation does not hold, since by adding a link (which can be viewed as adding capacity to a link with zero capacity), the performance of *all users* deteriorate at equilibrium. If we show that the *total* cost, or total price, is monotone decreasing in the capacity then a Braess paradox does not occur (with respect to the corresponding performance measure), since for at least one user, the cost (or price) improves with addition capacity.

In order to compare Nash equilibria corresponding to different parameters, it may seem desirable to make assumptions on the topology and costs such that under any throughput demand of users or any additional capacity, the equilibrium is unique. Indeed, some results on avoiding the Braess paradox (when adding capacity) have already been obtained in [13] under conditions that imply uniqueness of the equilibria. We do not make such assumptions, and our results allow us to compare the performance of any equilibrium in a system, with any other which is obtained by increasing the capacity or the demand appropriately.

We mention conditions that have been obtained in [13] under which Braess paradox can be avoided. These were obtained for two classes of problems. The first is that of identical users, i.e. systems in which the demands, the sources and the destinations of all users are the same. The second is that of simple users, defined as follows: a user is said to be simple if all of its flows are routed through paths of minimal delay (with link costs corresponding to M/M/1 type queues). The proposed methods for capacity addition studied in [13] are
1. Multiplying the capacity of each link by some constant factor $\alpha > 1$
2. Adding a link between the source and the destination.
The second upgrade shows to yield an improvement only in user price (not in the cost).

3. Impact of throughput variation on the equilibrium

In this section, we study the monotonicity of performance measure at equilibrium given by total price $\lambda_s = \sum_{i\in\mathcal{I}} \lambda_s^i$ and total cost $J = \sum_{i\in\mathcal{I}} J^i$ when the total demand increases. Under some assumption, the following study establishes that an increase of the total demand of users, results in an increase of the total price. For a fixed capacity $(c_l)_{l\in\mathcal{L}}$, we consider two throughput demands $(\tilde{r}^i)_{i\in\mathcal{I}}$ and $(\hat{r}^i)_{i\in\mathcal{I}}$ such that $\hat{r} = \sum_{i\in\mathcal{I}} \hat{r}^i < \tilde{r} = \sum_{i\in\mathcal{I}} \tilde{r}^i$, and let $\tilde{\lambda}^i$ and $\hat{\lambda}^i$ (resp., $\tilde{J}^i$ and $\hat{J}^i$) be the prices (resp., cost functions of type-A) of user i at the respective Nash equilibria $\tilde{X}$ and $\hat{X}$. We make the following observation.

Lemma 3.1 *There exists some path p^* between the source and destination such that $\tilde{x}_l > \hat{x}_l$ for all the links in that path.*

Proof. We construct a directed network $(\mathcal{N}', \mathcal{L}')$, where $\mathcal{N}' = \mathcal{N}$ and the set of links $\mathcal{L}'$ is constructed as follows:
1. For each link $l = (u, v) \in \mathcal{L}$, such that $\tilde{x}_l \geq \hat{x}_l$, we have a link $l' = (u, v) \in \mathcal{L}'$; to such a link l' we assign a (flow) value $z_{l'} = \tilde{x}_l - \hat{x}_l$.
2. for each link $l = (u, v)$, such that $\tilde{x}_l < \hat{x}_l$, we have a link $l' = (v, u) \in \mathcal{L}'$; to such a link we assign a (flow) value $z_{l'} = \hat{x}_l - \tilde{x}_l$.

It is easy to verify that the value $z_{l'}$ constitutes a nonnegative, directed flow in the network. Since $\hat{r} < \tilde{r}$, $z_{l'}$ must carry some flow (the amount of $\tilde{r} - \hat{r}$) from the source s to the destination d, this implies that there exists a path p^* from s to d, such that $z_{l'} > 0$ for all $l' \in p^*$. ∎

Assumption 1: We suppose that
1. $\tilde{x}_l^i > 0$ for all $l \in p^*$ and all $i \in \mathcal{I}$.
2. For all $l \in p^*$ for which $\hat{x}_l > 0$ all users send positive flows in the equilibrium $\hat{X}$, i.e. $\hat{x}_l^i > 0$ and all $i \in \mathcal{I}$.

This assumption is inspired by the (much stronger) assumption in [15] for uniqueness of Nash equilibrium, that states that if at equilibrium a flow on a link is positive then all users have positive flow on that link. We now state our first monotonicity result:

Proposition 3.1 *Let Assumption 1 hold, and consider cost functions J^i of type A. Consider two throughput demands $(\tilde{r}^i)_{i\in\mathcal{I}}$ and $(\hat{r}^i)_{i\in\mathcal{I}}$. Let $\tilde{X}$ and $\hat{X}$ be the equilibria associated to these demands, and $\tilde{\lambda}^i$ and $\hat{\lambda}^i$ user i's prices computed respectively at $\tilde{X}$ and $\hat{X}$. Then If $\hat{r} < \tilde{r}$ then $\hat{\lambda}_s < \tilde{\lambda}_s$, where $\tilde{\lambda}_s = \sum_{i\in\mathcal{I}} \tilde{\lambda}^i_s$ and $\hat{\lambda}_s = \sum_{i\in\mathcal{I}} \hat{\lambda}^i_s$.*

Proof. Consider now a link $l' = (u,v) \in p^*$. Since $z_{l'} > 0$ (the latter were defined in the proof of Lemma 3.1), either $\tilde{x}_{uv} > \hat{x}_{uv}$ or $\hat{x}_{vu} > \tilde{x}_{vu}$.
In the case where $\tilde{x}_{uv} > \hat{x}_{uv}$, for all $i \in \mathcal{I}$ we have, since $\tilde{X}$ and $\hat{X}$ are Nash equilibria, $\tilde{\lambda}^i_u - \tilde{\lambda}^i_v = \tilde{x}^i_{uv}\tilde{T}'_{uv} + \tilde{T}_{uv}$, and $\hat{\lambda}^i_u - \hat{\lambda}^i_v \leq \hat{x}^i_{uv}\hat{T}'_{uv} + \hat{T}_{uv}$, where $\tilde{T}$, $\hat{T}$, $\tilde{T}'$, and $\hat{T}'$ stands obviously for T_{uv} and T'_{uv} computed at $\tilde{x}_{uv}$ and $\hat{x}_{uv}$.
Summing over $i \in \mathcal{I}$, we obtain: $\tilde{\lambda}_u - \tilde{\lambda}_v = \tilde{x}_{uv}\tilde{T}'_{uv} + I\tilde{T}_{uv}$, and $\hat{\lambda}_u - \hat{\lambda}_v \leq \hat{x}_{uv}\hat{T}'_{uv} + I\hat{T}_{uv}$, where $\tilde{\lambda}_w = \sum_{i\in\mathcal{I}} \tilde{\lambda}^i_w$ and $\hat{\lambda}_w = \sum_{i\in\mathcal{I}} \hat{\lambda}^i_w$ for all $w \in \mathcal{N}$.
Since $\tilde{x}_{uv} > \hat{x}_{uv}$ and $\hat{T}_{uv} = T_{uv}(\hat{x}_{uv})$, then $\tilde{T}_{uv} > \hat{T}_{uv}$ and $\tilde{T}'_{uv} > \hat{T}'_{uv}$ (Ass. A3), hence we have $\tilde{\lambda}_u - \tilde{\lambda}_v = \tilde{x}_{uv}\tilde{T}'_{uv} + I\tilde{T}_{uv} > \hat{x}_{uv}\hat{T}'_{uv} + I\hat{T}_{uv} \geq \hat{\lambda}_u - \hat{\lambda}_v$. Thus,

$$\tilde{\lambda}_u - \hat{\lambda}_u > \tilde{\lambda}_v - \hat{\lambda}_v. \tag{2}$$

If $\hat{x}_{vu} > \tilde{x}_{vu}$, we have by symmetry that $\hat{\lambda}_v - \hat{\lambda}_u > \tilde{\lambda}_v - \tilde{\lambda}_u$, thus we obtain (2).
Define more precisely the path p^*, by $p^* = (s, u_1, u_2, ..., u_{n^*}, d)$, where u_k, $k = 1, 2, .., n^*$, is the k^{th} node after the source s on the path p^* and n^* is the number of nodes between the source s and the destination d. Hence, from (2) we have: $\tilde{\lambda}_s - \hat{\lambda}_s > \tilde{\lambda}_{u_1} - \hat{\lambda}_{u_1} > .... > \tilde{\lambda}_{u_{n^*}} - \hat{\lambda}_{u_{n^*}} > \tilde{\lambda}_d - \hat{\lambda}_d = 0$ $(\tilde{\lambda}_d = \hat{\lambda}_d = 0)$, and we conclude that $\tilde{\lambda}_s > \hat{\lambda}_s$. ∎

The following proposition gives sufficient conditions for obtaining the monotonicity of the total cost of type-C, when the total throughput demand of the users increases.

Proposition 3.2 *Consider cost functions J^i of type C. Consider two throughput demands $(\tilde{r}^i)_{i\in\mathcal{I}}$ and $(\hat{r}^i)_{i\in\mathcal{I}}$. Let $\tilde{X}$ and $\hat{X}$ the Nash equilibria associated to these throughput demands, and $\tilde{J}^i$ and $\hat{J}^i$ user i's costs computed respectively at $\tilde{X}$ and $\hat{X}$; assume that there exists a set of links $\hat{\mathcal{L}}_1 \subseteq \mathcal{L}$ such that $\hat{x}^i_l > 0$, $\forall i \in \mathcal{I}$ for $l \in \hat{\mathcal{L}}_1$, and $\hat{x}^i_l = 0$, $\forall i \in \mathcal{I}$ for $l \notin \hat{\mathcal{L}}_1$. Hence if $\frac{\tilde{r}}{\hat{r}} \geq I$ then $\hat{J} < \tilde{J}$.*

Proof. Consider any link $l = (u,v) \in \mathcal{L}_1$ $(x^i_l > 0, \forall i \in \mathcal{I})$, then we have from (1): $\forall i \in \mathcal{I}$, $\hat{\lambda}^i_u = \hat{x}^i_l\hat{T}'_{uv} + \hat{T}_{uv} + \hat{\lambda}^i_v$, and for any link $l = (u,v) \in \mathcal{L}$ for which $\hat{x}_l = 0$ $(l \notin \hat{\mathcal{L}}_1)$, we have: $\forall i \in \mathcal{I}$, $\hat{\lambda}^i_u \leq \hat{x}^i_l\hat{T}'_{uv} + \hat{T}_{uv} + \hat{\lambda}^i_v$. By summing over $i \in \mathcal{I}$, we obtain $\hat{\lambda}_u = \hat{x}_l\hat{T}'_{uv} + I\hat{T}_{uv} + \hat{\lambda}_v$ if $\hat{x}_l > 0$, and $\hat{\lambda}_u \leq \hat{x}_l\hat{T}'_{uv} + I\hat{T}_{uv} + \hat{\lambda}_v$. if $\hat{x}_l = 0$. Thus,

$$\hat{\lambda}_u = \frac{\hat{x}_l}{(c_l - \hat{x}_l)^2} + \frac{I}{c_l - \hat{x}_l} + \hat{\lambda}_v \text{ if } \hat{x}_l > 0, \quad \hat{\lambda}_u \leq \frac{\hat{x}_l}{(c_l - \hat{x}_l)^2} + \frac{I}{c_l - \hat{x}_l} + \hat{\lambda}_v, \text{ if } \hat{x}_l = 0. \tag{3}$$

Define the function V by

$$V((y_l)_{l\in\mathcal{L}}) = \sum_{l\in\mathcal{L}} \frac{y_l}{c_l - y_l} - (I-1)\sum_{l\in\mathcal{L}} \ln(c_l - y_l), \tag{4}$$

where $(y_l)_{l\in\mathcal{L}} \in \hat{S} := \{(y_l)_{l\in\mathcal{L}} \in \mathbb{R}^{|\mathcal{L}|} : 0 \leq y_l \leq c_l, l \in \mathcal{L}; \sum_{l\in Out(v)} y_l = \sum_{l\in In(v)} y_l + \hat{r}_v, v \in \mathcal{V}\}$, where $\hat{r}_s = \hat{r}$, $\hat{r}_d = -\hat{r}$ and $\hat{r}_v = 0$ for $v \neq s, d$.
Denote $(\hat{x}_l)_{l\in\mathcal{L}}$ the vector of total link flows at the Nash equilibrium $\hat{X}$. The condition (3) can be interpreted as Kuhn-Tucker condition for a single-user minimization of the function V, under the constraints $y \in \hat{S}$. This shows that the vector $(\hat{x}_l)_{l\in\mathcal{L}}$ is the unique

minimum of the function V.
Let $(\bar{x}_l)_{l\in\mathcal{L}} \in \mathbb{R}^{|\mathcal{L}|}$ defined by: $\bar{x}_l = \frac{\hat{r}}{\tilde{r}}\tilde{x}_l$, hence $(\bar{x}_l)_{l\in\mathcal{L}} \in \hat{S}$, and since $(\hat{x}_l)_{l\in\mathcal{L}}$ minimizes the V-function, we have:

$$\begin{aligned}
\sum_{l\in\mathcal{L}} \frac{\hat{x}_l}{c_l - \hat{x}_l} \quad &\leq \quad \sum_{l\in\mathcal{L}} \frac{\frac{\hat{r}}{\tilde{r}}\tilde{x}_l}{c_l - \frac{\hat{r}}{\tilde{r}}\tilde{x}_l} - (I-1)\sum_{l\in\mathcal{L}} \ln\Big(\frac{c_l - \frac{\hat{r}}{\tilde{r}}\tilde{x}_l}{c_l - \hat{x}_l}\Big) \\
&< \quad \sum_{l\in\mathcal{L}} \frac{\frac{\hat{r}}{\tilde{r}}\tilde{x}_l}{c_l - \frac{\hat{r}}{\tilde{r}}\tilde{x}_l} - (I-1)\sum_{l\in\mathcal{L}} \ln\Big(1 - \frac{\frac{\hat{r}}{\tilde{r}}\tilde{x}_l}{c_l}\Big).
\end{aligned}$$

Hence, in order to prove that $\hat{J}(\hat{X}) = \sum_{l\in\mathcal{L}} \frac{\hat{x}_l}{c_l - \hat{x}_l} < J(\tilde{X}) = \sum_{l\in\mathcal{L}} \frac{\tilde{x}_l}{c_l - \tilde{x}_l}$, it is enough to show that:

$$\frac{\tilde{x}_l}{c_l - \tilde{x}_l} - \frac{\frac{\hat{r}}{\tilde{r}}\tilde{x}_l}{c_l - \frac{\hat{r}}{\tilde{r}}\tilde{x}_l} + (I-1)\ln\Big(1 - \frac{\frac{\hat{r}}{\tilde{r}}\tilde{x}_l}{c_l}\Big) > 0,$$

which holds if $\tilde{r}/\hat{r} \geq I$ (see Appendix with $\alpha = \tilde{r}/\hat{r}$). $\blacksquare$

We recall the "all-positive flows" assumption from [15], that assumes that at equilibrium, for every link on which the total flow is positive, all users have strictly positive flows. Under this assumption, each link of the network satisfies the assumptions in Prop. 3.1 and 3.2. Hence we have the following result in this case.

Corollary 1 *Consider two throughput demands $(\tilde{r}^i)_{i\in\mathcal{I}}$ and $(\hat{r}^i)_{i\in\mathcal{I}}$, and let $\tilde{\lambda}^i$ and $\hat{\lambda}^i$ (resp., J^i and $\hat{J}^i$) be the prices (resp., the cost functions) of user i at the respective equilibria $\tilde{X}$ and $\hat{X}$. Assume that the "all-positive flows" assumption holds. Then:*
1. *For the cost functions of type-A, if $\hat{r} < \tilde{r}$ then $\hat{\lambda}_s < \tilde{\lambda}_s$.*
2. *For the cost functions of type-C, if $\frac{\tilde{r}}{\hat{r}} \geq I$ then $\hat{J} < \tilde{J}$.*

4. Impact of extra capacity on the equilibrium

In This section, we propose some methods for adding resources to general network that guarantee an improvement in performance so that the Braess paradox does not occur. The upgrade of general network, in terms of capacity can be obtained in different manners:
1. Multiplying the capacity of some specific links ($l \in \mathcal{L}$) by a constant factor $\alpha_l > 1$.
2. Adding a link between the source s and the destination d.

Consider an upgrade achieved by multiplying the capacity of each link $l \in \mathcal{L}$ by a constant factor $\alpha_l \geq 1$.

Proposition 4.1 *Suppose that the cost function $J^i(\cdot)$, $i \in \mathcal{I}$ are of type-C. Let $\tilde{c}$ and $\hat{c}$ be two capacity configurations such that $\hat{c}_l = \alpha_l \tilde{c}_l$ where $\alpha_l \geq 1$, and $\tilde{J}^i$ and $\hat{J}^i$ are the value of the cost functions of user i at the respective Nash equilibria $\tilde{X}$ and $\hat{X}$. Consider a set $\tilde{\mathcal{L}}_1 \subseteq \mathcal{L}$ defined by $\tilde{\mathcal{L}}_1 = \{l \in \mathcal{L}/\tilde{x}_l > 0\}$; assume that there exists a set of links $\hat{\mathcal{L}}_1 \subseteq \mathcal{L}$ such that $\hat{x}^i_l > 0,\ \forall i \in \mathcal{I}$ for $l \in \hat{\mathcal{L}}_1$, and $\hat{x}^i_l = 0,\ i \in \mathcal{I}$ for $l \notin \hat{\mathcal{L}}_1$. If $\alpha_l \geq I$ for all $l \in \tilde{\mathcal{L}}_1$ then $\hat{J} < \tilde{J}$.*

Proof. Using the procedure as in the proof of Prop. 3.2, we obtain:

$$\hat{\lambda}_u = \frac{\hat{x}_l}{(\alpha_l\tilde{c}_l - \hat{x}_l)^2} + \frac{I}{\alpha_l\tilde{c}_l - \hat{x}_l} + \hat{\lambda}_v \text{ if } \hat{x}_l > 0, \quad \hat{\lambda}_u \leq \frac{\hat{x}_l}{(\alpha_l\tilde{c}_l - \hat{x}_l)^2} + \frac{I}{\alpha_l\tilde{c}_l - \hat{x}_l} + \hat{\lambda}_v \text{ if } \hat{x}_l = 0 \quad (5)$$

Define the function V by

$$\hat{V}((y_l)_{l\in\mathcal{L}}) = \sum_{l\in\mathcal{L}} \frac{y_l}{\alpha_l \tilde{c}_l - y_l} - (I-1)\sum_{l\in\mathcal{L}} \ln(\alpha_l\tilde{c}_l - y_l),$$

where $(y_l)_{l\in\mathcal{L}} \in \hat{S}$ with

$$\hat{S} = \{(y_l)_{l\in\mathcal{L}} \in \mathbb{R}^{|\mathcal{L}|} : 0 \le y_l \le \alpha_l\tilde{c}_l, l \in \mathcal{L}; \sum_{l\in Out(v)} y_l = \sum_{l\in In(v)} y_l + \hat{r}_v, v \in \mathcal{V}\},$$

where $\hat{r}_s = \hat{r}$, $\hat{r}_d = -\hat{r}$ and $\hat{r}_v = 0$ for $v \neq s, d$.
Denote $(\hat{x}_l)_{l\in\mathcal{L}}$ the vector of total link flows at the Nash equilibrium $\hat{X}$. The condition (5) can be interpreted as Kuhn-Tucker condition for a single-user minimization of function V under constraints $y \in \hat{S}$. Then we can deduce that the vector $(\hat{x}_l)_{l\in\mathcal{L}}$ is the unique minimum of the function V, and since $(\tilde{x}_l)_{l\in\mathcal{L}} \in \hat{S}$, we have

$$\begin{aligned}\sum_{l\in\mathcal{L}} \frac{\hat{x}_l}{\alpha_l\tilde{c}_l - \hat{x}_l} &\le \sum_{l\in\mathcal{L}} \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l - \tilde{x}_l} - (I-1)\sum_{l\in\mathcal{L}} \ln(\frac{\alpha_l\tilde{c}_l - \tilde{x}_l}{\alpha_l\tilde{c}_l - \hat{x}_l}) \\ &< \sum_{l\in\mathcal{L}} \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l - \tilde{x}_l} - (I-1)\sum_{l\in\mathcal{L}} \ln(1 - \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l}) = \sum_{l\in\tilde{\mathcal{L}}_1} \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l - \tilde{x}_l} - (I-1)\sum_{l\in\tilde{\mathcal{L}}_1} \ln(1 - \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l}).\end{aligned}$$

Hence, in order to prove that $\hat{J}(\hat{X}) = \sum_{l\in\mathcal{L}} \frac{\hat{x}_l}{\alpha_l\tilde{c}_l - \hat{x}_l} < J(\tilde{X}) = \sum_{l\in\mathcal{L}} \frac{\tilde{x}_l}{\tilde{c}_l - \tilde{x}_l}$, it is enough to show that for all $l \in \tilde{\mathcal{L}}_1$:

$$\frac{\tilde{x}_l}{\tilde{c}_l - \tilde{x}_l} - \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l - \tilde{x}_l} + (I-1)\ln(1 - \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l}) > 0$$

which holds if $\alpha_l \ge I$ (see Appendix). ∎

Now consider an upgrade achieved by adding a link connecting the source and destination. The next result demonstrates that adding a link between the source and the destination, may lead to an increase of both the total price and the total cost.

Proposition 4.2 *Let $\hat{c}$ and $\tilde{c}$, resp., be the capacity configurations after and before the addition of a link $\hat{l}$ between the source s and destination d. Consider $\tilde{\lambda}^i_s$ and $\hat{\lambda}^i_s$ (resp., $\tilde{J}^i$ and $\hat{J}^i$) the prices (resp., the cost functions) of user i at the respective Nash equilibria $\tilde{X}$ and $\hat{X}$. We have*
1. *For the cost functions of type A, under assumption 1, if $\hat{x}_{\hat{l}} > 0$ then $\hat{\lambda}_s < \tilde{\lambda}_s$.*
2. *Assume that there exists a set of links $\hat{\mathcal{L}}_1 \subseteq \mathcal{L}$ such that $\{\hat{x}^i_l > 0, i \in \mathcal{I}\}$ for $l \in \hat{\mathcal{L}}_1$, and $\{\hat{x}^i_l = 0, i \in \mathcal{I}\}$ for $l \notin \hat{\mathcal{L}}_1$. For the cost functions of type C, we have*

$$\text{if } \hat{x}_{\hat{l}} \ge \hat{c}_{\hat{l}}\Big(1 - \prod_{l\in\mathcal{L}}(1 - \frac{\tilde{x}_l}{\tilde{c}_l})\Big) \text{ then } \hat{J} < \tilde{J},$$

where $\hat{x}_{\hat{l}}$ is the total flow on the link $\hat{l}$ at Nash equilibrium $\hat{X}$.

Proof
1. Consider the same network $(\mathcal{N}, \mathcal{L})$ with the initial capacity configuration $\tilde{c}$ and throughput demand $(\bar{r}^i)_{i\in\mathcal{I}}$ where $\bar{r}^i = r^i - \hat{x}^i_{\hat{l}}$ for all user $i \in \mathcal{I}$, and let $\bar{X}$ represent the Nash equilibrium associated to the throughput demand $(\bar{r}^i)_{i\in\mathcal{I}}$. From conditions (1) we have $\bar{x}^i_l = \hat{x}^i_l$ for all users i and $l \in \mathcal{L}$, and the Lagrange multipliers $\bar{\lambda}^i_u = \hat{\lambda}^i_u$, $i \in \mathcal{I}$

and $u \in \mathcal{N}$. Hence if $\hat{x}_{\hat{l}} > 0$, then $\bar{r} < r$ hence from Prop. 3.1 $\bar{\lambda}_s < \tilde{\lambda}_s$, thus $\hat{\lambda}_s < \tilde{\lambda}_s$.
2. If $\hat{x}_{\hat{l}} = 0$, by the above analysis, we show that $\hat{x}_l = \tilde{x}_l$, $\forall l \in \mathcal{L}$, hence $\hat{J} = \tilde{J}$.
If $\hat{x}_{\hat{l}} > 0$, then by using same procedure in Prop. (4.1), we show that the vector $(\hat{x}_l)_{l\in\mathcal{L}'}$ where $\mathcal{L}' = \mathcal{L} \cup \{\hat{l}\}$ is the unique minimum of the function:

$$\tilde{V}((y_l)_{l\in\mathcal{L}'}) = \sum_{l\in\mathcal{L}'} \frac{y_l}{\hat{c}_l - y_l} - (I-1)\sum_{l\in\mathcal{L}'} \ln(\hat{c}_l - y_l),$$

where $(y_l)_{l\in\mathcal{L}'} \in \hat{S}' := \{(y_l)_{l\in\mathcal{L}'} \in \mathbb{R}^{|\mathcal{L}|+1} : 0 \le y_l \le \hat{c}_l, l \in \mathcal{L}'; \sum_{l\in Out(v)} y_l = \sum_{l\in In(v)} y_l + \hat{r}_v, v \in \mathcal{V}\}$, where $\hat{r}_s = \hat{r}$, $\hat{r}_d = -\hat{r}$ and $\hat{r}_v = 0$ for $v \neq s, d$.
Let $(\bar{x}_l)_{l\in\mathcal{L}'} \in \mathbb{R}^{|\mathcal{L}|'}$ defined by: $\bar{x}_l = \tilde{x}_l$ for $l \in \mathcal{L}$ and $\bar{x}_{\hat{l}} = 0$. Clearly $(\bar{x}_l)_{l\in\mathcal{L}} \in \hat{S}$. Since $(\hat{x}_l)_{l\in\mathcal{L}}$ minimizes the V-function, we have:

$$\begin{aligned}
\sum_{l\in\mathcal{L}'} \frac{\hat{x}_l}{\hat{c}_l - \hat{x}_l} &\le \sum_{l\in\mathcal{L}} \frac{\tilde{x}_l}{\tilde{c}_l - \tilde{x}_l} - (I-1)\sum_{l\in\mathcal{L}} \ln(\frac{\tilde{c}_l - \tilde{x}_l}{\hat{c}_l - \hat{x}_l}) + (I-1)\ln(\frac{\hat{c}_{\hat{l}} - \hat{x}_{\hat{l}}}{\hat{c}_{\hat{l}}}) \\
&< \sum_{l\in\mathcal{L}} \frac{\tilde{x}_l}{\tilde{c}_l - \tilde{x}_l} - (I-1)\Big[\sum_{l\in\mathcal{L}} \ln(1 - \frac{\tilde{x}_l}{\tilde{c}_l}) - \ln(1 - \frac{\hat{x}_{\hat{l}}}{\hat{c}_{\hat{l}}})\Big]
\end{aligned}$$

To prove that $\hat{J} = \sum_{l\in\mathcal{L}'} \frac{\hat{x}_l}{\hat{c}_l - \hat{x}_l} < \tilde{J} = \sum_{l\in\mathcal{L}} \frac{\tilde{x}_l}{\tilde{c}_l - \tilde{x}_l}$, it is enough to remark that:

$$\sum_{l\in\mathcal{L}} \ln(1 - \frac{\tilde{x}_l}{\tilde{c}_l}) - \ln(1 - \frac{\hat{x}_{\hat{l}}}{\hat{c}_{\hat{l}}}) \ge 0, \text{ or } \hat{x}_{\hat{l}} \ge \hat{c}_{\hat{l}}\Big(1 - \prod_{l\in\mathcal{L}}(1 - \frac{\tilde{x}_l}{\tilde{c}_l})\Big)$$

which concludes the proof. ∎

Now, we consider a network $(\mathcal{N}, \mathcal{L})$ where there exists a link connecting the source and the destination. Later, we will derive sufficient conditions that guarantee an improvement in the performance when we increase the capacity of the link that connects the source s to the destination d. Denote by $\hat{l}$ the link connecting s and d.

Proposition 4.3 *Let $\tilde{c}$ and $\hat{c}$ be two capacity configurations such that $\hat{c}_l = \tilde{c}_l$ for $l \neq \hat{l}$ and $\hat{c}_{\hat{l}} = \alpha\tilde{c}_{\hat{l}}$ where $\alpha \in \mathbb{R}^+$. Consider $\tilde{\lambda}_s^i$ and $\hat{\lambda}_s^i$ (resp., $\tilde{J}^i$ and $\hat{J}^i$) the prices (resp., the cost functions) of user i at the respective Nash equilibria $\tilde{X}$ and $\hat{X}$. Then*
1. *For the cost functions of type B, under assumption 1, if $\alpha > 1$ and $\tilde{x}_{\hat{l}} > 0$ then $\hat{\lambda}_s < \tilde{\lambda}_s$.*
2. *Assume that there exists a set of links $\hat{\mathcal{L}}_1 \subseteq \mathcal{L}$ such that $\{\hat{x}_l^i > 0, i \in \mathcal{I}\}$ for $l \in \hat{\mathcal{L}}_1$, and $\{\hat{x}_l^i = 0, i \in \mathcal{I}\}$ for $l \notin \hat{\mathcal{L}}_1$. For the cost functions of type C, if $\alpha \ge I$ then $\hat{J} < \tilde{J}$.*

Proof
1. Assume that $\hat{x}_{\hat{l}} \le \tilde{x}_{\hat{l}}$, hence if $\tilde{x}_{\hat{l}} = 0$, then $\hat{x}_l = \tilde{x}_l$ for all other links $l \in \mathcal{L}$, and the price of each user are equal under both configurations.
If $\tilde{x}_{\hat{l}} > 0$, then we have $\forall i \in \mathcal{I}$, $\tilde{\lambda}_s^i = \tilde{x}_{\hat{l}}^i \tilde{T}_{\hat{l}}' + \tilde{T}_{\hat{l}}$. By summing over $i \in \mathcal{I}$, we obtain: $\tilde{\lambda}_s = \tilde{x}_{\hat{l}} \tilde{T}_{\hat{l}}' + I\tilde{T}_{\hat{l}}$. On the other hand, we have: $\hat{\lambda}_s \le \hat{x}_{\hat{l}}^i \hat{T}_{\hat{l}}' + I\hat{T}_{\hat{l}}$. Since $\hat{x}_{\hat{l}} \le \tilde{x}_{\hat{l}}$ and $\alpha > 1$, then from the last two equations we obtain $\hat{\lambda}_s < \tilde{\lambda}_s$.
Now assume that $\hat{x}_{\hat{l}} > \tilde{x}_{\hat{l}}$. Let us consider the two network that differ only by the presence or absence of link $\hat{l}$ that connects the source s and destination d. In both networks we have the same initial capacity configuration $\tilde{c}$ and the same set $\mathcal{I}$ of users, with respectively demands $\check{r}^i = r^i - \hat{x}_{\hat{l}}^i$ and $\bar{r}^i = r^i - \tilde{x}_{\hat{l}}^i$. Since $\hat{x}_{\hat{l}} > \tilde{x}_{\hat{l}}$ then $\check{r} = \sum_{i\in\mathcal{N}} \check{r}^i < \bar{r} = \sum_{i\in\mathcal{N}} \bar{r}^i$, hence from Prop. 3.1 we have: $\check{\lambda}_s < \bar{\lambda}_s$. On the other hand, for the network with demands $(\check{r}^i)_{i\in calI}$, it is easy to see that the conditions (1) are satisfied by the system flow

configuration $\check{X}$, with $\check{x}_l^i = \hat{x}_l^i$, $(\forall i \in \mathcal{I}, \forall l \in \mathcal{L})$, and $\check{\lambda}_u^i = \hat{\lambda}_u^i$ $(\forall u \in \mathcal{N})$. Similarly we conclude that the network with demands $\bar{r}^i$ has the system flow configuration $\bar{X}$, with $\bar{x}_l^i = \tilde{x}_l^i$ $(\forall i \in \mathcal{I}, \forall l \in \mathcal{L})$, and and $\bar{\lambda}_u^i = \tilde{\lambda}_u^i$, $(\forall u \in \mathcal{N})$. Hence from the fact that $\check{\lambda}_s < \bar{\lambda}_s$, we obtain $\hat{\lambda}_s < \tilde{\lambda}_s$.
2. Using the procedure as in Prop. 4.1, we show that $(\hat{x}_l)_{l\in\mathcal{L}}$ is the unique minimum of

$$\hat{V}((y_l)_{l\in\mathcal{L}}) = \sum_{l\in\mathcal{L}} \frac{y_l}{\hat{c}_l - y_l} - (I-1)\sum_{l\in\mathcal{L}} \ln(\hat{c}_l - y_l),$$

where $(y_l)_{l\in\mathcal{L}} \in \hat{S}$ with $\hat{S}' = \{(y_l)_{l\in\mathcal{L}'} \in \mathbb{R}^{|\mathcal{L}|} : 0 \le y_l \le \hat{c}_l, l \in \mathcal{L}; \sum_{l\in Out(v)} y_l = \sum_{l\in In(v)} y_l + \hat{r}_v, v \in \mathcal{V}\}$. Since $(\tilde{x}_l)_{l\in\mathcal{L}} \in \hat{S}$ and since $(\hat{x}_l)_{l\in\mathcal{L}}$ minimizes the $\hat{V}$-function,

$$\begin{aligned}\sum_{l\in\mathcal{L}} \frac{\hat{x}_l}{\hat{c}_l - \hat{x}_l} &\le \sum_{l\in\{\mathcal{L}\setminus\{\hat{l}\}\}} \frac{\tilde{x}_l}{\tilde{c}_l - \tilde{x}_l} - (I-1) \sum_{l\in\{\mathcal{L}\setminus\{\hat{l}\}\}} \ln(\frac{\tilde{c}_l - \tilde{x}_l}{\tilde{c}_l - \hat{x}_l}) \\ &\quad + \frac{\tilde{x}_{\hat{l}}}{\alpha\tilde{c}_{\hat{l}} - \tilde{x}_{\hat{l}}} - (I-1)\ln(\frac{\alpha\tilde{c}_{\hat{l}} - \tilde{x}_{\hat{l}}}{\alpha\tilde{c}_{\hat{l}} - \hat{x}_{\hat{l}}}) \\ &< \sum_{l\in\{\mathcal{L}\setminus\{\hat{l}\}\}} \frac{\tilde{x}_l}{\tilde{c}_l - \tilde{x}_l} - (I-1) \sum_{l\in\{\mathcal{L}\setminus\{\hat{l}\}\}} \ln(\frac{\tilde{c}_l - \tilde{x}_l}{\tilde{c}_l - \hat{x}_l}) + \frac{\tilde{x}_{\hat{l}}}{\alpha\tilde{c}_{\hat{l}} - \tilde{x}_{\hat{l}}} - (I-1)\ln(1 - \frac{\tilde{x}_{\hat{l}}}{\alpha\tilde{c}_{\hat{l}}}).\end{aligned}$$

To prove that $\hat{J} = \sum_{l\in\mathcal{L}} \frac{\hat{x}_l}{\hat{c}_l - \hat{x}_l} < \tilde{J} = \sum_{l\in\mathcal{L}} \frac{\tilde{x}_l}{\tilde{c}_l - \tilde{x}_l}$, it is enough to show that:

$$Q_\alpha(\tilde{x}_{\hat{l}}) = \frac{\tilde{x}_{\hat{l}}}{\tilde{c}_{\hat{l}} - \tilde{x}_{\hat{l}}} - \frac{\tilde{x}_{\hat{l}}}{\alpha\tilde{c}_{\hat{l}} - \tilde{x}_{\hat{l}}} + (I-1)\ln(1 - \frac{\tilde{x}_{\hat{l}}}{\alpha\tilde{c}_{\hat{l}}}) + (I-1)\sum_{l\in\{\mathcal{L}\setminus\{\hat{l}\}\}} \ln(\frac{\tilde{c}_l - \tilde{x}_l}{\tilde{c}_l - \hat{x}_l}) > 0$$

By using the same procedure as in the Appendix, we show that for $\alpha \ge I$, Q_α is strictly increasing, and since $Q_\alpha(0) \ge 0$ and $\tilde{x}_{\hat{l}} > 0$ then $Q_\alpha(\tilde{x}_{\hat{l}}) > 0$. ∎

Remark 1
For the first part of Prop. 4.3 , we can obtain some results in the case where $\tilde{x}_{\hat{l}} = 0$ by assuming that $\hat{x}_{\hat{l}} > 0$ (Prop. 4.2). In second part of Prop. 4.3, we can replace assumption $\tilde{x}_{\hat{l}} > 0$ by $\hat{x}_{\hat{l}} > 0$. Indeed, in this case we have $Q_\alpha(0) > 0$ and since Q_α is strictly increasing, then $Q_\alpha(\tilde{x}_{\hat{l}}) > 0$.

5. Experimental results

Let us now demonstrate the efficiency of the proposed capacity addition by means of a numerical example. Consider the example studied in [13], in which an addition of capacity may, in general, increase both the price and the cost of each and every user. In all cases below, we computed the equilibrium iteratively with relaxation (which has been proven for some topologies to converge to an equilibrium, see [1]) as follows:
1. Define a candidate solution $\{\tilde{x}(0)\}_{l\in\mathcal{L}}$ for the *total link flows* which is obtained by minimizing the function V defined in (4). The flow of each player i in the initial iteration is then defined as $\tilde{x}_l^i(0) = \tilde{x}_l(0) r^i [\sum_{j\in\mathcal{I}} r^j]^{-1}$.
2. At iteration $n > 0$, we first compute the best responses $\{\hat{x}_l^i(n)\}$ for each user i when all players other than i use $\{\tilde{x}_l^j(n-1)\}_{j\ne i,l}$.
3. The approximation of the equilibrium at step n is then given by $\tilde{x}_l^i(n) = \alpha\tilde{x}_l^i(n-1) + (1-\alpha)\hat{x}_l^i(n)$, for all $i \in \mathcal{I}$ and $l \in \mathcal{L}$. The procedure ends when $\tilde{x}(n)$ is sufficiently close to $\tilde{x}(n-1)$.

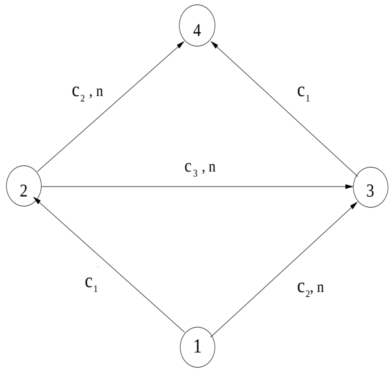

Figure 1. Network example

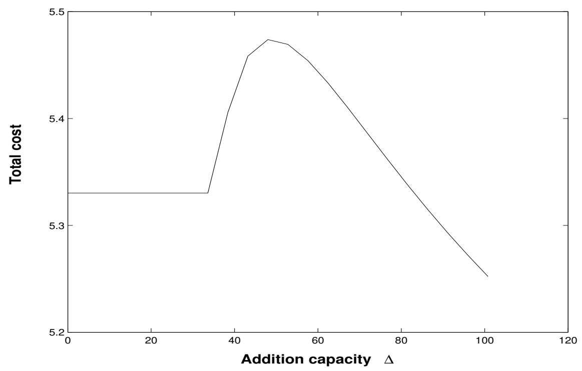

Figure 2. Total cost as a function of the added capacity in path (2,3)

Remark 2 Note that if the "all-positive flows" condition (defined before Corollary 1) holds at equilibrium, then $\{\tilde{x}_l(0)\}_l$ will already be the total link flow at equilibrium, and only the individual link flows have to be defined. Note also that if the users were identical, then by [15], there would be a unique equilibrium and it would be symmetric. Hence the condition of "all-positive flows" would indeed hold, and $\{\tilde{x}_l^i(0)\}_{i,l}$ would already correspond to the equilibrium. No further iteration is needed.

In all our experimentations below, it turned out that the condition of "all-positive flows" indeed was satisfied, so we could check that our algorithm indeed provided the correct value for the total link flows at equilibrium. The number of iterations that were required in all cases was around 20 (which leads to a difference between $\tilde{x}(n)$ and $\tilde{x}(n+1)$ of less than 10^{-5}), and we used $\alpha = 1/2$.

5.1. Braess paradox

Consider the network depicted in Fig. 1. Links (1,2) and (3,4) each have capacity $c_1 = 2.7$. Link (1,3) represents a path of n tandem links, each with capacity $c_2 = 27$. Similarly links (2,3) and (2,4) are paths of n consecutive links each with capacity $c_3 = 4.8$ and $c_4 = 27$ respectively. There are I users, each sending a flow r^i from node 1 to node 4.

We consider the scenario of the Braess paradox, where extra capacity is added to link (2,3). Fig. 2 shows, that the user cost as a function of the added capacity Δ in link (2,3), for $n = 54$, $I = 2$, $r^1 = 0.8$ and $r^2 = 1.2$. The figure indicates that the total cost increases when the additional capacity Δ increases, i.e., addition of capacity in link (2,3) leads to a degradation of performance of the network until the total capacity on the link reaches 53. Then it remains almost constant when the capacity is further increased (the cost slightly decreases at that region).

5.2. Multiplying the capacity of some specific links ($l \in \mathcal{L}$) by a constant factor.

We use the method proposed in Prop. 4.1 for efficiently adding resources to this network. Fig. 3 shows the total cost as a function of added capacity Δ, for $c_1 = 2.7 + \frac{\Delta}{6}$, $c_2 = 27 + \frac{\Delta}{4}$ and $c_3 = 4.8 + \frac{\Delta}{3}$. Fig. 3 indicates that the total cost decreases when the additional capacity Δ increases. Hence the Braess paradox is indeed avoided.

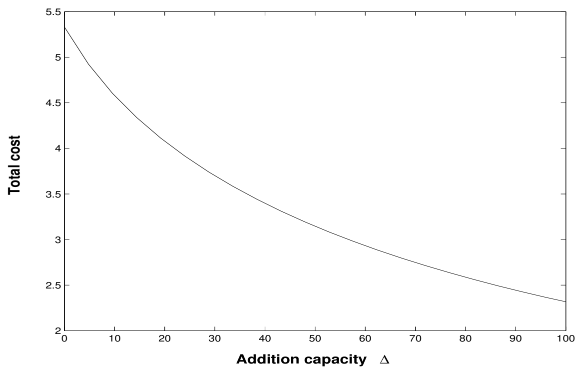

Figure 3. Total cost as a function of the added capacity in all links

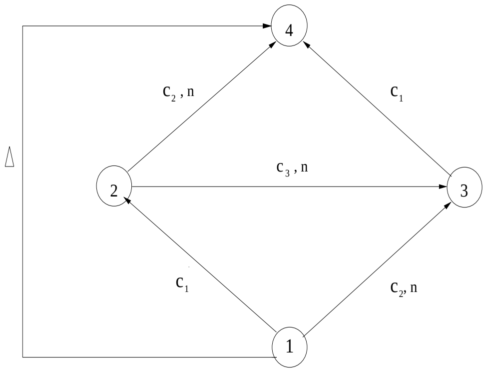

Figure 4. New network

5.3. Adding a link between source 1 and destination 4

Consider an upgrade as proposed in Prop. 4.2 and 4.3, i.e., the upgrade achieved by adding a link connecting source 1 and destination 4. The results in Prop. 4.2 and 4.3 suggest that yet another good design practice is to focus the upgrades on direct connections between source and destination; and figure 5 and 6 illustrate that indeed this approach decreases the total price and the total cost.

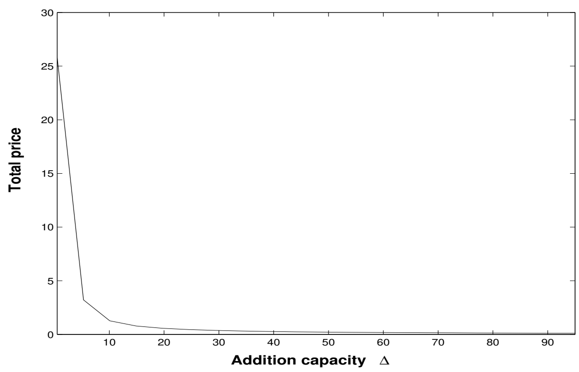

Figure 5. Total price as a function of the added capacity in link (1,4)

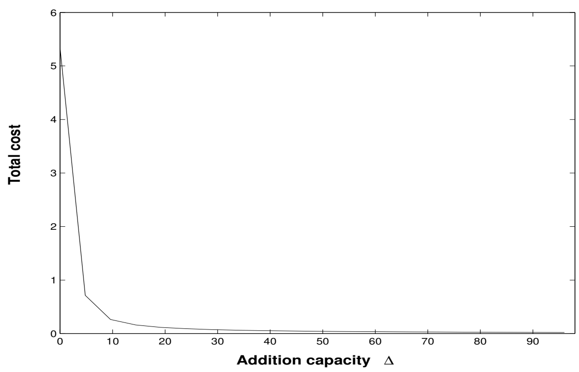

Figure 6. Total cost as a function of the added capacity in link (1,4)

Appendix

In this appendix, we analyze a function $H_\alpha : [0, \alpha\bar{c}) \to \mathbb{R}$ defined by:

$$H_\alpha(x) = \frac{x}{\bar{c} - x} - \frac{x}{\alpha c - x} + (I - 1)\ln(1 - \frac{x}{\alpha}c), \tag{6}$$

where $\alpha \geq 1$ and $\bar{c}$ is a constant positive. More precisely, we wish to determine α such that H_α is positive for every x in $[0, \alpha\bar{c})$. By remarking that $H(0) = 0$, $\forall\alpha$, it is enough to determine α such that

$$\frac{\partial H_\alpha}{\partial x} = \frac{c}{(c-x)^2} - \frac{\alpha c}{(\alpha c - x)^2} - (I-1)\frac{1}{\alpha c - x} > 0.$$

This last inequality is equivalent to

$$\frac{c}{(c-x)^2} > \frac{I\alpha c - (I-1)x}{(\alpha c - x)^2}$$

which is equivalent to $(\alpha^2 - I\alpha)c^3 + cx^2(3 - I\alpha - 2I) > c^2x(2\alpha - 2I\alpha - I + 1) - (I-1)x^3$. If $\alpha \geq I$, then it is enough to show that $cx(3 - I\alpha - 2I) > c^2(2\alpha - 2I\alpha - I + 1) - (I-1)x^2$ Since $x^2 + c^2 \geq 2cx$, it is sufficient to verify that: $c^2(1 + 3I\alpha - 4\alpha) > x^2(I\alpha - 1)$, or $c^2(\alpha(I - 2) + 1) > 0$, which trivially holds.

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