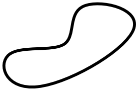
# 4 - Stability and Topological Inference

Clément Maria

clement.maria@inria.fr
www-sop.inria.fr/members/Clement.Maria/

MPRI 2023–2024 / 2.14.1 Computational Geometry and Topology

# Homology Inference Problem



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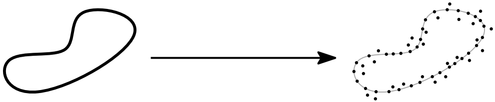

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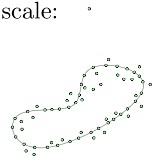

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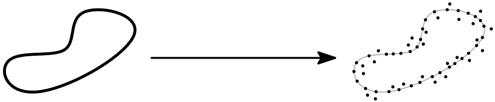

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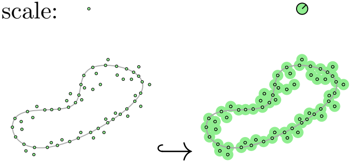

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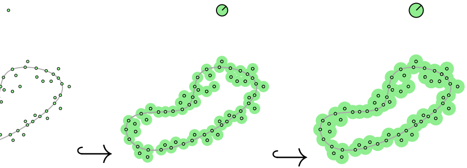

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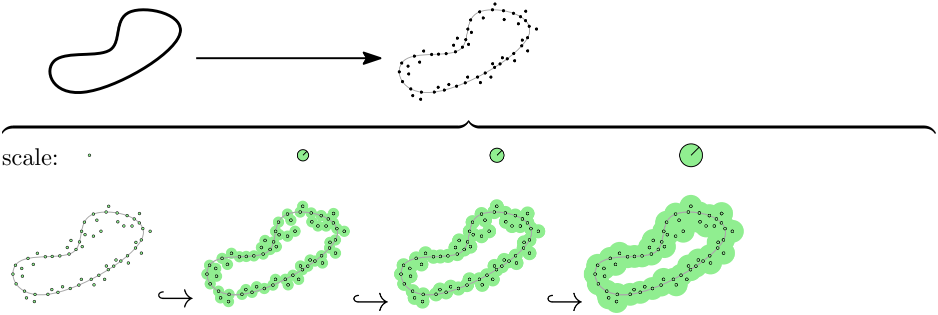

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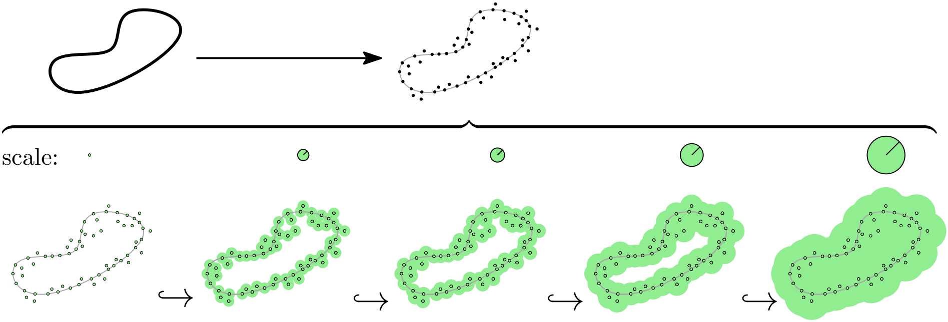

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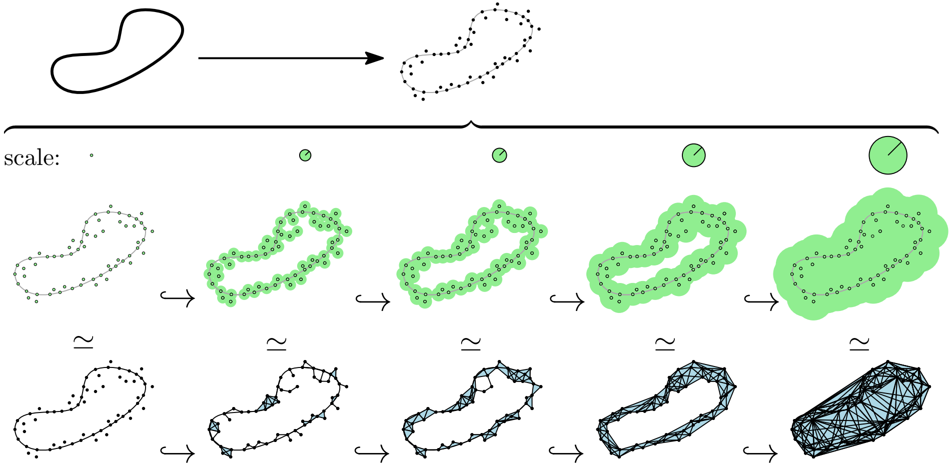

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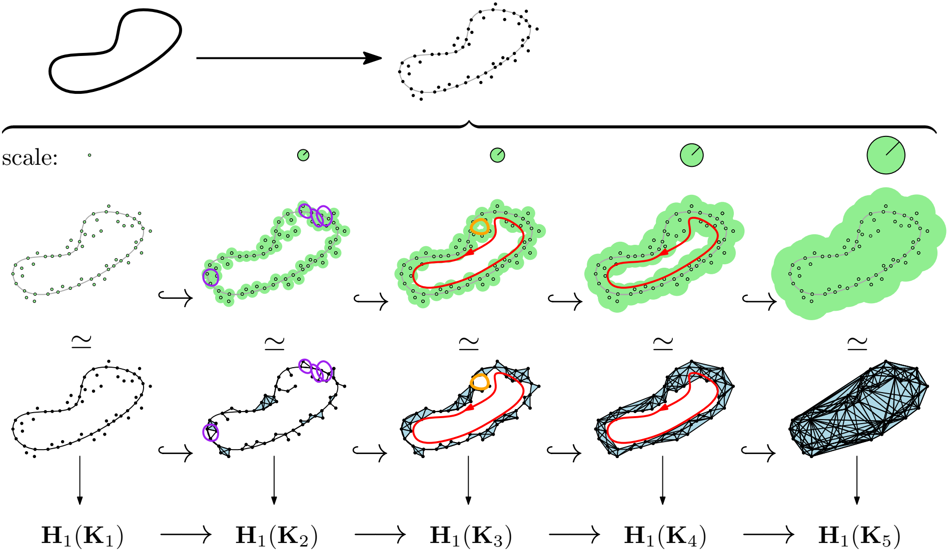

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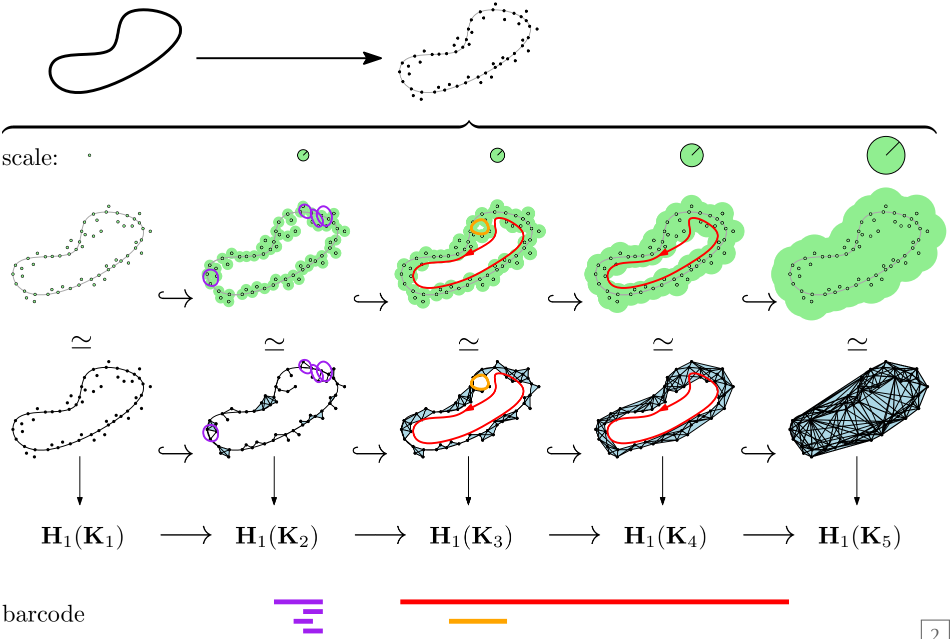


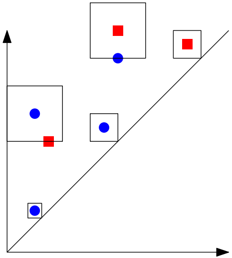
# Stability Theorems

Let $\mathrm{D} = \{(b_i, d_i)\}_{i \in I} \cup \{x = y\}$ and $\mathrm{D}' = \{(b'_j, d'_j)\}_{j \in J} \cup \{x = y\}$ be two persistence diagrams, $b, d \in \mathbb{R}$.

The bottleneck distance $\mathrm{d}_B(\mathrm{D}, \mathrm{D}')$ between D and D′ is:

$$\mathrm{d}_B(\mathrm{D}, \mathrm{D}') := \inf_{\substack{\Phi:\mathrm{D}\to\mathrm{D}' \\ \text{bijection}}} \sup_{p \in \mathrm{D}} ||p - \Phi(p)||_\infty$$

Defined even when $|I| \neq |J|$ by sending points to the diagonal.



**Theorem (Stability on a simplicial complex)**

*Let $f, g \colon \mathbf{K} \to \mathbb{R}$ be two functions on a same simplicial complex $\mathbf{K}$, inducing filtrations $\mathbf{K}_\alpha = f^{-1}((-\infty; \alpha])$ and $\mathbf{K}'_\gamma = g^{-1}((-\infty; \gamma])$. Then*

$$\mathrm{d}_B(\mathrm{D}(f), \mathrm{D}(g)) \leq ||f - g||_\infty.$$

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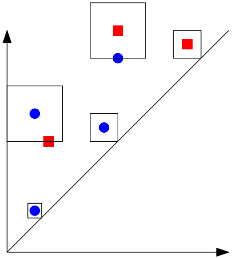


**Theorem (Stability on general space)**

*Let $f, g \colon M \to \mathbb{R}$ be two functions on a same metric space $M$, satisfying some "tameness" conditions. If $\exists \varepsilon \geq 0$ s.t. $\forall r \in \mathbb{R}$ $f^{-1}(-\infty; r] \subseteq g^{-1}(-\infty; r + \varepsilon]$ and $g^{-1}(-\infty; r] \subseteq f^{-1}(-\infty; r + \varepsilon]$,*

*then* $\mathrm{d}_B(\mathrm{D}(f), \mathrm{D}(g)) \leq \varepsilon$.

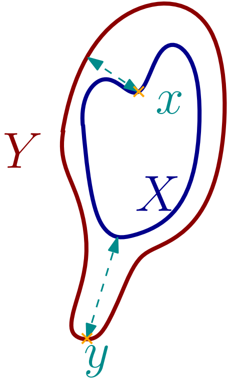
# Distance between spaces

## Definition (Hausdorff distance)

The Hausdorff distance between two non-empty subsets $X, Y$ of a metric space $(M, \mathrm{d})$ is:

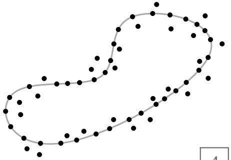
$$\mathrm{d}_H(X,Y) := \max\left\{\sup_{x\in X}\mathrm{d}(x,Y), \sup_{y\in Y}\mathrm{d}(X,y)\right\},$$

where,
$\mathrm{d}(x,Y) = \inf_{y\in Y}\mathrm{d}(x,y)$ and $\mathrm{d}(X,y) = \inf_{x\in X}\mathrm{d}(x,y)$.




## Definition ($\varepsilon$-sample)

Let $K \subset \mathbb{R}^D$ be a compact set. An $\varepsilon$-sample of $K$, for some $\varepsilon \geq 0$, is a finite set of points $P$ such that $\mathrm{d}_H(P,K) \leq \varepsilon$.



# Sampling compacts

Now consider compacts $K$ in Euclidean space $\mathbb{R}^D$. Any compact $K$ defines a function:

$$\mathrm{d}_K \colon \mathbb{R}^D \to \mathbb{R}, \quad x \mapsto \mathrm{d}_K(x) := \mathrm{d}(x, K).$$

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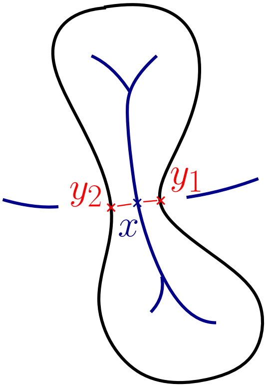
**Definition (Critical point)**

A point $x \in \mathbb{R}^D$ is a critical point for $d_K$ if there exist distinct points $y_1, y_2 \in \mathbf{K}$ such that:

$$d(x, y_1) = d(x, y_2) = d(x, K).$$

The set of critical points is called the medial axis of $K$.



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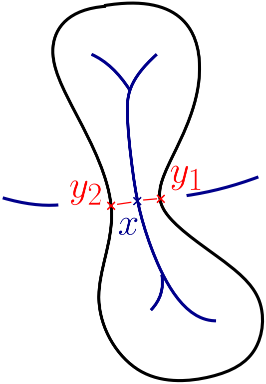
**Definition (Reach of a compact)**

The reach of a compact $K$ is:

$$\text{reach}\,\mathbf{K} := \inf\{d_K(x) : x \text{ critical point for } d_K\}.$$



# Sampling, distance function approach

**Lemma**

*Let $P$ be an $\varepsilon$-sample of a compact $K$. Then,*

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Because $P$ is an $\varepsilon$-sample (and $K$ is compact), there exists a $p_0 \in P$ s.t. $\mathrm{d}(p_0, y_0) \leq \varepsilon$.

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Consequently, by triangle inequality:

$$\mathrm{d}_P(x) \leq \mathrm{d}(x, p_0) \leq \mathrm{d}(x, y_0) + \mathrm{d}(y_0, p_0) \leq \mathrm{d}_K(x) + \varepsilon.$$

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And vice versa, exchanging $K$ and $P$.

# Reconstruction theorem

**Theorem (Reconstruction)**

*Let $K \subset \mathbb{R}^D$ be a compact set, and $\varepsilon > 0$ be such that:*

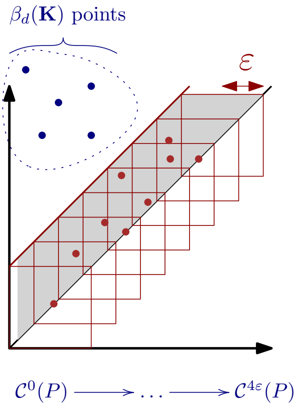
$$0 < 4\varepsilon < \text{reach}\, \mathbf{K}.$$

*Let $P$ be an $\varepsilon$-sample of $K$. Then,*

- *there is a one-to-one correspondence between the points of the persistence diagram of the Cech filtration:*

$$\check{\mathcal{C}}^0(P) \longrightarrow \ldots \longrightarrow \check{\mathcal{C}}^{4\varepsilon}(P)$$

  *that are at least $\varepsilon$-away from the diagonal $\Delta$ in $\infty$ norm, and the homology features of $K$.*



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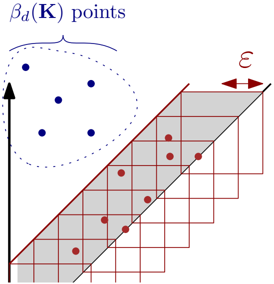


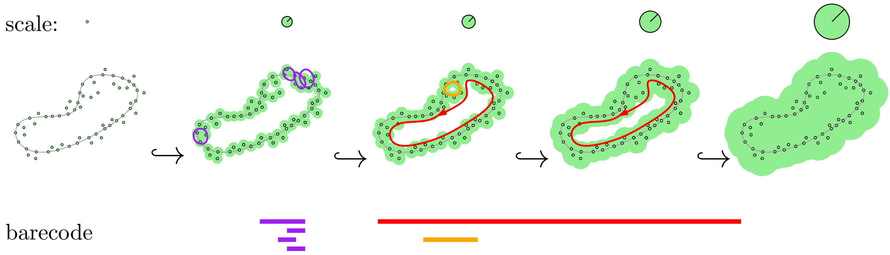

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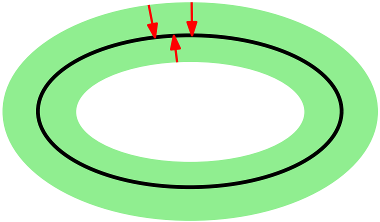
$\longrightarrow$ by projecting each $x \in \mathrm{d}_K^{-1}(-\infty; r] \setminus K$ to its unique nearest neighbor on $K$.



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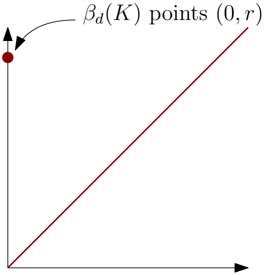
Consequently, for all $0 < r_1 \leq r_2 < \mathrm{reach}(K)$ we have

$$\mathbf{H}(d_K^{-1}(-\infty; r_1]) \xrightarrow{\text{id}} \mathbf{H}(d_K^{-1}(-\infty; r_2])$$



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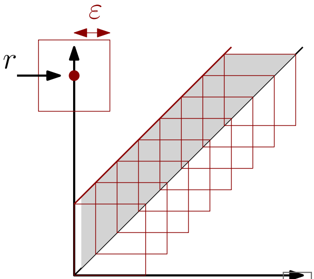
For $\mathrm{reach}(K) > r > 4\varepsilon > 0$, the radius $r$ $||\cdot||_\infty$-ball in $\mathbb{R}^2$ around $(0, r)$, and the $||\cdot||_\infty$-band of width $\varepsilon$ around the diagonal are disjoint.



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## Definition (Flag complex)

Let $G = (V, E)$ be a graph. The flag complex induced by $G$ is the abstract simplicial complex $\mathcal{F}(G)$ satisfying:

- the vertices of $\mathcal{F}(G)$ are $V$,
- $\sigma = \{v_0, \ldots, v_d\} \in \mathcal{F}(G)$ iff $v_0, \ldots, v_d$ form a clique in $G$ (i.e., an induced complete subgraph of $G$).

It is the maximal simplicial complex with $G$ as 1-skeleton.

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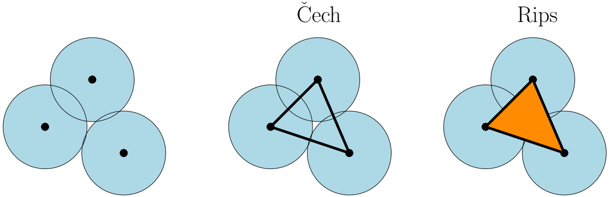
Computing intersections of balls in $\mathbb{R}^D$ is *expensive* (high arithmetic complexity): Approximate the Čech complex instead.

**Definition (Rips complex)**

Let $P = \{x_1, \ldots, x_n\}$ be a set of points in $\mathbb{R}^N$. The neighbor graph of threshold $r$ of $P$ is the graph $N_r(P)$ with:

- vertices the points in $P$, and
- any two points $x_i, x_j$, $i \neq j$ are connected by an edge iff $||x_i - x_j|| \leq r$.

The Rips complex of threshold $r$, denoted by $\mathcal{R}^r(P)$, is the flag complex $\mathcal{F}(N_{2r}(P))$.



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Computation: Simply compute pairwise distances (low arithmetic complexity predicate), then expand the flag complex combinatorial

$\sim O(\#\text{points}^2) +$ linear in the number of cliques/size of the complex.

# Rips vs Čech

**Lemma**

*For a point cloud $P$ in Euclidean space, and any $r > 0$:*

$$\check{\mathcal{C}}^r(P) \subseteq \mathcal{R}^r(P) \subseteq \check{\mathcal{C}}^{2r}(P).$$

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$\boxed{\check{\mathcal{C}}^r(P) \subseteq \mathcal{R}^r(P)}$. Let $\sigma = \{x_0, \ldots, x_k\} \in \check{\mathcal{C}}^r(P)$, $d \geq 1$, then for any $i \neq j$, $0 \leq i, j \leq k$ we have $\mathcal{B}_r(x_i) \cap \mathcal{B}_r(x_j) \neq \emptyset \Rightarrow ||x_i - x_j|| \leq 2r$. Hence $x_0, \ldots, x_k$ form a clique in $N_{2r}(P)$ and $\sigma \in \mathcal{R}^r(P)$.

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$\boxed{\mathcal{R}^r(P) \subseteq \check{\mathcal{C}}^{2r}(P)}$. Let $\sigma = \{x_0, \ldots, x_k\} \in \mathcal{R}^r(P)$, then $||x_i - x_j|| \leq 2r, \forall i, j$. Let $z$ be the barycenter of $x_0, \ldots, x_k$ : $z$ belongs to the convex hull of the points.

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Consequently, for all $i$:

$$||z - x_i|| \leq \mathrm{diam}(CH(x_0, \ldots, x_k)) \leq \max_{i,j} ||x_i - x_j|| \leq 2r.$$

In conclusion, $z \in \bigcap_{i=0 \ldots k} \mathcal{B}_{2r}(x_i) \neq \emptyset$ and $\sigma \in \check{\mathcal{C}}^{2r}(P)$.

# Ranks in sequence of morphisms

**Lemma**

*Consider the following sequence of vector spaces and morphisms:*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E \xrightarrow{\eta} F$$

*such that* $\quad \mathrm{rk}\,(\,A \xrightarrow{\eta\circ\ldots\circ\alpha} F\,) = \mathrm{rk}\,(\,C \xrightarrow{\gamma} D\,) = k, \quad$ *then:*

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- $\boxed{B \longrightarrow C \longrightarrow D \longrightarrow E}$ $\operatorname{rk}(B \longrightarrow E) \leq \operatorname{rk}(C \longrightarrow D) = k$.

# Cech persistence, a closer look

**Lemma**

*Let $P$ be an $\varepsilon$-sample of a compact $K$ of* $\mathrm{reach}(K) > 0$ *in* $\mathbb{R}^N$*. Then, for any* $\varepsilon_0 > \varepsilon$*, and* $\alpha$*, such that* $\varepsilon \leq \alpha < \mathrm{reach}(K) - 2\varepsilon_0$*, we have:*

$$\mathrm{rk}\ \left(\ \mathbf{H}_d(\check{\mathcal{C}}^{\alpha}(P)) \longrightarrow \mathbf{H}_d(\check{\mathcal{C}}^{\alpha+\varepsilon_0}(P))\ \right) = \dim \mathbf{H}_d(K)$$

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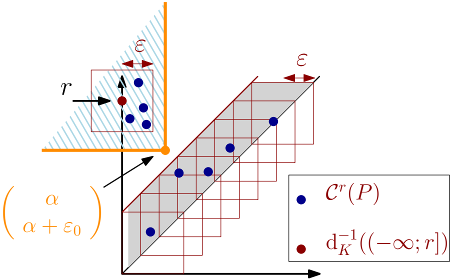

Theorem

*Let $P$ be an $\varepsilon$-sample of a compact $K$ s.t. $0 < \varepsilon < \frac{\mathrm{reach}(K)}{8}$. The homology of $K$ can be read off the persistence diagram of $\mathcal{R}^r(P)$ for $r \geq \varepsilon + 2\varepsilon_0$, for any $\varepsilon < \varepsilon_0 < \frac{\mathrm{reach}(K)}{8}$.*

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$$\check{\mathcal{C}}^{\varepsilon} \longrightarrow \mathcal{R}^{\varepsilon} \longrightarrow \underbrace{\check{\mathcal{C}}^{\varepsilon+\varepsilon_0}}_{\varepsilon+\varepsilon_0>2\varepsilon} \longrightarrow \check{\mathcal{C}}^{\varepsilon+2\varepsilon_0} \longrightarrow \mathcal{R}^{\varepsilon+2\varepsilon_0} \longrightarrow \check{\mathcal{C}}^{2\varepsilon+4\varepsilon_0}$$

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Now, with $2\varepsilon + 4\varepsilon_0 < \text{reach}(K)$,

$$\begin{aligned} \text{rk}\ \left(\mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon}) \to \mathbf{H}_d(\check{\mathcal{C}}^{2\varepsilon+4\varepsilon_0})\right) &= \text{rk}\ (\mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon+\varepsilon_0}) \to \mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon+2\varepsilon_0}) \\ &= \dim \mathbf{H}_d(\mathbf{K}). \end{aligned}$$

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In conclusion, $\dim \mathbf{H}_d(K)$ is exactly the number of points in the persistence diagram of the Rips filtration (up to $r = \varepsilon + 2\varepsilon_0$) in the upper left quadrant based at $(\varepsilon, \varepsilon + 2\varepsilon_0)$.

# Morse filtration

**Definition**

Let $\mathbf{K}$ be a complex, equipped with a filtration,

$$\emptyset := \mathbf{K}_0 \xrightarrow{\mathfrak{S}_1} \mathbf{K}_1 \xrightarrow{\mathfrak{S}_2} \mathbf{K}_2 \xrightarrow{\mathfrak{S}_3} \ldots \xrightarrow{\mathfrak{S}_m} \mathbf{K}_m = \mathbf{K}$$

where $\mathfrak{S}_i$ contains at least one simplex.
A Morse filtration for the filtration of $\mathbf{K}$ is a collection of Morse matchings $(X_i \sqcup T_i \sqcup S_i, \omega_i \colon T_i \to S_i)$ for each $\mathbf{K}_i$, $i = 0 \ldots m$, satisfying:

$$X_i \subseteq X_{i+1}, \qquad T_i \subseteq T_{i+1}, \qquad S_i \subseteq S_{i+1},$$

$$\omega_{i+1}\big|_{T_i} = \omega_i, \qquad \partial^{X_{i+1}}\big|_{X_i} = \partial^{X_i}, \qquad \forall i = 0 \ldots m-1.$$

# Morse filtration

**Theorem**

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*and a Morse filtration* $(X_i \sqcup T_i \sqcup S_i, \omega_i \colon T_i \to S_i)$, $i = 0 \dots m$:

$$\begin{gathered} X_i \subseteq X_{i+1}, \qquad T_i \subseteq T_{i+1}, \qquad S_i \subseteq S_{i+1}, \\ \omega_{i+1}\big|_{T_i} = \omega_i, \qquad \partial^{X_{i+1}}\big|_{X_i} = \partial^{X_i}, \qquad \forall i = 0 \dots m-1. \end{gathered}$$

*Then the following persistence modules:*

$$\mathbf{H}(\mathbf{K}_0, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_1, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_2, \partial) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_m, \partial)$$

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*are isomorphic.*

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*Sketch of proof*:

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implies that:

$$(\mathbf{C}(X_0), \partial^{X_0}) \xrightarrow{\subseteq} (\mathbf{C}(X_1), \partial^{X_1}) \xrightarrow{\subseteq} (\mathbf{C}(X_2), \partial^{X_2}) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} (\mathbf{C}(X_m), \partial^{X_m})$$

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is well-defined filtration of complexes inducing the persist. module:

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*Sketch of proof*:

For each *i* there is a chain equivalence:

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where each chain map commutes with inclusion:

$$\begin{array}{ccc} \mathbf{C}(\mathbf{K}_i) & \xrightarrow{\subseteq} & \mathbf{C}(\mathbf{K}_{i+1}) \\ {\scriptstyle\phi_i}\downarrow & & \downarrow{\scriptstyle\phi_{i+1}} \\ \mathbf{C}(X_i) & \xrightarrow{\subseteq} & \mathbf{C}(X_{i+1}) \end{array} \qquad \begin{array}{ccc} \mathbf{C}(\mathbf{K}_i) & \xrightarrow{\subseteq} & \mathbf{C}(\mathbf{K}_{i+1}) \\ {\scriptstyle\psi_i}\uparrow & & \uparrow{\scriptstyle\psi_{i+1}} \\ \mathbf{C}(X_i) & \xrightarrow{\subseteq} & \mathbf{C}(X_{i+1}) \end{array}$$

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$\longrightarrow$ enough prove commutativity for addition of a Morse pair $(\tau,\sigma)$:

$$(\mathbf{C}(X),\partial^{X}) \xrightarrow{\phi} (\mathbf{C}(X'),\partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X),\partial^{X})$$

with $X' = X - \{\sigma,\tau\}$ as used previously (same chain maps).

# Topology Inference Problem