Derivation Graphs, Greediness, and Bounded-treewidth in the Context of Existential Rules

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Preliminaries

Existential Rule: $\forall\vec{x}\forall\vec{y}\ (\ \phi(\vec{x},\vec{y}) \rightarrow \exists\vec{z}\ \psi(\vec{y},\vec{z})\ )$

Example:

$$\rho = male(x) \wedge human(x) \rightarrow \exists z(female(z) \wedge parent(z,x))$$

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Example: Let $h(x) = \mathit{Joe}$.

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$\mathcal{D} = F_0, (\rho, h, F_1)$.

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Query Entailment: Let F be a fact and $\mathcal{R}$ be a set of existential rules. Given query $Q = \{p(\vec{t_1}), \ldots, p(\vec{t_n})\}$, does $(F, \mathcal{R}) \models Q$?

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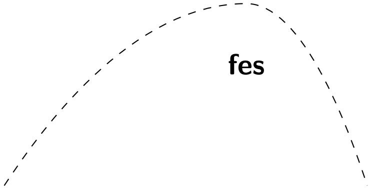

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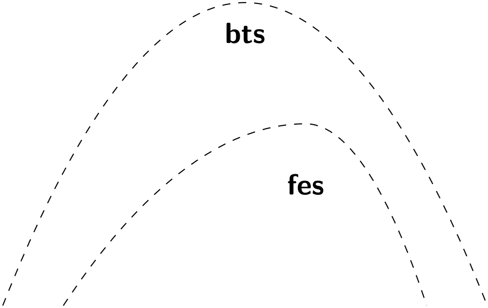

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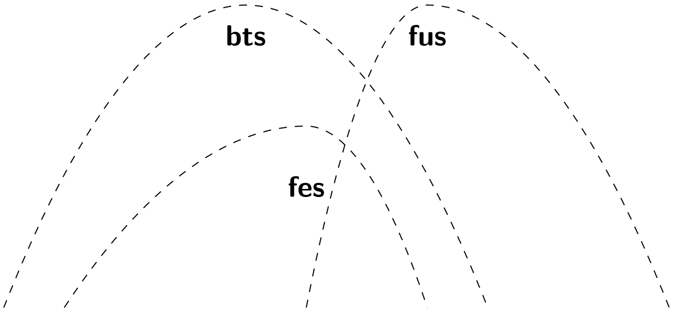

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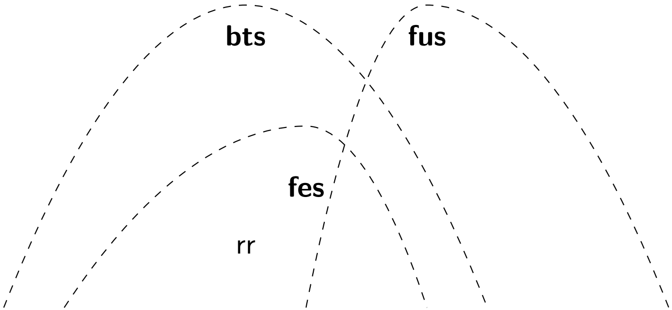

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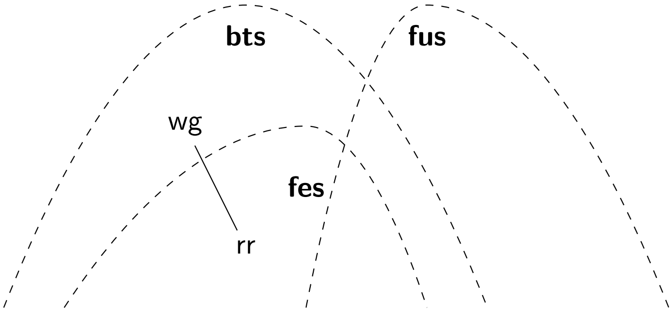

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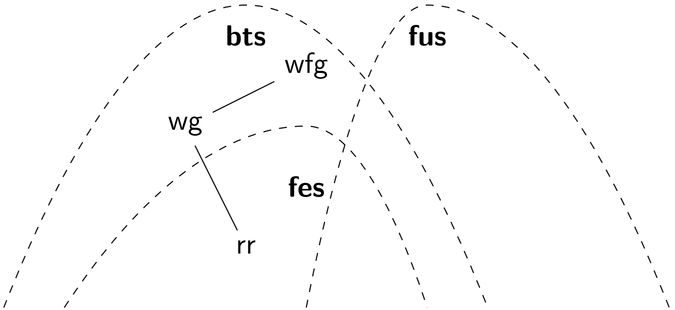

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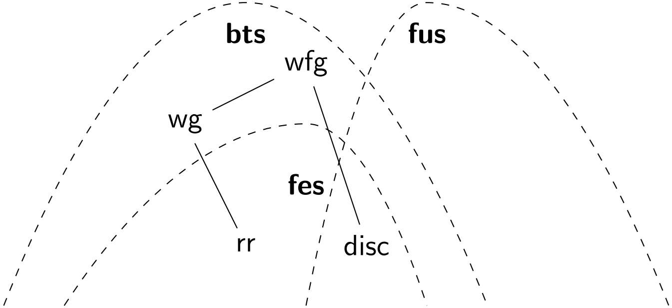

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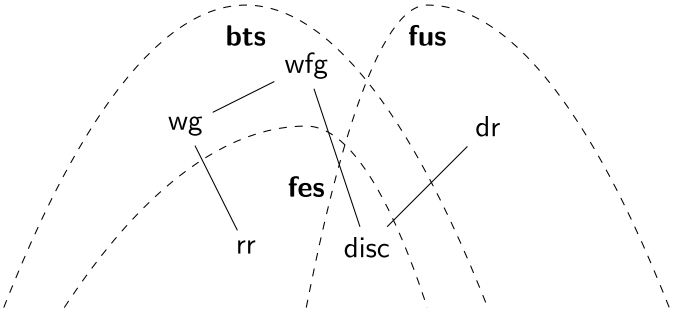

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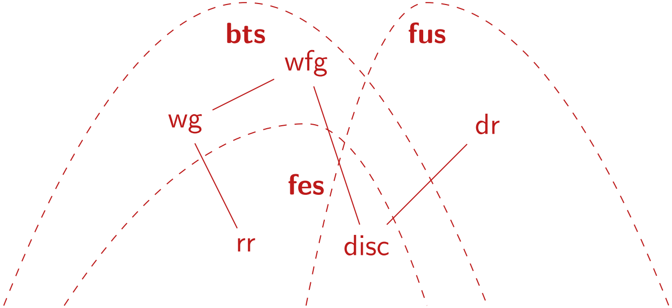

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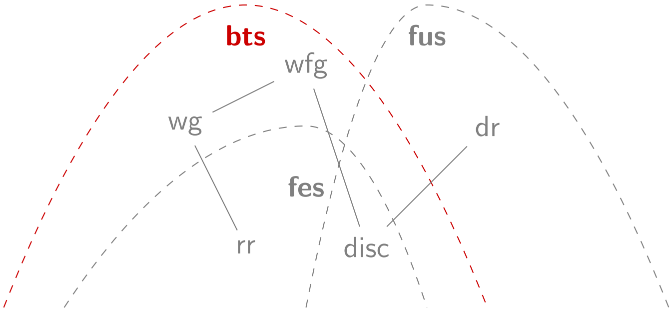

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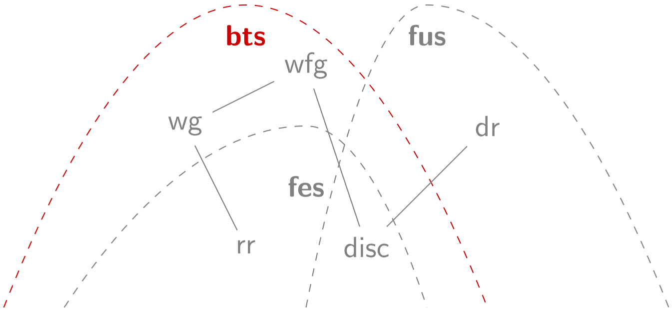

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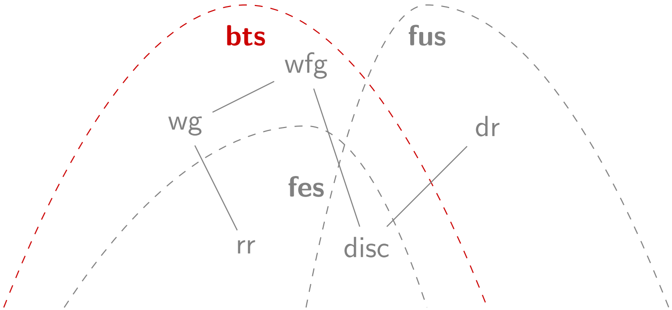

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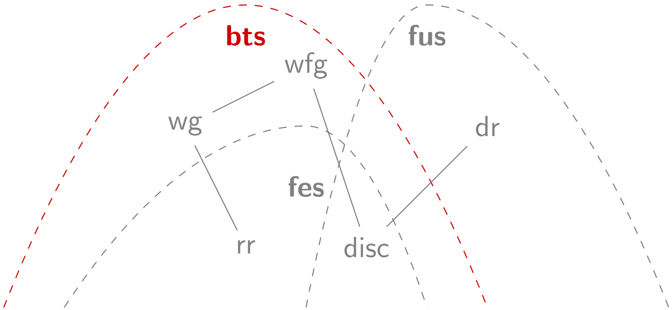

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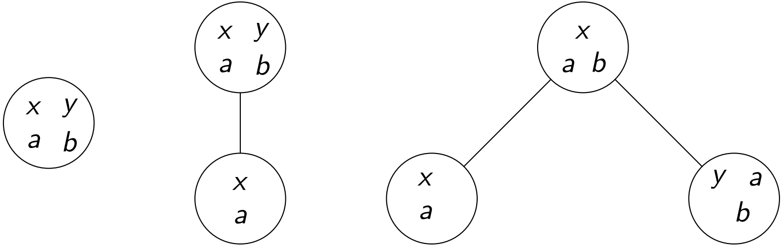
Example: Let $F = \{p(x, a), q(x, a, b), r(y, a, b)\}$



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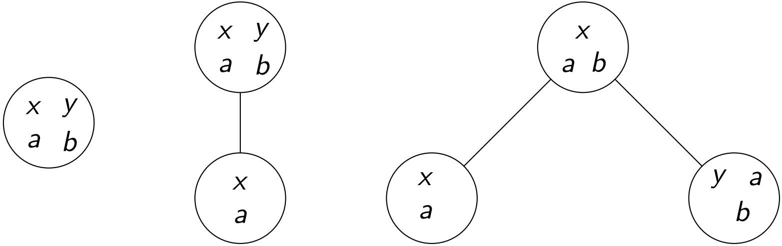

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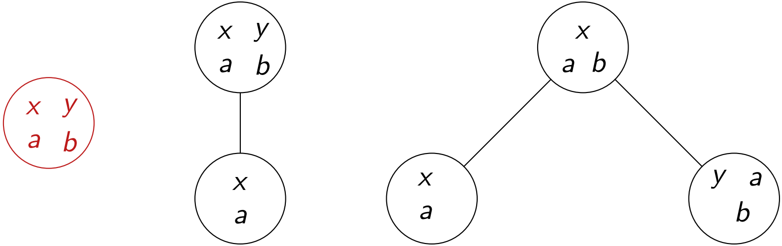

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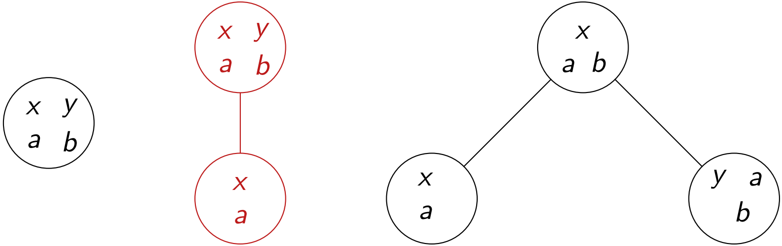

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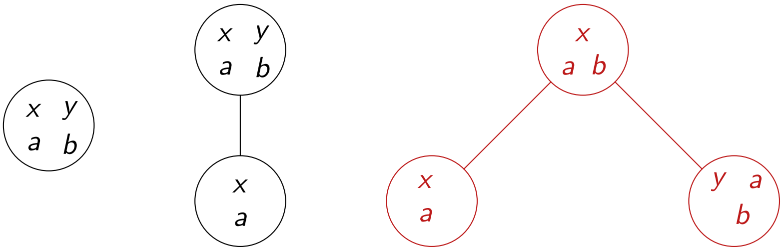

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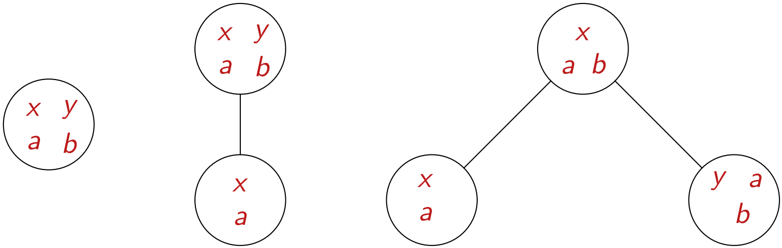

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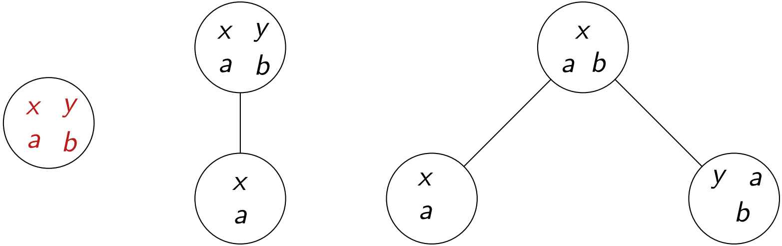

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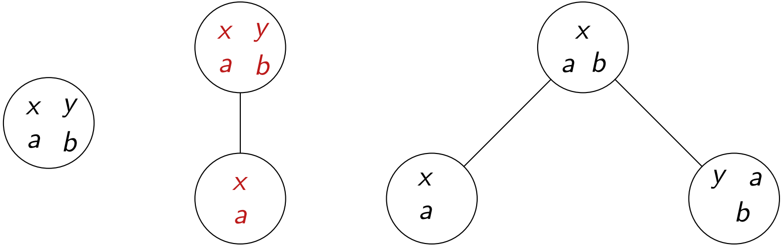

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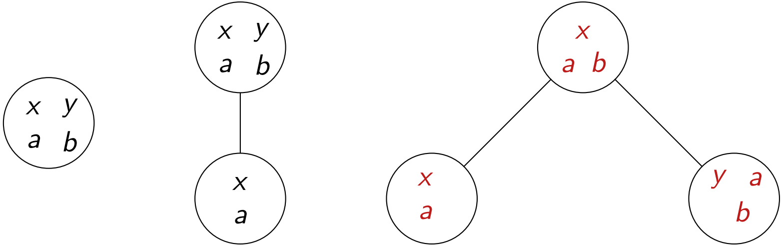

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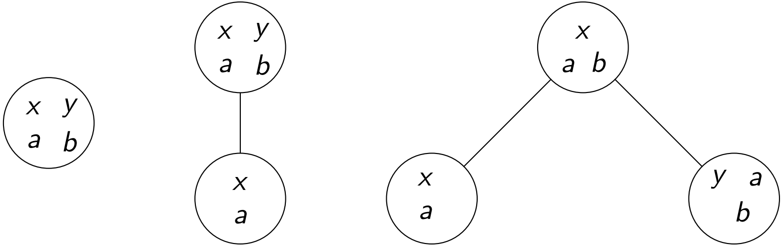

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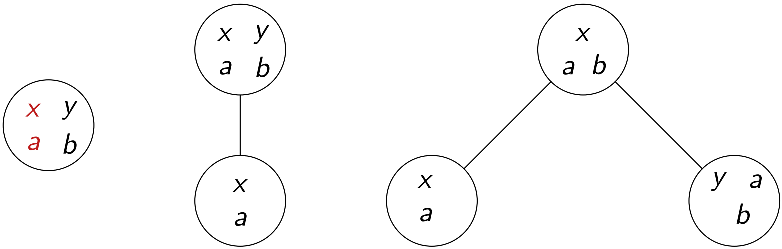

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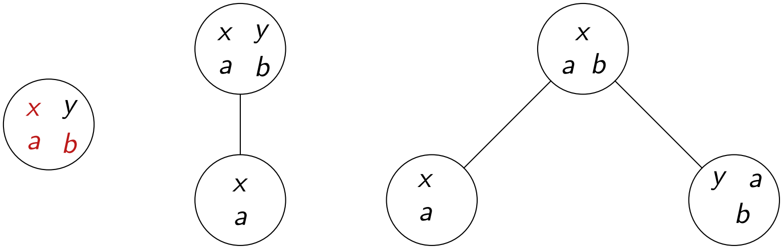

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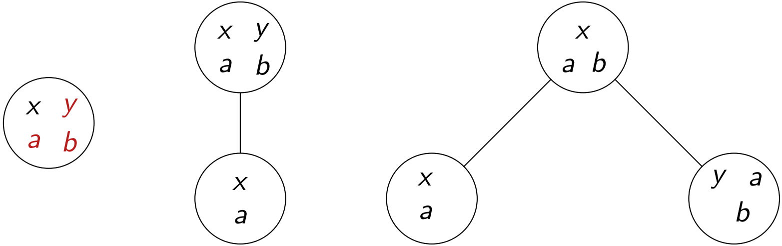

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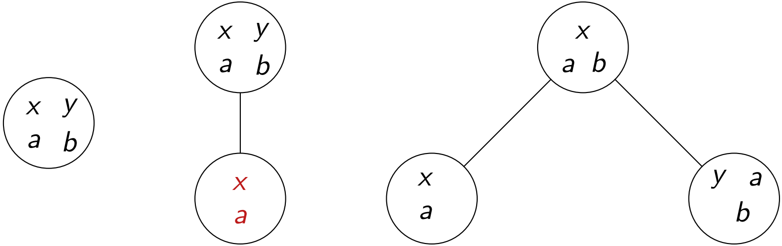

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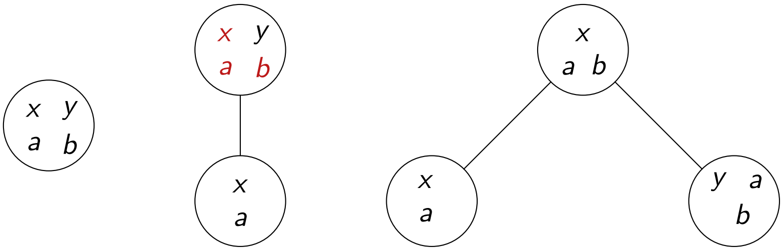

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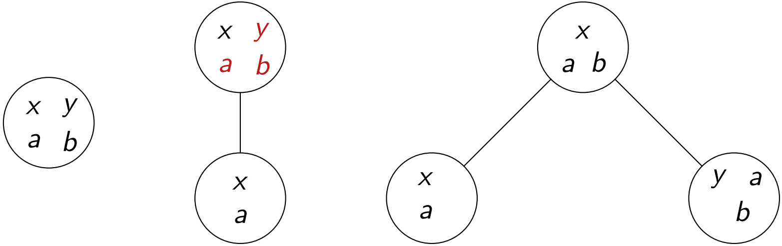

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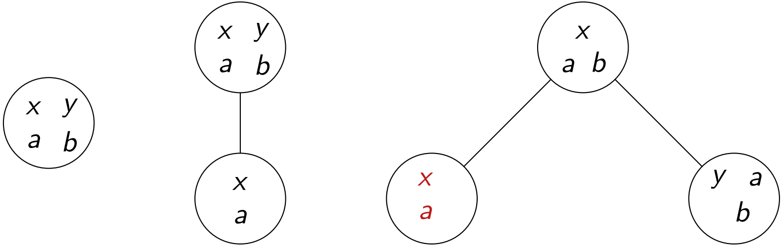

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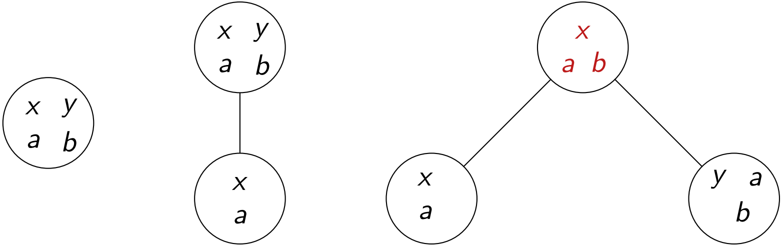

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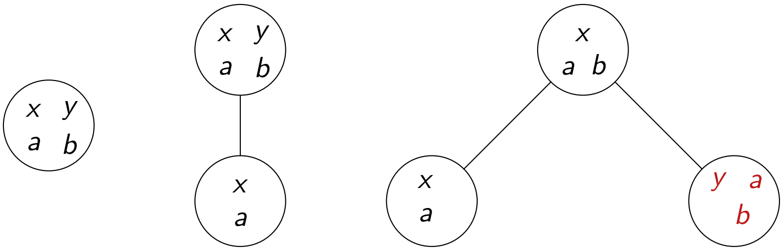

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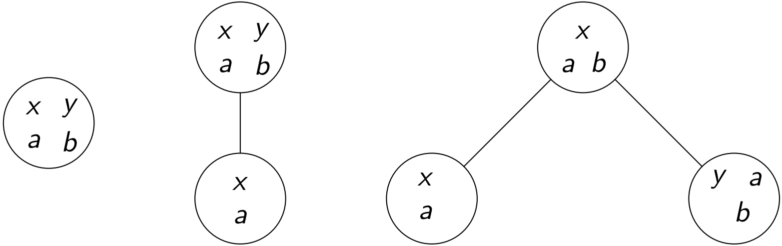

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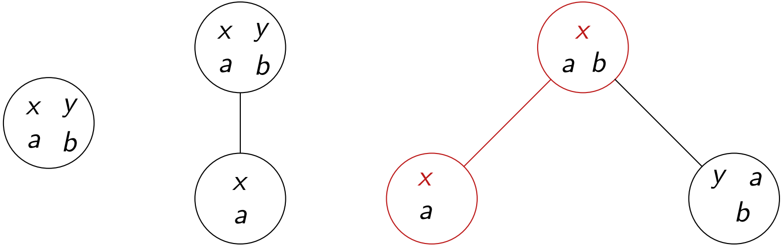

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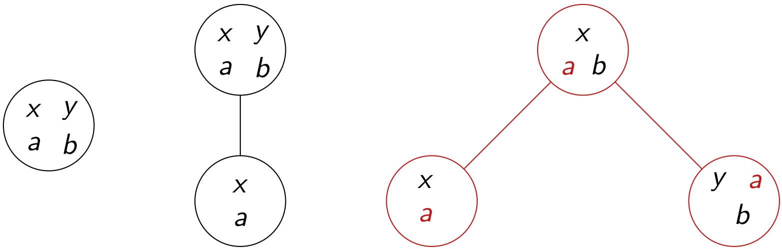

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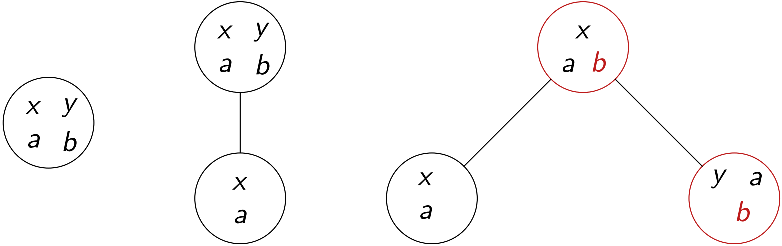

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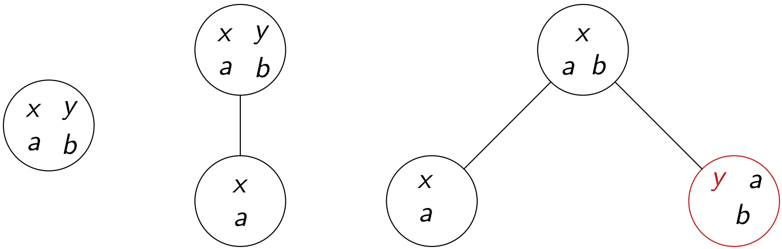

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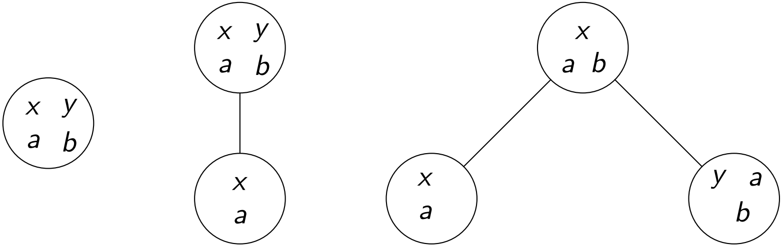

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Treewidth: Smallest width among all tree decompositions.

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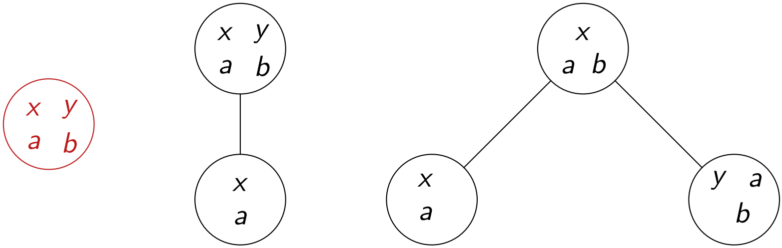

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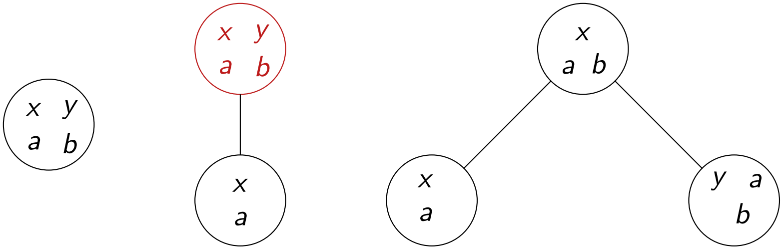

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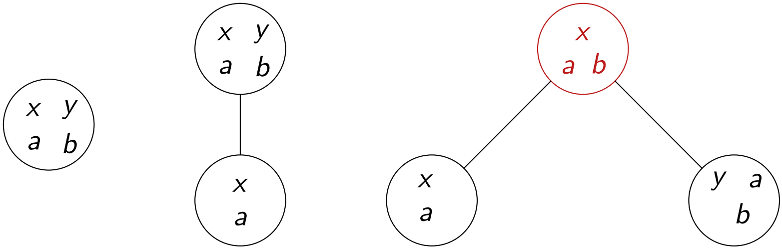

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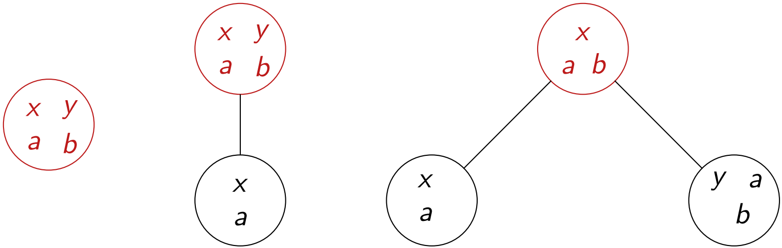

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The Importance of Treewidth

Theorem (Courcelle, 1990)

The first-order theory of a class of first-order structures with bounded treewidth is decidable.

Bounded Treewidth Sets: A rule set $\mathcal{R}$ is **bts** iff for any fact F, $(F, \mathcal{R})$ has a universal model of bounded treewidth.

Universal Model: A model for a FO theory that has a homomorphism into every other model.

Theorem (Baget et al. 2011)

If $\mathcal{R}$ is **bts**, *then query entailment is decidable.*

Alternative Formulation of BTS: **adbts**

All-Derivation Bounded Treewidth Sets: $\mathcal{R}$ is **adbts** iff for any F, there exists an $n \in \mathbb{N}$, if F' is derivable from F, then $tw(F') \leq n$.

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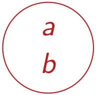

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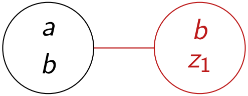

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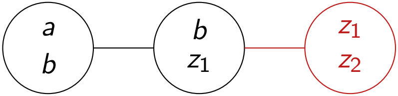

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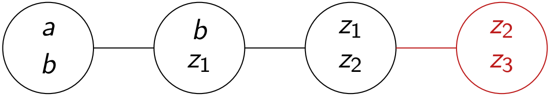

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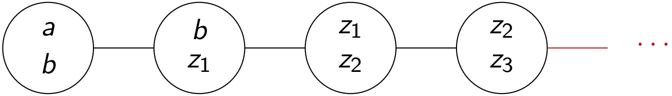

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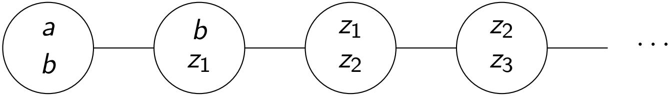

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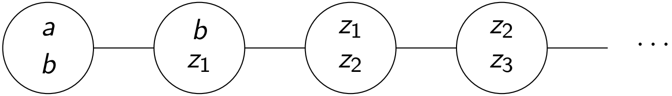

Finding: **adbts** is a *proper* subset of **bts**.

Theorem

adbts $\subsetneq$ **bts**

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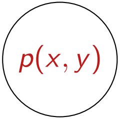

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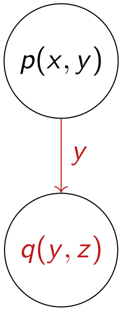

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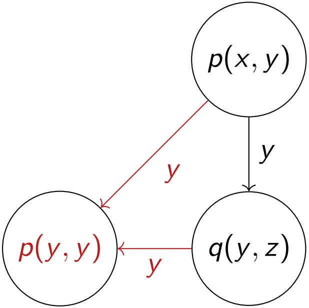

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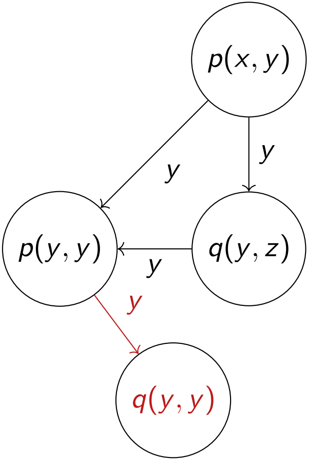

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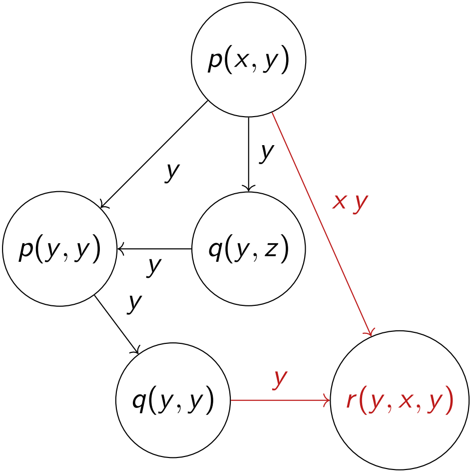
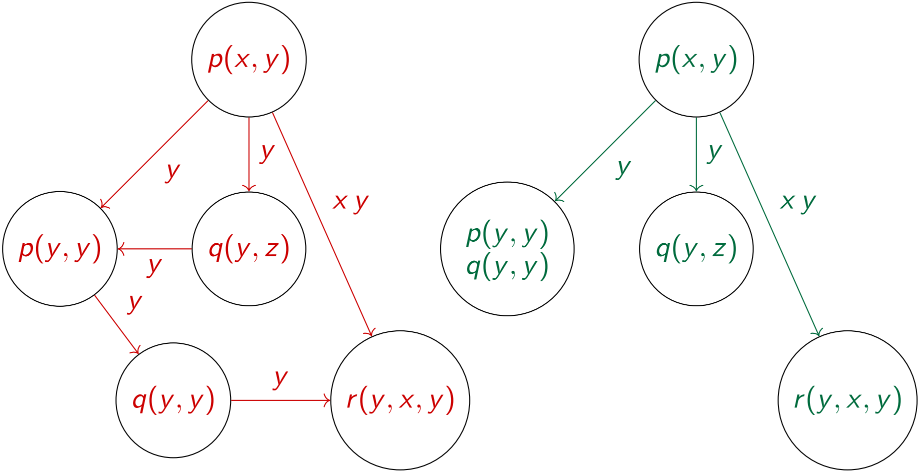


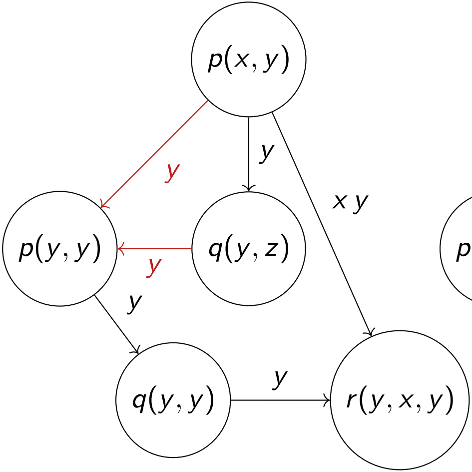
Derivation Graphs and Tree Decompositions

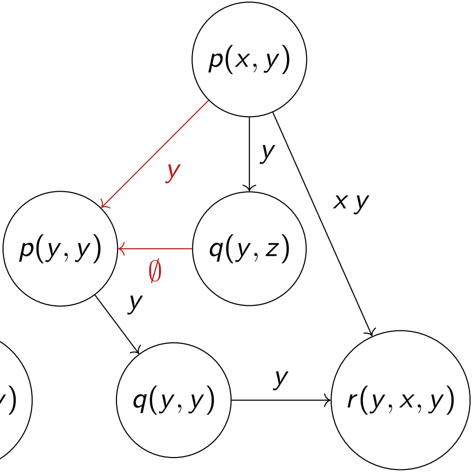
Idea (Baget et al. 2011): For a rule set $\mathcal{R}$, if derivation graph $\rightsquigarrow$ tree for any fact, then $\mathcal{R}$ is BTS.



Reduction Operations

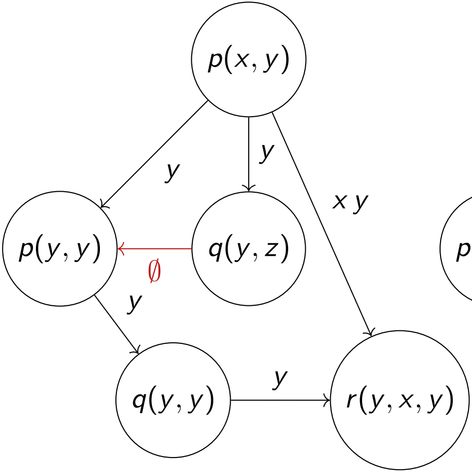
Term Removal: If a term labels two converging arcs, it may be removed from one.

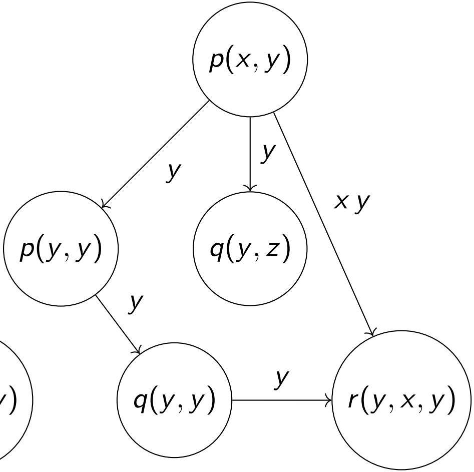




Reduction Operations

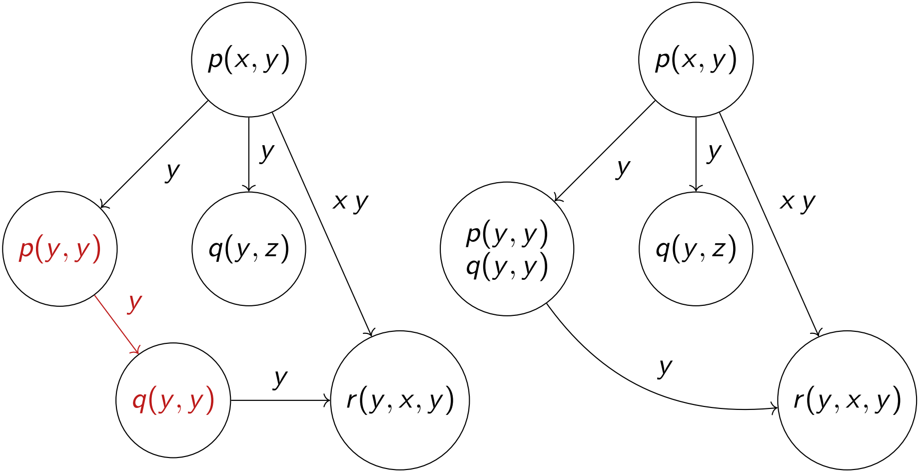
Arc Removal: If an arc is labelled with $\emptyset$, then it may be deleted.





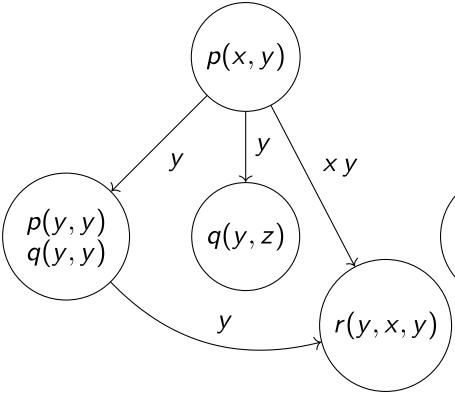
Reduction Operations

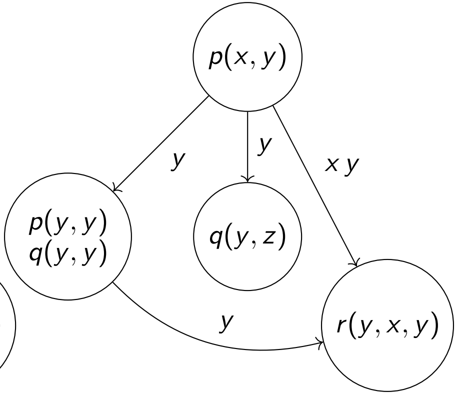
Arc Contraction: If a node X sees a node Y and $terms(Y) \subseteq terms(X)$, then they can be merged.



Reduction Operations

Cycle Removal: If there are arcs (X, Z) and (Y, Z) with an "ancestor" W such that $L(X, Z) \cup L(Y, Z) \subseteq terms(W)$, then (1) delete (X, Z), (Y, Z), (2) add (W, Z), and (3) let $L(W, Z) = Lab(X, Z) \cup L(Y, Z)$.





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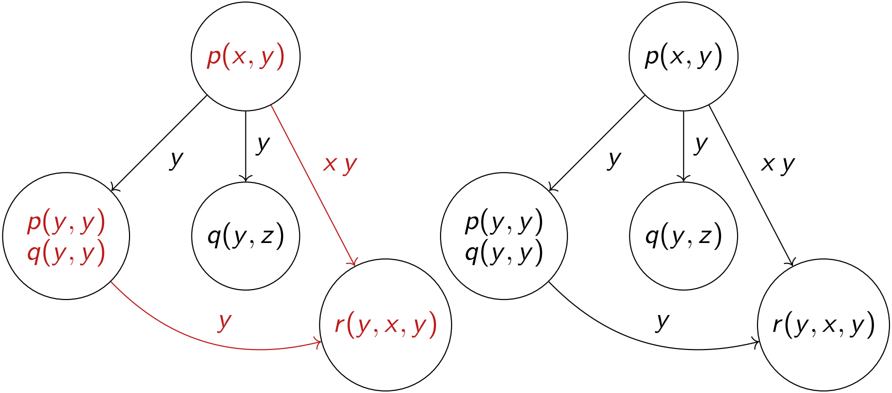

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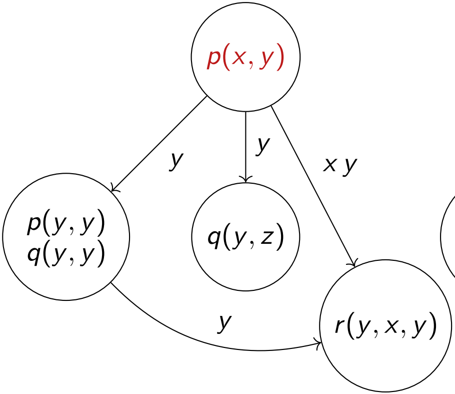

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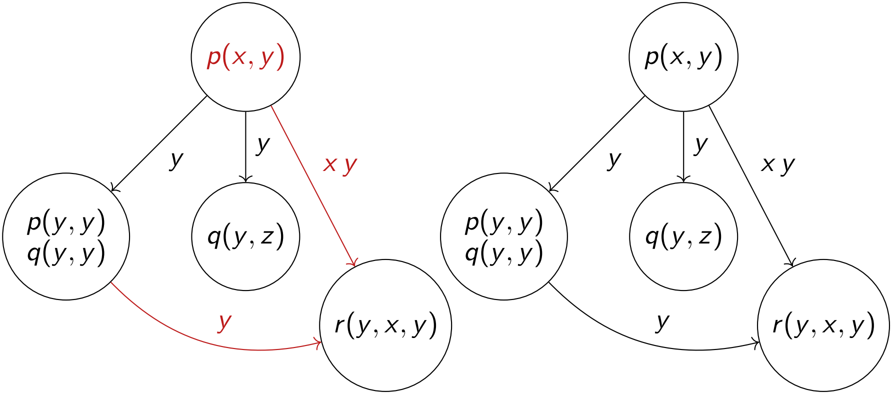

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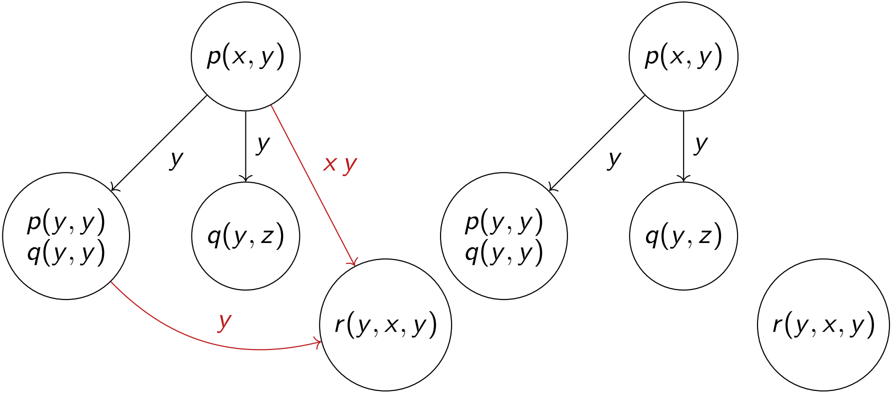

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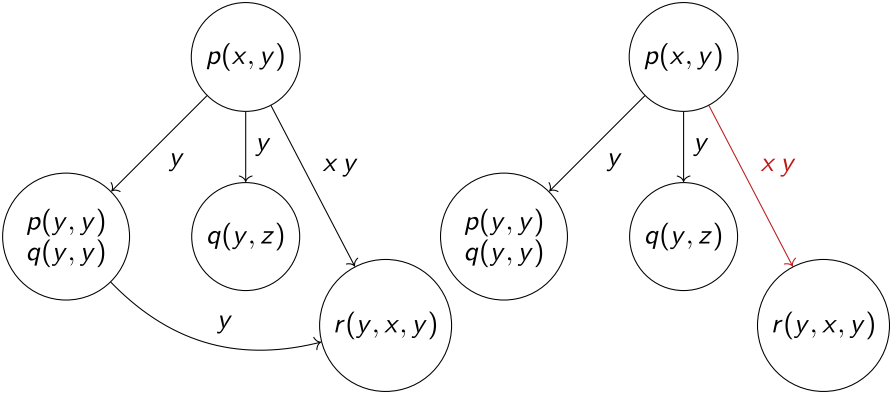

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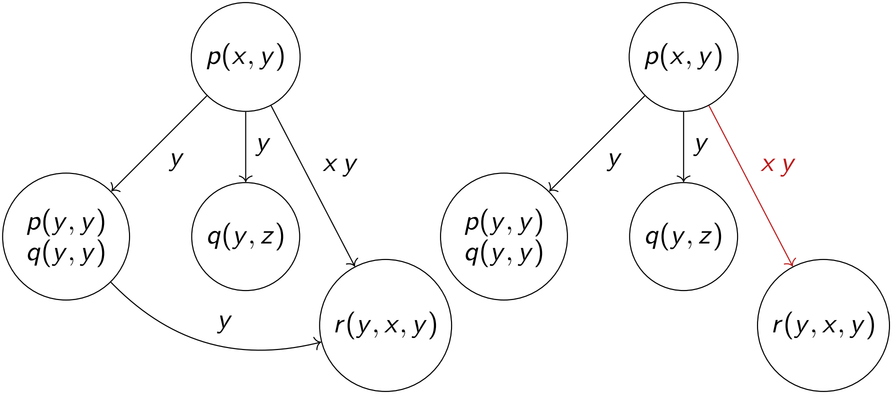

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Theorem (Baget et al. 2011)

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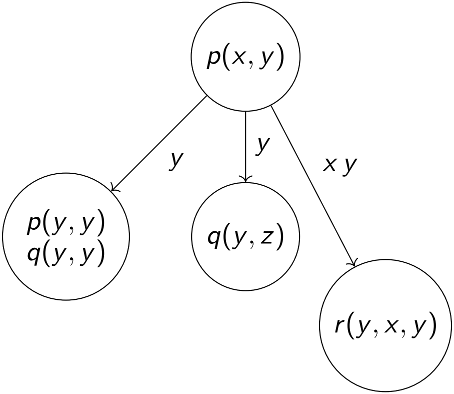

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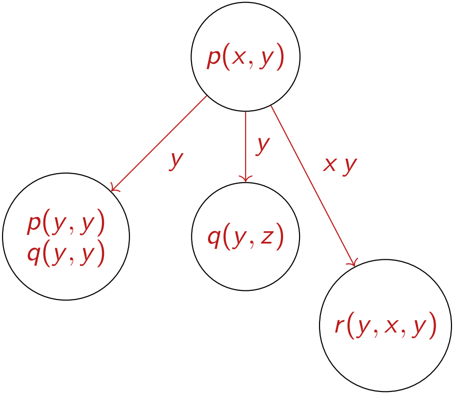

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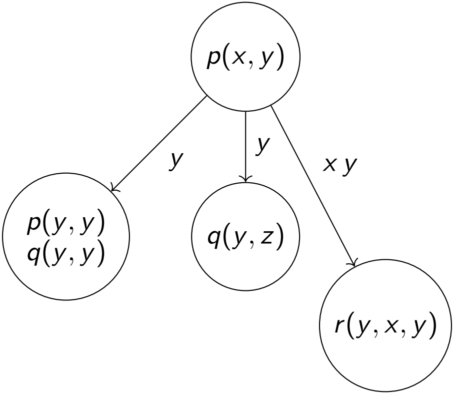

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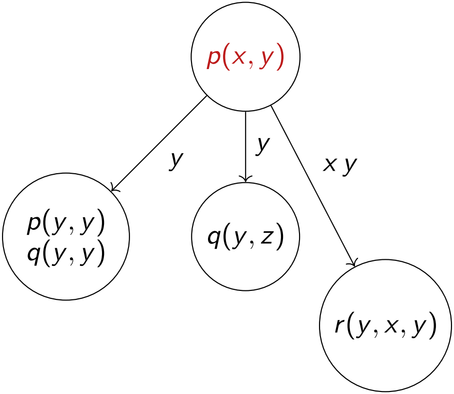

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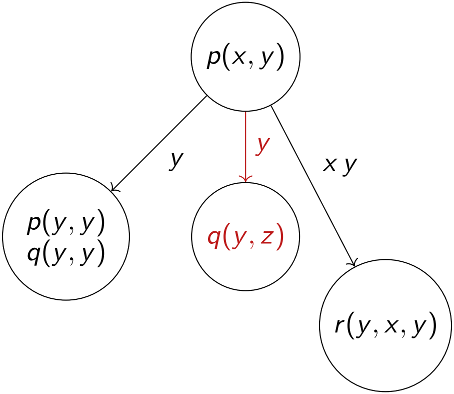

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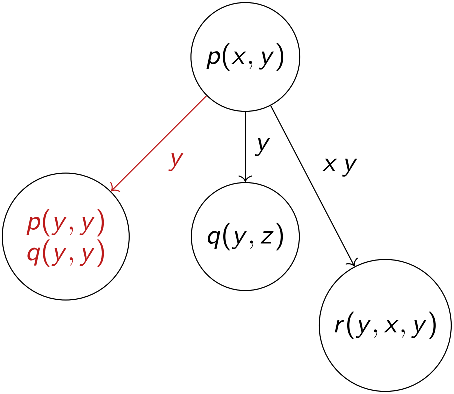

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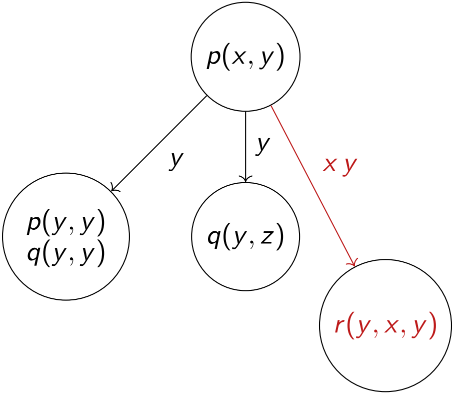

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Theorem

- **adgs** $\subset$ **wadgs** $\subseteq$ **adbts** $\subset$ **bts**
- *If $\mathcal{R}$ is* **adgs** *or* **wadgs***, then query entailment is decidable.*

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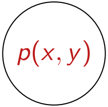

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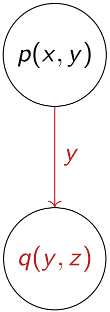

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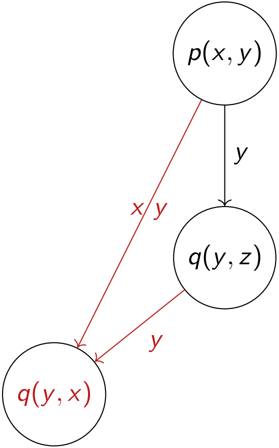

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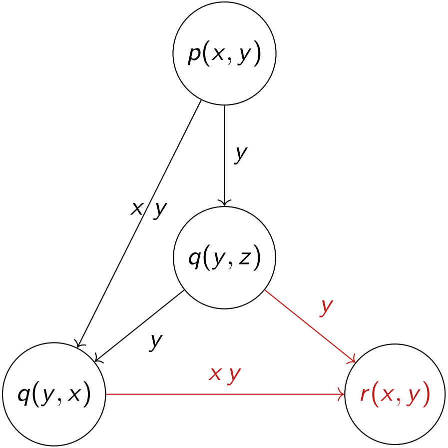

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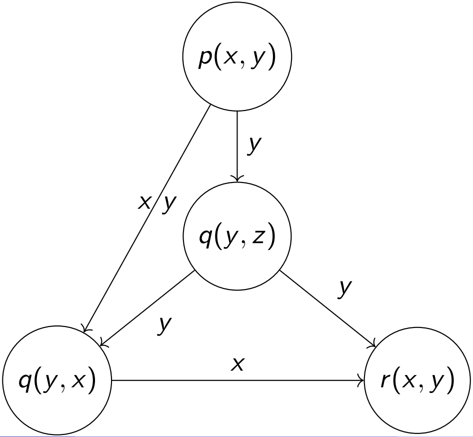
Cycle Removal: If there are arcs (X, Z) and (Y, Z) with an "ancestor" W such that $L(X, Z) \cup L(Y, Z) \subseteq \mathit{terms}(W)$, then (1) delete (X, Z), (Y, Z), (2) add (W, Z), and (3) let $L(W, Z) = \mathit{Lab}(X, Z) \cup L(Y, Z)$.



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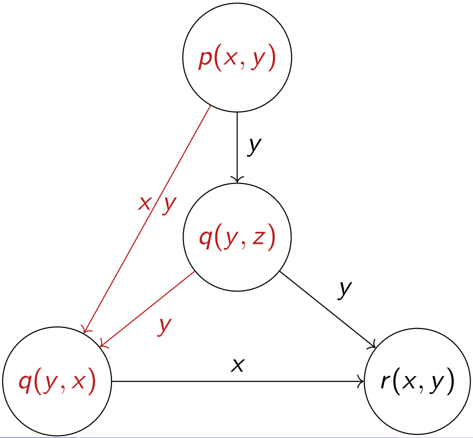

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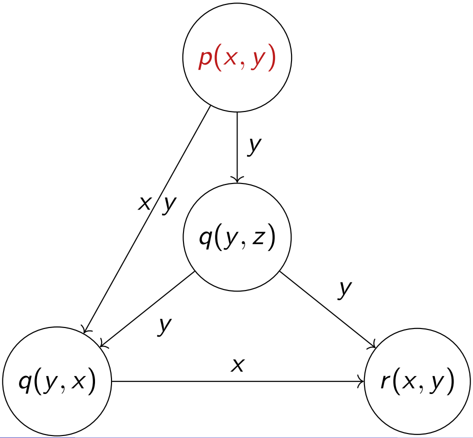

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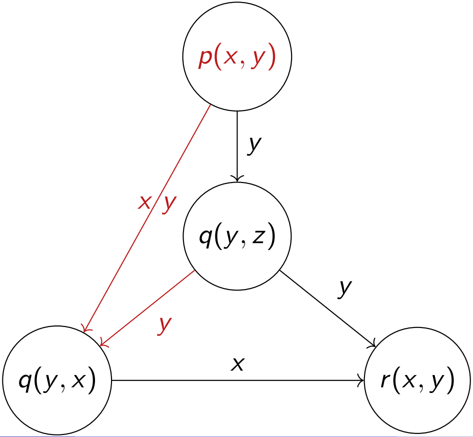

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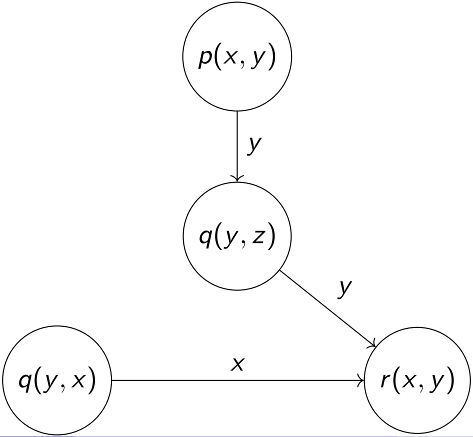

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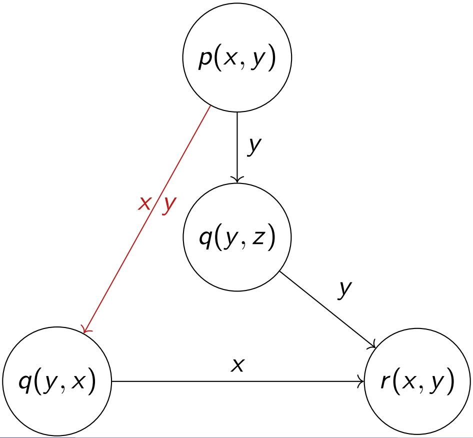

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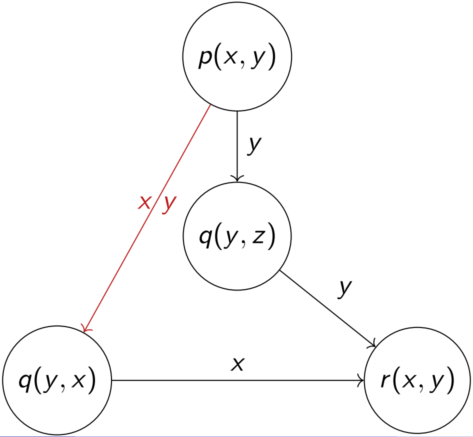

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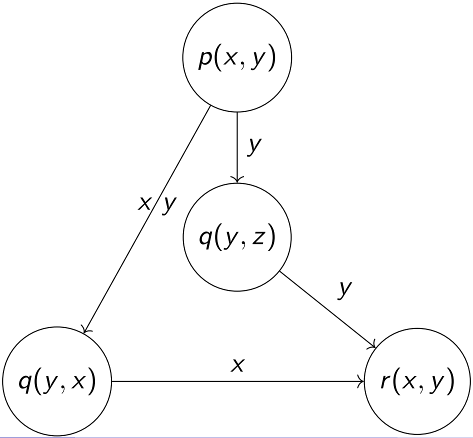

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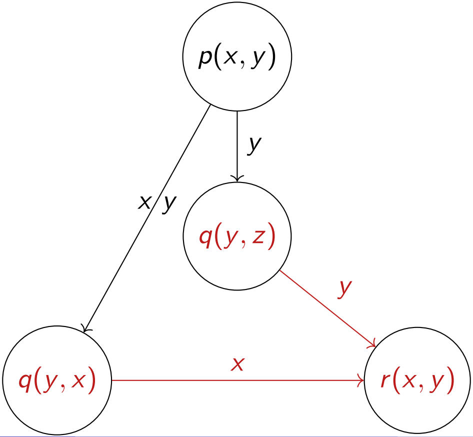

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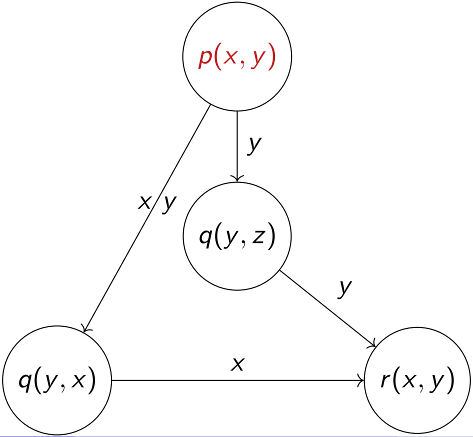

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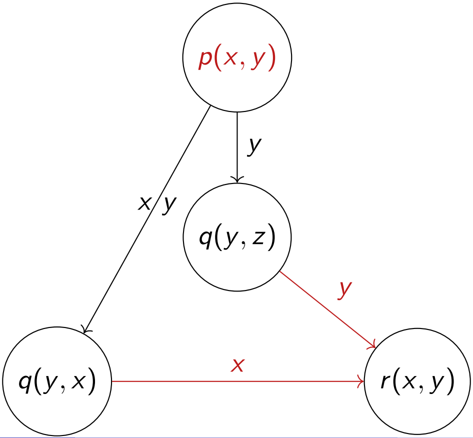

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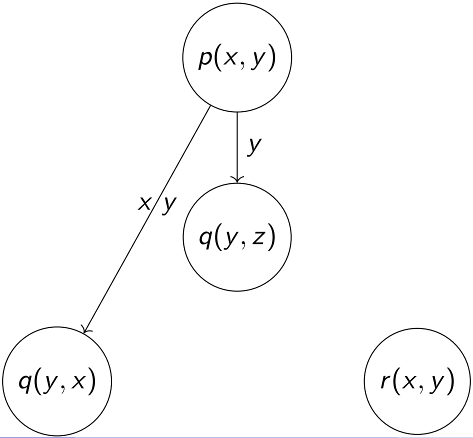

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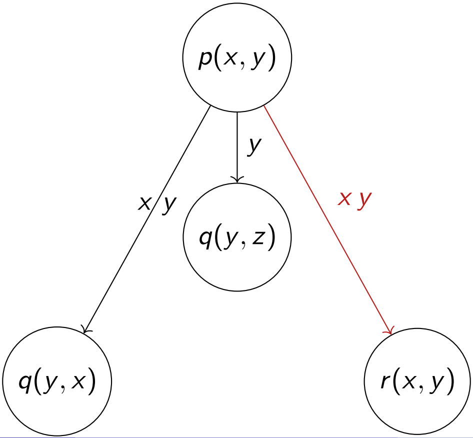

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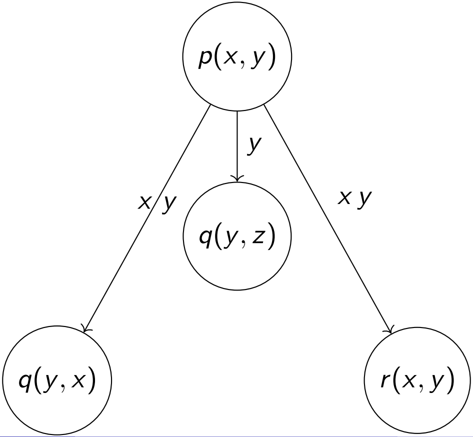

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Dependency (Baget 2004): A rule ρ' depends on a rule ρ, if there exists a fact F such that applying ρ to F gives a fact F' that ρ' can be applied to (and was not previously applicable to).

Example: Let $F = \{p(a, b)\}$

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Graph of Rule Dependencies (Baget 2004): A directed graph showing all dependencies between rules.

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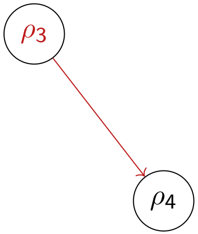

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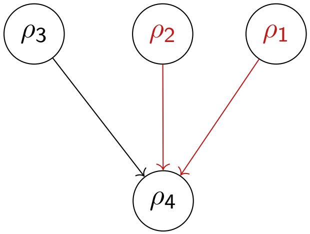

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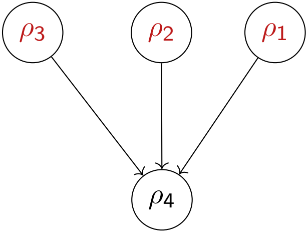

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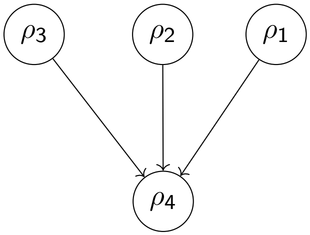

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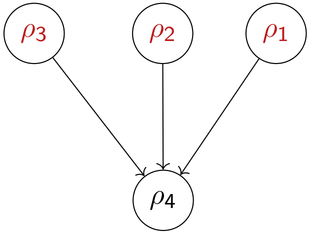

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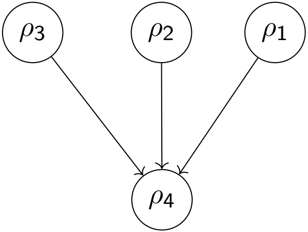

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The Observation: We can permute rule applications!

$$F_0, \ldots, (\rho, h_{i+1}, F_{i-1} \cup (F_{i+1} \setminus F_i)), (\rho_i, h_i, F_{i+1}), \ldots, (\rho_n, h_n, F_n)$$

Re-writing Derivations via Permutations

Lemma (Permutation Lemma)

Suppose we have a derivation of the following form:

$$\mathcal{D} := F_0, \ldots, (\rho_i, h_i, F_{i+1}), (\rho_{i+1}, h_{i+1}, F_{i+2}), \ldots, (\rho_{n-1}, h_{n-1}, F_n)$$

If ρ_{i+1} does not depend on ρ_i, then the following is a derivation as well:

$$\mathcal{D}' := F_0, \ldots, (\rho_{i+1}, h_{i+1}, F_i'), (\rho_i, h_i, F_{i+2}), \ldots, (\rho_{n-1}, h_{n-1}, F_n)$$

where $F_i' = F_i \cup (F_{i+2} - F_{i+1})$.

A Useful Rule Set for Non-Greediness

$\mathcal{R} = \{$

1. $p(x) \to q(x, y)$
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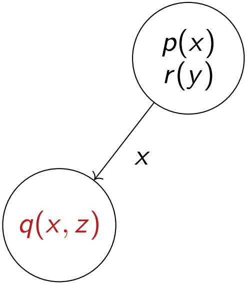

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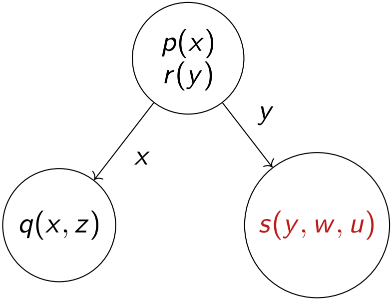

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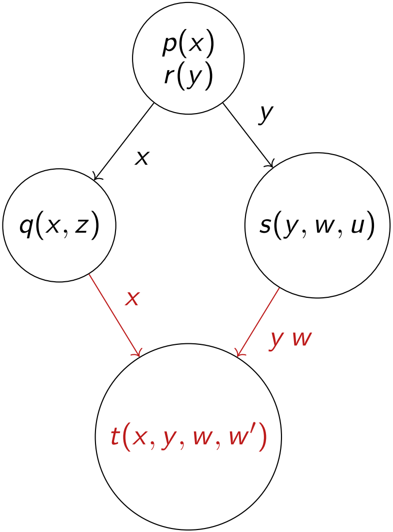

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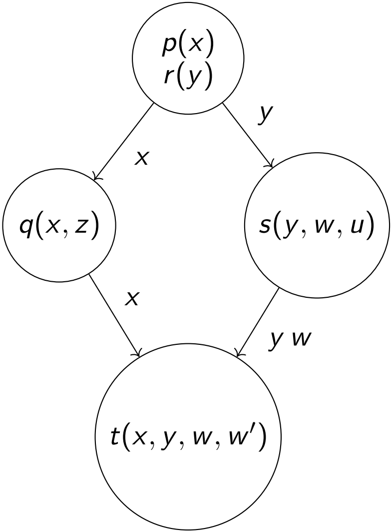

$\mathcal{R}$ is not greedy!

But is $\mathcal{R}$ Weakly Greedy?!

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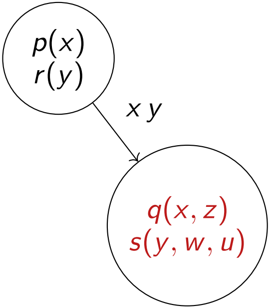

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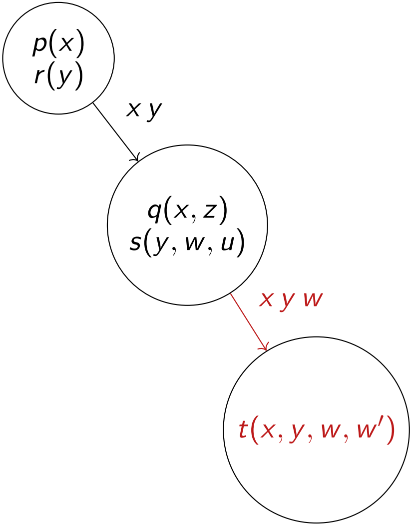

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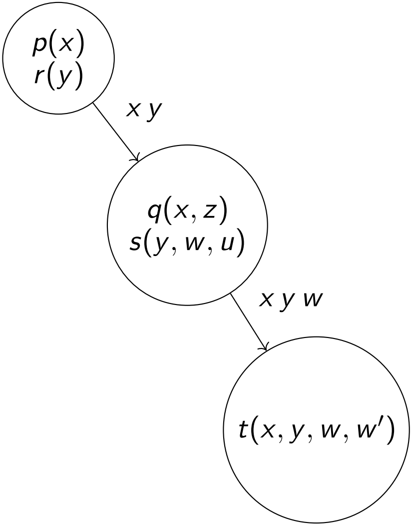

Perhaps $\mathcal{R}$ is weakly greedy!

Showing $\mathcal{R}$ is weakly greedy

Idea 1: Use 3 to simulate 1 and 2.

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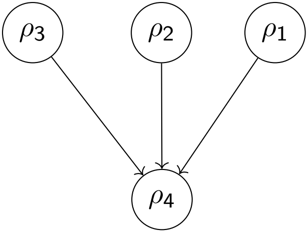

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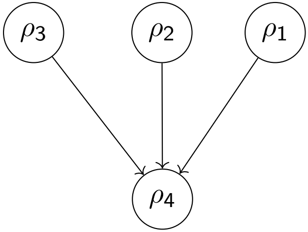

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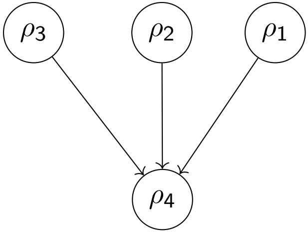

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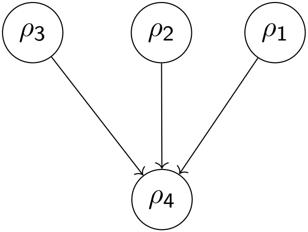

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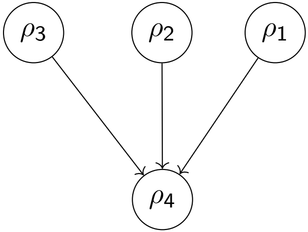
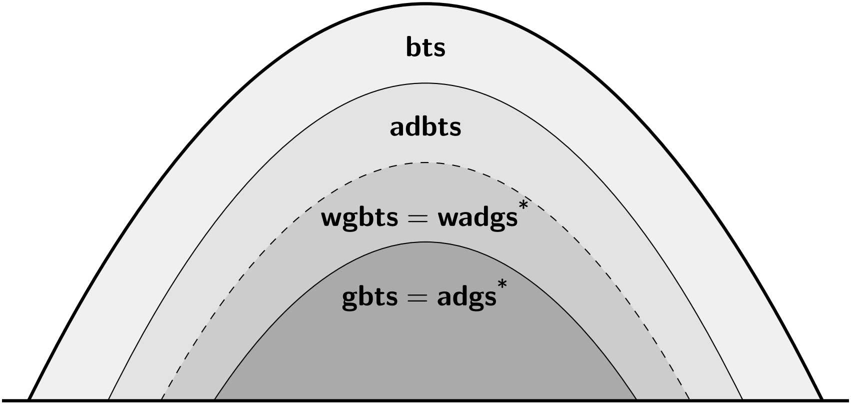


Idea 2: Permutation Lemma $\rightsquigarrow$

(i) Permute instances of 1 or 2 backward to instances of 2 or 1

(ii) Replace 1,2 or 2,1 instances with 3.

Conclusion



*Note: Proof must be checked in detail.