4 - Homology Inference

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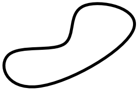
Homology Inference Problem



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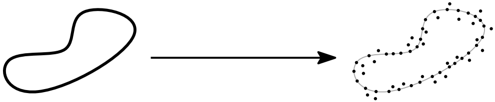

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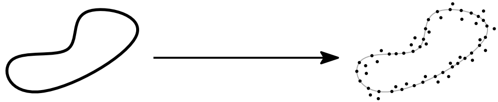

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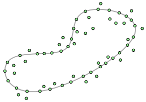

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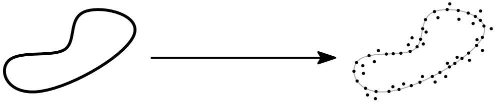

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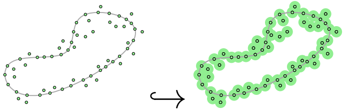

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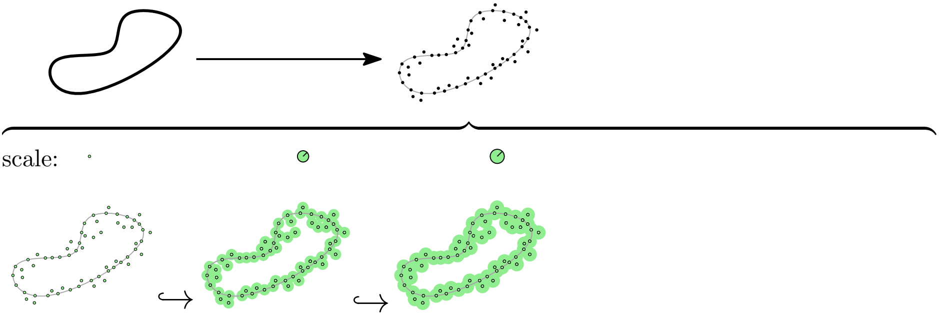

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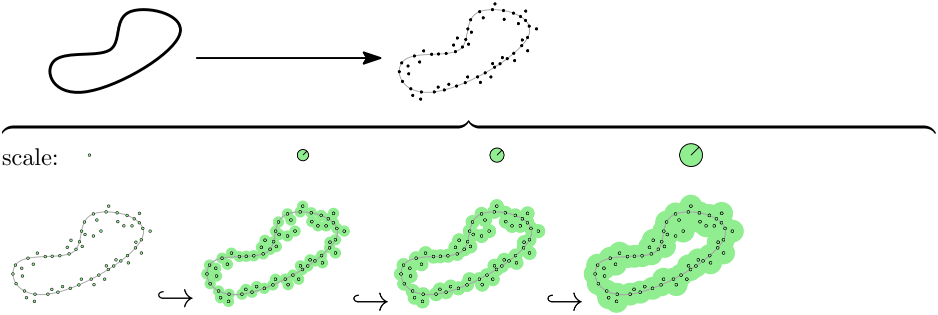

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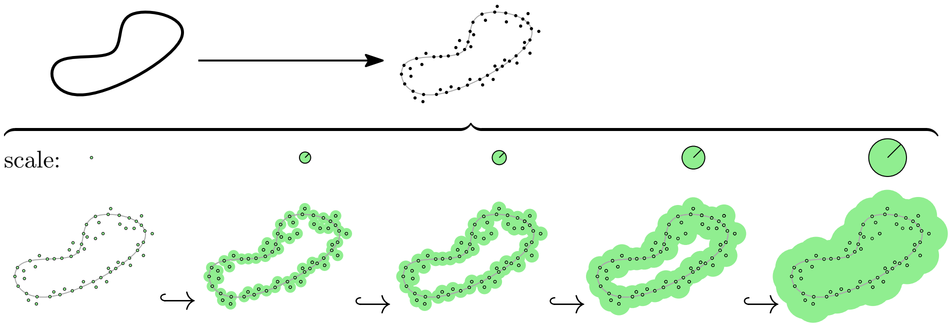

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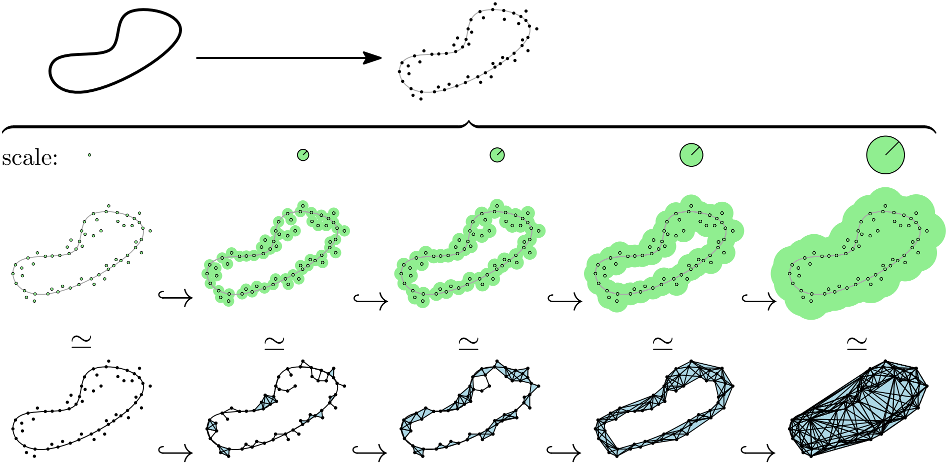

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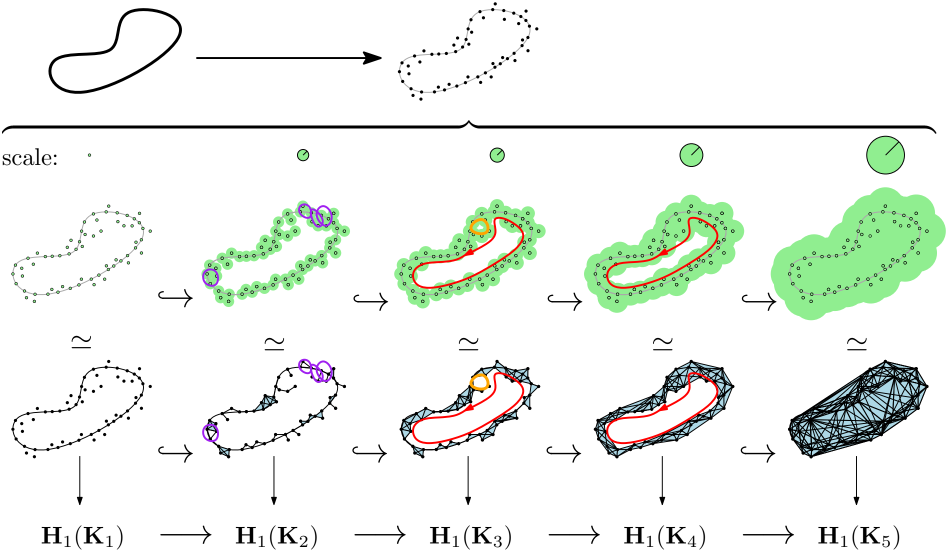

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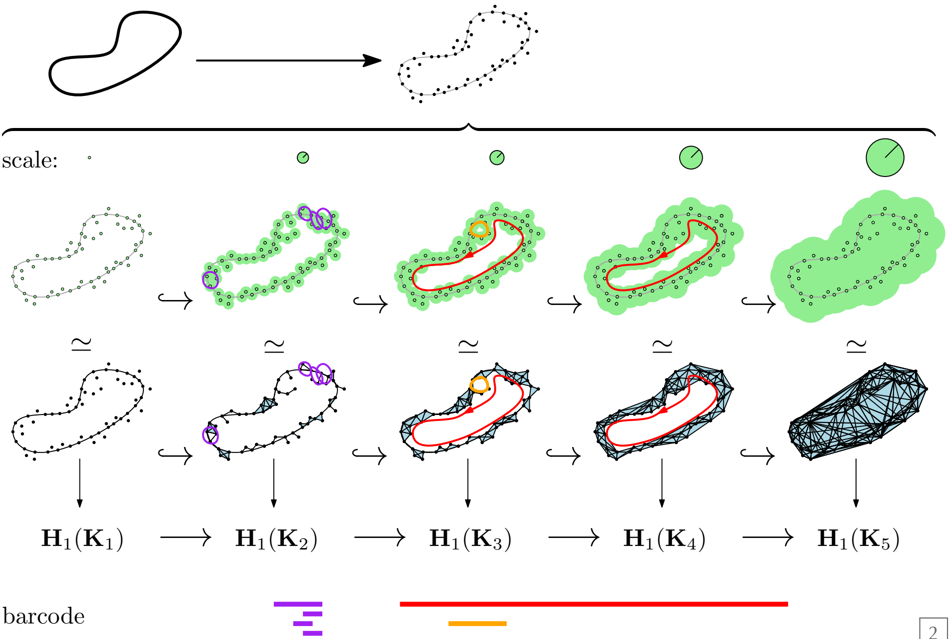
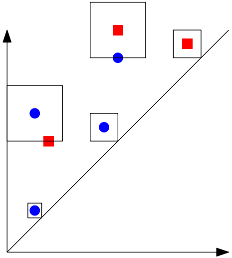


Stability Theorems

Let $\mathrm{D} = \{(b_i, d_i)\}_{i \in I} \cup \{x = y\}$ and $\mathrm{D}' = \{(b'_j, d'_j)\}_{j \in J} \cup \{x = y\}$ be two persistence diagrams, $b, d \in \mathbb{R}$.

The bottleneck distance $\mathrm{d}_B(\mathrm{D}, \mathrm{D}')$ between D and D′ is:

$$\mathrm{d}_B(\mathrm{D}, \mathrm{D}') := \inf_{\substack{\Phi:\mathrm{D}\to\mathrm{D}' \\ \text{bijection}}} \sup_{p \in \mathrm{D}} ||p - \Phi(p)||_\infty$$



Defined even when $|I| \neq |J|$ by sending points to the diagonal.

Theorem (Stability on a simplicial complex)

Let $f, g \colon \mathbf{K} \to \mathbb{R}$ be two functions on a same simplicial complex $\mathbf{K}$, inducing filtrations $\mathbf{K}_\alpha = f^{-1}((-\infty; \alpha])$ and $\mathbf{K}'_\gamma = g^{-1}((-\infty; \gamma])$. Then

$$\mathrm{d}_B(\mathrm{D}(f), \mathrm{D}(g)) \leq ||f - g||_\infty.$$

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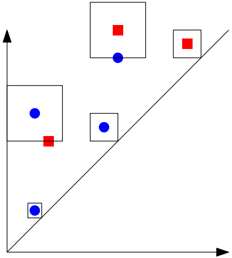


Theorem (Stability on general space)

Let $f, g\colon M \to \mathbb{R}$ be two functions on a same metric space M, satisfying some "tameness" conditions. If $\exists \varepsilon \geq 0$ s.t. $\forall r \in \mathbb{R}$
$f^{-1}(-\infty; r] \subseteq g^{-1}(-\infty; r + \varepsilon]$ and $g^{-1}(-\infty; r] \subseteq f^{-1}(-\infty; r + \varepsilon]$,

then $\mathrm{d}_B(\mathrm{D}(f), \mathrm{D}(g)) \leq \varepsilon$.

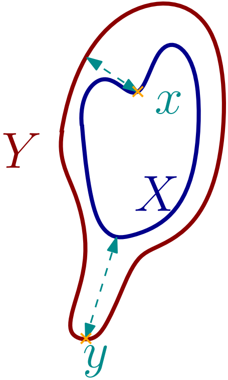
Distance between spaces

Definition (Hausdorff distance)

The Hausdorff distance between two non-empty subsets X, Y of a metric space (M, d) is:

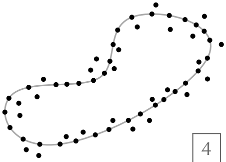
$$\mathrm{d}_H(X, Y) := \max \left\{ \sup_{x \in X} \mathrm{d}(x, Y), \sup_{y \in Y} \mathrm{d}(X, y) \right\},$$

where,
$\mathrm{d}(x, Y) = \inf_{y \in Y} \mathrm{d}(x, y)$ and $\mathrm{d}(X, y) = \inf_{x \in X} \mathrm{d}(x, y)$.



Definition (ε-sample)

Let $K \subset \mathbb{R}^D$ be a compact set. An ε-sample of K, for some $\varepsilon \geq 0$, is a finite set of points P such that $\mathrm{d}_H(P, K) \leq \varepsilon$.



Sampling compacts

Now consider compacts K in Euclidean space $\mathbb{R}^D$. Any compact K defines a function:

$$d_K \colon \mathbb{R}^D \to \mathbb{R}, \quad x \mapsto d_K(x) = d(x, K).$$

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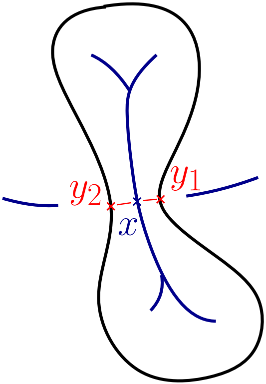
Definition (Critical point)

A point $x \in \mathbb{R}^D$ is a critical point for d_K if there exist distinct points $y_1, y_2 \in \mathbf{K}$ such that:

$$d(x, y_1) = d(x, y_2) = d(x, K).$$

The set of critical points is called the medial axis of K.



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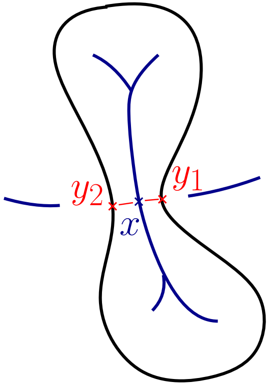
Definition (Reach of a compact)

The reach of a compact K is:

$$\mathrm{reach}\,\mathbf{K} := \inf\{\mathrm{d}_K(x) : x \text{ critical point for } \mathrm{d}_K\}.$$



Sampling, distance function approach

Lemma

Let P be an ε-sample of a compact K. Then,

$$||\mathrm{d}_K - \mathrm{d}_P||_\infty \leq \varepsilon.$$

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Consequently, by triangle inequality:

$$\mathrm{d}_P(x) \leq \mathrm{d}(x, p_0) \leq \mathrm{d}(x, y_0) + \mathrm{d}(y_0, p_0) \leq \mathrm{d}_K(x) + \varepsilon.$$

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And vice versa, exchanging K and P.

Reconstruction theorem

Theorem (Reconstruction)

Let $K \subset \mathbb{R}^D$ be a compact set, and $\varepsilon > 0$ be such that:

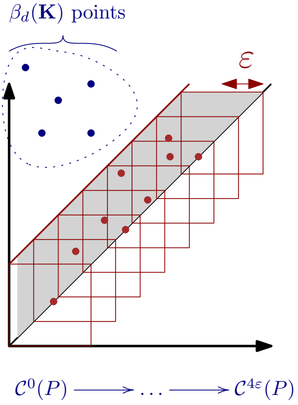
$$0 < 4\varepsilon < \text{reach}\,\mathbf{K}.$$

Let P be an ε-sample of K. Then,

- *there is a one-to-one correspondence between the points of the persistence diagram of the Čech filtration:*

$$\check{\mathcal{C}}^0(P) \longrightarrow \ldots \longrightarrow \check{\mathcal{C}}^{4\varepsilon}(P)$$

  that are at least ε-away from the diagonal Δ in ∞ norm, and the homology features of K.



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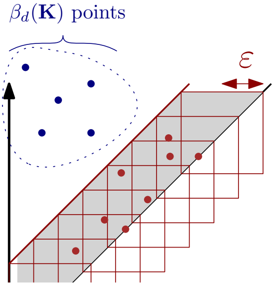

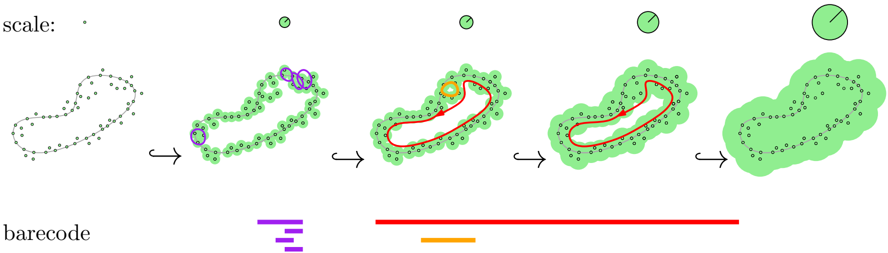

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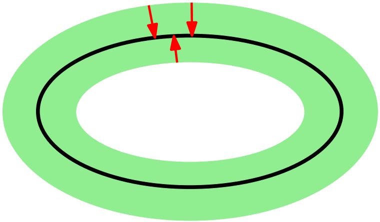
$\longrightarrow$ by projecting each $x \in d_K^{-1}(-\infty; r] \setminus K$ to its unique nearest neighbor on K.



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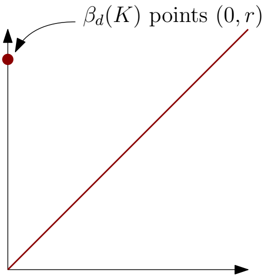
Consequently, for all $0 < r_1 \leq r_2 < \mathrm{reach}(K)$ we have

$$\mathbf{H}(d_K^{-1}(-\infty; r_1]) \xrightarrow{\text{id}} \mathbf{H}(d_K^{-1}(-\infty; r_2])$$



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(2) Consider the persistent homology of

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Apply the (continuous version of the) stability theorem:

$$\mathrm{d}_B(\mathrm{D}(\mathrm{d}_k), \mathrm{D}(\check{\mathcal{C}}(P))) \leq \varepsilon.$$

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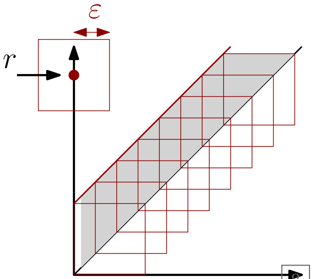
For $\mathrm{reach}(K) > r > 4\varepsilon > 0$, the radius r $||\cdot||_\infty$-ball in $\mathbb{R}^2$ around $(0, r)$, and the $||\cdot||_\infty$-band of width ε around the diagonal are disjoint.



Rips complex

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Definition (Flag complex)

Let $G = (V, E)$ be a graph. The flag complex induced by G is the abstract simplicial complex $\mathcal{F}(G)$ satisfying:

- the vertices of $\mathcal{F}(G)$ are V,
- $\sigma = \{v_0, \ldots, v_d\} \in \mathcal{F}(G)$ iff $v_0, \ldots, v_d$ form a clique in G (i.e., an induced complete subgraph of G).

It is the maximal simplicial complex with G as 1-skeleton.

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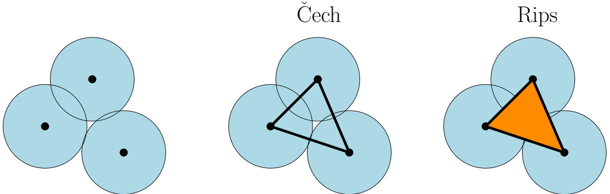
Computing intersections of balls in $\mathbb{R}^D$ is *expensive* (high arithmetic complexity): Approximate the Čech complex instead.

Definition (Rips complex)

Let $P = \{x_1, \ldots, x_n\}$ be a set of points in $\mathbb{R}^N$. The neighbor graph of threshold r of P is the graph $N_r(P)$ with:

- vertices the points in P, and
- any two points x_i, x_j, $i \neq j$ are connected by an edge iff $||x_i - x_j|| \leq r$.

The Rips complex of threshold r, denoted by $\mathcal{R}^r(P)$, is the flag complex $\mathcal{F}(N_{2r}(P))$.



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Computation: Simply compute pairwise distances (low arithmetic complexity predicate), then expand the flag complex combinatorial

$$\sim O(\#\text{points}^2) + \text{ linear in the number of cliques/size of the complex.}$$

Rips vs Čech

Lemma

For a point cloud P in Euclidean space, and any $r > 0$:

$$\check{\mathcal{C}}^r(P) \subseteq \mathcal{R}^r(P) \subseteq \check{\mathcal{C}}^{2r}(P).$$

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$\boxed{\check{\mathcal{C}}^r(P) \subseteq \mathcal{R}^r(P)}$. Let $\sigma = \{x_0, \ldots, x_k\} \in \check{\mathcal{C}}^r(P)$, $d \geq 1$, then for any $i \neq j$, $0 \leq i, j \leq k$ we have $\mathcal{B}_r(x_i) \cap \mathcal{B}_r(x_j) \neq \emptyset \Rightarrow ||x_i - x_j|| \leq 2r$. Hence $x_0, \ldots, x_k$ form a clique in $N_{2r}(P)$ and $\sigma \in \mathcal{R}^r(P)$.

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Consequently, for all i:

$$||z - x_i|| \leq \mathrm{diam}(CH(x_0, \ldots, x_k)) \leq \max_{i,j} ||x_i - x_j|| \leq 2r.$$

In conclusion, $z \in \bigcap_{i=0\ldots k} \mathcal{B}_{2r}(x_i) \neq \emptyset$ and $\sigma \in \check{\mathcal{C}}^{2r}(P)$.

Ranks in sequence of morphisms

Lemma

Consider the following sequence of vector spaces and morphisms:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E \xrightarrow{\eta} F$$

such that $\quad \mathrm{rk}\left(A \xrightarrow{\eta\circ\ldots\circ\alpha} F \right) = \mathrm{rk}\left(C \xrightarrow{\gamma} D \right) = k, \quad$ *then:*

$$\mathrm{rk}\left(B \xrightarrow{\delta\circ\gamma\circ\beta} E \right) = k.$$

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such that $\operatorname{rk}\left(A \xrightarrow{\eta\circ\ldots\circ\alpha} F \right) = \operatorname{rk}\left(C \xrightarrow{\gamma} D \right) = k,$ *then:*

$$\operatorname{rk}\left(B \xrightarrow{\delta\circ\gamma\circ\beta} E \right) = k.$$

Note that in any $U \to V \to W \to X$, $\operatorname{rk}\left(U \to X \right) \leq \operatorname{rk}\left(V \to W \right)$.

Ranks in sequence of morphisms

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Cech persistence, a closer look

Lemma

Let P be an ε-sample of a compact K of $\mathrm{reach}(K) > 0$ in $\mathbb{R}^N$. Then, for any $\varepsilon_0 > \varepsilon$, and α, such that $\varepsilon \leq \alpha < \mathrm{reach}(K) - 2\varepsilon_0$, we have:

$$\mathrm{rk}\left(\mathbf{H}_d(\check{\mathcal{C}}^{\alpha}(P)) \longrightarrow \mathbf{H}_d(\check{\mathcal{C}}^{\alpha+\varepsilon_0}(P)) \right) = \dim \mathbf{H}_d(K)$$

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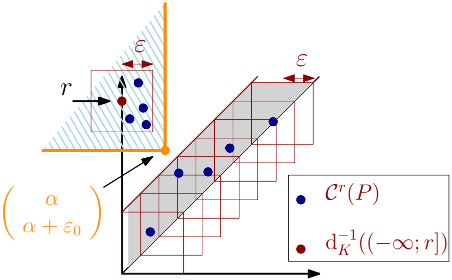

Theorem

Let P be an ε-sample of a compact K s.t. $0 < \varepsilon < \frac{\mathrm{reach}(K)}{8}$. The homology of K can be read off the persistence diagram of $\mathcal{R}^r(P)$ for $r \geq \varepsilon + 2\varepsilon_0$, for any $\varepsilon < \varepsilon_0 < \frac{\mathrm{reach}(K)}{8}$.

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$$\check{\mathcal{C}}^{\varepsilon} \longrightarrow \mathcal{R}^{\varepsilon} \longrightarrow \underbrace{\check{\mathcal{C}}^{\varepsilon+\varepsilon_0}}_{\varepsilon+\varepsilon_0>2\varepsilon} \longrightarrow \check{\mathcal{C}}^{\varepsilon+2\varepsilon_0} \longrightarrow \mathcal{R}^{\varepsilon+2\varepsilon_0} \longrightarrow \check{\mathcal{C}}^{2\varepsilon+4\varepsilon_0}$$

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Now, with $2\varepsilon + 4\varepsilon_0 < \mathrm{reach}(K)$,

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In conclusion, $\dim \mathbf{H}_d(K)$ is exactly the number of points in the persistence diagram of the Rips filtration (up to $r = \varepsilon + 2\varepsilon_0$) in the upper left quadrant based at $(\varepsilon, \varepsilon + 2\varepsilon_0)$.

Morse filtration

Definition

Let $\mathbf{K}$ be a complex, equipped with a filtration,

$$\emptyset := \mathbf{K}_0 \xrightarrow{\mathfrak{S}_1} \mathbf{K}_1 \xrightarrow{\mathfrak{S}_2} \mathbf{K}_2 \xrightarrow{\mathfrak{S}_3} \dots \xrightarrow{\mathfrak{S}_m} \mathbf{K}_m = \mathbf{K}$$

where $\mathfrak{S}_i$ contains at least one simplex.
A Morse filtration for the filtration of $\mathbf{K}$ is a collection of Morse matchings $(X_i \sqcup T_i \sqcup S_i, \omega_i \colon T_i \to S_i)$ for each $\mathbf{K}_i$, $i = 0 \dots m$, satisfying:

$$X_i \subseteq X_{i+1}, \qquad T_i \subseteq T_{i+1}, \qquad S_i \subseteq S_{i+1},$$

$$\omega_{i+1}\big|_{T_i} = \omega_i, \qquad \partial^{X_{i+1}}\big|_{X_i} = \partial^{X_i}, \qquad \forall i = 0 \dots m-1.$$

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and a Morse filtration $(X_i \sqcup T_i \sqcup S_i, \omega_i \colon T_i \to S_i)$, $i = 0 \ldots m$:

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$$\omega_{i+1}\big|_{T_i} = \omega_i, \qquad \partial^{X_{i+1}}\Big|_{X_i} = \partial^{X_i}, \qquad \forall i = 0 \ldots m-1.$$

Then the following persistence modules:

$$\mathbf{H}(\mathbf{K}_0, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_1, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_2, \partial) \xrightarrow{\subseteq} \ldots \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_m, \partial)$$
$$\mathbf{H}(X_0, \partial^{X_0}) \xrightarrow{\subseteq} \mathbf{H}(X_1, \partial^{X_1}) \xrightarrow{\subseteq} \mathbf{H}(X_2, \partial^{X_2}) \xrightarrow{\subseteq} \ldots \xrightarrow{\subseteq} \mathbf{H}(X_m, \partial^{X_m})$$

are isomorphic.

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The following persistence modules are isomorphic:

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Sketch of proof:

$$X_i \subseteq X_{i+1}, \quad T_i \subseteq T_{i+1}, \quad S_i \subseteq S_{i+1}, \quad \omega_{i+1}\big|_{T_i} = \omega_i, \quad \partial^{X_{i+1}}\Big|_{X_i} = \partial^{X_i},$$

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implies that:

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is well-defined filtration of complexes inducing the persist. module:

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Sketch of proof:

For each i there is a chain equivalence:

$$(\mathbf{C}(\mathbf{K}_i), \partial) \xrightarrow{\phi_i} (\mathbf{C}(X_i), \partial^{X_i}) \xrightarrow{\psi_i} (\mathbf{C}(\mathbf{K}_i), \partial^{X_i})$$

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where each chain map commutes with inclusion:

$$\begin{array}{ccc} \mathbf{C}(\mathbf{K}_i) & \xrightarrow{\subseteq} & \mathbf{C}(\mathbf{K}_{i+1}) \\ \phi_i \downarrow & & \downarrow \phi_{i+1} \\ \mathbf{C}(X_i) & \xrightarrow{\subseteq} & \mathbf{C}(X_{i+1}) \end{array} \qquad \begin{array}{ccc} \mathbf{C}(\mathbf{K}_i) & \xrightarrow{\subseteq} & \mathbf{C}(\mathbf{K}_{i+1}) \\ \psi_i \uparrow & & \uparrow \psi_{i+1} \\ \mathbf{C}(X_i) & \xrightarrow{\subseteq} & \mathbf{C}(X_{i+1}) \end{array}$$

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$\longrightarrow$ enough prove commutativity for addition of a Morse pair (τ,σ):

$$(\mathbf{C}(X),\partial^{X}) \xrightarrow{\phi} (\mathbf{C}(X'),\partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X),\partial^{X})$$

with $X' = X - \{\sigma,\tau\}$ as used previously (same chain maps).

Topology Inference Problem