Properties of Equilibria in Competitive Routing with Several User Types [1]

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Abstract

In recent years there has been a growing interest in models for routing in networks in which decisions are taken in a non-cooperative way. Instead of a single decision maker, one considers a number of decision makers having each its own utility to maximize by routing its own flow. This gives rise to a non-cooperative game whose solution concept is the Nash equilibrium. In the special case where each decision maker wishes to find a minimal path for each routed job (a packet) then the solution concept is the Wardrop equilibrium. It is well known that equilibria may exhibit inefficiencies and paradoxical behavior, such as the famous Braess paradox in which the addition of a link to a network results in worse performance to all users. This raises the challenge of how to upgrade the network so that it indeed results in improved performance. We present guidelines for the allocation of extra resources in a manner that Braess' Paradox is avoided, and the performance of every user is improved.

1 Introduction

We consider the problem of routing, in which the performance measure to be minimized is some general cost (e.g. expected delay). We assume that some objects (packets or sessions) are routed over shortest paths computed in terms of that cost. A routing approach in which each packet follows a shortest delay path has been advocated in Ad-hoc networks [11], in which, the large amount of mobility of users and of the routers requires to update the routes frequently; it has further been argued that by minimizing the delay of each packet, we minimize re-sequencing delays, that may be harmful in real time applications, but also in data communications (e.g. TCP/IP connections). When the above type of routing approach is used then the expected load at different links in the network can be predicted as an equilibrium which can be computed in a way similar to Wardrop equilibria that arise in road traffic [21] (it is known to exist and to be unique under general assumptions on the topology and on the cost [20, p. 74-75]). We study in this paper some properties of the equilibrium. In particular, we are interested in the impact of the demand of link capacities and of the topology on the performance measures at equilibrium. This has a particular significance for the network administrator or designer when it comes to upgrading the network.

A frequently used heuristic approach for upgrading a network is through *Bottleneck Analysis.* A system bottleneck is defined as "a resource or service facility whose capacity seriously limits the performance of the entire system" [14, p. 13]. Bottleneck analysis consists of adding capacity to identified bottlenecks until they cease to be bottlenecks. In a non-cooperative framework, however, this approach may have devastating effects; adding capacity to a bottleneck link may cause delays of all users to increase. The problem has already been identified in the road-traffic context by Braess [4] (see also [8]), and have further been studied in the context of queuing networks, communication networks and computer load balancing in [3, 5, 6, 7, 18, 13, 15, 17]. The Braess paradox illustrates that the network designer or service providers have to take into consideration the reaction of non-cooperative users to their decisions. Some upgrading guidelines have been proposed in [1, 17] so as to avoid the Braess paradox or to obtain better performance. They considered not only the framework of the Wardrop equilibrium, but also the Nash-equilibrium concept in which a finite number of service providers try to minimize the average delays (or cost) for all the flow generated by its subscribers. The results obtained for the Wardrop equilibrium were restricted to a particular cost representing the delay of a M/M/1 queue at each link. In this paper we extend the above results to general costs. We further consider a more general routing structure (between paths and not just between links) and allow for several classes of users (so that the cost of a path or of a link may depend on the class). For guidelines for avoiding Braess paradox in the setting of Nash equilibrium see [1].

The paper is organized as follows: In the next section (Section 2), we present the network model, we define the concept of Wardrop equilibrium, and formulate the problem. In Section 3 we then present a framework for an equilibrium that allows for different costs for different classes of users. In Section 4 we then present a sufficient conditions for the monotonicity of performance measures when the demands increase. This allows us then to study in Section 5 methods for capacity addition. In section 6, we demonstrate the efficiency of the proposed capacity addition by means of a numerical example in **BCMP** queuing network.

2 Problem formulation and notation

We consider an open network model that consists of a set $\mathbb{M}$ containing M nodes, and a set $\mathbb{L}$ containing L links. We call the unit that has to be routed a "job". It may stand for a packet (in a packet switched network) or to a whole session (if indeed all packets of a session follow the same path). The network is utilized by infinitely many jobs that have to choose their routing. Jobs are classified into K different classes (we denote $\mathbb{K}$ the set of classes). For example, in the context of road traffic a class represents the set of a given type of vehicles, such as buses, trucks, cars or bicycles. In the context of telecommunications a class may represent the jobs sent by all the users of a given service provider. We assume that jobs do not change their

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class while passing through the network. We suppose that the jobs of a given class k may arrive in the system at some possible points, and leave the system at some different possible points. Nevertheless the origin and destination points of a given job are determined when the job arrives in the network, and cannot change while in the system. We call a pair of one origin and one destination points an *O-D* pair. A job with a given *O-D* pair (od) arrives in the system at node o and leaves it at node d after visiting a series of nodes and links, which we refer to as a *path*.

In many previous papers [19, 15], routing could be done at each node. In this paper we follow the approach in which a job of class k with O-D pair (od) has to choose one of a given finite set of paths (see also [12, 20]). We suppose that the routing decision scheme is completely decentralized: each single job has to decide among a set of possible paths that connect the *O-D* pair of that job so as to minimize the cost of that job. The solution concept is the Wardrop equilibrium [21].

Let l denote a link of the network connecting a pair of nodes and let p denote a path, consisting of a sequence of links connecting an *O-D* pair of nodes. Let W^k denote the set of *O-D* pairs for the jobs of class k. Denote also W the union $W = \bigcup_k W^k$. The set of paths connecting the *O-D* pair $w \in W^k$ is denoted by $\mathcal{P}^k_w$ and the entire set of paths in the network for the jobs of class k by $\mathcal{P}^k$. There are n^k_p paths in the networks for jobs of class k and n_p paths of all jobs. Let y^k_l denote the flow of class k on link l and let x^k_p denote the non-negative flow of class k on path p. The relationship between the link loads by class and the path flows is: $y^k_l = \sum_{p \in \mathcal{P}^k} x^k_p \delta_{lp}$ where $\delta_{lr} = 1$, if link l is contained in path p, and 0, otherwise. Let μ^k_l the service rate of class k at link l. Hence, the utilization of link l for class k is given by $\rho^k_l = y^k_l / \mu^k_l$ and the total utilization on link l is: $\rho_l = \sum_{k \in I\!\!K} \rho^k_l$.. Let r^k_w denote the demand of class k for *O-D* pair w, where the following conditions are satisfied: $r^k_w = \sum_{p \in \mathcal{P}^k_w} x^k_p, \quad \forall k, \quad \forall w,$. In addition, let x_p denote the total flow on path p, where

$$x_p = \sum_{k \in I\!\!K} x^k_p, \quad \forall p \in \mathcal{P}.$$

We group the class path flows into the n^k_p-dimensional column vector $\mathbf{X}$ with components: $[x^1_{r_1}, \ldots, x^K_{r_{n^k_p}}]^T$; We refer to such a vector as a *flow configuration.* We also group total path flow into a n_p-dimensional column vector $\mathbf{x}$ with components: $[x_{r_1}, \ldots, x_{r_{n_p}}]^T$. We call this vector the *total path flow vector.* A flow configuration $\mathbf{X}$ is said feasible, if it satisfies for each *O-D* pair $w \in W^k$,

$$\sum_{p \in \mathcal{P}^k_w} x^k_p = r^k_w. \tag{1}$$

For a feasible flow configuration $\mathbf{X}$, let $T^k_p(\mathbf{X})$ denote the travel cost incurred by a job of class k for using the path p if the flow configuration resulting from the routing of each job is $\mathbf{X}$.

3 Wardrop equilibrium for multiclass network

Each individual job of class k with *O-D* pair w, chooses its routing through the system, by means of the choice of a path $p \in \mathcal{P}^k_w$. A flow configuration $\mathbf{X}$ follows from the choices of each of the infinitely many jobs. A flow configuration $\mathbf{X}$ will be said to be a *Wardrop equilibrium* or *individually optimal*, if none of the jobs has any incentive to change unilaterally its decision. In the context of multi-class this can be defined as

Definition 1 *A feasible flow configuration (satisfying (1)) $\boldsymbol{X}$, is a Wardrop equilibrium for the multi-class problem if for any class k, any $w \in W^k$ and any path $p \in \mathcal{P}^k_w$ we have*

$$\begin{cases} T^k_p(\boldsymbol{X}) \geq \lambda^k_w & \text{if } x^k_p = 0, \\ T^k_p(\boldsymbol{X}) = \lambda^k_w & \text{if } x^k_p \geq 0, \end{cases} \tag{2}$$

where $\lambda^k_w = \min_{p \in \mathcal{P}^k_w} T^k_p(\boldsymbol{X})$. The minimal cost λ^k_w will be referred to as "the travel cost" associated to class k and O-D pair w.

We need one of the following assumptions on the costs:

Assumption A

1- There exists a function T_p that depends only upon the total flow vector $\mathbf{x}$ (and not on the flow sent by each class), such that the average cost per flow unit, for jobs of class k, can be written as $T^k_p(\mathbf{X}) = c^k T_p(\mathbf{x})$, $\forall p \in \mathcal{P}^k$, where c^k is some class dependent positive constant.
2- T_p is positive, continuous and strictly increasing. We denote $\mathbf{T}' = (T_{p_1}, T_{p_2}, \ldots)'$ the vector of functions T_p.

Assumption B

1- The average cost per flow unit for jobs of class k that passes through path $p \in \mathcal{P}^k$ is:

$$T^k_p(\mathbf{X}) = \sum_{l \in I\!\!L} \frac{\delta_{lp}}{\mu^k_l} T_l(\rho_l),$$

where $T_l(\rho_l)$ is the weighted cost per unit flow in link l (the function T_l does not depend on the class k).
2- $T_l(.)$ is positive, continuous and strictly increasing.
3- μ^k_l can be represented as μ_l / c^k where c^k is some class dependent positive constant, and $0 < \mu_l$ is finite.

We denote $v_w = \sum_{k \in I\!\!K} c^k r^k_w$ the weighted total demand for *O-D* pair w.

Lemma 1 *Consider a cost vector $\boldsymbol{T}$ satisfying Assumption $\boldsymbol{A}$ or $\boldsymbol{B}$. Then the Wardrop conditions (1) and (2) become: For all k, all $w \in W^k$ and all $p \in \mathcal{P}^k_w$,*

$$\begin{array}{l} T^k_p(\boldsymbol{X}) \geq \lambda^k_w, \text{ if } x_p = 0, \\ T^k_p(\boldsymbol{X}) = \lambda^k_w, \text{ if } x_p > 0. \end{array} \tag{3}$$

Moreover, the ratio λ^k_w / c^k is independent of class k, so that we can define λ_w by $\lambda_w := \lambda^k_w / c^k$.

Proof. Consider first the case of a cost vector $\mathbf{T}$ that satisfies Assumption $\mathbf{A}$. Let $w \in W$ and $p \in \mathcal{P}^k$. If $x_p = 0$ then $x^k_p = 0 \ \forall k \in I\!\!K$. The first part of (3) follows from the first part of (2). Suppose that $x_p > 0$ and, by contradiction, that there exists $\bar{k} \in I\!\!K$ such that

$$T^{\bar{k}}_p(\mathbf{X}) = T_p(\mathbf{x}) c^{\bar{k}} > \lambda^{\bar{k}}_w. \tag{4}$$

Since $x_p > 0$, there exit $k_0 \in I\!\!K$ such that $x^{k_0}_p > 0$. From the second part of (2), we have

$$T^{k_0}_p(\mathbf{X}) = T_p(\mathbf{x}) c^{k_0} = \lambda^{k_0}_w. \tag{5}$$

Because $r_w^{\bar{k}} > 0$, there exists $p' \in \mathcal{P}_w^{\bar{k}}$ such that $x_{p'}^{\bar{k}} > 0$. Then, from (2) we get

$$T_{p'}^{\bar{k}}(\mathbf{X}) = T_{p'}(\mathbf{x})c^{\bar{k}} = \lambda_w^{\bar{k}}. \tag{6}$$

It follows from (4) and (6) that

$$T_p(\mathbf{x}) > T_{p'}(\mathbf{x}). \tag{7}$$

Since $\lambda_w^{k_0} \leq T_{p'}(\mathbf{x})c^{k_0}$, from (5), we obtain $T_p(\mathbf{x}) \leq T_{p'}(\mathbf{x})$, which contradicts (7). This establishes (3).
For any $w \in W$, let $p \in \bigcup_k \mathcal{P}_w^k$ be a path such that $x_p > 0$. From (3), it follows that for any class k such that $p \in \mathcal{P}_w^k$, $T_p(\mathbf{x}) = T_p^k(\mathbf{X})/c^k = \lambda_w^k/c^k$. The second part of Lemma 1 follows, since the terms in the above equation do not depend on k. The proof for a cost function vector satisfying assumption **B** follows along similar lines. $\diamond$

4 Impact of throughput variation

We now study the impact of a variation of the demands r_w^k of some class k on the cost vector $\mathbf{T}(\mathbf{X})$ at the Wardrop equilibrium $\mathbf{X}$. Out results extend those of [9] where a simpler cost structure was considered: for any class k, the cost for using a path was the sum of link costs along that path, and the link costs did not depend on k.

Theorem 1 ([10, Th 1]) *Consider two throughput demand profiles $(\tilde{r}_w^k)_{(w,k)}$ and $(\hat{r}_w^k)_{(w,k)}$. Let $\tilde{\mathbf{X}}$ and $\hat{\mathbf{X}}$ be the Wardrop equilibria associated to these throughput demands, and let $\tilde{\lambda}_w^k$ and $\hat{\lambda}_w^k$ be the class k's travel cost associated to these two equilibria.*
1- For a cost vector $\boldsymbol{T}$ satisfying Assumption $\boldsymbol{A}$, if $\hat{r}_{\overline{w}} < \tilde{r}_{\overline{w}}$, for some $\overline{w} \in W$ and $\hat{r}_w = \tilde{r}_w$ for all $w \neq \overline{w}$, then $\hat{\lambda}_{\overline{w}}^k < \tilde{\lambda}_{\overline{w}}^k$ $\forall k \in I\!\!K$.
2- For a cost vector $\boldsymbol{T}$ satisfying Assumption $\boldsymbol{B}$, if $\hat{v}_{\overline{w}} < \tilde{v}_{\overline{w}}$, for some $\overline{w} \in W$ and $\hat{v}_w = \tilde{v}_w$ for all $w \neq \overline{w}$, then $\hat{\lambda}_{\overline{w}}^k < \tilde{\lambda}_{\overline{w}}^k$ $\forall k \in I\!\!K$.

Remark 1 *In the case where all classes ship flow from a common source s to a common destination d i.e. $\mathcal{P}^k = \{(sd)\}, \forall k$, Theorem 1 establishes the monotonicity of performance (given by travel cost $\lambda_{(sd)}^k$) at Wardrop equilibrium for all $k \in I\!\!K$, when the demands of classes increases.*

5 Avoiding the Braess paradox

The purpose of this section is to provide some methods for adding resources to a general network, with one source s and one destination d, that guarantee improvement in performance. This would guarantee in particular that the Braess paradox does not occur. For some given network with one source and one destination, the designer problem is to distribute some additional capacity among the links of the network so as to improve the performances at the (Wardrop) equilibrium. Adding capacity in the network can be done by several ways. Among them,
(1) by adding a new direct path from s to d,
(2) by improving an existing direct path,
(3) by improving all the paths connecting s to d.

We first consider (1). The added path could be in fact a whole new network, provided that it is disjoint with the previous network; it may also have new sources and destinations in addition to s and d and new traffic from new classes that use these new sources and destinations. The next theorem shows that this may lead to a decrease of the costs of all paths used at equilibrium in the original network.

Theorem 2 *Consider a cost vector $\mathbf{T}$ that satisfies Assumption $\boldsymbol{A}$ or $\boldsymbol{B}$. Let $\hat{\boldsymbol{X}}$ and $\tilde{\boldsymbol{X}}$ be the Wardrop equilibria after and before the addition of a direct path $\hat{p}$ from s to d. Consider $\tilde{\lambda}_{(sd)}^k$ and $\hat{\lambda}_{(sd)}^k$ the travel costs for class k respectively at $\tilde{\boldsymbol{X}}$ and $\hat{\boldsymbol{X}}$. Then, $\hat{\lambda}_{(sd)}^k \leq \tilde{\lambda}_{(sd)}^k$, $\forall k \in I\!\!K$. Moreover the last inequality is strict if $\hat{x}_{\hat{p}} > 0$.*

Proof. Consider the same network $(I\!\!M, I\!\!L)$ with the initial service rate configuration $\tilde{\mu}$ and throughput demand $(\bar{r}_{(sd)}^k)_{k \in I\!\!K}$ where $\bar{r}_{(sd)}^k = r_{(sd)}^k - \hat{x}_{\hat{p}}^k$ for all class $k \in I\!\!K$. Let $\bar{\mathbf{X}}$ represent the Wardrop equilibrium associated to this new throughput demand and $\bar{\lambda}_{(sd)}^k$ the travel cost for class k at Wardrop equilibrium $\bar{\mathbf{X}}$. From Conditions (1) and (2) we have $\bar{\lambda}_{(sd)}^k = \hat{\lambda}_{(sd)}^k$, $\forall k \in I\!\!K$. If $x_{\hat{p}} = 0$, then $\bar{\lambda}_{(sd)}^k = \tilde{\lambda}_{(sd)}^k$, which implies that $\hat{\lambda}_{(sd)}^k = \tilde{\lambda}_{(sd)}^k$.
Assume, then, that $\hat{x}_{\hat{p}} > 0$. We have $\bar{r}_{(sd)} < r_{(sd)}$ (which will be used for Assumption **A**) and $\bar{v}_{(sd)} < v_{(sd)}$ (which will be used for Assumption **B**), following Theorem 1, we conclude that $\hat{\lambda}_{(sd)}^k = \bar{\lambda}_{(sd)}^k < \tilde{\lambda}_{(sd)}^k$, for all $k \in I\!\!K$ and this completes the proof. $\diamond$

We now examine the second way of adding capacity to the network: the improvement of an existing direct path. We consider a network $(I\!\!M, I\!\!L)$ that contains a direct path, $\hat{p}$, from s to d that can be used by the jobs of all classes.

Theorem 3 *Let $\boldsymbol{T}$ a cost vector satisfying Assumptions $\boldsymbol{A}$. We consider an improvement of the path $\hat{p}$ so that the cost associated to this path is smaller for all classes, i.e. $\hat{T}_{\hat{p}}(\boldsymbol{x}) < \tilde{T}_{\hat{p}}(\boldsymbol{x})$. Let $\hat{\boldsymbol{X}}$ and $\tilde{\boldsymbol{X}}$ the Wardrop equilibria respectively after and before this improvement. Consider $\tilde{\lambda}_{(sd)}^k$ and $\hat{\lambda}_{(sd)}^k$ the travel cost of class k at equilibria. Then $\hat{\lambda}_{(sd)}^k \leq \tilde{\lambda}_{(sd)}^k$, $\forall k \in I\!\!K$. Moreover the inequality is strict if $\hat{x}_{\hat{p}} > 0$ or $\tilde{x}_{\hat{p}} > 0$.*

Proof. From Lemma 1 we have

$$\begin{aligned} \lambda_{(sd)} = T_p(\mathbf{x}), &\quad x_p > 0, \\ \lambda_{(sd)} \leq T_p(\mathbf{x}), &\quad x_p = 0, \\ \sum_{p \in \cup_k \mathcal{P}_{(sd)}^k} x_p = r_{(sd)}, &\quad x_p \geq 0, \ p \in \mathcal{P}^k. \end{aligned} \tag{8}$$

We know from Theorems 3.2 and 3.14 in [20] that $\hat{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ must satisfy the variational inequalities

$$\hat{\mathbf{T}}'(\hat{\mathbf{x}})^T(\mathbf{x} - \hat{\mathbf{x}}) \geq 0, \quad \forall \mathbf{x} \text{ that satisfies (8)}, \tag{9}$$
$$\tilde{\mathbf{T}}'(\hat{\mathbf{x}})^T(\mathbf{x} - \tilde{\mathbf{x}}) \geq 0, \quad \forall \mathbf{x} \text{ that satisfies (8)}. \tag{10}$$

By adding (9) with $\mathbf{x} = \tilde{\mathbf{x}}$ and (10) with $\mathbf{x} = \hat{\mathbf{x}}$, we obtain $[\hat{\mathbf{T}}'(\hat{\mathbf{x}}) - \tilde{\mathbf{T}}'(\tilde{\mathbf{x}})][\hat{\mathbf{x}} - \tilde{\mathbf{x}}] \leq 0$, thus

$$[\hat{\mathbf{T}}'(\hat{\mathbf{x}}) - \tilde{\mathbf{T}}'(\hat{\mathbf{x}}) + \tilde{\mathbf{T}}'(\hat{\mathbf{x}}) - \tilde{\mathbf{T}}'(\tilde{\mathbf{x}})][\hat{\mathbf{x}} - \tilde{\mathbf{x}}] \leq 0,$$

and

$$[\hat{\mathbf{T}}'(\hat{\mathbf{x}}) - \tilde{\mathbf{T}}'(\hat{\mathbf{x}})][\hat{\mathbf{x}} - \tilde{\mathbf{x}}] \leq [\hat{\mathbf{T}}'(\hat{\mathbf{x}}) - \tilde{\mathbf{T}}'(\tilde{\mathbf{x}})][\hat{\mathbf{x}} - \tilde{\mathbf{x}}] < 0.$$

Since the costs of other paths are unchanged, *i.e.* $\hat{T}_p = \tilde{T}_p$ for all $p \neq \hat{p}$, the last inequality becomes $(\hat{T}_{\hat{p}}(\hat{\mathbf{x}}) - \tilde{T}_{\hat{p}}(\hat{\mathbf{x}})(\hat{x}_{\hat{p}} - \tilde{x}_{\hat{p}}) < 0$ if $\hat{\mathbf{x}} \neq \tilde{\mathbf{x}}$. Since $\hat{T}_{\hat{p}}(\hat{\mathbf{x}}) < \tilde{T}_{\hat{p}}(\hat{\mathbf{x}})$, we have

$$\hat{x}_{\hat{p}} > \tilde{x}_{\hat{p}} \text{ if } \hat{\mathbf{x}} \neq \tilde{\mathbf{x}}. \tag{11}$$

Now we have two cases:
1- If $\hat{\mathbf{x}} = \tilde{\mathbf{x}}$ and since $\hat{T}_{\hat{p}}(\mathbf{x}) \neq \tilde{T}_{\hat{p}}(\mathbf{x})$ $\forall \mathbf{x}$, it follows that $\hat{x}_{\hat{p}} = \tilde{x}_{\hat{p}} = 0$, which implies that $\hat{\lambda}^k_{(sd)} = \tilde{\lambda}^k_{(sd)}$.
2- If $\hat{\mathbf{x}} \neq \tilde{\mathbf{x}}$, then from (11) we have $\hat{x}_{\hat{p}} > \tilde{x}_{\hat{p}}$. Consider now two networks that differ only by the presence or absence of the direct path $\hat{p}$ from s to d. In both networks we have the same initial capacity configuration and the same set $I\!K$ of classes, with respectively demands $\check{r}^k_{(sd)} = r^k_{(sd)} - \hat{x}^k_{\hat{p}}$ and $\bar{r}^k_{(sd)} = r^k_{(sd)} - \tilde{x}^k_{\hat{p}}$. Let $\check{\lambda}^k_{(sd)}$ and $\bar{\lambda}^k_{(sd)}$ the travel cost of class k associated to these throughput demands. Since $\hat{x}_{\hat{p}} > \tilde{x}_{\hat{p}}$ then $\check{r}_{(sd)} < \bar{r}_{(sd)}$, and from Theorem 1 we have

$$\forall k \in I\!K, \quad \check{\lambda}^k_{(sd)} < \bar{\lambda}^k_{(sd)}. \tag{12}$$

On the other hand, for the network with demands $(\check{r}^k_{(sd)})_{k \in K}$, it is easy to see that the equilibria conditions (1) and (2) are satisfied by the system flow configuration $\check{\mathbf{X}}$, with $\check{\lambda}^k_{(sd)} = \hat{\lambda}^k_{(sd)}$. Similarly we conclude that the network with demands $(\check{r}^k_{(sd)})_{k \in I\!K}$ has the system flow configuration $\bar{\mathbf{X}}$, with $\bar{\lambda}^k_{(sd)} = \tilde{\lambda}^k_{(sd)}$. Hence from (12) we obtain $\hat{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$.

Theorem 4 *Consider a cost function vector that satisfies Assumptions **B**. Let $\hat{\mu}^k_l$ and $\tilde{\mu}^k_l$, respectively, be the service rate configurations after and before adding the capacity to the path $\hat{p}$, i.e $\hat{\mu}_l > \tilde{\mu}_l$ for $l \in \hat{p}$ and $\hat{\mu}_l = \tilde{\mu}_l$ for $l \notin \hat{p}$. Let $\hat{\boldsymbol{X}}$ and $\tilde{\boldsymbol{X}}$, respectively, the Wardrop equilibria after and before this improvement. Let $\hat{\lambda}^k_{(sd)}$ and $\tilde{\lambda}^k_{(sd)}$ the travel cost of class k at the equilibria. Then $\hat{\lambda}^k_{(sd)} \leq \tilde{\lambda}^k_{(sd)}$, $\forall k \in I\!K$. Moreover the inequality is strict if $\hat{x}_{\hat{p}} > 0$ or $\tilde{x}_{\hat{p}} > 0$.*

Proof. Note that if there exists a link l_1 that belongs to the path $\hat{p}$, such that $\hat{\rho}_{l_1} < \tilde{\rho}_{l_1}$, then $\hat{\rho}_l < \tilde{\rho}_l$ for each link that belongs to the path $\hat{p}$. Indeed, since $\hat{p}$ is a direct path, then $x^k_{\hat{p}} = x^k_l$ $\forall l \in \hat{p}$.
Assume, then, that $\hat{\rho}_l \leq \tilde{\rho}_l$ for $l \in \hat{p}$. We have two possibilities. First, if $\tilde{x}_{\hat{p}} = 0$, then $\hat{\lambda}^k_{(sd)} = \tilde{\lambda}^k_{(sd)}$ for all $k \in I\!K$. Second, if $\tilde{x}_{\hat{p}} > 0$, then we have

$$\tilde{\lambda}_{(sd)} = \sum_{l \in \hat{p}} \frac{\tilde{T}_l(\tilde{\rho}_l)}{\tilde{\mu}_l} \text{ and } \hat{\lambda}_{(sd)} \leq \sum_{l \in \hat{p}} \frac{\tilde{T}_l(\hat{\rho}_l)}{\hat{\mu}_l}.$$

Since $T_l(.)$ is strictly increasing and $\tilde{\mu}_l < \hat{\mu}_l$ for all $l \in \hat{p}$, we have $\hat{\lambda}_{(sd)} \leq \sum_{l \in \hat{p}} \frac{\tilde{T}_l(\hat{\rho}_l)}{\hat{\mu}_l} < \sum_{l \in \hat{p}} \frac{\tilde{T}_l(\tilde{\rho}_l)}{\tilde{\mu}_l} = \tilde{\lambda}_{(sd)}$. It follows that $\hat{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$ for all $k \in I\!K$.
Now assume that $\hat{\rho}_l > \tilde{\rho}_l$ for $l \in \hat{p}$. Let us consider the two networks that differ only by the presence or absence of the direct path $\hat{p}$ from s to d. In both networks we have the same initial capacity configuration and the same set $I\!K$ of classes, with respective demands $\check{r}^k_{(sd)} = r^k_{(sd)} - \hat{x}^k_{\hat{p}}$ and $\bar{r}^k_{(sd)} = r^k_{(sd)} - \tilde{x}^k_{\hat{p}}$. Let $\check{\lambda}^k_{(sd)}$ and $\bar{\lambda}^k_{(sd)}$ the travel costs of class k associated to these throughput demands. Since $\hat{\rho}_l > \tilde{\rho}_l$ and $\hat{\mu}_l > \tilde{\mu}_l$ for $l \in \hat{p}$, we have

$$\begin{aligned}
\bar{v}_{(sd)} - \check{v}_{(sd)} &> \sum_{k \in I\!K} c^k \bar{r}^k_{(sd)} - c^k \check{r}^k_{(sd)} \\
&= \sum_{k \in I\!K} c^k (r^k_{(sd)} - \tilde{x}^k_{\hat{p}}) - c^k (r^k_{(sd)} - \hat{x}^k_{\hat{p}}) \\
&= \sum_{k \in I\!K} c^k (\hat{x}^k_{\hat{p}}) - c^k (\tilde{x}^k_{\hat{p}}) = \sum_{l \in \hat{p}} (\hat{\mu}_l \hat{\rho}_{\hat{l}} - \tilde{\mu}_l \tilde{\rho}_l) > 0.
\end{aligned}$$

From Theorem 1, we conclude that $\check{\lambda}^k_{(sd)} < \bar{\lambda}^k_{(sd)}$ for all $k \in I\!K$. Proceeding as in the proof of Theorem 3, we obtain $\hat{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$ for all $k \in I\!K$. $\diamond$

We examine the last way of adding capacity to the network, i.e. the addition of capacity on all the paths connecting s to d. Consider a network $(I\!M, I\!L)$ and a cost vector $\mathbf{T}$ that satisfies Assumption **A** or **B**. We consider the improvement of the capacity of all path so that the following holds:

$$\hat{T}^k_p(\mathbf{X}) = \frac{1}{\alpha} \tilde{T}^k_p(\frac{\mathbf{X}}{\alpha}), \text{ with } \alpha > 1. \tag{13}$$

We observe that for any $\alpha > 1$, $\hat{T}^k_p(\mathbf{X}) = \frac{1}{\alpha} \tilde{T}^k_p(\frac{\mathbf{X}}{\alpha}) < \tilde{T}^k_p(\frac{\mathbf{X}}{\alpha}) < \tilde{T}^k_p(\mathbf{X})$.

Theorem 5 *Consider a cost vector $\boldsymbol{T}$ that satisfies Assumption **A** or **B**. Let $\tilde{\boldsymbol{X}}$ and $\hat{\boldsymbol{X}}$ be the Wardrop equilibria associated respectively to cost functions $\tilde{T}^k_p$ and $\hat{T}^k_p$. Consider $\tilde{\lambda}^k_{(sd)}$ and $\hat{\lambda}^k_{(sd)}$ the travel cost of class k at the respective Wardrop equilibria $\tilde{\boldsymbol{X}}$ and $\hat{\boldsymbol{X}}$. Then $\hat{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$, $\forall k \in I\!K$.*

Proof. We consider now the network $(I\!M, I\!L)$, with travel costs $\tilde{T}^k_p$ and throughput demands $\bar{r}^k_{(sd)} = r^k_{(sd)}/\alpha$, $k \in I\!K$. Let $\bar{\lambda}^k_{(sd)}$ the travel cost of class k associated to these throughput demands. At equilibrium $\hat{\mathbf{X}}$, by redefining the cost and path flows as $\alpha \hat{\lambda}^k_{(sd)}$ and $(1/\alpha)\hat{x}^k_p$, respectively, it is straightforward to show that changing the demands from $r^k_{(sd)}$ to $\bar{r}^k_{(sd)}$ using the cost functions $\tilde{T}^k_p(\mathbf{X})$ is equivalent to changing the cost functions from $\tilde{T}^k_p(\mathbf{X})$ to $\tilde{T}^k_p(\mathbf{X}/\alpha)$ using the demands $r^k_{(sd)}$. Hence the corresponding travel costs are $\bar{\lambda}^k_{(sd)} = \alpha \hat{\lambda}^k_{(sd)}$. On the other hand, we have $\bar{r}_{(sd)} = r_{(sd)}/\alpha < r_{(sd)}$ and $\bar{v}_{(sd)} = v_{(sd)}/\alpha < v_{(sd)}$ hence from Theorem 1, $\bar{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$, $\hat{\lambda}^k_{(sd)} = \bar{\lambda}^k_{(sd)}/\alpha < \bar{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$, which concludes the proof. $\diamond$

6 An open BCMP queuing network

We study an example of networks consisting entirely of **BCMP** [2] queuing networks (BCMP stand for the initial of the authors).

6.1 BCMP queuing network

We consider an open **BCMP** queuing network model that consists of L service links. Each service center contains either a single-server queue with the processor-sharing (**PS**) discipline. We assume that the service

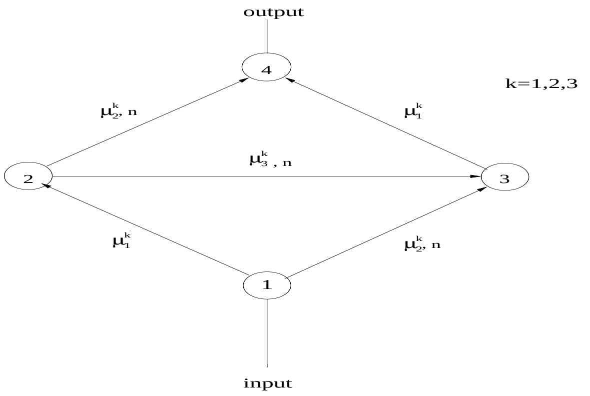

Figure 1: Network

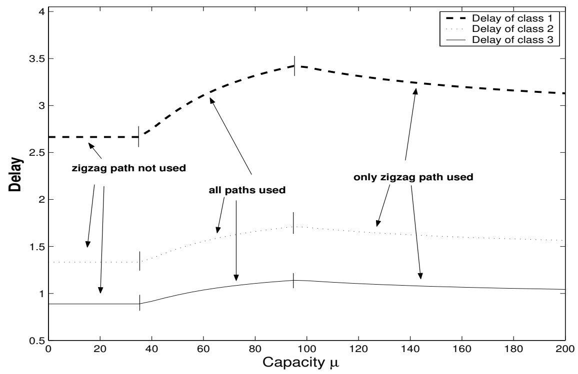

Figure 2: Delay of each class as a function of the added capacity in path (2,3)

rate of each single server is state independent. Jobs are classified into K different classes. The arrival process of jobs of each class forms a Poisson process and is independent of the state of the system. Let us denote the state of the network by $\mathbf{n} = (n_1, n_2, ..., n_L)$ where $\mathbf{n}_l = (\mathbf{n}_l^1, \mathbf{n}_l^2, .., \mathbf{n}_l^K)$ and $n_l = \sum_{l \in L} n_l^k$ where n_l^k denotes the total number of jobs of class k visiting link l. For an open queuing network [2], the equilibrium probability of the network state $\mathbf{n}$ has the following product form:

$$p(\mathbf{n}) = \prod_{l \in L} \frac{p_l(\mathbf{n_l})}{G_l},$$

where G_l is a normalizing constant. Let $E[n_l^k]$ be the average number of class k jobs at link l. We have [2] $E[n_l^k] = \rho_l^k/(1-\rho_l)$. and hence by Little's formula,

$$T_l^k = \frac{E[n_l^k]}{x_l^k} = \frac{1/\mu_l^k}{(1-\rho_l)},$$

from which the average delay of a class k job that flows through path-class $p \in \mathcal{P}^k$ is given by

$$T_p^k = \sum_{l \in L} \delta_{lp} T_l^k = \sum_{l \in L} \delta_{lp} \frac{1/\mu_l^k}{(1-\rho_l)}.$$

We assume that μ_l^k can be represented by μ_l/c^k, hence the average delays satisfy assumption **B**.

6.2 Braess paradox

Consider the networks shown in Figure 1. Packets are classified into *three* different classes. Links (1,2) and (3,4) have each the following service rates: $\mu_1^1 \mu_1$, $\mu_1^2 = 2\mu_1$ and $\mu_1^3 = 3\mu_1$ where $\mu_1 = 2.7$. Link (1,3) represents a path of n tandem links, each with the service rates: $\mu_2^1 = \mu_2$, $\mu_2^2 = 2\mu_2$ and $\mu_2^3 = 3\mu_2$ with $\mu_2 = 27$. Similarly link (2,4) is a path made of n consecutive links, each with service rates: $\mu_2^1 = 27$, $\mu_2^2 = 54$ and $\mu_2^3 = 81$. Link (2,3) is path of n consecutive links each with service rate of each class $\mu_3^1 = \mu$, $\mu_3^2 = 2\mu$ and $\mu_3^2 = 3\mu$ where μ varies from 0 (absence of the link) to infinity. We denote $x_{p_1}^k$ the left flow of class k using links (1,2) and (2,4), $x_{p_2}^k$ the right flow of class k using links (1,3) and (3,4), and x_{p_3} the zigzag flow of class k using links (1,2), (2,3) and (3,4). The total cost for each class is given by $\mathbf{T}^k = x_{p_1}^k T_{p_1}^k + x_{p_2}^k T_{p_2}^k + x_{p_3}^k T_{p_3}^k$,, where $x_{p_1}^k + x_{p_1}^k + x_{p_1}^k = r^k$.

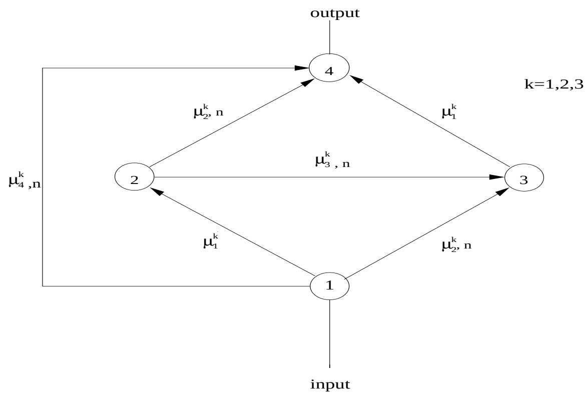

Figure 3: New network

We first consider the scenario where additional capacity μ is added to path (2,3), for $n = 54$, $r^1 = 0.6$, $r^2 = 1.6$ and $r^3 = 1.8$. In Fig. 2 we observe that no traffic uses the zigzag path for $0 \le \mu \le 36.28$. For $36.28 \le \mu \le 96.49$, all three paths are used. For $\mu > 96.49$, all traffic uses the zigzag path. For μ between 36.28 and 96.49, the delay is, paradoxically, worse than it would be without the zigzag path. The delay of class 1 (*resp.* 2,3) decreases to 2.85 (*resp.* 1.42, 0.95) as μ goes to infinity.

6.3 Adding a direct path

Now we use the method proposed in Theorem 2 and Theorem 3, i.e., the upgrade achieved by adding a direct path connecting source 1 and destination 4. Theorems 2 and 3 suggest that yet another good design practice is to focus the upgrades on direct connections between source and destination; and figure 4 illustrates that indeed this approach decreases the delay of each class.

6.4 Multiplying the capacity of all links ($l \in \mathbb{L}$) by a constant factor $\alpha > 1$.

Now we use the method proposed in Theorem 5 for efficiently adding resources to this network. Figure 6 shows the delay of each class as a function of the additional capacity μ where $\mu = (\alpha - 1)(2\mu_1 + 2\mu_2 + \mu 3)$ with $\mu_1 = 2.7$, $\mu_2 = 27$ and $\mu_3 = 40$. Figure 6 indicates that the delay of each class decreases when the additional capacity μ increases. Hence the Braess paradox is indeed avoided.

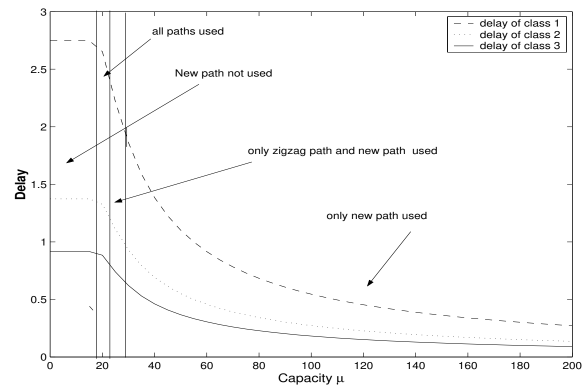

Figure 4: Delay as a function of the added capacity in path (1,4)

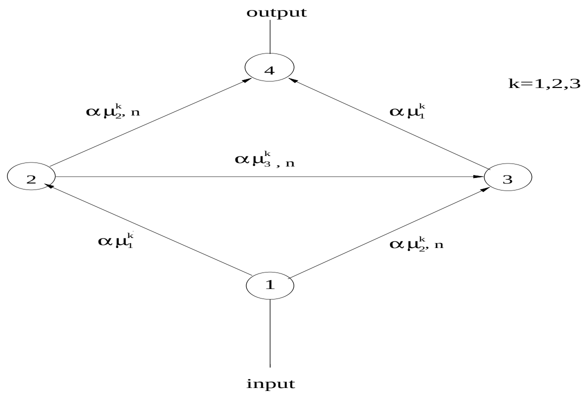

Figure 5: New network

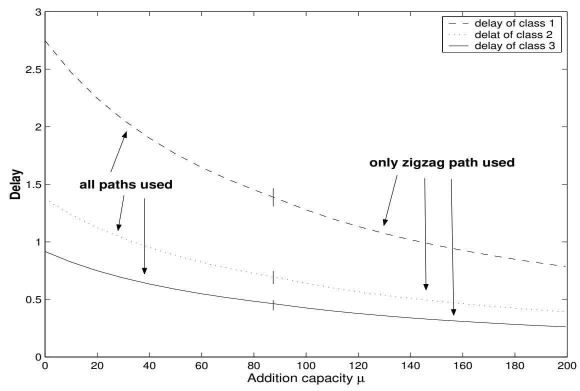

Figure 6: Delay of each class as a function of the added capacity in all links

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