A Parametric Graph Obstructions View Point on Width Parameters

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The case of treewidth and minor monotone parameters

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Graph properties

- Is a graph G acyclic ? $\rightarrow$ forests
- Can a graph G be embedded in the plane without edge crossing ? $\rightarrow$ planar graphs
- Is the treewidth of a graph at most k ? $\rightarrow \mathbf{tw}(G) \leq k$
- Can the vertices of a graph G be 3-colored such that no adjacent vertices get the same color? $\rightarrow \chi(G) \leq 3$
- Does a graph contain a subset of k pairwise on-adjacent vertices ? $\rightarrow \alpha(G) \leq k$

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What makes a graph property easy to decide ?

When does a graph property has a succinct description ?

Graph properties and quasi-ordering relations

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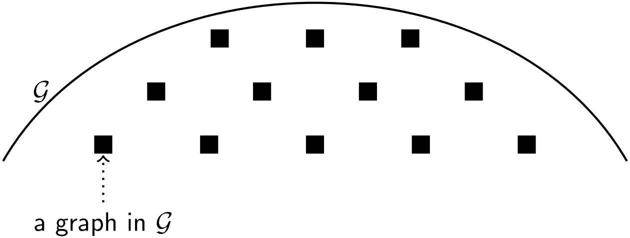

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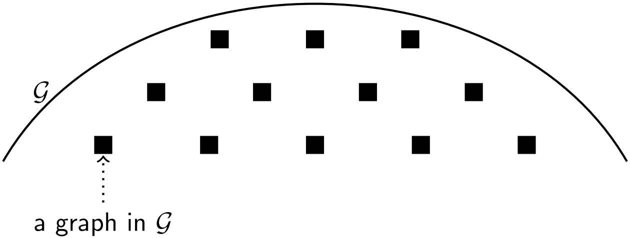

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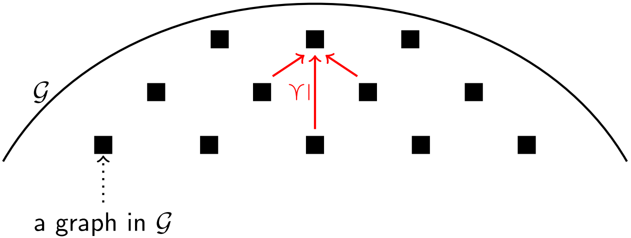

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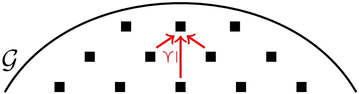
Examples of the classic quasi-orderings

- $\preceq_{\mathrm{i.sg}}$: induced subgraph $\rightarrow$ vertex deletion
- $\preceq_{\mathrm{sg}}$: subgraph $\rightarrow$ vertex / edge deletion
- $\preceq_{\mathrm{m}}$: minor $\rightarrow$ vertex / edge deletion + edge contraction



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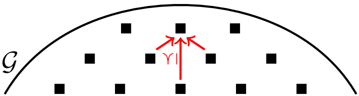
Observation: If a graph parameter $\mathbf{p}$ is $\preceq$-monotone, then the graph property

$$\mathcal{G}_{\mathbf{p}}^{k} = \{G \in \mathcal{G}_{\text{all}} \mid \mathbf{p}(G) \leq k\} \text{ is } \preceq\text{-closed.}$$



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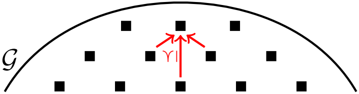
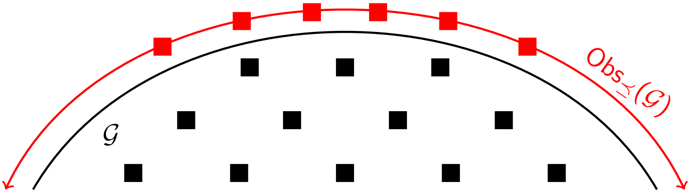
$$\mathcal{G}^k_{\mathbf{p}} = \{G \in \mathcal{G}_{\text{all}} \mid \mathbf{p}(G) \leq k\} \text{ is } \preceq\text{-closed.}$$

- $\mathcal{G}^k_{\mathbf{tw}}$ is $\preceq_{\text{i.sg}}$-closed and $\preceq_{\text{m}}$-closed
- $\mathcal{G}^k_{\alpha}$ is $\preceq_{\text{i.sg}}$-closed but not $\preceq_{\text{m}}$-closed (⇝ consider the C_4)



Obstructions of closed graph properties

The $\preceq$-obstruction set of a graph class $\mathcal{G}$, denoted $\mathrm{Obs}_{\preceq}(\mathcal{G})$ is the set of $\preceq$-minimal graphs not belonging to $\mathcal{G}$.



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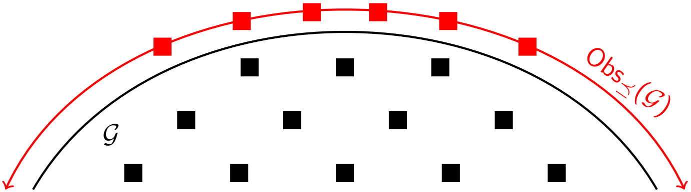
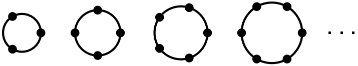


$\rightarrow$ $\mathrm{Obs}_{\preceq_{\mathrm{i.sg}}}(\mathcal{G}_{\mathrm{forest}})$



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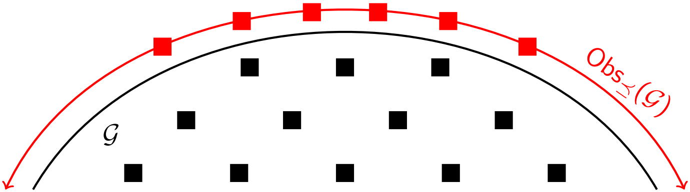

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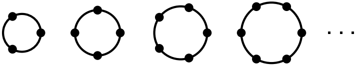

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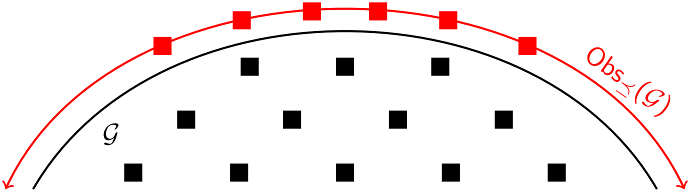
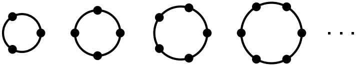
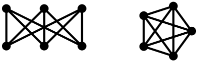
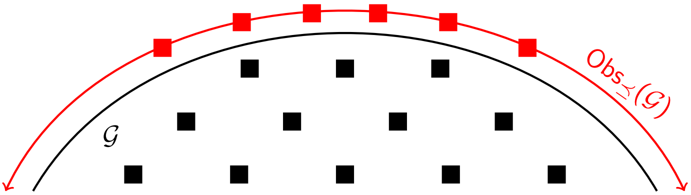


$\rightarrow$ $\mathrm{Obs}_{\preceq_{\mathrm{i.sg}}}(\mathcal{G}_{\mathrm{forest}})$ 

$\rightarrow$ $\mathrm{Obs}_{\preceq_{\mathrm{m}}}(\mathcal{G}_{\mathrm{forest}})$ 

$\rightarrow$ $\mathrm{Obs}_{\preceq_{\mathrm{m}}}(\mathcal{G}_{\mathrm{planar}})$  [Kuratowski Theorem]

Membership problem to closed graph properties



Observation: If $\mathrm{Obs}_{\leq}(\mathcal{G})$ is finite and if we can test whether $H \leq G$ in poly($|G|$)-time, then
for every graph G, we can decide the membership to $\mathcal{G}$ in poly($|G|$)-time.

The menbership problem to $\preceq_{\mathsf{m}}$-closed graph properties

Robertson & Seymour Theorem [2004]
The minor relation $\preceq_{\mathsf{m}}$ is a well-quasi-ordering.

$\rightsquigarrow$ in every infinite sequence $\langle G_i \rangle_{i\in\mathbb{N}} = \langle G_1, G_2, G_3 \ldots \rangle$ of graphs,
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Theorem [RS'95, KKR'12, KPS'24]

Given two graphs H and G, we can decide if $H \preceq_{\mathsf{m}} G$ in time

$$f(|H|) \cdot |G|^{1+o(1)}.$$

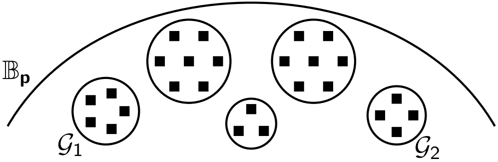
$\rightsquigarrow$ $\boxed{\text{The membership problem to any minor-closed graph property } \mathcal{G} \text{ is decidable in almost linear time.}}$

$\preceq$-Class properties

- Let $\mathcal{G}$ be a $\preceq_{\mathrm{m}}$-closed class of graphs and $\mathbf{p}$ be $\preceq_{\mathrm{m}}$-monotone parameter. Does there exist $c \in \mathbb{N}$ such that for every graph $G \in G$, $\mathbf{p}(G) \leq c$?

$$\mathbb{B}_{\mathbf{p}} = \{\mathcal{G} \subseteq \mathcal{G}_{\mathrm{all}} \text{ and } \mathcal{G} \preceq_{\mathrm{m}}\text{-closed} \mid \exists c \in \mathbb{N}, \forall G \in \mathcal{G}, \mathbf{p}(G) \leq c\}$$

$\leadsto$ We may consider $\mathbf{p} = \mathbf{tw}$.



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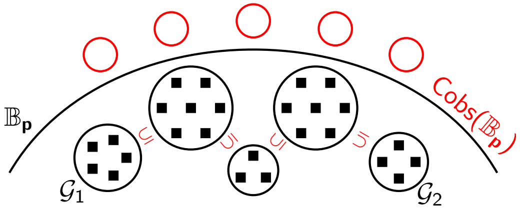

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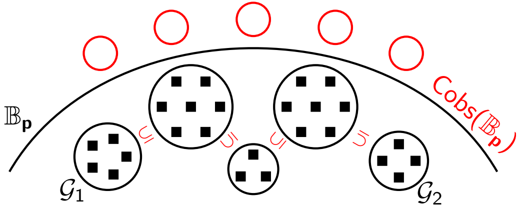
$\rightsquigarrow$ We may consider $\mathbf{p} = \mathbf{tw}$.

- Let $\mathcal{G}$ be a class of graphs and Π is a problem. Is Π polynomial-time tractable on $\mathcal{G}$?

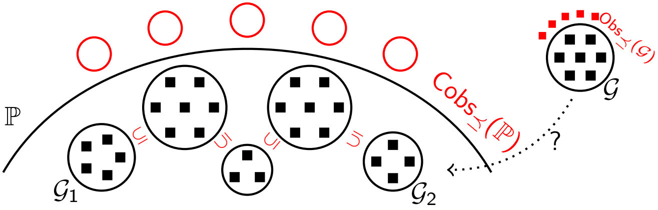
$$\mathbb{P}_{\Pi} = \{\mathcal{G} \subseteq \mathcal{G}_{\mathsf{all}} \mid \Pi \text{ is polynomial-time tractable on } \mathcal{G}\}$$

$\rightsquigarrow$ We may consider Π = Maximum Independent Set.



Membership problem to $\preceq$-class property

- Let $\mathcal{G}$ be a graph class and $\mathbb{P}$ be a $\preceq$-class property.
- Does $\mathcal{G}$ belongs to $\mathbb{P}$?



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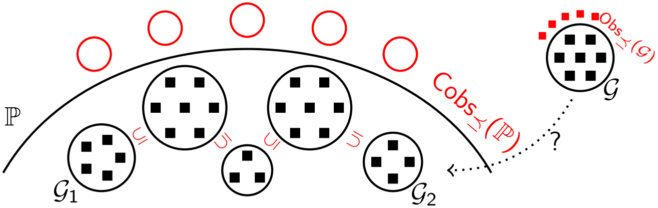

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⇝ What makes a $\preceq$-class property easy to decide ?

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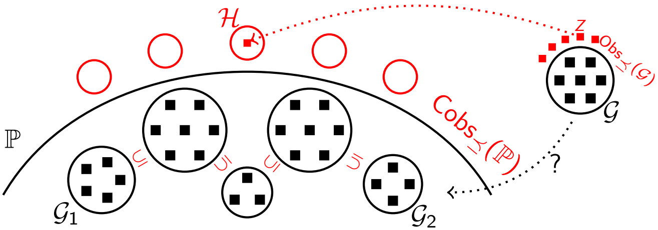

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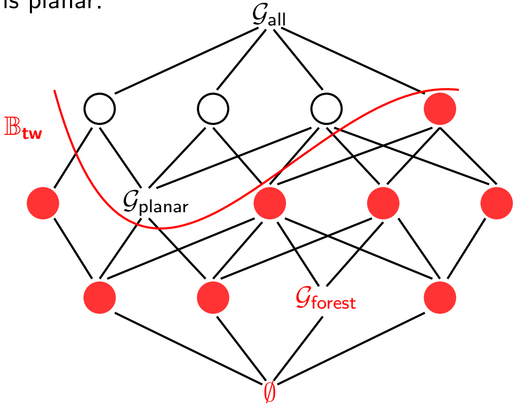
Observation

$\mathcal{G} \in \mathbb{P}$ if anf only if, for every $\mathcal{H} \in \mathsf{Cobs}_{\preceq}(\mathbb{P})$, there exists $Z \in \mathsf{Obs}_{\preceq}(\mathcal{G})$ such that $Z \in \mathcal{H}$.

Bounded tree-width $\preceq_m$-class property

Grid theorem [Robertson & Seymour, 1986]
Let $\mathcal{G}_H$ be the class of graphs such that $\mathrm{Obs}_{\preceq_m} = \{H\}$, then $\mathcal{G}_H \in \mathbb{P}_{\mathbf{tw}}$ if and only if H is planar.



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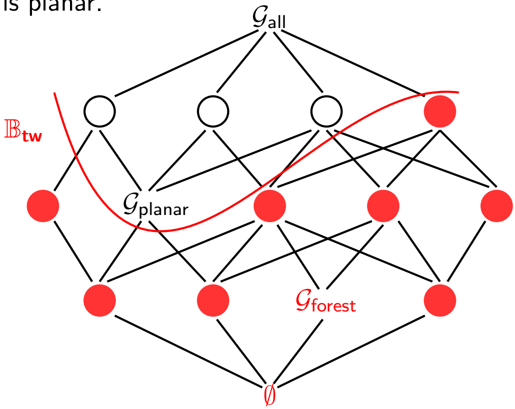

$\rightsquigarrow$ membership of $\mathcal{G}$ to $\mathbb{B}_{\mathbf{tw}}$: check if $\mathrm{Obs}_{\preceq_m}(\mathcal{G})$ contains a planar graph.

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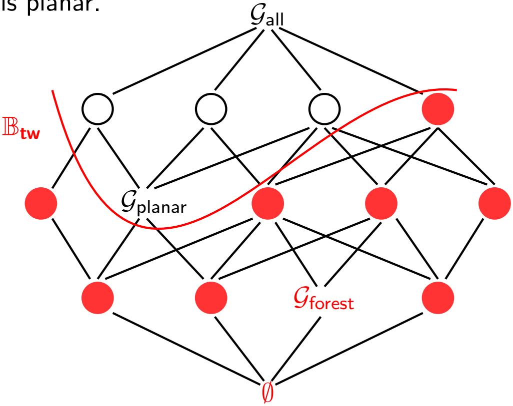


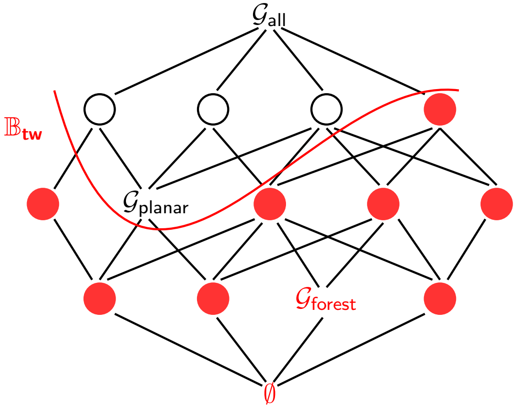
$\rightsquigarrow$ membership of $\mathcal{G}$ to $\mathbb{B}_{\mathbf{tw}}$: check if $\mathrm{Obs}_{\preceq_m}(\mathcal{G})$ contains a planar graph.

Observation: Planar graphs is the only (inclusion) minimal $\preceq_m$-closed class of unbounded treewdith.

$\mathbb{P}_{\mathsf{MIS}}$ – Max Independent Set tractable $\preceq_m$-classes

Theorem [Folkore]: MIS is NP-complete on planar graphs

Courcelle's Theorem: For every class $\mathcal{G} \in \mathbb{P}_{\mathbf{tw}}$, MIS is in P.



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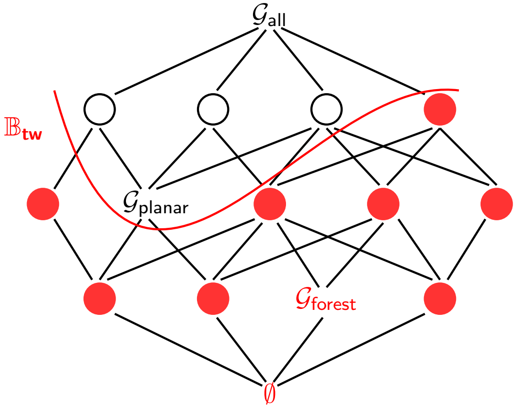

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$\mathbb{P}_{\text{MIS}}$ – Max Independent Set tractable $\preceq_{\text{i.sg}}$-classes

What about the $\preceq_{\text{i.sg}}$ relation ?

Theorem [Alekseev 2004]
Let $\mathcal{G}$ be a $\preceq_{\text{i.sg}}$-closed graph such that $\text{Obs}_{\preceq_{\text{i.sg}}}(\mathcal{G})$ is finite.
Then $\mathcal{G} \notin \mathbb{P}_{\text{MIS}}$ if and only if it includes a boundary class for MIS.

Theorem [Alekseev 2004]
If $\text{P} \neq \text{NP}$, then $\mathcal{G}_{\text{forest}}$ is the unique $\preceq_{\text{i.sg}}$-closed boundary class for MIS.

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For some class properties, the answer is positive:

- $\rightsquigarrow$ By Ramsey's Theorem, the classes $\mathcal{G}_{\text{edgeless}}$ and $\mathcal{G}_{\text{complete}}$ are the only minimal $\preceq_{\text{i.sg}}$-closed classes not in $\mathbb{P}_{\text{size}}$.

Testing membership of $\preceq_m$-closed class properties

- Let $\mathcal{G}$ be a graph $\preceq_m$-closed class and $\mathbb{P}$ be a $\preceq_m$-class property.
- Does $\mathcal{G}$ belongs to $\mathbb{P}$?

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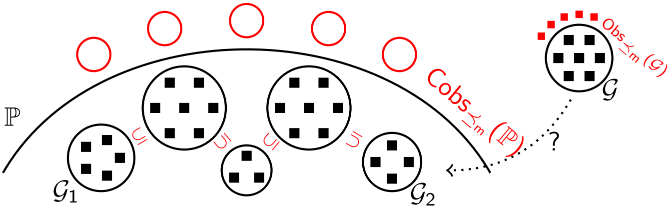

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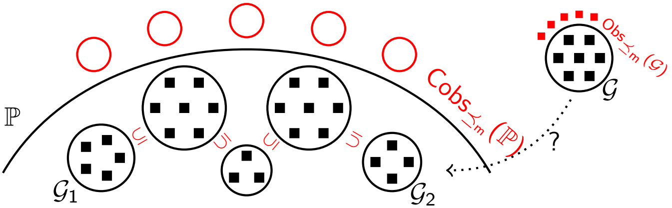

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Theorem [Jancar 1999, Marcone 2001]
If $\preceq_{\mathrm{m}}$ is a ω^2-well-quasi-ordering on graphs, then for every $\mathbb{P}$

$$\mathrm{Cobs}_{\preceq_{\mathrm{m}}}(\mathbb{P}) \text{ is finite}.$$

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$\rightsquigarrow$ Whether $\preceq_{\mathsf{m}}$ is a ω^2-well-quasi-ordering on graphs is a wide open problem in order theory.

The parametric framework

Objective:

Prove the finiteness of $\mathrm{Cobs}_{\preceq_{\mathrm{m}}}(\mathbb{B}_{\mathbf{p}})$ for $\mathbf{p}$ a $\preceq_{\mathrm{m}}$-monotone parameter.

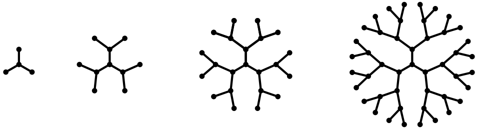
The parametric framework

Objective:

$$\boxed{\text{Prove the finiteness of } \mathsf{Cobs}_{\preceq_{\mathsf{m}}}(\mathbb{B}_{\mathbf{p}}) \text{ for } \mathbf{p} \text{ a } \preceq_{\mathsf{m}}\text{-monotone parameter.}}$$

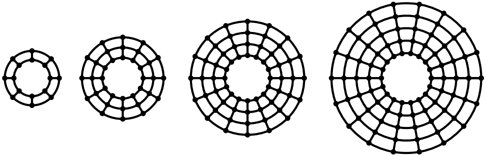
A $\preceq_{\mathsf{m}}$-parametric graph is a $\preceq_{\mathsf{m}}$-increasing sequence of graphs $\mathscr{G} = \langle G_i \rangle_{i \in \mathbb{N}}$.

$\rightsquigarrow$ The parametric ternary trees $\mathscr{T} = \langle T_i \rangle_{i \in \mathbb{N}}$



Observation: every tree is a minor of some ternary tree.

$\rightsquigarrow$ The parametric annulus grids $\mathscr{G} = \langle G_i \rangle_{i \in \mathbb{N}}$



Observation: every planar graph is a minor of some annulus grid.

The parametric framework - theorem template

Theorem: Given a $\preceq_{\mathsf{m}}$-monotone parameter $\mathbf{p}$, there is

- a finite set $\{\mathscr{H}^1, \mathscr{H}^2, \ldots, \mathscr{H}^c\}$ of $\preceq_{\mathsf{m}}$-parametric graphs;
- a function $f : \mathbb{N} \to \mathbb{N}$;

such that, for every $k \in \mathbb{N}$ and every graph G

1. if $p(G) \leq k$, then for every $i \in [c]$, $\mathscr{H}^i_{f(k)} \not\preceq_{\mathsf{m}} G$;
2. if for every $i \in [c]$, $\mathscr{H}^i_{f(k)} \not\preceq_{\mathsf{m}} G$, then $\mathbf{p}(G) \leq f(k)$.

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$\rightsquigarrow$ $\mathbf{p}(G)$ is small if and only if G excludes a member of every parametric graph $\mathscr{H}^i$.

$\rightsquigarrow$ The set $\{\mathscr{H}^1, \mathscr{H}^2, \ldots, \mathscr{H}^c\}$ is called the $\preceq_{\mathrm{m}}$-universal obstruction for $\mathbf{p}$

Universal obstructions - examples

- Let $\mathscr{G}$ be the parametric annulus grids. Then $\{\mathscr{G}\}$ is the universal obstrruction of **tw**.
- Let $\mathscr{T}$ be the parametric ternary trees. Then $\{\mathscr{T}\}$ is the universal obstrruction of **pw**.

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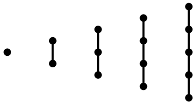

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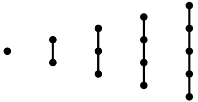
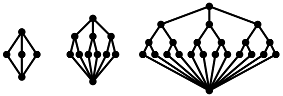
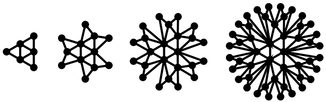


- The biconnected pathwidth is characterized by a set of 2 universal obstructions.





The parametric framework

Theorem [Protopapas, P., Thilikos]
If $\preceq$ is an ω^2-well-quasi-ordering on graphs, then every $\preceq$-monotone parameter $\mathbf{p}$ has a $\preceq$ -universal obstruction.

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$\rightsquigarrow$ There exists a finite set $\{\mathscr{H}^1, \mathscr{H}^2, \dots, \mathscr{H}^c\}$ of $\preceq$-parametric graphs such that:

A $\preceq$-closed class of graph $\mathcal{G}$ belongs to $\mathbb{B}_{\mathbf{p}}$

if and only if

$\mathcal{G}$ does not contain all members of $\mathscr{H}^1$ and of $\mathscr{H}^2$ …and of $\mathscr{H}^c$

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Theorem [Protopapas, P., Thilikos]:
If $\preceq$ is an ω^2-well-quasi-ordering on graphs, then for every $\preceq$-class property $\mathbb{P}$, there exists graph $\preceq$-monotone parameters $\mathbf{p}_1$, $\mathbf{p}_2$, ..., $\mathbf{p}_\ell$ (with $\preceq$-universal obstructions) such that

$$\mathbb{P} = \{\mathcal{G} \subseteq \mathcal{G}_{\mathsf{all}} \mid \mathcal{G} \in \mathbb{B}_{\mathbf{p}_1} \text{ or } \mathcal{G} \in \mathbb{B}_{\mathbf{p}_2} \text{ or } \ldots \mathcal{G} \in \mathbb{B}_{\mathbf{p}_\ell}\}$$

Revisiting Erdös-Pósa theory by means of class properties

Theorem [Erdös-Pósa 1965]
Every graph G contains

- either k disjoint cycles
- or a set S of $k \log k$ vertices such that $G - S$ is acyclic.

Revisiting Erdös-Pósa theory by means of class properties

Theorem [Erdös-Pósa 1965]

For every $G \in \mathcal{G}_{\text{all}}$, G contains

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Revisiting Erdös-Pósa theory by means of class properties

Theorem [Erdös-Pósa 1965]
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$\rightsquigarrow$ $\mathcal{G}_{\text{forest}}$-modulator

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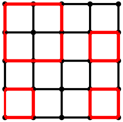

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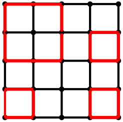

Definition

Let $\mathcal{H}$ and $\mathcal{G}$ be two $\preceq_{\text{m}}$-closed classes.

$(\mathcal{H}, \mathcal{G})$ is an Erdös-Pósa pair if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that every graph $G \in \mathcal{G}$ contains

- either an $\mathcal{H}$-barrier of size k
- or an $\mathcal{H}$-modulator of size $f(k)$.

The Erdös-Pósa class property

Let $\mathcal{H}$ be a $\preceq_{\mathsf{m}}$-closed class. Is $(\mathcal{H}, \mathcal{G}_{\mathsf{all}})$ an Erdös-Pósa pair ?

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Theorem [Robertson & Seymour, 1986]

$(\mathcal{H}, \mathcal{G}_{\mathsf{all}})$ an Erdös-Pósa pair if and only if $\mathcal{H}$ has bounded treewidth.

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Let $\mathcal{H}$ be a $\preceq_{\mathsf{m}}$-closed target class. Then

$$\mathbb{EP}_{\mathcal{H}} = \{\mathcal{G} \mid \mathcal{G} \text{ is } \preceq_{\mathsf{m}}\text{-closed and } (\mathcal{H}, \mathcal{G}) \text{ is an Erdös-Pósa pair}\}$$

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Observation: Suppose that $\mathcal{H} \in \mathbb{B}_{\mathbf{tw}}$, then

- $\rightsquigarrow$ $\mathbb{EP}_{\mathcal{H}}$ is the set of all $\preceq_{\mathsf{m}}$-closed classes;
- $\rightsquigarrow$ $\mathsf{Cobs}_{\preceq_{\mathsf{m}}}(\mathbb{EP}_{\mathcal{H}}) = \emptyset$.

Class obstructions of Erdös-Pósa class properties

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Theorem [Protopapas, P. Thilikos, Wiederrecht, 2024]
For every $\preceq_{\mathsf{m}}$-closed class $\mathcal{H}$, we have

$$|\mathsf{Cobs}_{\preceq_{\mathsf{m}}}(\mathbb{EP}_{\mathcal{H}})| \leq 2^{2^{O(\ell(h))}}$$

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Suppose that:

$\rightsquigarrow$ $\mathcal{H} = \{K_4\}$-$\preceq_{\mathsf{m}}$-free graphs, since K_4 is planar, $\mathcal{H} \in \mathbb{B}_{\mathbf{tw}}$ and thereby $\mathsf{Cobs}_{\preceq_{\mathsf{m}}}(\mathbb{EP}_{\mathcal{H}}) = \emptyset$;

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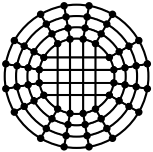
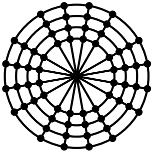
- $\rightsquigarrow$ $\mathcal{H} = \{K_5\}$-$\preceq_{\mathsf{m}}$-free graphs, then the class obstructions are

$\mathcal{G}_{\text{torus}}$ and $\mathcal{G}_{\text{projective}}$





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Theorem [Protopapas, P. Thilikos, Wiederrecht, 2024]
For every $\preceq_{\mathsf{m}}$-closed class $\mathcal{H}$, we have

$$|\mathsf{Cobs}_{\preceq_{\mathsf{m}}}(\mathbb{EP}_{\mathcal{H}})| \leq 2^{2^{O(\ell(h))}}$$

The proof relies on

- a constructive proof of

  Theorem [Liu 2022] for every $\mathcal{H}$ and $\mathcal{G}$, $(\mathcal{H}, \mathcal{G})$ is an $\frac{1}{2}$-Erdös-Pósa pair;

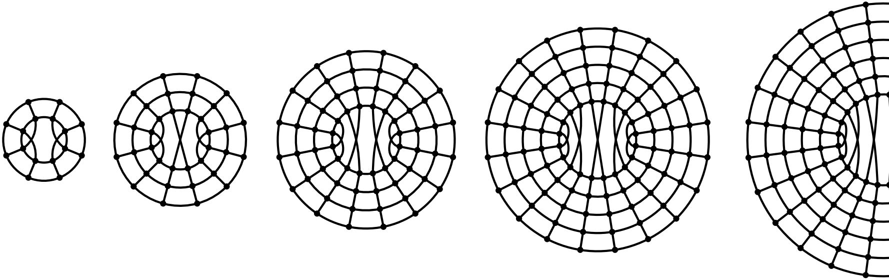
- a notion of $\mathcal{H}$-**tw** measuring tree-decomposability into graphs belonging to $\mathcal{H}$;

  In a $\mathcal{H}$-tree-decomposition, the leaves bags may belong to $\mathcal{H}$ and be of unbounded size.

- similar arguments than those used for the grid-minor theorem.

Papers

- C. Paul, E. Protopapas, D. M. Thilikos.
  Universal Obstructions of Graph Parameters.
  CoRR abs/2304.14121 (2023) – submitted to D.A.M.
- C. Paul, E. Protopapas, D. M. Thilikos.
  Graph Parameters, Universal Obstructions, and WQO.
  CoRR abs/2304.03688 (2023) – submitted to Order
- C. Paul, E. Protopapas, D. M. Thilikos, S. Wiederrecht.
  Delineating Half-Integrality of the Erdös-Pósa Property for Minors: the Case of Surfaces.
  CoRR abs/2406.16647 (2024) – ICALP 2024
- C. Paul, E. Protopapas, D. M. Thilikos, S. Wiederrecht.
  Obstructions to Erdös-Pósa Dualities for Minors.
  CoRR abs/2407.09671 (2024) – FOCS 2024
- E. Protopapas.
  Universal Obstructions of Graph Parameters.
  Phd, Montpellier University (2025).



Thanks !