Stable vectorization of Multiparameter Persistent Homology using Signed Barcodes as Measures

joint work with L.Scoccola, M. Carrière, S. Oudot, and M.Botnan

Young Topologists Meeting

EPFL

Inria

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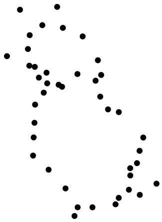
FRANCE 2030

Topological Data Analysis introduction

Persistent homology, for Čech's filtration



$\mathcal{K}_1$

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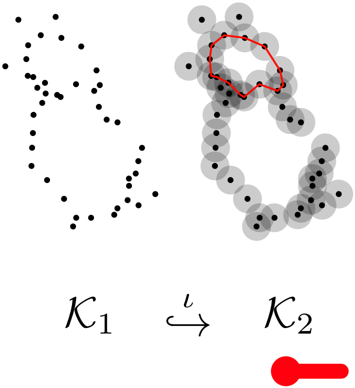

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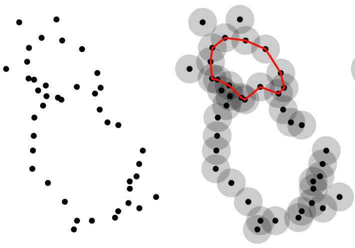

$$\mathcal{K}_1 \overset{\iota}{\hookrightarrow} \mathcal{K}_2 \overset{\iota}{\hookrightarrow} \mathcal{K}_3$$

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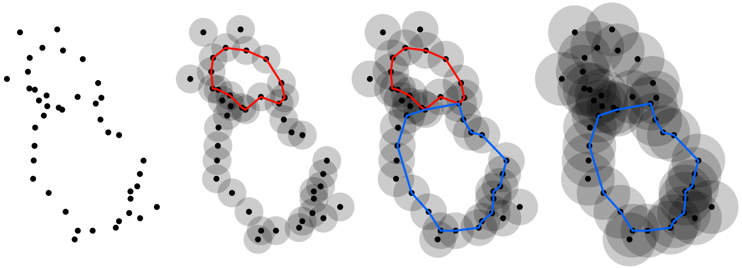

$$\mathcal{K}_1 \overset{\iota}{\hookrightarrow} \mathcal{K}_2 \overset{\iota}{\hookrightarrow} \mathcal{K}_3 \overset{\iota}{\hookrightarrow} \mathcal{K}_4$$

Topological Data Analysis introduction

Formally

Topological Filtration

$$X_t = \{x \in \mathbb{R}^n : d(x,P) \leq t\} = \bigcup_{p \in P} B(x,t)$$

$$X = \left\{ (P = X_0) \overset{\iota}{\hookrightarrow} \cdots \overset{\iota}{\hookrightarrow} X_{r_1} \overset{\iota}{\hookrightarrow} \cdots \overset{\iota}{\hookrightarrow} X_{r_2} \overset{\iota}{\hookrightarrow} \cdots \right\}$$

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Persistent Homology or Persistent Module

$$PH_\bullet(X) = \left\{ H_\bullet(X_{r_0}) \overset{\iota_\star}{\to} \cdots \overset{\iota_\star}{\to} H_\bullet(X_{r_1}) \overset{\iota_\star}{\to} \cdots \overset{\iota_\star}{\to} H_\bullet(X_{r_2}) \overset{\iota_\star}{\to} \cdots \right\}$$

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In general, a Persistent Module is a family of *vector spaces* $\mathbb{V} = (V_t)_{t \in \mathbb{R}}$, with *linear maps* $v_{s \to t} : V_s \to V_t$,

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$$v_{s \to r} \circ v_{t \to s} = v_{t \to r} \quad \text{and} \quad v_{t \to t} \equiv \mathrm{id}_{V_t}.$$

Pillars of Topological Data Analysis

Theorem (Krull,Remak,Schmidt,Azumaya,Gabriel)

Let M be a pointwise finite dimensional persistence module. Then, M is interval-decomposable, i.e.,

$$M \simeq \bigoplus_i k_{[b_i,d_i)}.$$

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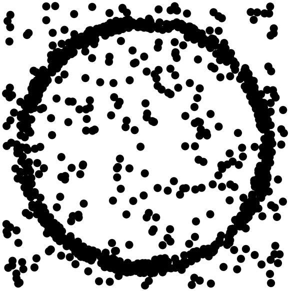
Theorem (Cohen-Steiner, Edelsbrunner, Harer)

If X is a triangulable space, with continuous tame functions $f, g\colon X \to \mathbb{R}$,

$$d_I(H_\star(X_f), H_\star(X_g)) \leq \|f - g\|_\infty$$

where $X_h := (\{x \in X \mid h(x) \leq t\})_{t\in\mathbb{R}}$.

Easy multi-persistence motivation : noise



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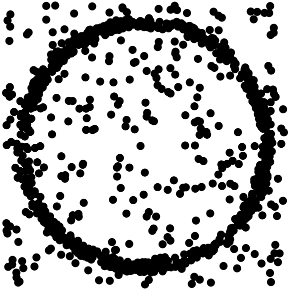

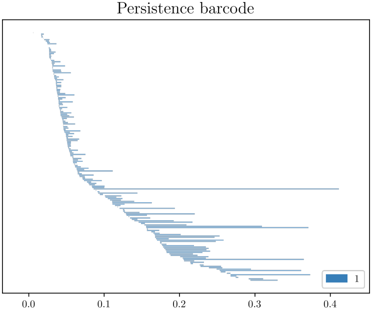

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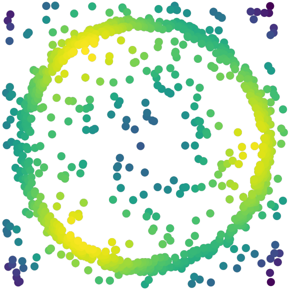
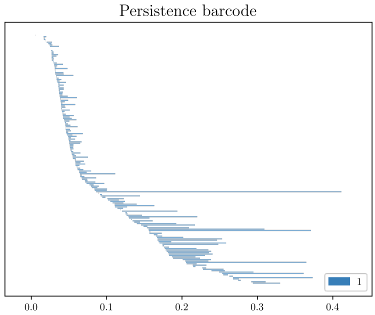
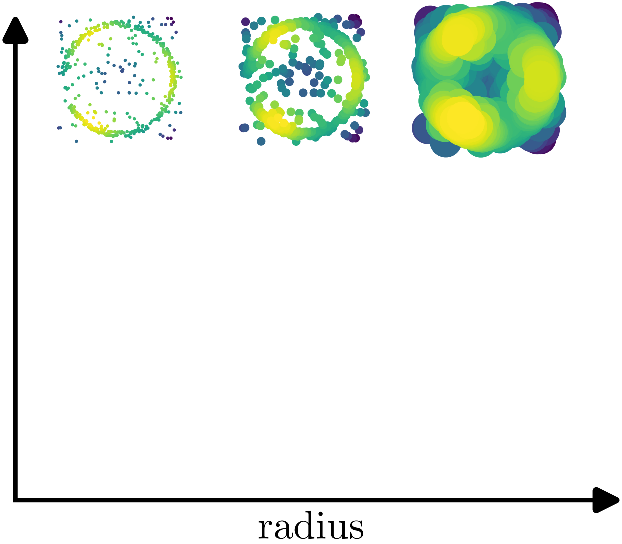




2-persistence with noise 101



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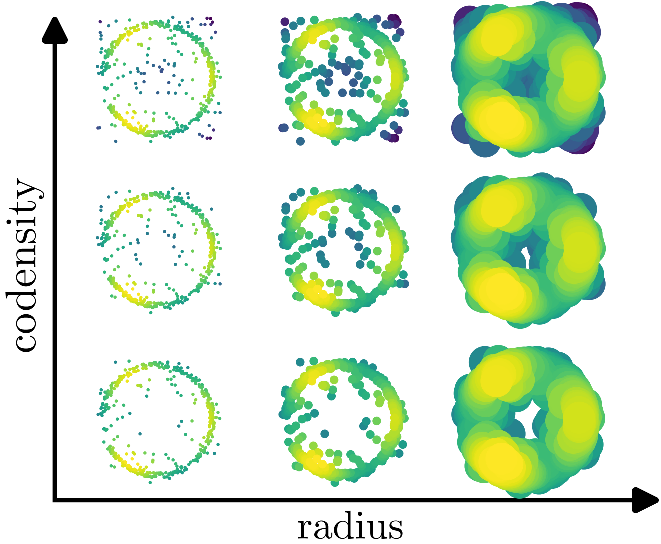

2-persistence with noise 101

$$
\begin{array}{ccccccccc}
\mathcal{K}_1^k & \xhookrightarrow{\iota} & \mathcal{K}_2^k & \xhookrightarrow{\iota} & \mathcal{K}_3^k & \xhookrightarrow{\iota} & \cdots & \xhookrightarrow{\iota} & \mathcal{K}_n^k \\
\iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow \\
\cdots & \xhookrightarrow{\iota} & \cdots & \xhookrightarrow{\iota} & \cdots & \xhookrightarrow{\iota} & \cdots & \xhookrightarrow{\iota} & \cdots \\
\iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow \\
\mathcal{K}_1^2 & \xhookrightarrow{\iota} & \mathcal{K}_2^2 & \xhookrightarrow{\iota} & \mathcal{K}_3^2 & \xhookrightarrow{\iota} & \cdots & \xhookrightarrow{\iota} & \mathcal{K}_n^2 \\
\iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow \\
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\end{array}
$$

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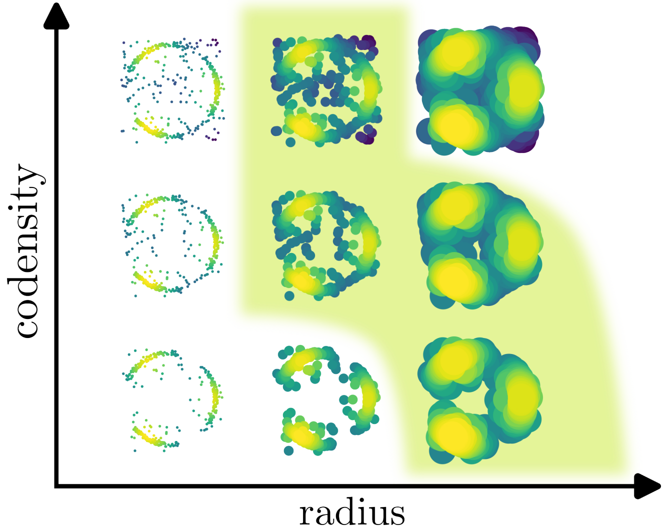

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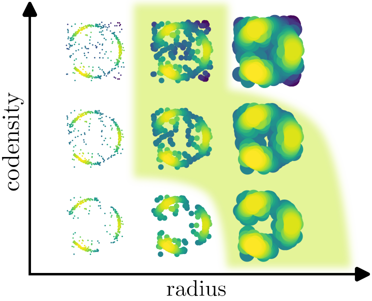

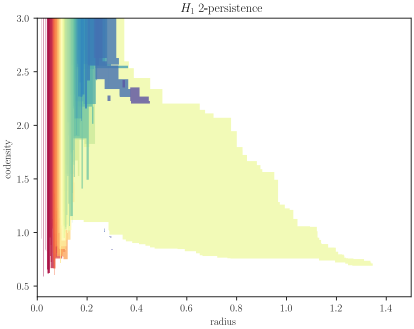

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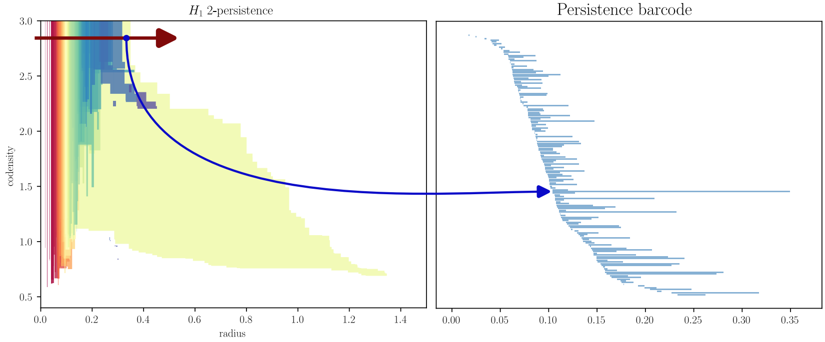
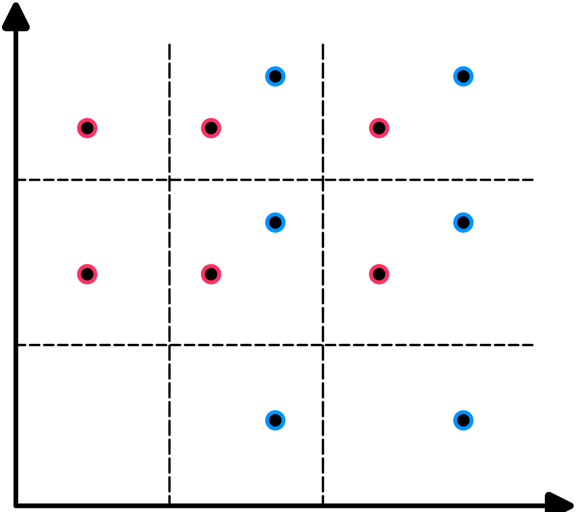


Multiparameter curse

Connected components



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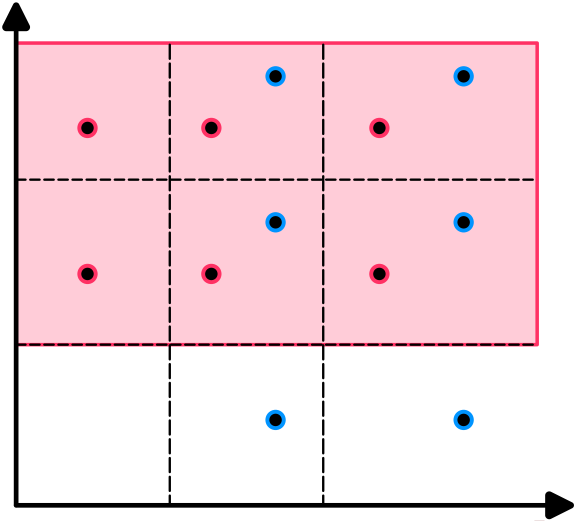

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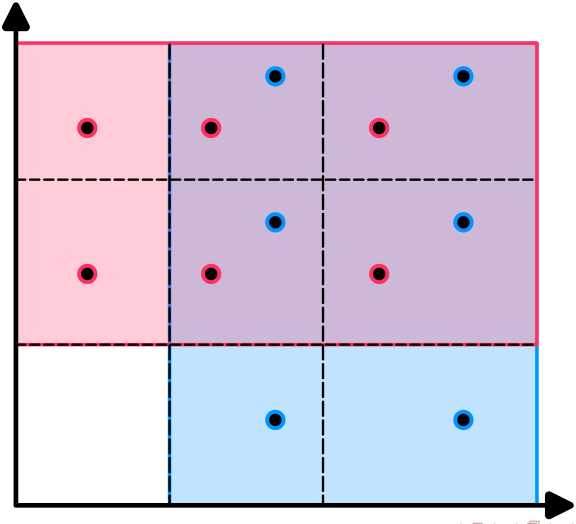

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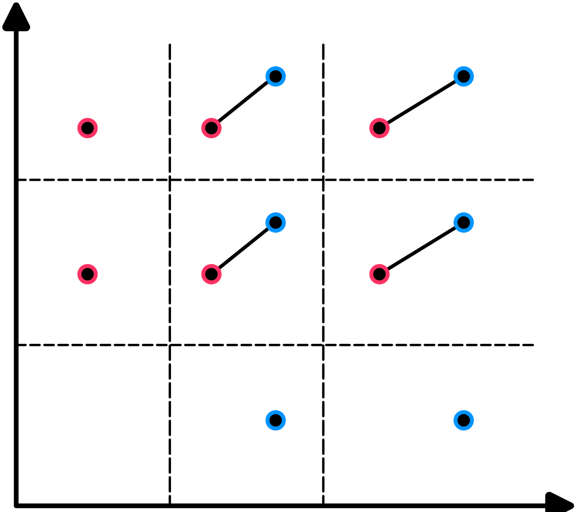

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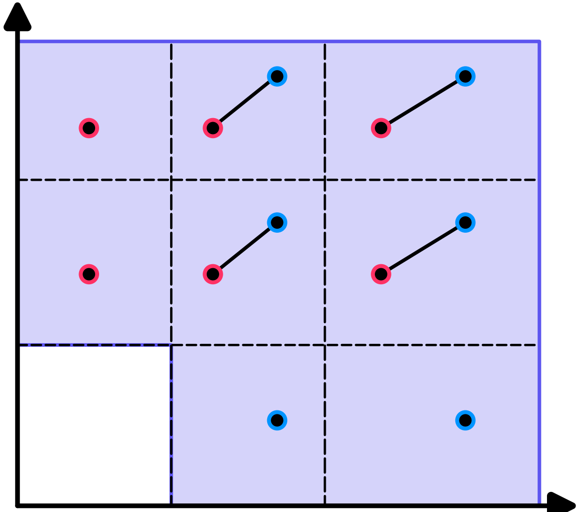

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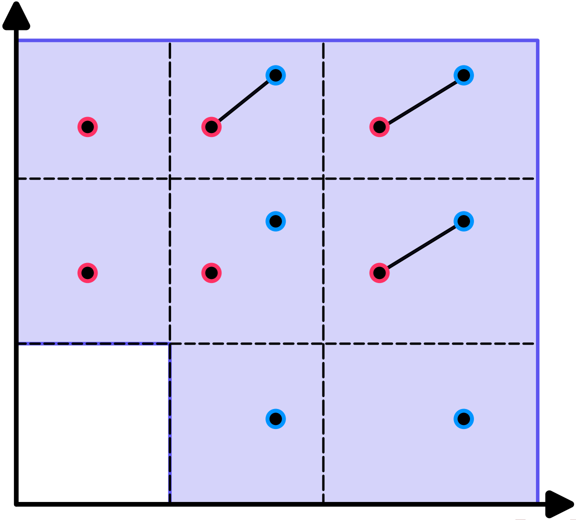

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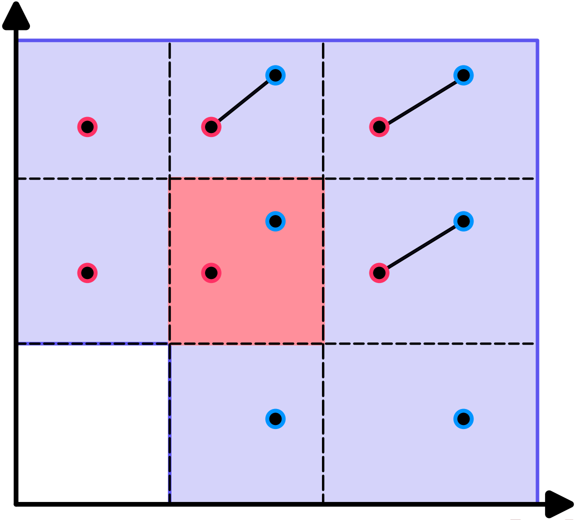
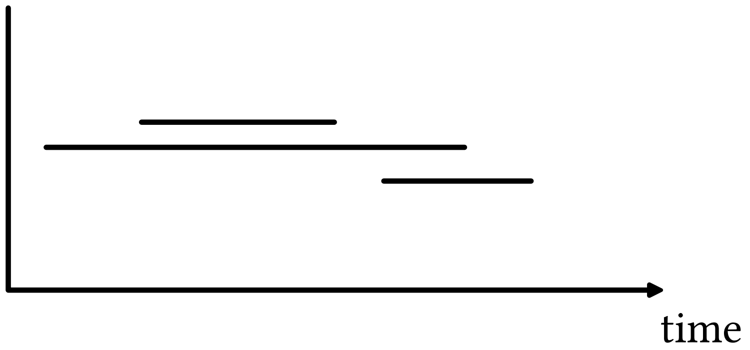


Let's forget about this matching : 1-parameter pictures



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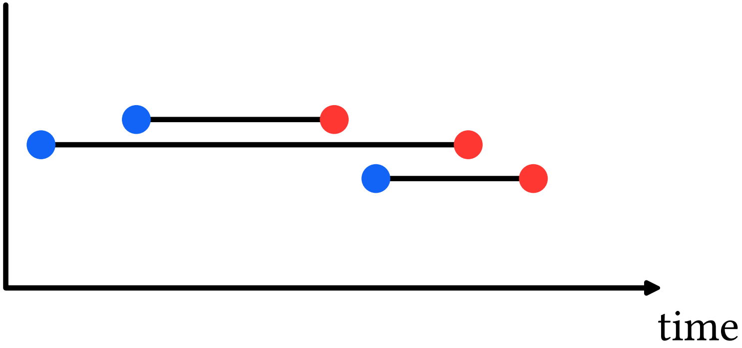

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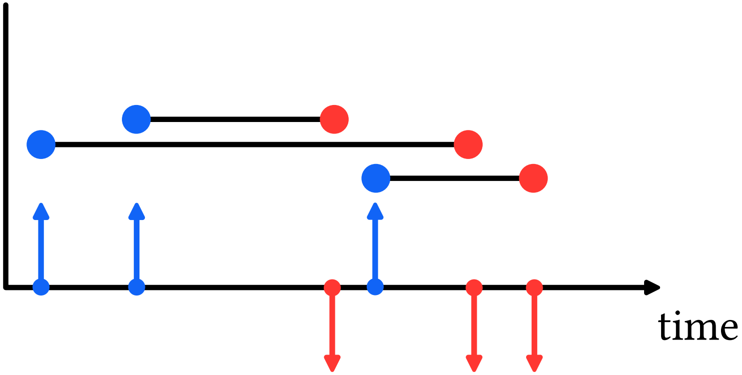

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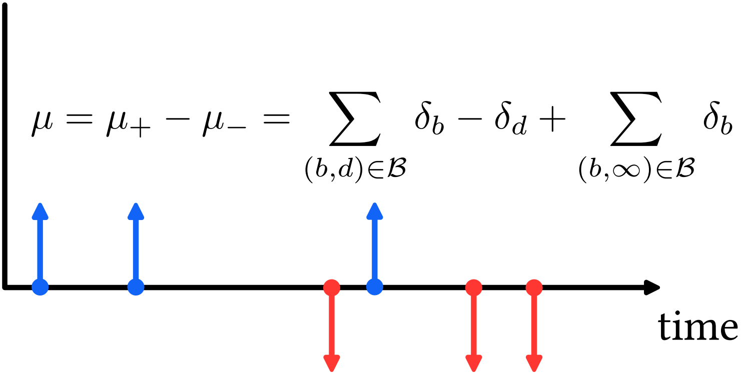
$$\mu = \mu_+ - \mu_- = \sum_{(b,d)\in\mathcal{B}} \delta_b - \delta_d + \sum_{(b,\infty)\in\mathcal{B}} \delta_b$$



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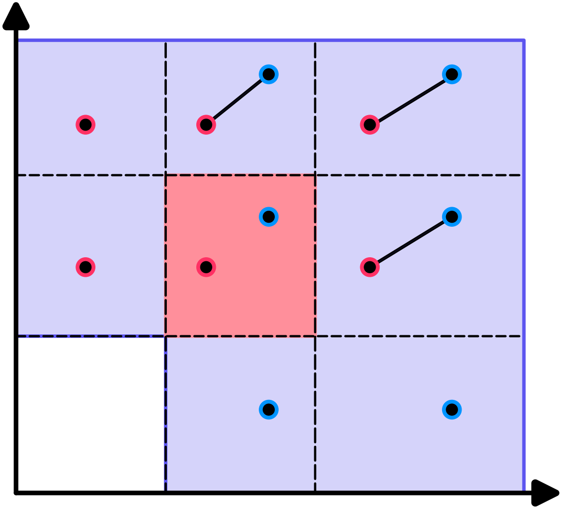

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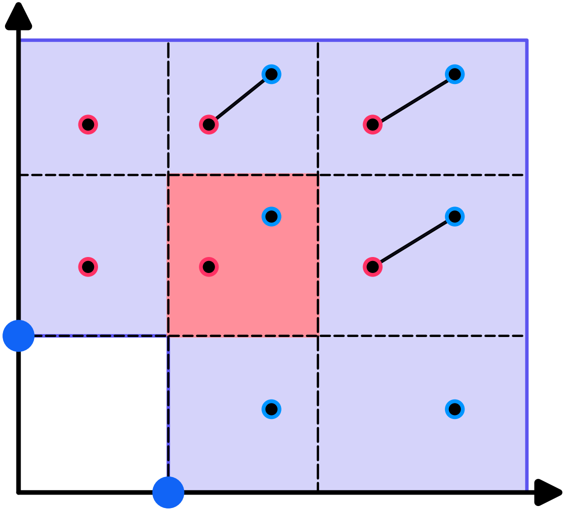

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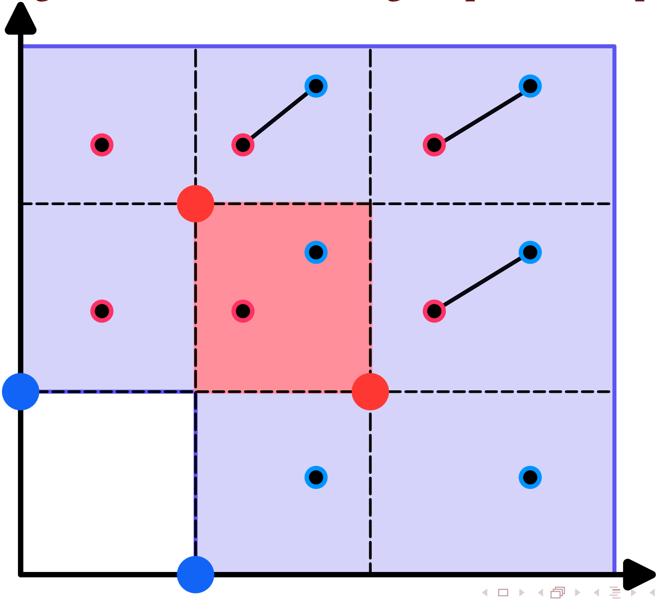

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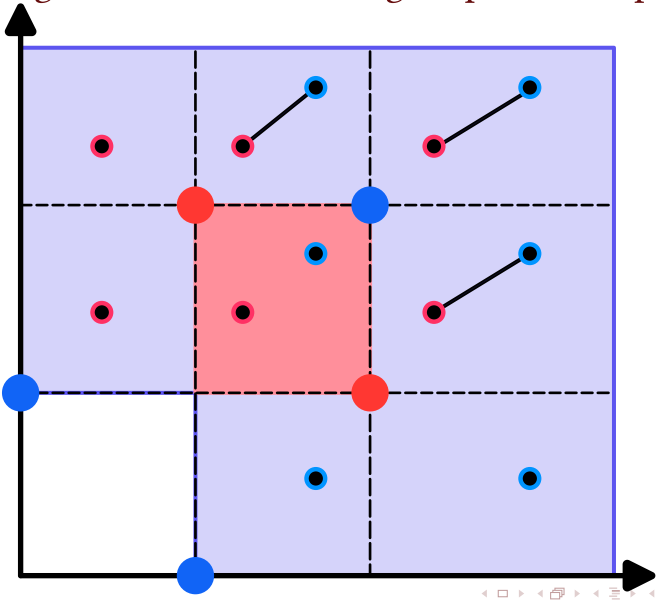

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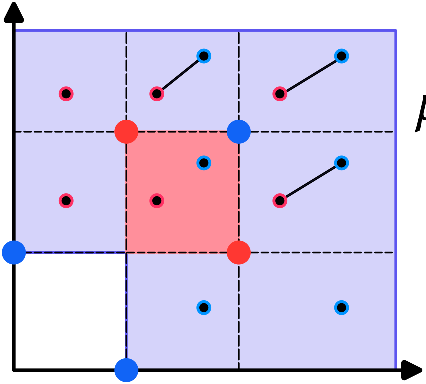

$$\mu = \delta_{(1,0)} + \delta_{(0,1)} - \delta_{(2,1)} - \delta_{(1,2)} + \delta_{(2,2)}$$

Re-extension to Multiparameter Persistence

Definition (Hilbert decomposition signed measure)

For all *finitely presentable* multiparameter persistence module M, there *exists* a *unique discrete radon* measure $\boldsymbol{\mu}_M$ such that

$$\forall x \in \mathbb{R}^n, \quad \dim(M_x) = \mu_M\left(\{y \in \mathbb{R}^n : y \leq x\}\right)$$

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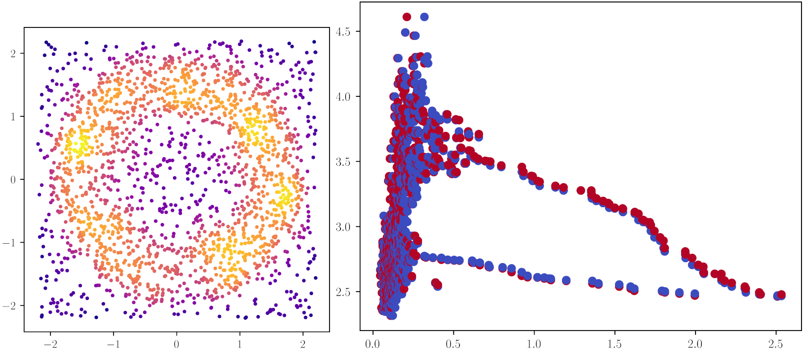
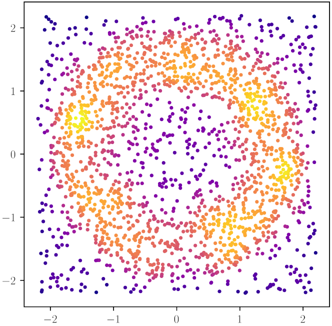


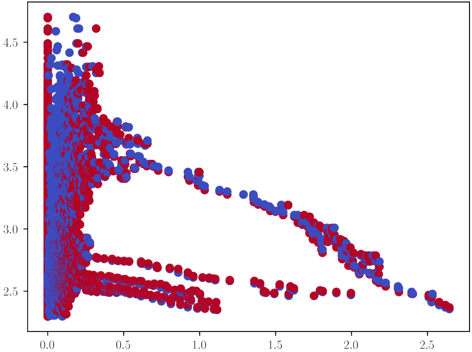
Re-extension to Multiparameter Persistence

Definition (Euler decomposition signed measure)

For a *finite* multi-filtered simplicial complex (S, f),

$$\mu_{\chi(f)} := \sum_{i \in \mathbb{N}} (-1)^i \mu_{H_i(f)}.$$



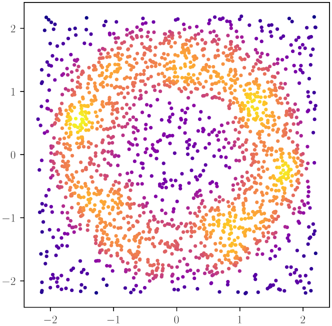


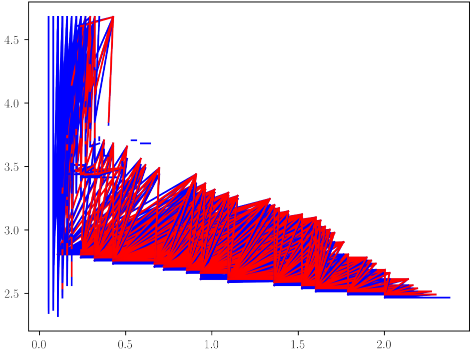
Re-extension to Multiparameter Persistence

Definition (Rank decomposition signed measure)

For all *finitely presentable* multiparameter persistence module M, there *exists* a *unique discrete radon* measure $\boldsymbol{\mu}_M$ such that

$$\forall x, y \in \mathbb{R}^n, \quad \dim(M(x \leq y)) = \mu_M\left(\{z \in \mathbb{R}^n : z \leq x\} \times \{z \in \mathbb{R}^n : z \leq y\}\right)$$





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$$\|\boldsymbol{\mu}\| := d_{W^1}(\mu_+, \mu_-) = \min_{\sigma \in \mathfrak{S}_{|\mu_+|}} \left\{ \sum_{1 \leq i \leq |\mu_+|} \left\| x_i - y_{\sigma(i)} \right\|_1 \right\}$$

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If $\boldsymbol{\nu}$ is another such measure, then

$$\boldsymbol{\mu} - \boldsymbol{\nu} = (\mu_+ + \nu_-) - (\mu_- + \nu_+).$$

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In particular, we can consider *Wasserstein* norms on signed measures

$$\|\boldsymbol{\mu} - \boldsymbol{\nu}\|_1^K := d_{W^1}(\mu_+ + \nu_-, \mu_- + \nu_+)$$

Stability

Theorem

Let $\mathbf{n} \in \mathbb{N}$, S be a finite simplicial complex and (S, f), (S, g) two $\mathbf{n}$-filtrations.

1. *For $\mathbf{n} \in \{1, 2\}$ and $i \in \mathbb{N}$,*

$$\left\| \mu_{H_i(f)} - \mu_{H_i(g)} \right\|_1^K \leq \mathbf{n} \cdot \|f - g\|_1$$

2. *For all $\mathbf{n} \in \mathbb{N}$,*

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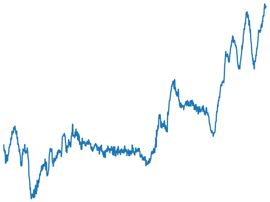
$$\|f - g\|_1 = \sum_{\sigma \in S} \|f(\sigma) - g(\sigma)\|_1 .$$

Convolutions



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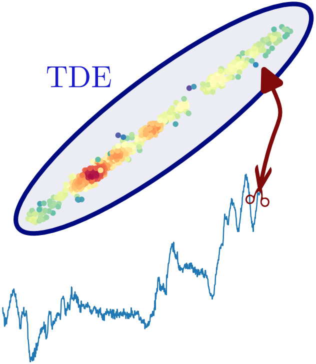

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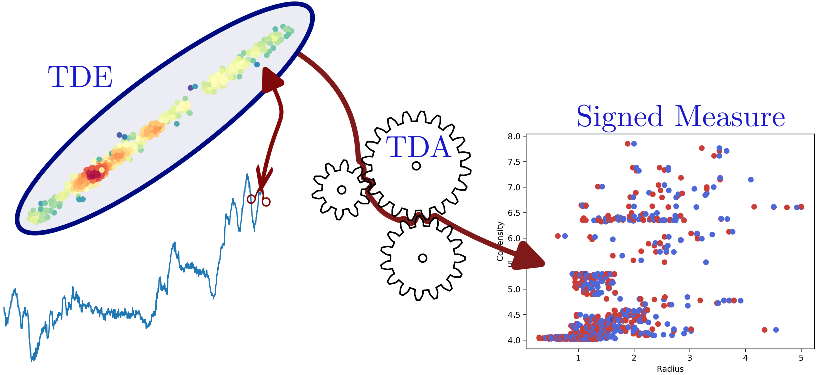

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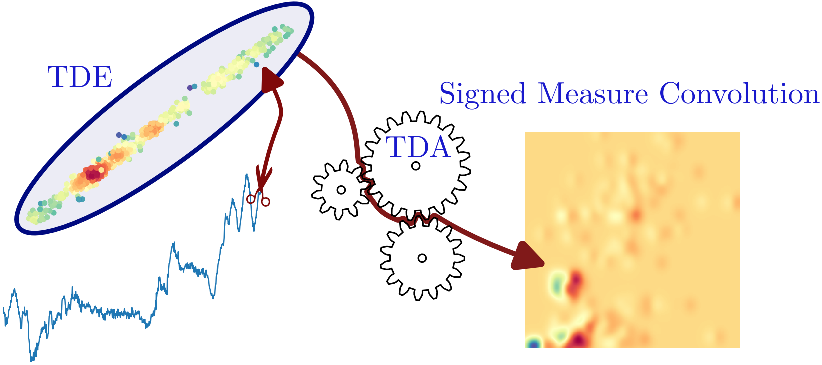

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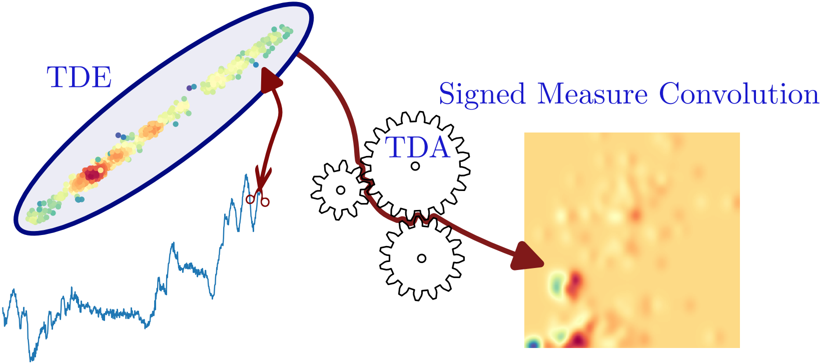
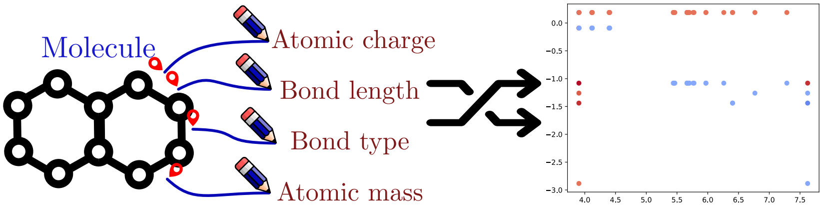


Proposition

Let $K:\mathbb{R}^n\to\mathbb{R}$ be a kernel satisfying $\|K_x-K_y\|_2\lesssim\|x-y\|_2$. Then if $\mu,\nu\in\mathcal{M}$ have the same total mass,

$$\|K*\mu-K*\nu\|_2\lesssim\|\mu-\nu\|_2^K$$

Not limited to 2 parameters :)



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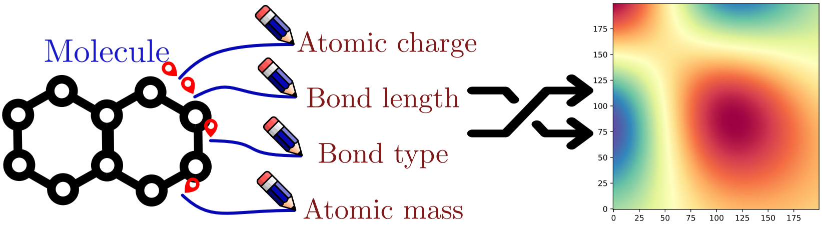

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Definition (Sliced Wasserstein on signed measures)

For any measure α on S^{n-1}, and $\mu, \nu \in \mathcal{M}_0$, define

$$\mathrm{SW}^{\alpha}(\mu,\nu) := \int \left\| \pi^{\theta}_{*}\mu - \pi^{\theta}_{*}\nu \right\|_1^K \mathrm{d}\alpha(\theta), \text{ and, } k^{\alpha}_{\mathrm{SW}} = \exp(-\mathrm{SW}^{\alpha}(\mu,\nu)).$$

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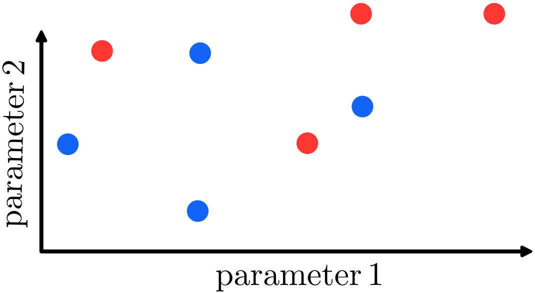

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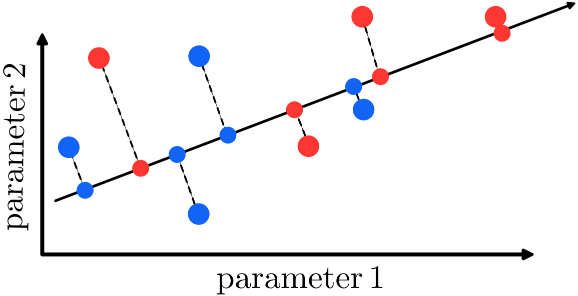

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Proposition

For any $n \in \mathbb{N}$, there exists a Hilbert space $\mathcal{H}$ and a map $\Phi^{\alpha}_{\mathrm{SW}} : \mathcal{M}_0(\mathbb{R}^n) \to \mathcal{H}$ such that, for any $\mu, \nu \in \mathcal{M}_0$,

$$\|\Phi^{\alpha}_{\mathrm{SW}}(\mu) - \Phi^{\alpha}_{\mathrm{SW}}(\nu)\|_{\mathcal{H}} \leq 2\alpha(S^{n-1}) \|\mu - \nu\|_2^K$$

and

$$k^{\alpha}_{\mathrm{SW}}(\mu,\nu) = \langle \Phi^{\alpha}_{\mathrm{SW}}(\mu), \Phi^{\alpha}_{\mathrm{SW}}(\nu)\rangle_{\mathcal{H}}$$

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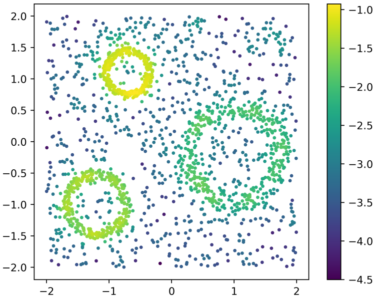
Code

- *Python* library, available as a गुढी GUDHI Geometry Understanding in Higher Dimensions extension at : `https://github.com/DavidLapous/multipers-signed-measure`

Another motivation for multipers



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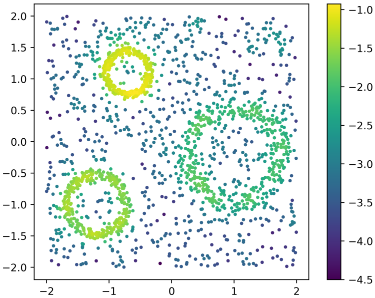

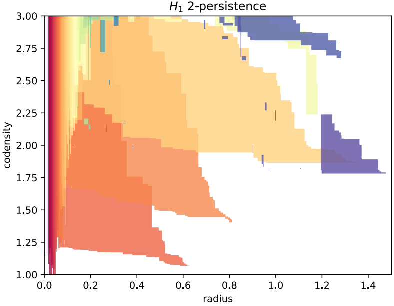

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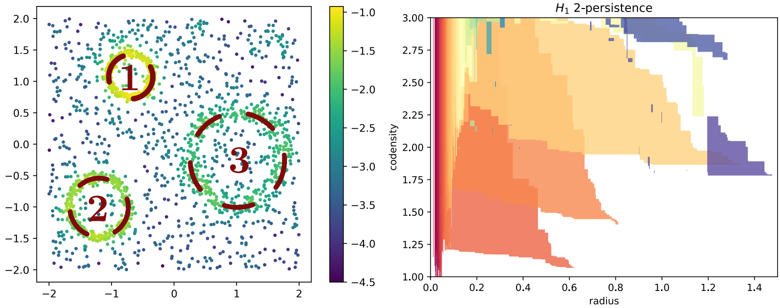

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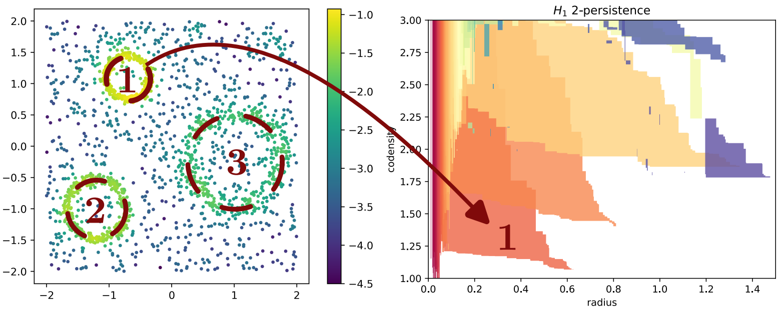

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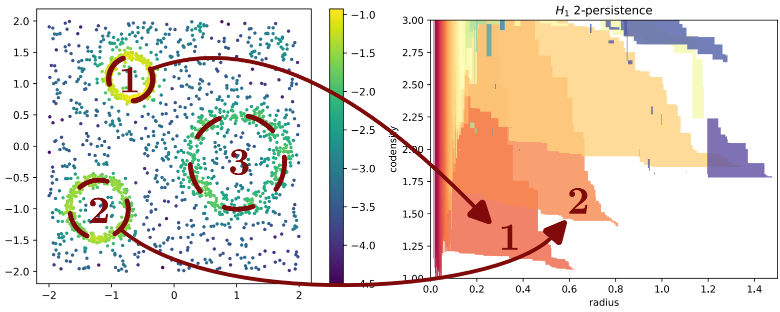

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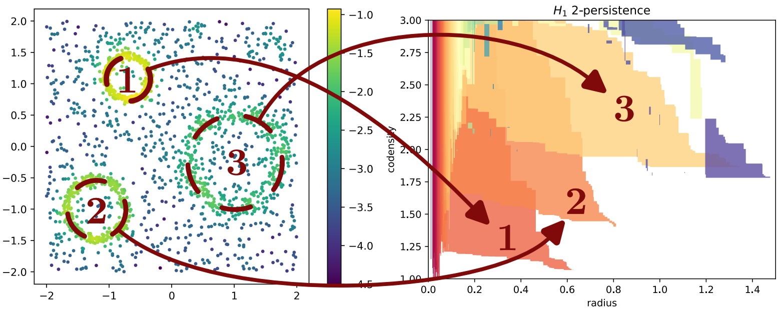

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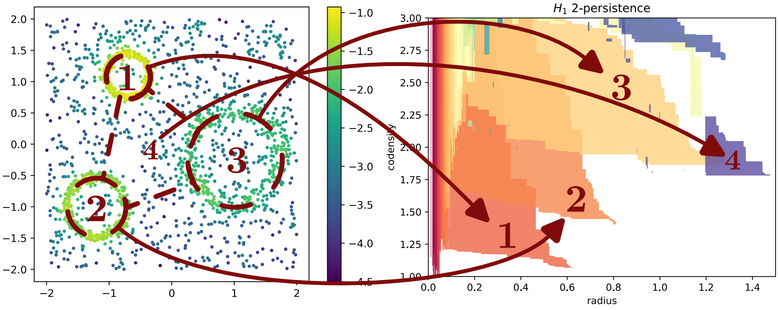

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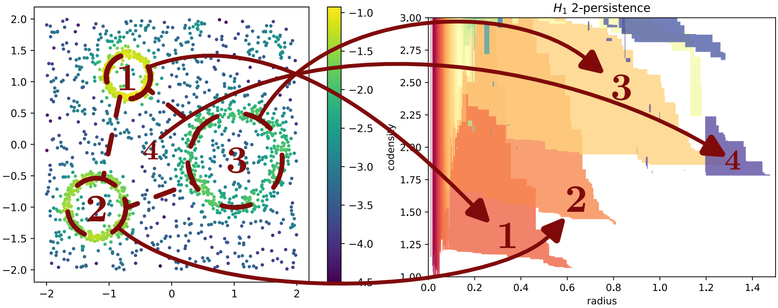
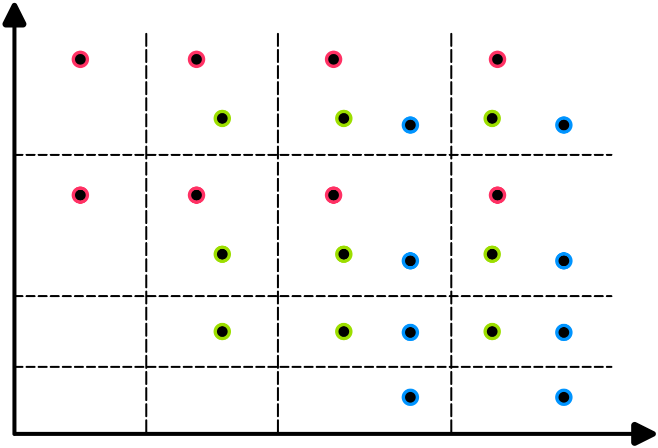


At every scale, *and at **every** concentrations !*

Another indecomposable



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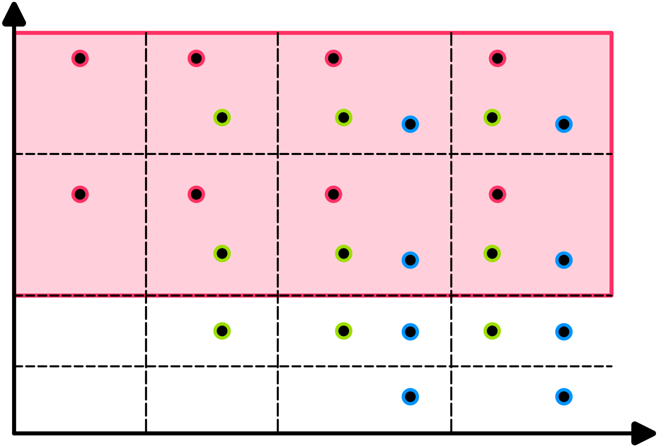

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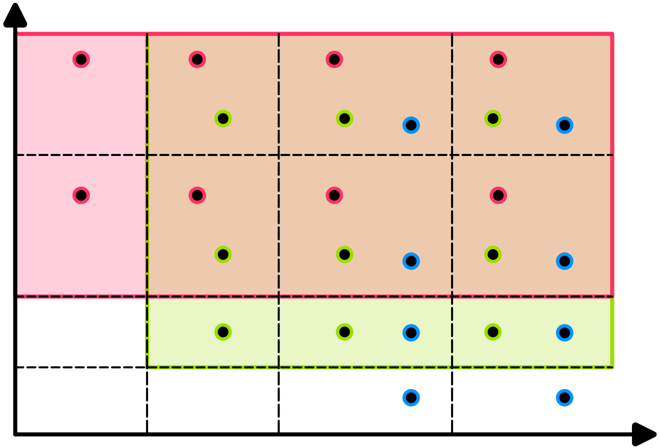

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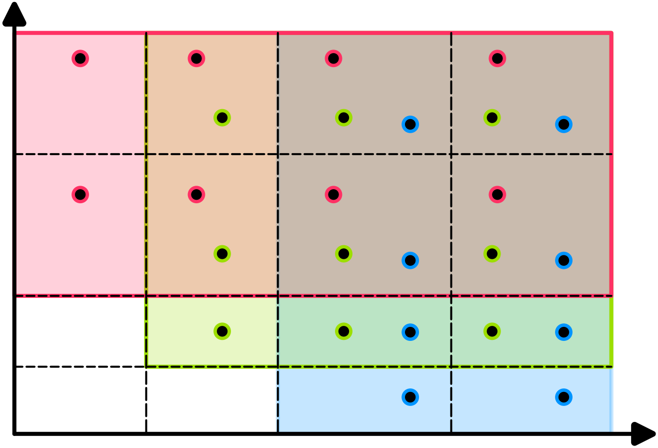

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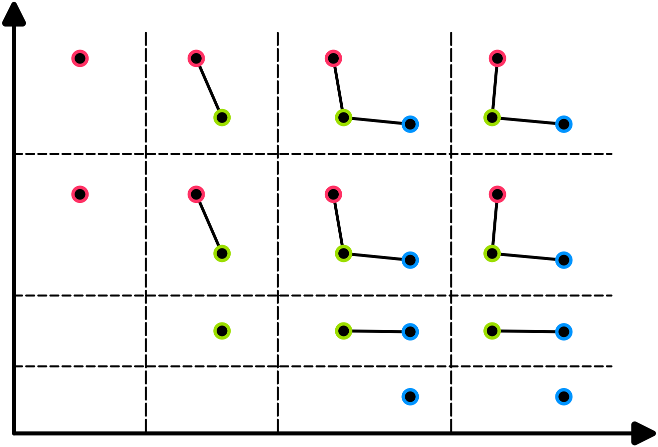

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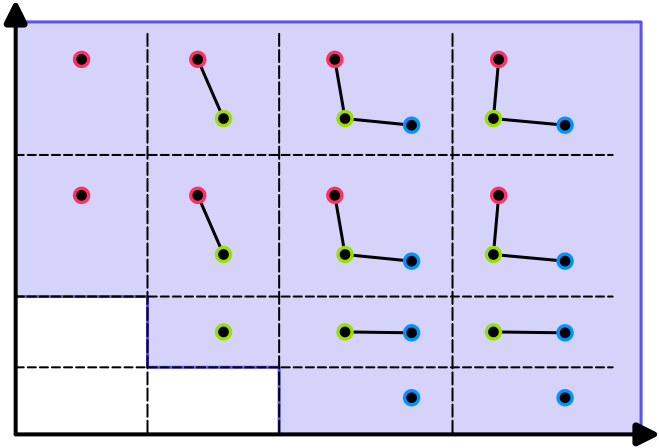

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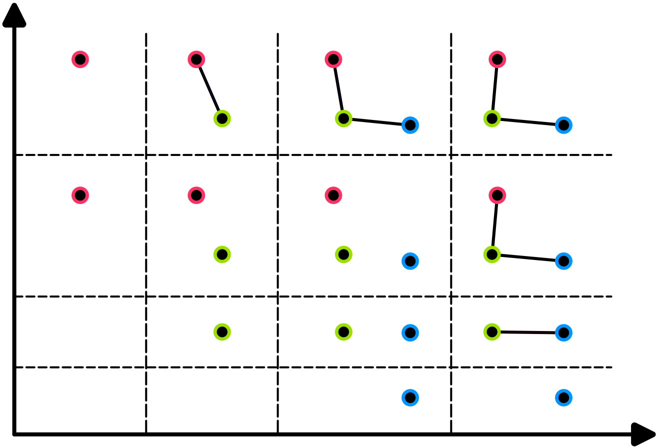

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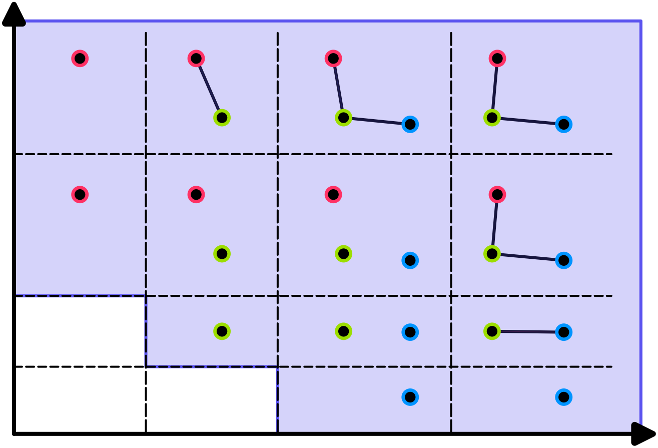

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