Model Checking Disjoint-Paths Logic
on Topological-Minor-Free Graph Classes

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General conditions that imply the automatic derivation of efficient algorithms.

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Does G contain H as a subgraph?

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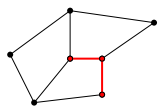

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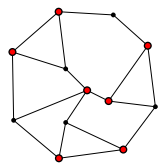

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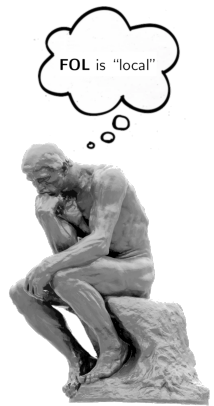

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- *Connectivity*.

$$\forall x \; \forall y \; \neg \left(\exists V \;\; \left(x \in V \;\wedge\; y \notin V \;\wedge \left(\forall u \; \forall v \; \mathbf{adj}(u, v) \implies (u \in V \Leftrightarrow v \in V) \right) \right) \right)$$

Model-checking

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We can test whether an **FO**-formula φ is satisfied on a graph G ($G \models \varphi$), in time $|G|^{\mathcal{O}(|\varphi|)}$.

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3-Colorability is NP-complete.

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- Our notion of efficiency:

Fixed-Parameter Tractability, i.e., algorithms running in time $f(|\varphi|) \cdot |G|^{\mathcal{O}(1)}$.

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Every **FO** property can be tested in (almost) linear time on nowhere dense graph classes.
[Grohe, Kreutzer, & Siebertz, 2017]

(C)MSO$_1$: bounded cliquewidth. [Courcelle, Makowski, & Rotics, 2000] + [Oum & Seymour, 2006]

FO:

Sparse classes:

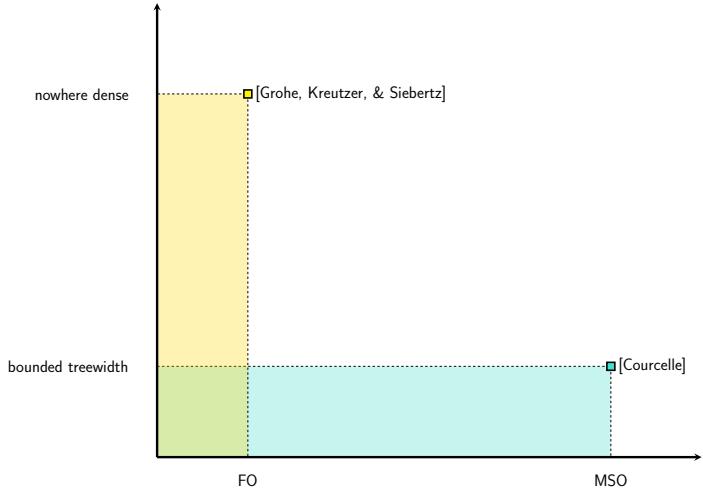
locally bounded treewidth [Frick & Grohe, 2001]
excluding a minor [Flum & Grohe, 2001]
bounded degree [Seese, 1996] locally excluding a minor [Dawar, Grohe, & Kreutzer, 2007]
bounded expansion [Dvořák, Kráľ, & Thomas, 2011]
nowhere dense [Grohe, Kreutzer, & Siebertz, 2017]

bounded twinwidth* [Bonnet, Kim, Thomassé, & Watrigant, 2022]
* given an identification sequence with the input.

Dense classes:

structurally bounded degree [Gajarský, Hliněný, Lokshtanov, Obdržálek, & Ramanujan, 2016]
structurally bounded expansion [Gajarský, Kreutzer, Nešetřil, Ossona de Mendez, Pilipczuk, Siebertz, & Toruńczyk, 2018]
structurally nowhere dense [Dreier, Mählmann, Siebertz, 2023]

structurally bounded local cliquewidth [Bonnet, Dreier, Gajarský, Kreutzer, Mählmann, Simon, & Toruńczyk, 2022]



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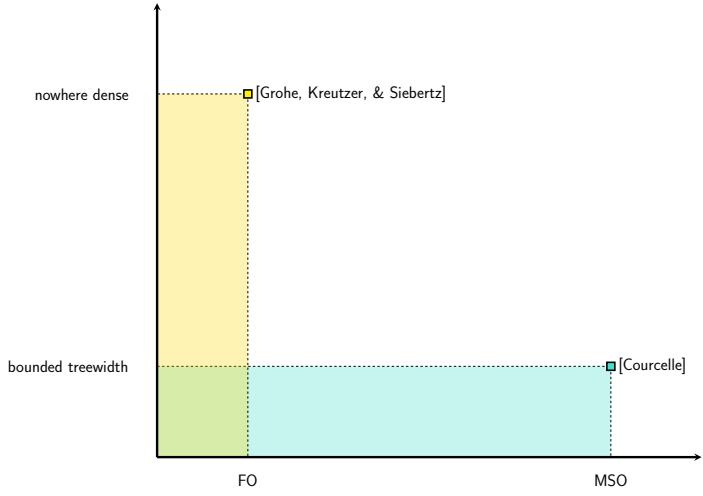
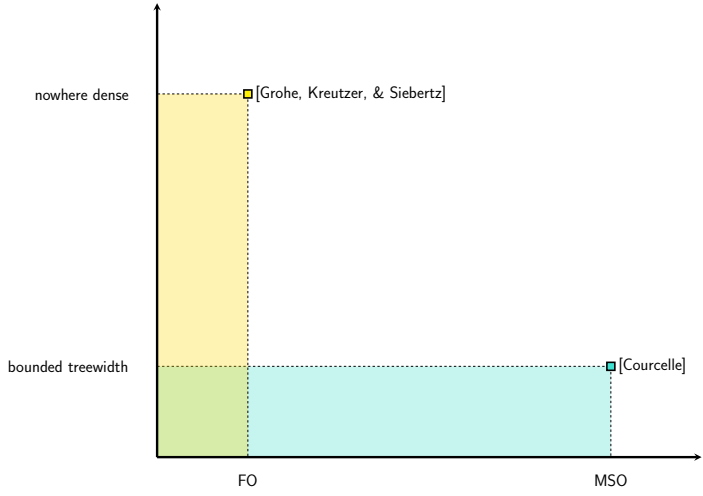


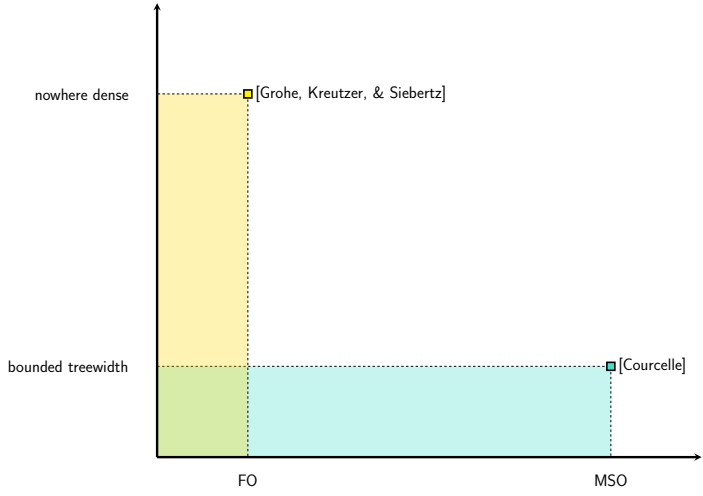
Assuming ETH:

- No FPT algorithm for **MSO** on unbounded treewidth classes.



Assuming ETH:

- No FPT algorithm for **MSO** on unbounded treewidth classes.
- No FPT algorithm for **FO** on subgraph-closed somewhere dense classes.



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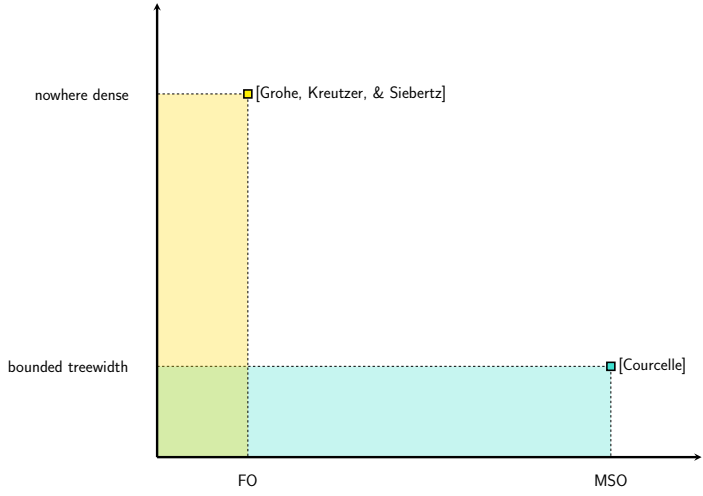

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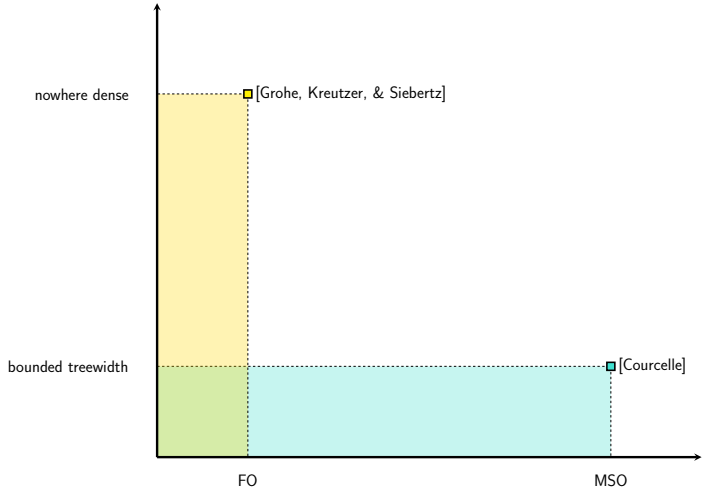
- *Meta-algorithmics* of **FO**: *counting predicates*, *transitive-closure operators*, *fixed-point operators*, *successor-invariant formulas*, *FO-interpretability*,...



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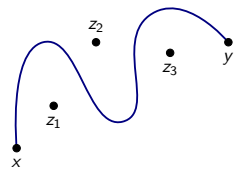
- **Consider logics between FO and MSO**.

Separator Logic & (Scattered) Disjoint-Paths Logic

Separator logic (FO+conn):
[Schirrmacher, Siebertz, & Vigny, 2021]
[Bojańczyk, 2021]



Additional predicate $\mathrm{conn}_k(x, y, z_1, \ldots, z_k)$:

There is an (x, y)-path that avoids $z_1, \ldots, z_k$.

Can express:
k-connectivity. "$\forall x \forall y \forall z_1 \ldots \forall z_{k-1} \mathrm{conn}_{k-1}(x, y, z_1, \ldots, z_{k-1})$"
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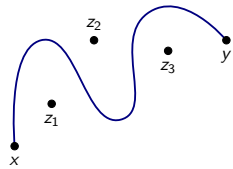

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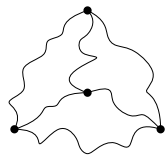

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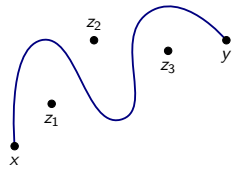

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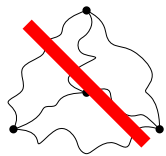

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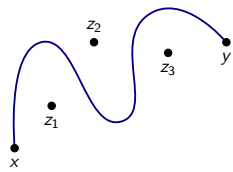

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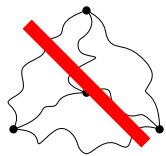

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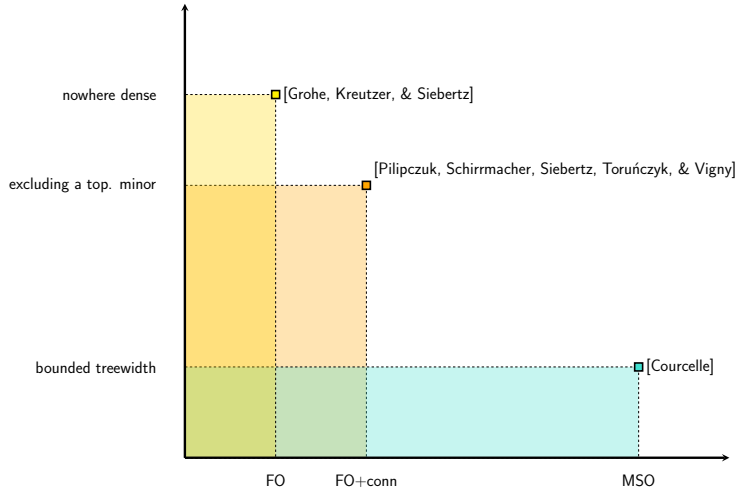

Every FO+conn property can be tested in cubic time on graphs excluding a topological minor*.
[Pilipczuk, Schirrmacher, Siebertz, Toruńczyk, & Vigny, 2022]

- Model checking FO+conn on graphs **not** excluding a top. minor: as hard as FO on general graphs.

FO+DP

[Schirrmacher, Siebertz, & Vigny, 2021]

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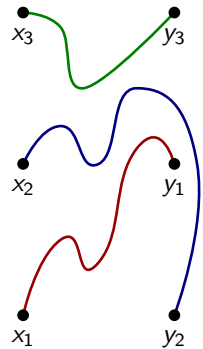

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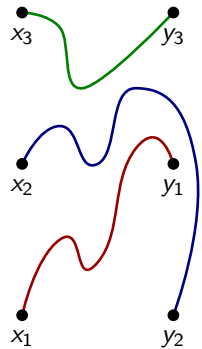

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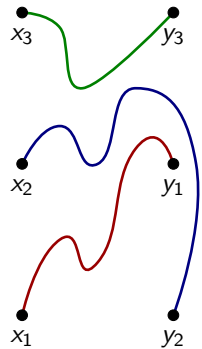

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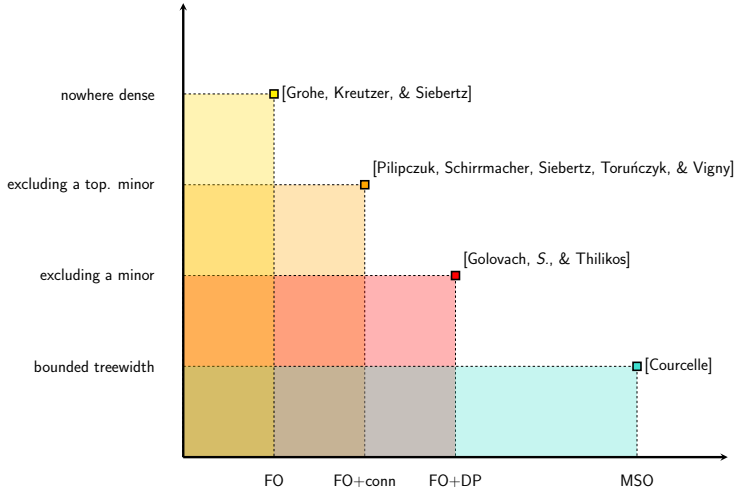

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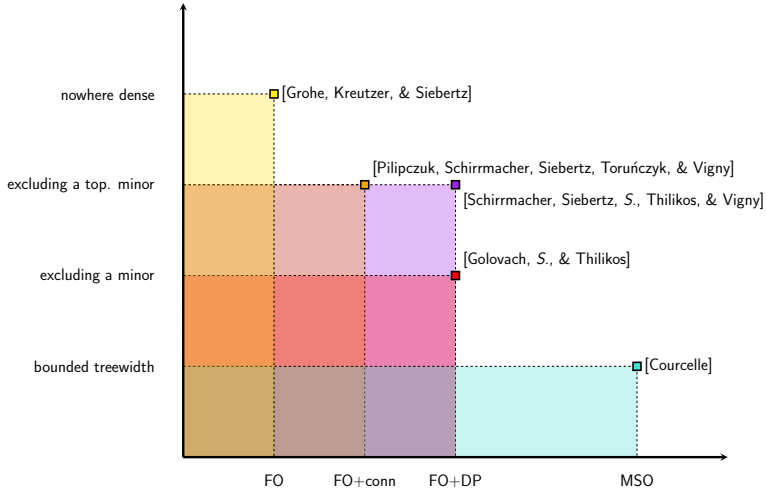
[Golovach, *S.*, & Thilikos, 2023]



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[Schirrmacher, Siebertz, *S.*, Thilikos, & Vigny, 2023+]

FO+SDP

Scattered disjoint paths predicates:

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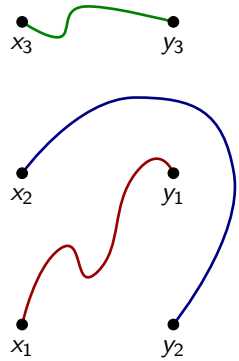
s.t. no two vertices of two distinct paths are within distance $\leq s$.



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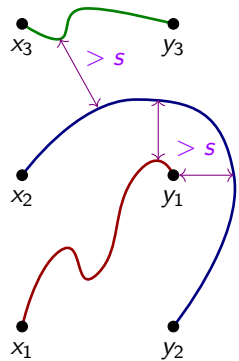
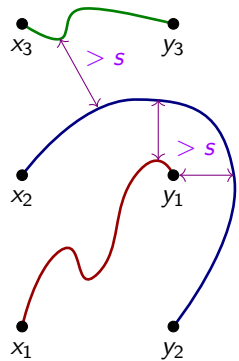


FO+SDP

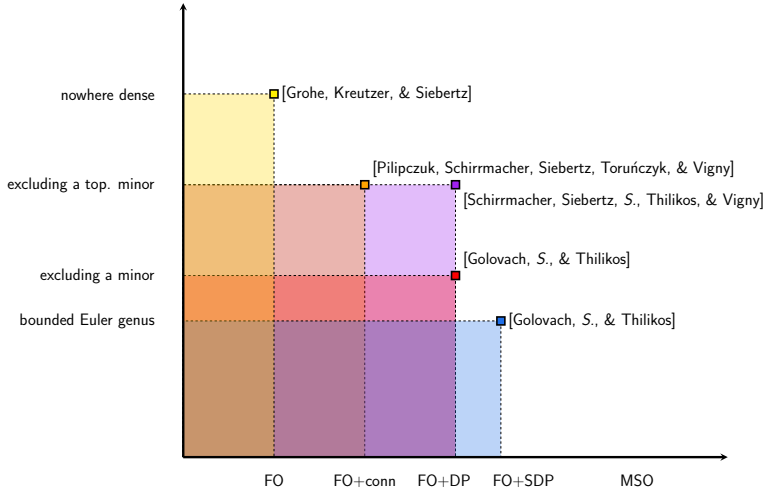
Scattered disjoint paths predicates:

s-$\mathrm{dp}_k(x_1, y_1, \ldots, x_k, y_k)$

There are pairwise vertex-disjoint paths
between x_i and y_i, for every $i \in \{1, \ldots, k\}$
s.t. no two vertices of two distinct paths are within distance $\leq s$.



$\mathrm{dp}_k(x_1, y_1, \ldots, x_k, y_k) = 0\text{-}\mathrm{dp}_k(x_1, y_1, \ldots, x_k, y_k)$



Every FO+SDP property can be tested in quadratic time on graphs of bounded Euler genus.
[Golovach, *S.*, & Thilikos, 2023]

More logics?

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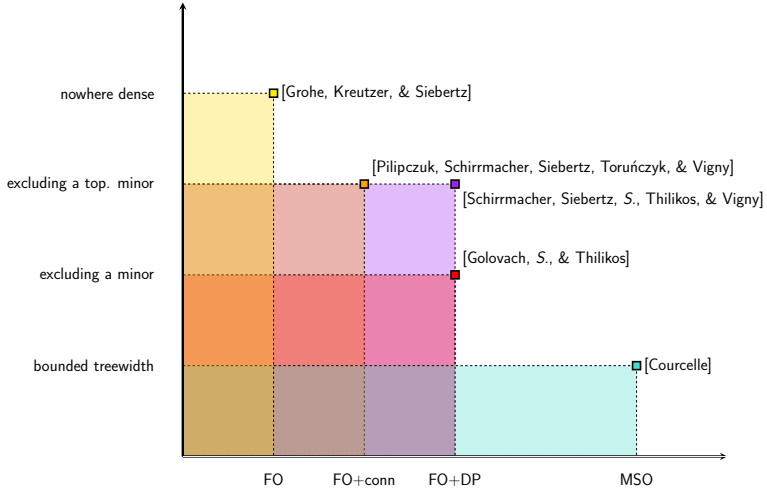
nowhere dense
excluding a top. minor
excluding a minor
bounded treewidth
[Grohe, Kreutzer, & Siebertz]
[Pilipczuk, Schirrmacher, Siebertz, Toruńczyk, & Vigny]
[Schirrmacher, Siebertz, S., Thilikos, & Vigny]
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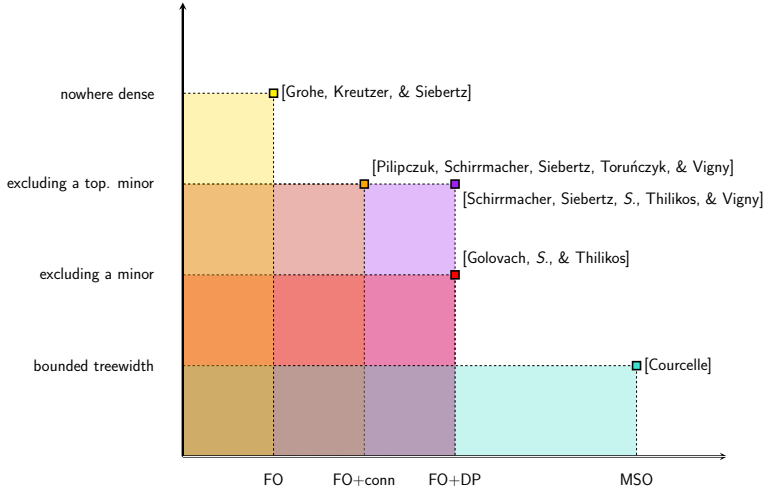

Compound logic(s) $\tilde{\Theta}^{(s)dp} = \begin{cases} \text{sentences expressing the recursive removal of MSO-expressible} \\ \text{and bounded treewidth modulators to a FO+(S)DP property.} \end{cases}$

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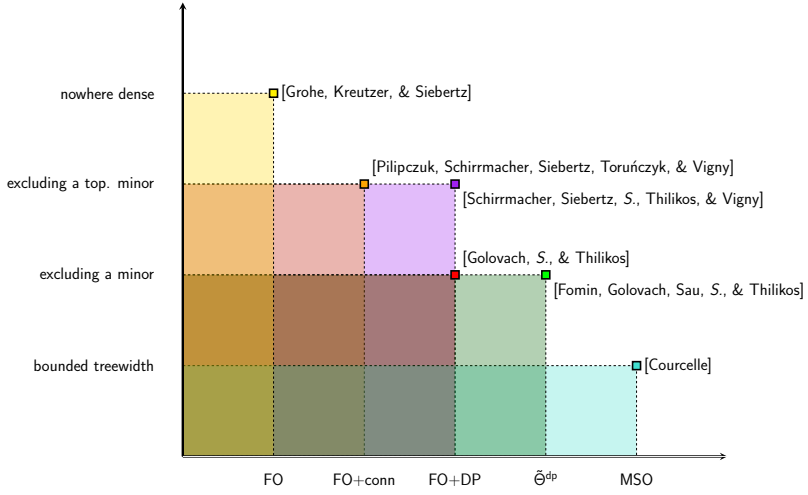

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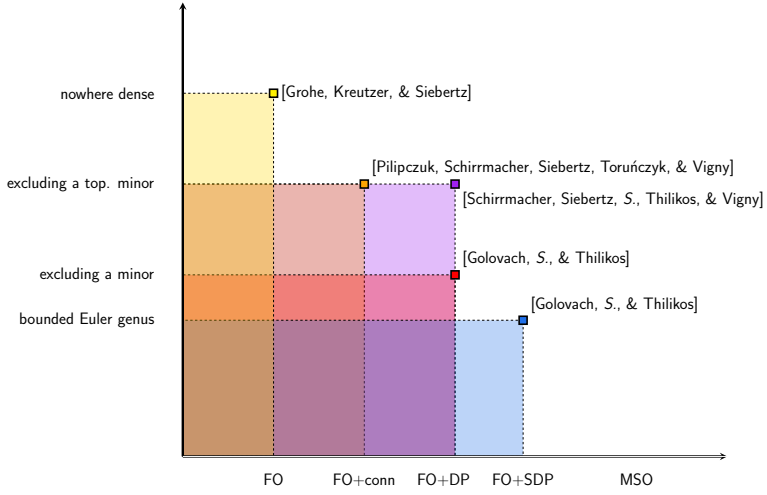

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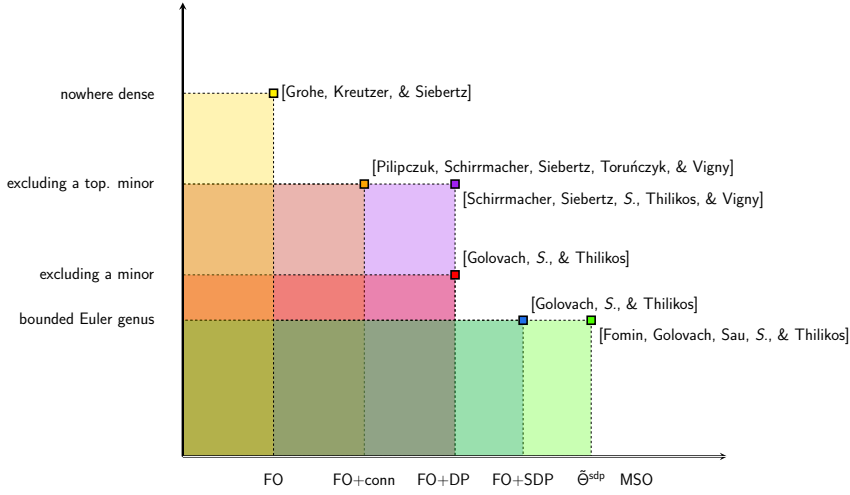

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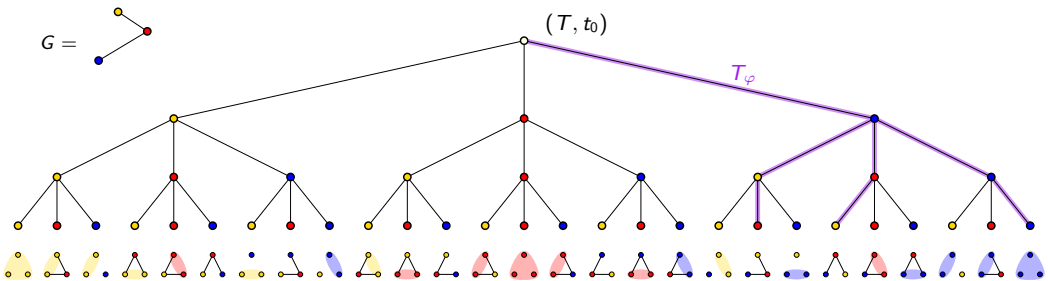
Discussion of the proof

Model checking game [Hinttikka, 1982].
a.k.a. *game trees*, *evaluation trees*, *morphism trees*,...

Example:



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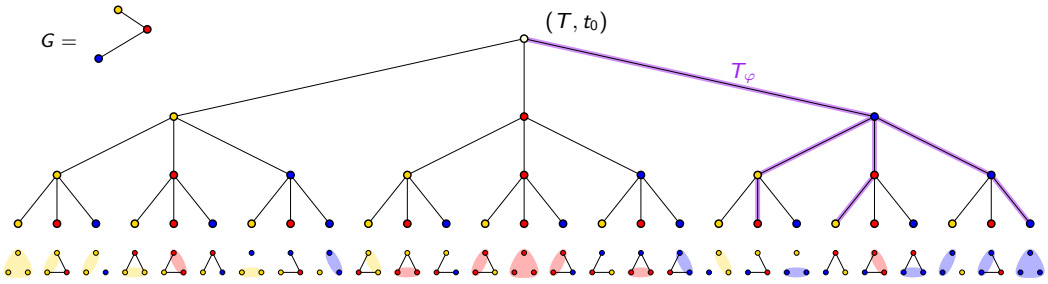

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- For a graph of size n and a formula with q quantifiers, the tree is of size n^q.

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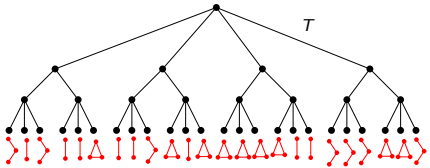

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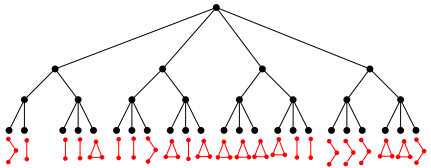

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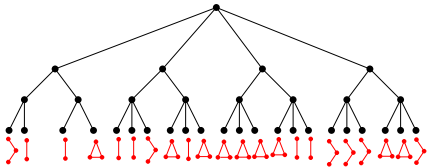

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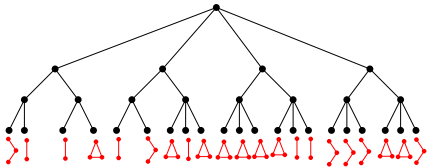

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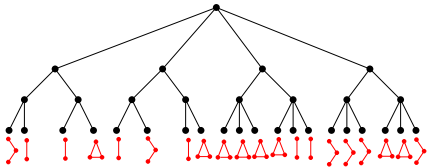

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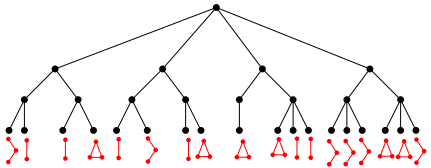

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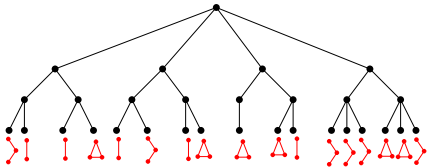

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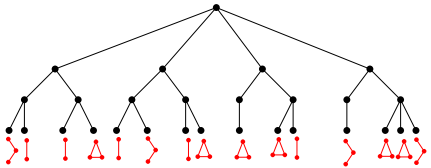

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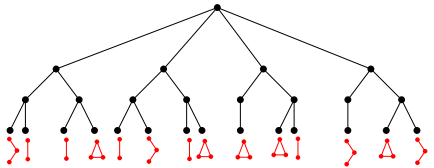

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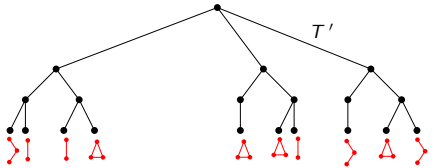

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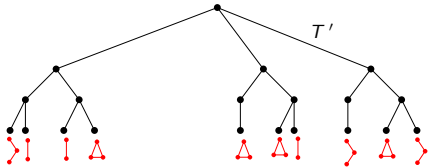

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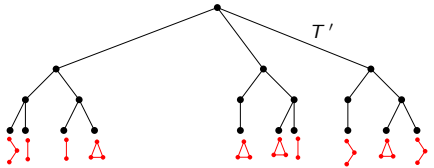

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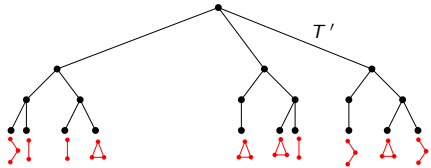

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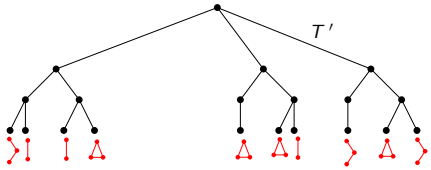

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Large treewidth

Flat Wall Theorem

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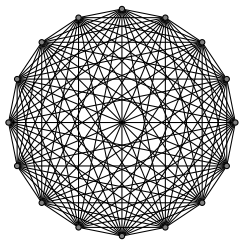
Large clique minor

Large *"flat" grid-like* induced subgraph



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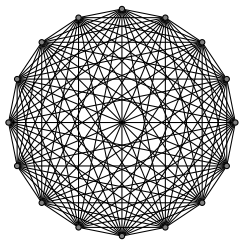

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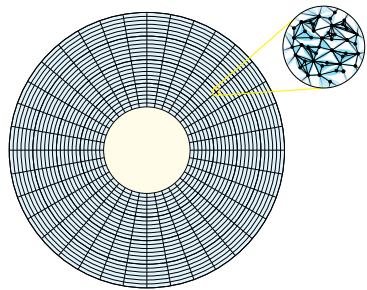

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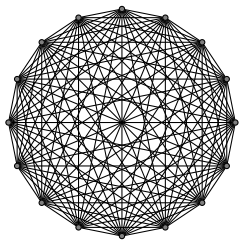

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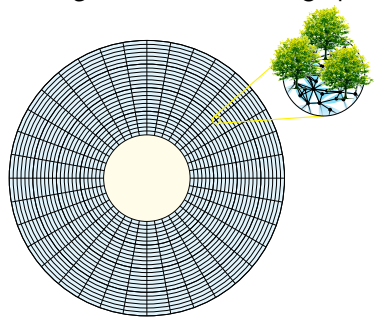

How this helps to compute the pattern?

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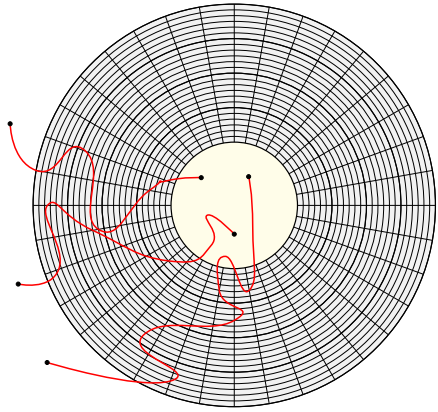

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There is a function $f : \mathbb{N} \to \mathbb{N}$ such that if
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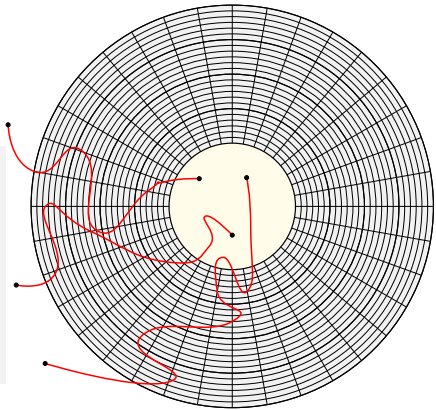

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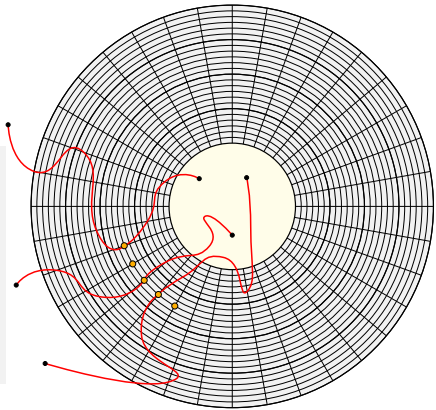

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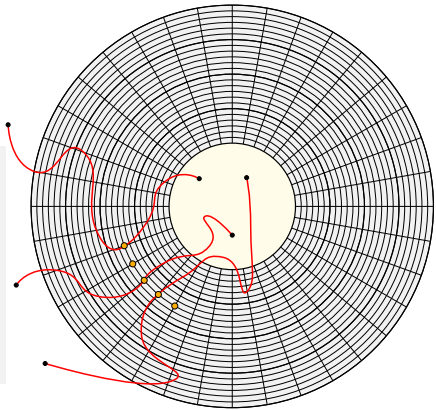

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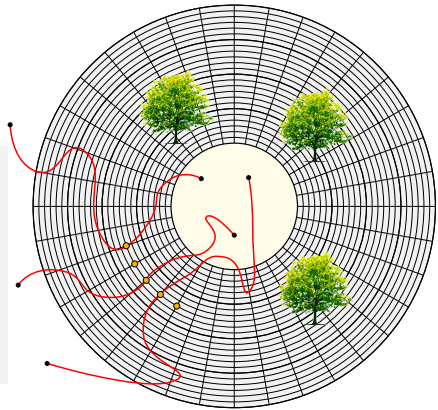

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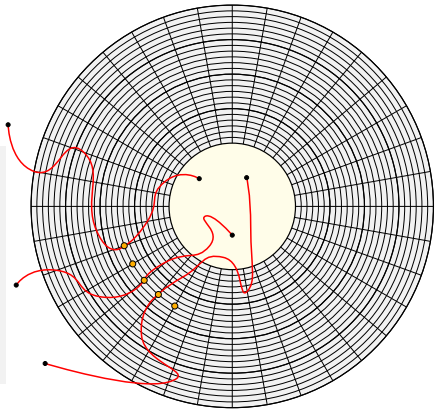

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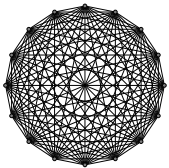
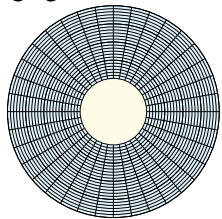
- Same partial patterns $\underset{\textbf{Combing}}{\Longrightarrow}$ same (global) patterns.

Large treewidth

Large clique minor



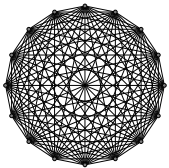
Large *grid-like "flat" structure*



- Compute partial patterns in *bounded treewidth part*.

Large treewidth

Large clique minor



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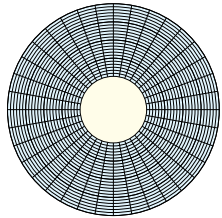

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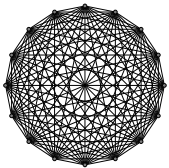

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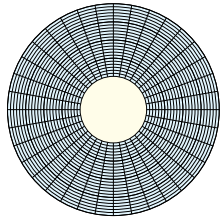

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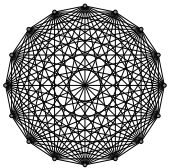

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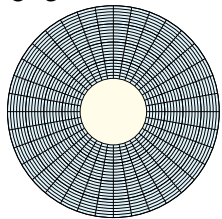

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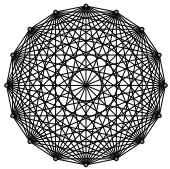

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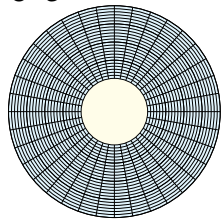

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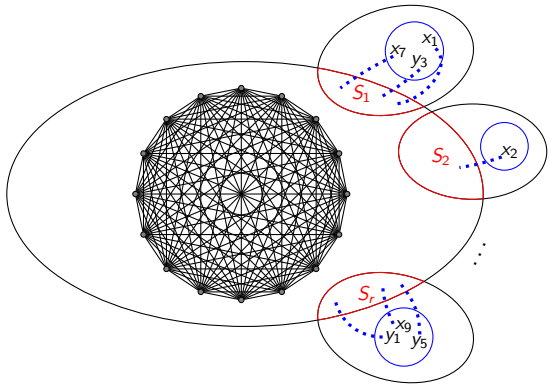

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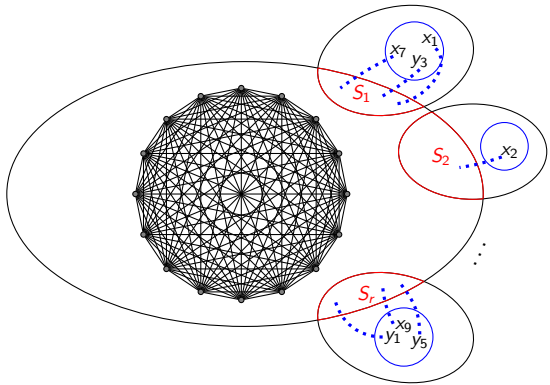

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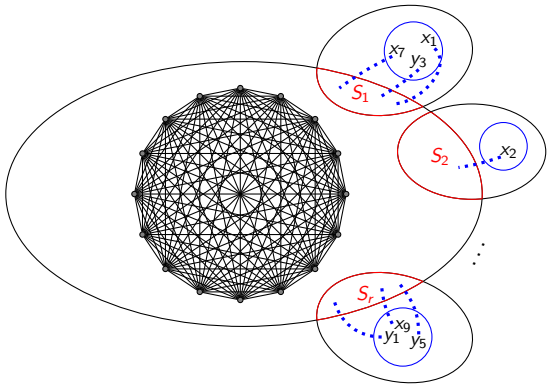

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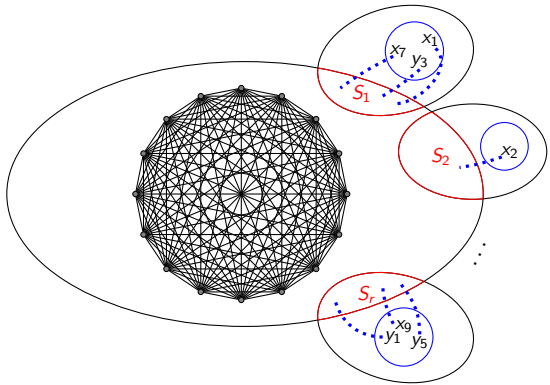
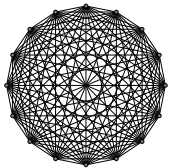
FO+DP collapses to FO



Large treewidth

Large clique minor



- Extra assumption: unbreakability.
- FO+DP collapses to FO.
- Using FO-machinery, compute pattern.

[Schirrmacher, Siebertz, *S.*, Thilikos, & Vigny, 2023+]

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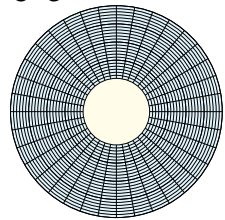

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Wrapping up the proof. [Schirrmacher, Siebertz, *S.*, Thilikos, & Vigny, 2023+]

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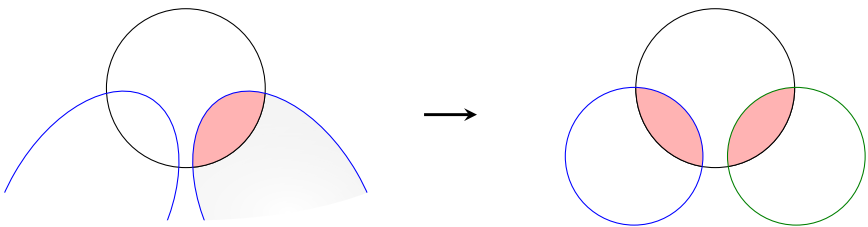
Dynamic programming on a tree-decomposition in unbreakable bags.
[Cygan, Lokshtanov, Pilipczuk, Pilipczuk, and Saurabh, 2014]



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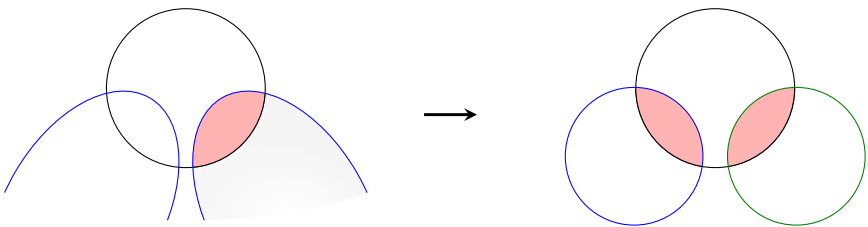

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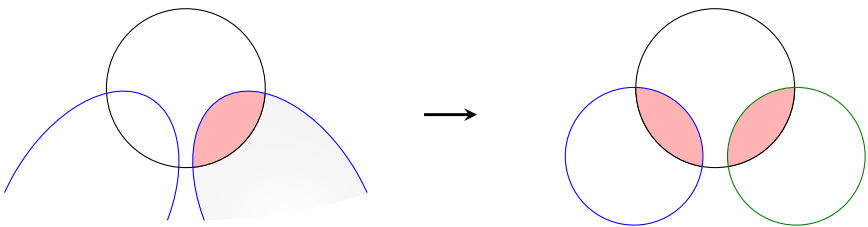

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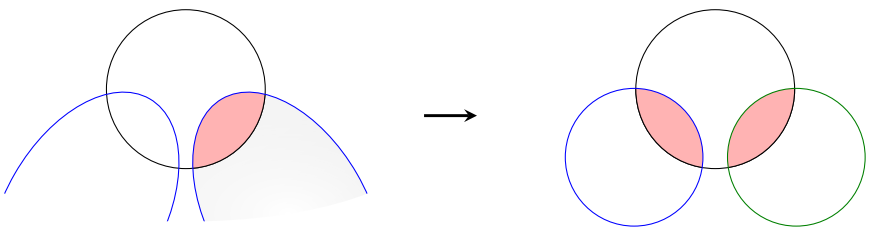

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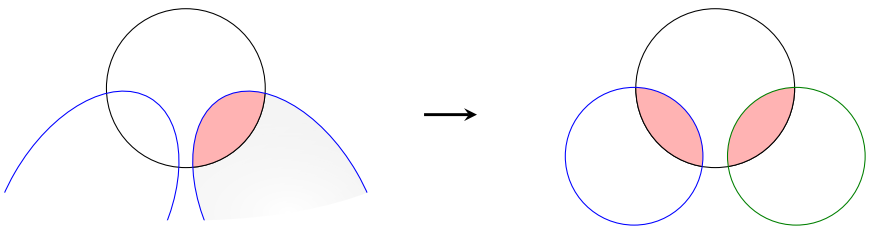
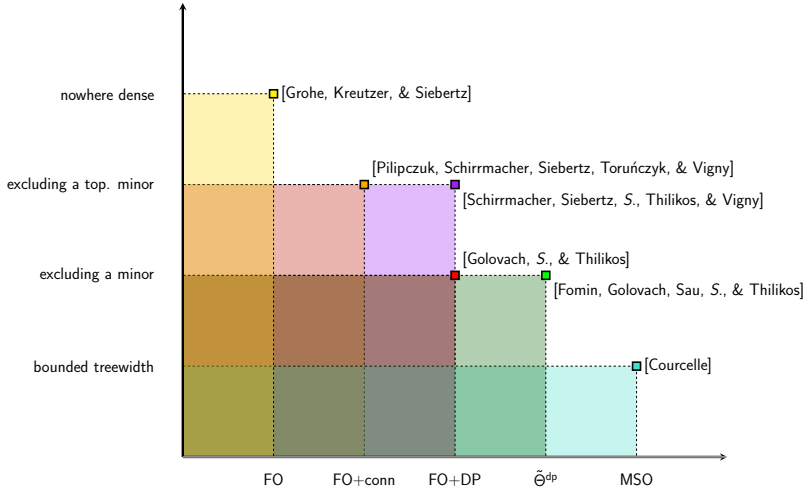
- Combine solutions bottom-up: Possible **only if the input graph excludes some topological minor.**




nowhere dense
excluding a top. minor
excluding a minor
bounded treewidth
[Grohe, Kreutzer, & Siebertz]
[Pilipczuk, Schirrmacher, Siebertz, Toruńczyk, & Vigny]
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[Courcelle]
FO
FO+conn
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MSO

Thank you!