Decidability of Quasi-Dense Modal Logics

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Introduction

Setting up the stage

Considered logic

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K4 with any MRPs

M. Zakharyaschev. Canonical formulas for K4. Part I: Basic results (1992).

Our paper

Quasi-Dense Modal Logic

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is any MRP of the form $k \to k_+$ where $k < k_+$.

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Quasi-Dense Modal Logic
is any extension of K with a finite number of Quasi-Dense MRPs.

In this paper

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Quasi-Dense Modal Logics can be decided in EXPSPACE.

A glimpse into the proof

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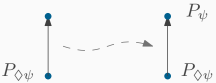

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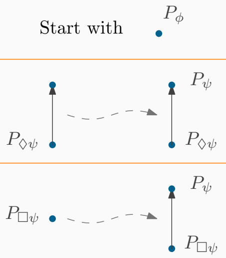

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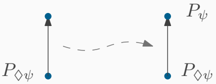

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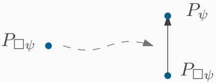

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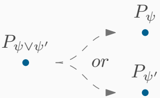

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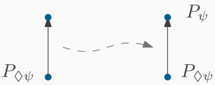

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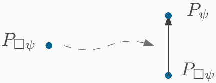

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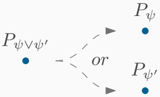

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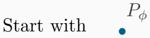

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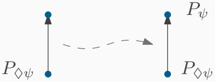

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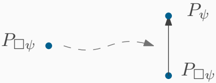

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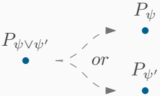

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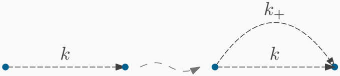

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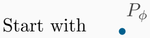

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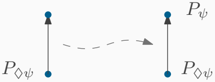

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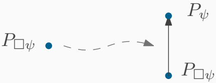

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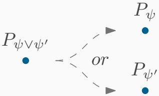

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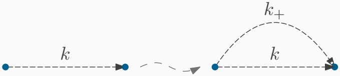

If P_p and $P_{\neg p}$ appear over a node derive nullary predicate contr.

Satisfiability via the procedure

We have that:

$\langle \varphi, P \rangle$ is satisfiable *iff* The procedure non-deterministically derives a structure that does not contain contr.

A quick example

$\varphi = \Box\Box p$

$P = \{2 \to 3\}$

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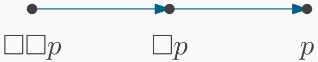

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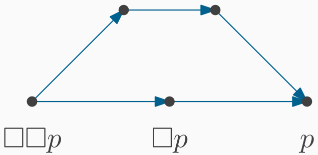

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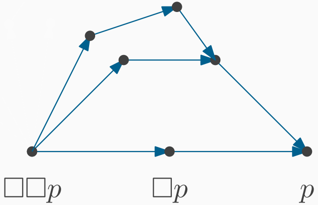

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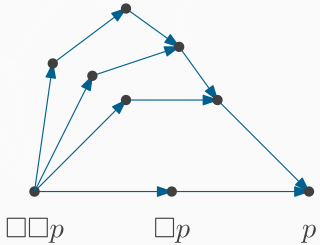

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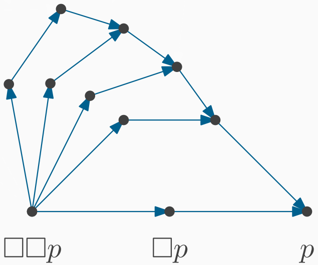

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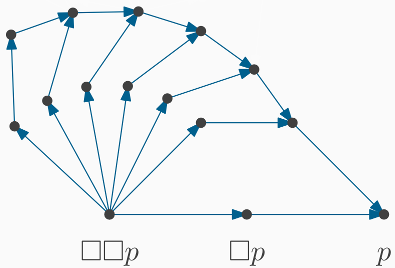

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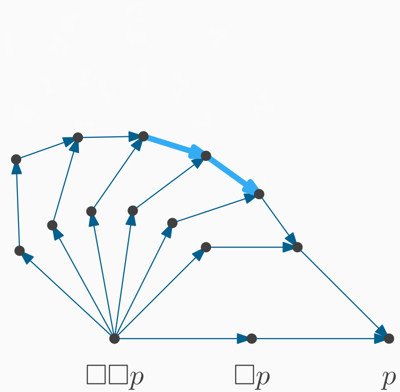

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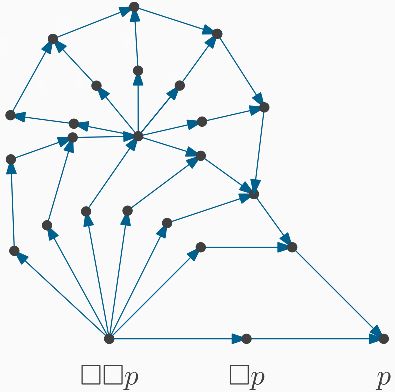

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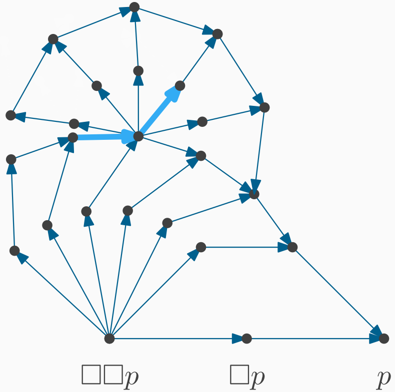

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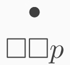

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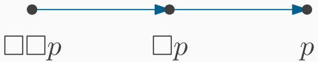

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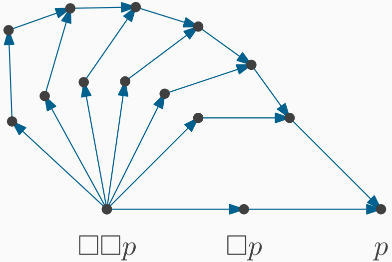

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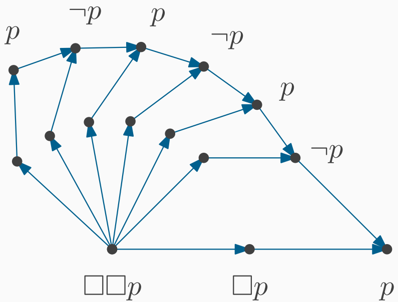

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Thank you!

Questions?