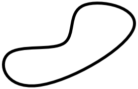
CGL 4 - Homology Inference

Clément Maria

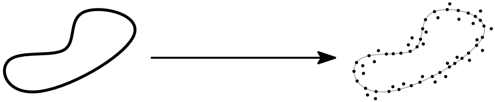
INRIA Sophia Antipolis-Méditerranée

MPRI 2020-2021

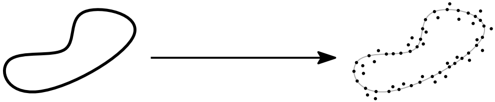
Homology Inference Problem



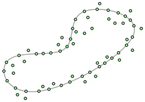
Homology Inference Problem



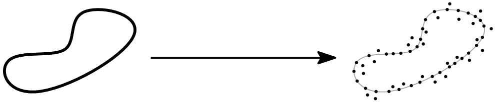
Homology Inference Problem



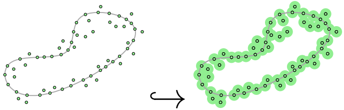
scale:



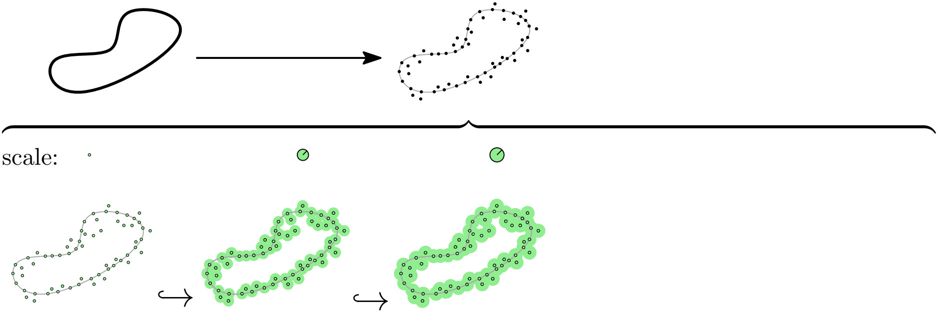
Homology Inference Problem



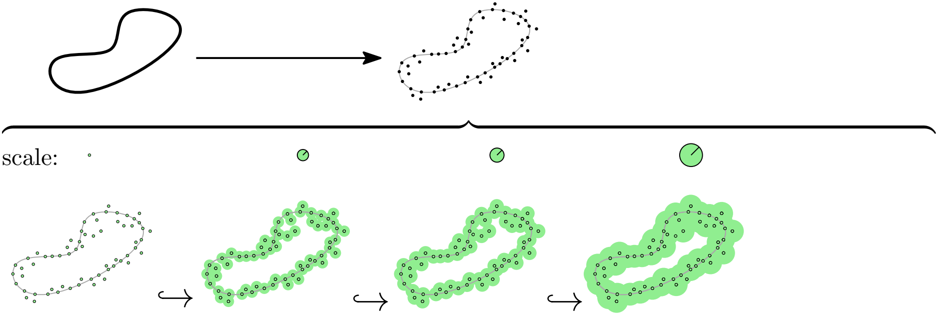
scale:



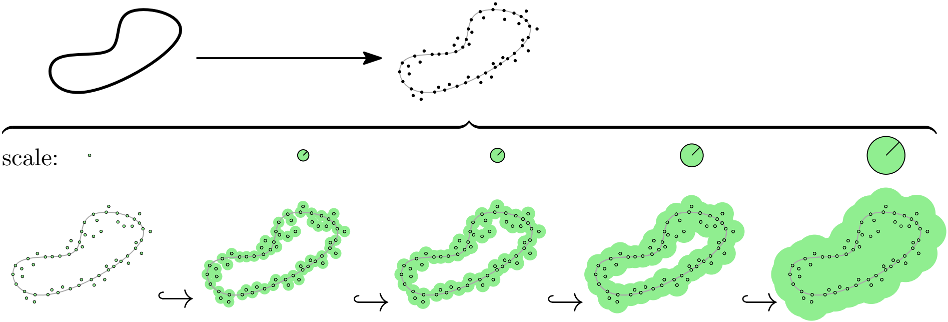
Homology Inference Problem



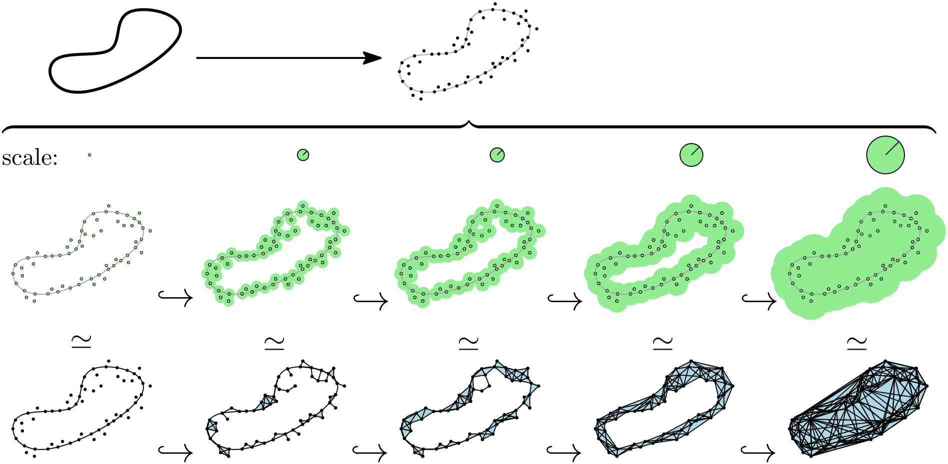
Homology Inference Problem



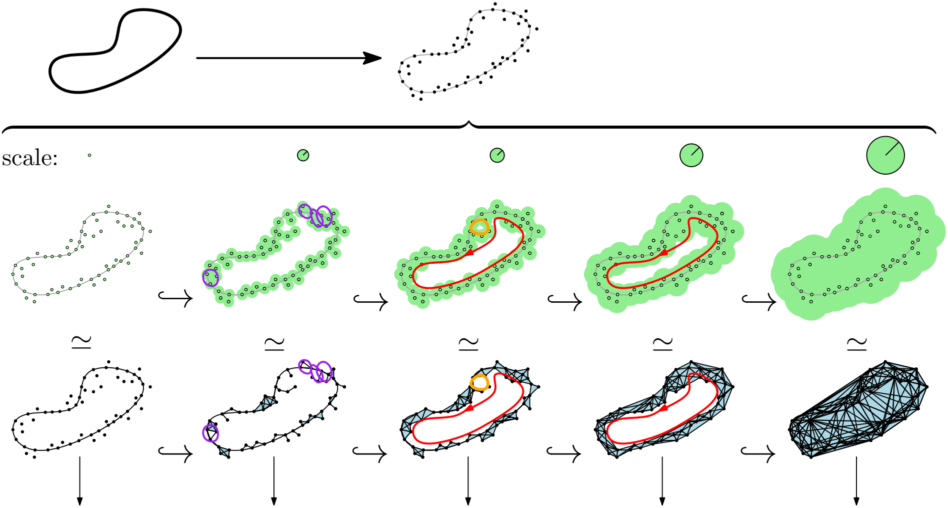
Homology Inference Problem



Homology Inference Problem

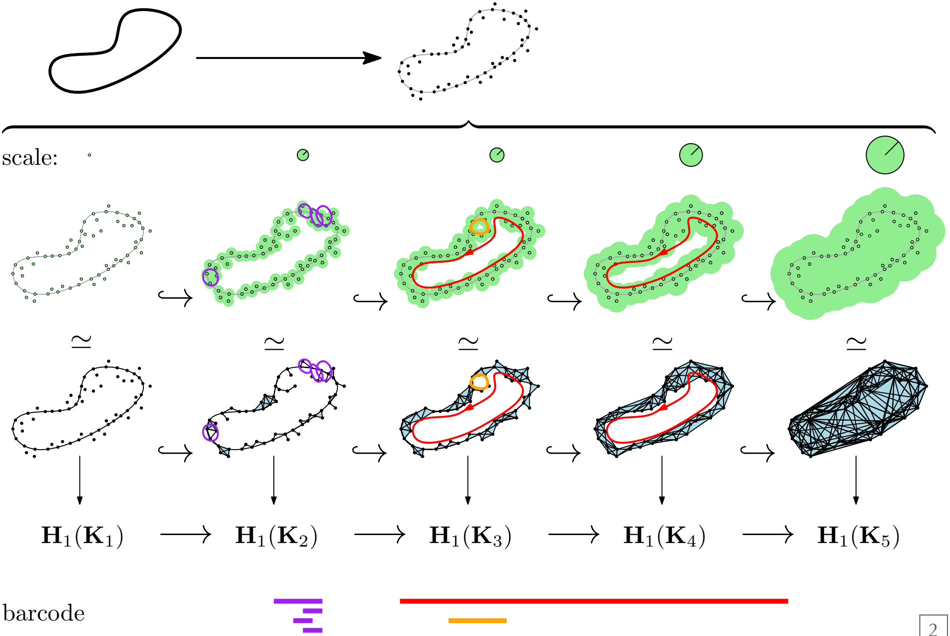


Homology Inference Problem



$$\mathbf{H}_1(\mathbf{K}_1) \longrightarrow \mathbf{H}_1(\mathbf{K}_2) \longrightarrow \mathbf{H}_1(\mathbf{K}_3) \longrightarrow \mathbf{H}_1(\mathbf{K}_4) \longrightarrow \mathbf{H}_1(\mathbf{K}_5)$$

Homology Inference Problem



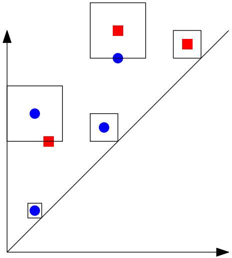
Stability Theorems

Let $\mathrm{D} = \{(b_i, d_i)\}_{i \in I} \cup \{x = y\}$ and $\mathrm{D}' = \{(b'_j, d'_j)\}_{j \in J} \cup \{x = y\}$ be two persistence diagrams, $b, d \in \mathbb{R}$.

The bottleneck distance $\mathrm{d}_B(\mathrm{D}, \mathrm{D}')$ between D and D′ is:

$$\mathrm{d}_B(\mathrm{D}, \mathrm{D}') := \inf_{\substack{\Phi : \mathrm{D} \to \mathrm{D}' \\ \text{bijection}}} \sup_{p \in \mathrm{D}} ||p - \Phi(p)||_\infty$$

Defined even when $|I| \neq |J|$ by sending points to the diagonal.



Theorem (Stability on a simplicial complex)

Let $f, g \colon \mathbf{K} \to \mathbb{R}$ be two functions on a same simplicial complex $\mathbf{K}$, inducing filtrations $\mathbf{K}_\alpha = f^{-1}((-\infty; \alpha])$ and $\mathbf{K}'_\gamma = g^{-1}((-\infty; \gamma])$. Then

$$\mathrm{d}_B(\mathrm{D}(f), \mathrm{D}(g)) \leq ||f - g||_\infty.$$

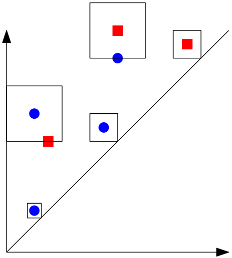
Stability Theorems

Let $\mathrm{D} = \{(b_i, d_i)\}_{i\in I} \cup \{x = y\}$ and $\mathrm{D}' = \{(b'_j, d'_j)\}_{j\in J} \cup \{x = y\}$ be two persistence diagrams, $b, d \in \mathbb{R}$.

The bottleneck distance $\mathrm{d}_B(\mathrm{D}, \mathrm{D}')$ between D and D′ is:

$$\mathrm{d}_B(\mathrm{D}, \mathrm{D}') := \inf_{\substack{\Phi:\mathrm{D}\to\mathrm{D}' \\ \text{bijection}}} \sup_{p\in\mathrm{D}} ||p-\Phi(p)||_\infty$$

Defined even when $|I| \neq |J|$ by sending points to the diagonal.



Theorem (Stability on general space)

Let $f, g\colon M \to \mathbb{R}$ be two functions on a same metric space M, satisfying some "tameness" conditions. If $\exists \varepsilon \geq 0$ *s.t.* $\forall r \in \mathbb{R}$
$f^{-1}(-\infty; r] \subseteq g^{-1}(-\infty; r+\varepsilon]$ *and* $g^{-1}(-\infty; r] \subseteq f^{-1}(-\infty; r+\varepsilon]$,

then $\mathrm{d}_B(\mathrm{D}(f), \mathrm{D}(g)) \leq \varepsilon$.

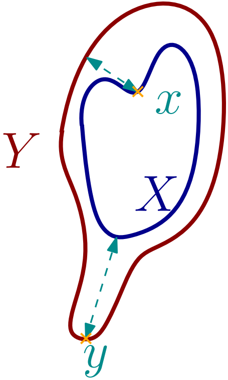
Distance between spaces

Definition (Hausdorff distance)

The Hausdorff distance between two non-empty subsets X, Y of a metric space (M, d) is:

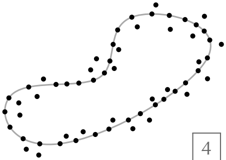
$$\mathrm{d}_H(X, Y) := \max\left\{\sup_{x\in X} \mathrm{d}(x, Y), \sup_{y\in Y} \mathrm{d}(X, y)\right\},$$

where,
$\mathrm{d}(x, Y) = \inf_{y\in Y} \mathrm{d}(x, y)$ and $\mathrm{d}(X, y) = \inf_{x\in X} \mathrm{d}(x, y)$.



Definition (ε-sample)

Let $K \subset \mathbb{R}^D$ be a compact set. An ε-sample of K, for some $\varepsilon \geq 0$, is a finite set of points P such that $\mathrm{d}_H(P, K) \leq \varepsilon$.



Sampling compacts

Now consider compacts K in Euclidean space $\mathbb{R}^D$. Any compact K defines a function:

$$\mathrm{d}_K\colon \mathbb{R}^D \to \mathbb{R}, \quad x \mapsto \mathrm{d}_K(x) = \mathrm{d}(x, K).$$

Sampling compacts

Now consider compacts K in Euclidean space $\mathbb{R}^D$. Any compact K defines a function:

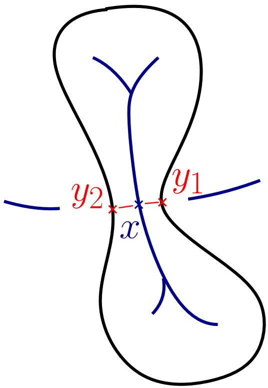
$$d_K\colon \mathbb{R}^D \to \mathbb{R}, \quad x \mapsto d_K(x) = d(x, K).$$

Definition (Critical point)

A point $x \in \mathbb{R}^D$ is a critical point for d_K if there exist distinct points $y_1, y_2 \in \mathbf{K}$ such that:

$$d(x, y_1) = d(x, y_2) = d(x, K).$$

The set of critical points is called the medial axis of K.



Sampling compacts

Now consider compacts K in Euclidean space $\mathbb{R}^D$. Any compact K defines a function:

$$\mathrm{d}_K\colon \mathbb{R}^D \to \mathbb{R}, \quad x \mapsto \mathrm{d}_K(x) = \mathrm{d}(x, K).$$

Definition (Critical point)

A point $x \in \mathbb{R}^D$ is a critical point for d_K if there exist distinct points $y_1, y_2 \in \mathbf{K}$ such that:

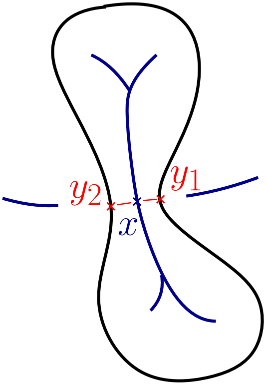
$$\mathrm{d}(x, y_1) = \mathrm{d}(x, y_2) = \mathrm{d}(x, K).$$

The set of critical points is called the medial axis of K.

Definition (Reach of a compact)

The reach of a compact K is:

$$\text{reach}\,\mathbf{K} := \inf\{\mathrm{d}_K(x) : x \text{ critical point for } \mathrm{d}_K\}.$$



Sampling, distance function approach

Lemma

Let P be an ε-sample of a compact K. Then,

$$||\mathrm{d}_K - \mathrm{d}_P||_\infty \leq \varepsilon.$$

Sampling, distance function approach

Lemma

Let P be an ε-sample of a compact K. Then,

$$||\mathrm{d}_K - \mathrm{d}_P||_\infty \leq \varepsilon.$$

For any $x \in \mathbb{R}^D$, let $y_0 := \arg\min_{y \in K} \mathrm{d}(x, y)$ be the nearest neighbor of x on K.

Sampling, distance function approach

Lemma

Let P be an ε-sample of a compact K. Then,

$$||\mathrm{d}_K - \mathrm{d}_P||_\infty \leq \varepsilon.$$

For any $x \in \mathbb{R}^D$, let $y_0 := \arg\min_{y \in K} \mathrm{d}(x, y)$ be the nearest neighbor of x on K.
Because P is an ε-sampling (and K is compact), there exists a $\exists p_0 \in P$ s.t. $\mathrm{d}(p_0, y_0) \leq \varepsilon$.

Sampling, distance function approach

Lemma

Let P be an ε-sample of a compact K. Then,

$$||\mathrm{d}_K - \mathrm{d}_P||_\infty \leq \varepsilon.$$

For any $x \in \mathbb{R}^D$, let $y_0 := \arg\min_{y \in K} \mathrm{d}(x, y)$ be the nearest neighbor of x on K.
Because P is an ε-sampling (and K is compact), there exists a $\exists p_0 \in P$ s.t. $\mathrm{d}(p_0, y_0) \leq \varepsilon$.
Consequently, by triangle inequality:

$$\mathrm{d}_P(x) \leq \mathrm{d}(x, p_0) \leq \mathrm{d}(x, y_0) + \mathrm{d}(y_0, p_0) \leq \mathrm{d}_K(x) + \varepsilon.$$

Sampling, distance function approach

Lemma

Let P be an ε-sample of a compact K. Then,

$$||\mathrm{d}_K - \mathrm{d}_P||_\infty \leq \varepsilon.$$

For any $x \in \mathbb{R}^D$, let $y_0 := \arg\min_{y \in K} \mathrm{d}(x, y)$ be the nearest neighbor of x on K.
Because P is an ε-sampling (and K is compact), there exists a $\exists p_0 \in P$ s.t. $\mathrm{d}(p_0, y_0) \leq \varepsilon$.
Consequently, by triangle inequality:

$$\mathrm{d}_P(x) \leq \mathrm{d}(x, p_0) \leq \mathrm{d}(x, y_0) + \mathrm{d}(y_0, p_0) \leq \mathrm{d}_K(x) + \varepsilon.$$

And vice versa, exchanging K and P.

Reconstruction theorem

Theorem (Reconstruction)

Let $K \subset \mathbb{R}^D$ be a compact set, and $\varepsilon > 0$ be such that:

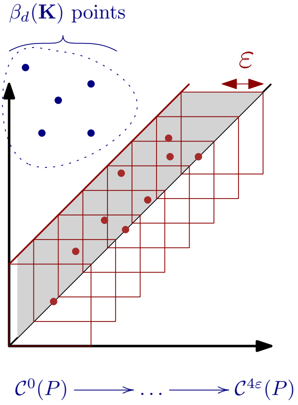
$$0 < 4\varepsilon < \text{reach}\,\mathbf{K}.$$

Let P be an ε-sample of K. Then,

- *there is a one-to-one correspondence between the points of the persistence diagram of the Čech filtration:*

$$\check{\mathcal{C}}^0(P) \longrightarrow \ldots \longrightarrow \check{\mathcal{C}}^{4\varepsilon}(P)$$

that are at least ε-away from the diagonal Δ in ∞ norm, and the homology features of K.



Reconstruction theorem

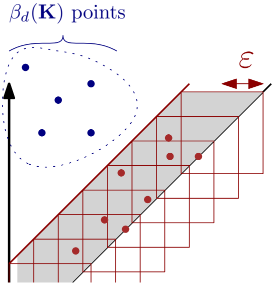
Theorem (Reconstruction)

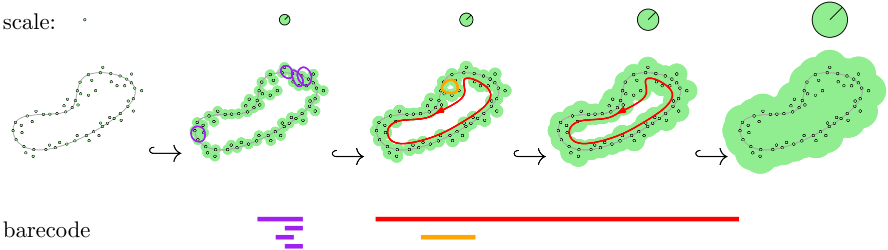
Let $K \subset \mathbb{R}^D$ be a compact set, and $\varepsilon > 0$ be such that:

$$0 < 4\varepsilon < \text{reach}\,\mathbf{K}.$$

Let P be an ε-sample of K. Then,

- *there is a one-to-one correspondence between the points of the persistence diagram of the Čech filtration:*





Reconstruction theorem

Sketch of proof:

(1) Consider the "persistent homology of $\mathrm{d}_K^{-1}(-\infty; r]$" (assuming d_K is tame and its persistent homology well-defined).

Reconstruction theorem

Sketch of proof:

(1) Consider the "persistent homology of $\mathrm{d}_K^{-1}(-\infty; r]$" (assuming d_K is tame and its persistent homology well-defined).

For all r, such that $0 < r < \mathrm{reach}(K)$, the space $\mathrm{d}_K^{-1}(-\infty; r]$ deformation retracts into K, and they have same homology.

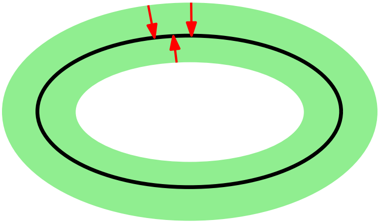
Reconstruction theorem

Sketch of proof:

(1) Consider the "persistent homology of $\mathrm{d}_K^{-1}(-\infty; r]$" (assuming d_K is tame and its persistent homology well-defined).

For all r, such that $0 < r < \mathrm{reach}(K)$, the space $\mathrm{d}_K^{-1}(-\infty; r]$ deformation retracts into K, and they have same homology.

$\longrightarrow$ by projecting each $x \in \mathrm{d}_K^{-1}(-\infty; r] \setminus K$ to its unique nearest neighbor on K.



Reconstruction theorem

Sketch of proof:

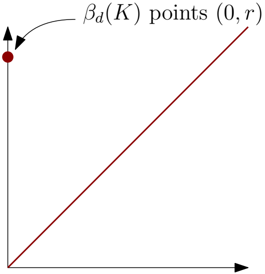
(1) Consider the "persistent homology of $d_K^{-1}(-\infty; r]$" (assuming d_K is tame and its persistent homology well-defined).

For all r, such that $0 < r < \mathrm{reach}(K)$, the space $d_K^{-1}(-\infty; r]$ deformation retracts into K, and they have same homology.

$\longrightarrow$ by projecting each $x \in d_K^{-1}(-\infty; r] \setminus K$ to its unique nearest neighbor on K.

Consequently, for all $0 < r_1 \leq r_2 < \mathrm{reach}(K)$ we have

$$\mathbf{H}(d_K^{-1}(-\infty; r_1]) \xrightarrow{\text{id}} \mathbf{H}(d_K^{-1}(-\infty; r_2])$$



Reconstruction theorem

Sketch of proof:

(2) Consider the persistent homology of

$$\mathrm{d}_P^{-1}(-\infty; r] = \bigcup_{p\in P} \mathcal{B}_r(p) \simeq |\check{\mathcal{C}}^r(P)| \text{ (Nerve lemma).}$$

Reconstruction theorem

Sketch of proof:

(2) Consider the persistent homology of

$$\mathrm{d}_P^{-1}(-\infty; r] = \bigcup_{p \in P} \mathcal{B}_r(p) \simeq |\check{\mathcal{C}}^r(P)| \text{ (Nerve lemma).}$$

Now, we have $||\mathrm{d}_K - \mathrm{d}_P||_\infty \leq \varepsilon$ by triangle inequality.

Reconstruction theorem

Sketch of proof:

(2) Consider the persistent homology of

$$\mathrm{d}_P^{-1}(-\infty; r] = \bigcup_{p \in P} \mathcal{B}_r(p) \simeq |\check{\mathcal{C}}^r(P)| \text{ (Nerve lemma).}$$

Now, we have $||\mathrm{d}_K - \mathrm{d}_P||_\infty \leq \varepsilon$ by triangle inequality.

Apply the (continuous version of the) stability theorem:

$$\mathrm{d}_B(\mathrm{D}(\mathrm{d}_k), \mathrm{D}(\check{\mathcal{C}}(P))) \leq \varepsilon.$$

Reconstruction theorem

Sketch of proof:

(2) Consider the persistent homology of

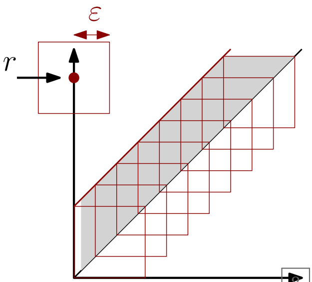
$$\mathrm{d}_P^{-1}(-\infty; r] = \bigcup_{p \in P} \mathcal{B}_r(p) \simeq |\check{\mathcal{C}}^r(P)| \text{ (Nerve lemma).}$$

Now, we have $||\mathrm{d}_K - \mathrm{d}_P||_\infty \leq \varepsilon$ by triangle inequality.

Apply the (continuous version of the) stability theorem:

$$\mathrm{d}_B(\mathrm{D}(\mathrm{d}_k), \mathrm{D}(\check{\mathcal{C}}(P))) \leq \varepsilon.$$

For $\mathrm{reach}(K) > r > 4\varepsilon > 0$, the radius r $|| \cdot ||_\infty$-ball in $\mathbb{R}^2$ around $(0, r)$, and the $|| \cdot ||_\infty$-band of width ε around the diagonal are disjoint.



Rips complex

Computing intersections of balls in $\mathbb{R}^D$ is *expensive* (high arithmetic complexity): Approximate the Čech complex instead.

Rips complex

Computing intersections of balls in $\mathbb{R}^D$ is *expensive* (high arithmetic complexity): Approximate the Čech complex instead.

Definition (Flag complex)

Let $G = (V, E)$ be a graph. The flag complex induced by G is the abstract simplicial complex $\mathcal{F}(G)$ satisfying:

- the vertices of $\mathcal{F}(G)$ are V,
- $\sigma = \{v_0, \ldots, v_d\} \in \mathcal{F}(G)$ iff $v_0, \ldots, v_d$ form a clique in G (i.e., an induced complete subgraph of G).

It is the maximal simplicial complex with G as 1-skeleton.

Rips complex

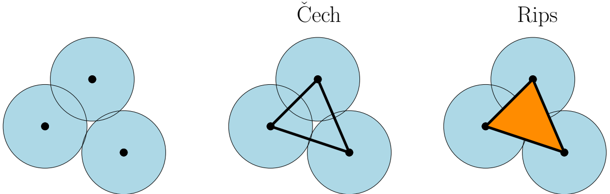
Computing intersections of balls in $\mathbb{R}^D$ is *expensive* (high arithmetic complexity): Approximate the Čech complex instead.

Definition (Rips complex)

Let $P = \{x_1, \ldots, x_n\}$ be a set of points in $\mathbb{R}^N$. The neighbor graph of threshold r of P is the graph $N_r(P)$ with:

- vertices the points in P, and
- any two points x_i, x_j, $i \neq j$ are connected by an edge iff $||x_i - x_j|| \leq r$.

The Rips complex of threshold r, denoted by $\mathcal{R}^r(P)$, is the flag complex $\mathcal{F}(N_{2r}(P))$.



Rips complex

Computing intersections of balls in $\mathbb{R}^D$ is *expensive* (high arithmetic complexity): Approximate the Čech complex instead.

Definition (Rips complex)

Let $P = \{x_1, \ldots, x_n\}$ be a set of points in $\mathbb{R}^N$. The neighbor graph of threshold r of P is the graph $N_r(P)$ with:

- vertices the points in P, and
- any two points x_i, x_j, $i \neq j$ are connected by an edge iff $||x_i - x_j|| \leq r$.

The Rips complex of threshold r, denoted by $\mathcal{R}^r(P)$, is the flag complex $\mathcal{F}(N_{2r}(P))$.

Computation: Simply compute pairwise distances (low arithmetic complexity predicate), then expand the flag complex combinatorial

$$\sim O(\#\text{points}^2) + \text{ linear in the number of cliques/size of the complex.}$$

Rips vs Čech

Lemma

For a point cloud P in Euclidean space, and any $r > 0$:

$$\check{\mathcal{C}}^r(P) \subseteq \mathcal{R}^r(P) \subseteq \check{\mathcal{C}}^{2r}(P).$$

Rips vs Čech

Lemma

For a point cloud P in Euclidean space, and any $r > 0$:

$$\check{\mathcal{C}}^r(P) \subseteq \mathcal{R}^r(P) \subseteq \check{\mathcal{C}}^{2r}(P).$$

$\boxed{\check{\mathcal{C}}^r(P) \subseteq \mathcal{R}^r(P)}$. Let $\sigma = \{x_0, \ldots, x_k\} \in \check{\mathcal{C}}^r(P)$, $d \geq 1$, then for any $i \neq j$, $0 \leq i, j \leq k$ we have $\mathcal{B}_r(x_i) \cap \mathcal{B}_r(x_j) \neq \emptyset \Rightarrow ||x_i - x_j|| \leq 2r$. Hence $x_0, \ldots, x_k$ form a clique in $N_{2r}(P)$ and $\sigma \in \mathcal{R}^r(P)$.

Rips vs Čech

Lemma

For a point cloud P in Euclidean space, and any $r > 0$:

$$\check{\mathcal{C}}^r(P) \subseteq \mathcal{R}^r(P) \subseteq \check{\mathcal{C}}^{2r}(P).$$

$\boxed{\check{\mathcal{C}}^r(P) \subseteq \mathcal{R}^r(P)}$. Let $\sigma = \{x_0, \ldots, x_k\} \in \check{\mathcal{C}}^r(P)$, $d \geq 1$, then for any $i \neq j$, $0 \leq i, j \leq k$ we have $\mathcal{B}_r(x_i) \cap \mathcal{B}_r(x_j) \neq \emptyset \Rightarrow ||x_i - x_j|| \leq 2r$. Hence $x_0, \ldots, x_k$ form a clique in $N_{2r}(P)$ and $\sigma \in \mathcal{R}^r(P)$.

$\boxed{\mathcal{R}^r(P) \subseteq \check{\mathcal{C}}^{2r}(P)}$. Let $\sigma = \{x_0, \ldots, x_k\} \in \mathcal{R}^r(P)$, then $||x_i - x_j|| \leq 2r, \forall i, j$. Let z be the barycenter of $x_0, \ldots, x_k$: z belongs to the convex hull of the points.

Rips vs Čech

Lemma

For a point cloud P in Euclidean space, and any $r > 0$:

$$\check{\mathcal{C}}^r(P) \subseteq \mathcal{R}^r(P) \subseteq \check{\mathcal{C}}^{2r}(P).$$

$\boxed{\check{\mathcal{C}}^r(P) \subseteq \mathcal{R}^r(P)}$. Let $\sigma = \{x_0, \ldots, x_k\} \in \check{\mathcal{C}}^r(P)$, $d \geq 1$, then for any $i \neq j$, $0 \leq i, j \leq k$ we have $\mathcal{B}_r(x_i) \cap \mathcal{B}_r(x_j) \neq \emptyset \Rightarrow ||x_i - x_j|| \leq 2r$. Hence $x_0, \ldots, x_k$ form a clique in $N_{2r}(P)$ and $\sigma \in \mathcal{R}^r(P)$.

$\boxed{\mathcal{R}^r(P) \subseteq \check{\mathcal{C}}^{2r}(P)}$. Let $\sigma = \{x_0, \ldots, x_k\} \in \mathcal{R}^r(P)$, then $||x_i - x_j|| \leq 2r, \forall i, j$. Let z be the barycenter of $x_0, \ldots, x_k$: z belongs to the convex hull of the points.

Consequently, for all i:

$$||z - x_i|| \leq \operatorname{diam}(CH(x_0, \ldots, x_k)) \leq \max_{i,j} ||x_i - x_j|| \leq 2r.$$

In conclusion, $z \in \bigcap_{i=0\ldots k} \mathcal{B}_{2r}(x_i) \neq \emptyset$ and $\sigma \in \check{\mathcal{C}}^{2r}(P)$.

Ranks in sequence of morphisms

Lemma

Consider the following sequence of vector spaces and morphisms:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E \xrightarrow{\eta} F$$

such that $\quad \mathrm{rk}\,(\, A \xrightarrow{\eta\circ\ldots\circ\alpha} F \,) = \mathrm{rk}\,(\, C \xrightarrow{\gamma} D \,) = k,$ $\quad$ *then:*

$$\mathrm{rk}\,(\, B \xrightarrow{\delta\circ\gamma\circ\beta} E \,) = k.$$

Ranks in sequence of morphisms

Lemma

Consider the following sequence of vector spaces and morphisms:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E \xrightarrow{\eta} F$$

such that $\operatorname{rk}\left(A \xrightarrow{\eta\circ\ldots\circ\alpha} F \right) = \operatorname{rk}\left(C \xrightarrow{\gamma} D \right) = k,$ *then:*

$$\operatorname{rk}\left(B \xrightarrow{\delta\circ\gamma\circ\beta} E \right) = k.$$

Note that in any $U \to V \to W \to X$, $\operatorname{rk}\left(U \to X \right) \leq \operatorname{rk}\left(V \to W \right)$.

Ranks in sequence of morphisms

Lemma

Consider the following sequence of vector spaces and morphisms:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E \xrightarrow{\eta} F$$

such that $\operatorname{rk}\left(A \xrightarrow{\eta\circ\ldots\circ\alpha} F \right) = \operatorname{rk}\left(C \xrightarrow{\gamma} D \right) = k,$ *then:*

$$\operatorname{rk}\left(B \xrightarrow{\delta\circ\gamma\circ\beta} E \right) = k.$$

Note that in any $U \to V \to W \to X$, $\operatorname{rk}\left(U \to X \right) \leq \operatorname{rk}\left(V \to W \right)$.
Consequently,

- $\boxed{A \longrightarrow B \longrightarrow E \longrightarrow F}$ $\quad k = \operatorname{rk}\left(A \longrightarrow F \right) \leq \operatorname{rk}\left(B \longrightarrow E \right),$

Ranks in sequence of morphisms

Lemma

Consider the following sequence of vector spaces and morphisms:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E \xrightarrow{\eta} F$$

such that $\quad \mathrm{rk}\left(A \xrightarrow{\eta\circ\ldots\circ\alpha} F \right) = \mathrm{rk}\left(C \xrightarrow{\gamma} D \right) = k,$ $\quad$ *then:*

$$\mathrm{rk}\left(B \xrightarrow{\delta\circ\gamma\circ\beta} E \right) = k.$$

Note that in any $U \to V \to W \to X$, $\mathrm{rk}\left(U \to X \right) \leq \mathrm{rk}\left(V \to W \right)$.
Consequently,

- $\boxed{A \longrightarrow B \longrightarrow E \longrightarrow F}$ $\quad k = \mathrm{rk}\left(A \longrightarrow F \right) \leq \mathrm{rk}\left(B \longrightarrow E \right)$,
- $\boxed{B \longrightarrow C \longrightarrow D \longrightarrow E}$ $\quad \mathrm{rk}\left(B \longrightarrow E \right) \leq \mathrm{rk}\left(C \longrightarrow D \right) = k$.

Cech persistence, a closer look

Lemma

Let P be an ε-sample of a compact K of $\mathrm{reach}(K) > 0$ in $\mathbb{R}^N$. Then, for any $\varepsilon_0 > \varepsilon$, and α, such that $\varepsilon \leq \alpha < \mathrm{reach}(K) - 2\varepsilon_0$, we have:

$$\mathrm{rk}\ \left(\ \mathbf{H}_d(\check{\mathcal{C}}^{\alpha}(P)) \longrightarrow \mathbf{H}_d(\check{\mathcal{C}}^{\alpha+\varepsilon_0}(P))\ \right) = \dim \mathbf{H}_d(K)$$

Cech persistence, a closer look

Lemma

Let P be an ε-sample of a compact K of $\operatorname{reach}(K) > 0$ in $\mathbb{R}^N$. Then, for any $\varepsilon_0 > \varepsilon$, and α, such that $\varepsilon \le \alpha < \operatorname{reach}(K) - 2\varepsilon_0$, we have:

$$\operatorname{rk}\left(\mathbf{H}_d(\check{\mathcal{C}}^{\alpha}(P)) \longrightarrow \mathbf{H}_d(\check{\mathcal{C}}^{\alpha+\varepsilon_0}(P)) \right) = \dim \mathbf{H}_d(K)$$

$\operatorname{rk}\left(\mathbf{H}_d(\check{\mathcal{C}}^{\alpha}(P)) \longrightarrow \mathbf{H}_d(\check{\mathcal{C}}^{\alpha+\varepsilon_0}(P)) \right)$
counts the number of points (x, y)
of the persistence diagram of the
filtration:

$$\check{\mathcal{C}}^{0}(P) \longrightarrow \dots \longrightarrow \check{\mathcal{C}}^{\operatorname{reach}(K)-\nu}(P)$$

with $x \le \alpha$ and $y \ge \alpha + \varepsilon_0$.

Cech persistence, a closer look

Lemma

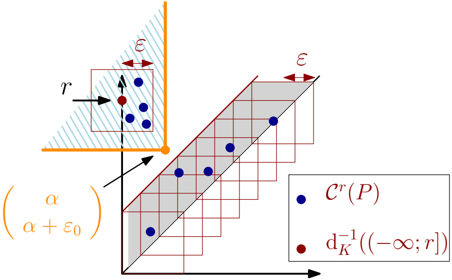
Let P be an ε-sample of a compact K of $\mathrm{reach}(K) > 0$ in $\mathbb{R}^N$. Then, for any $\varepsilon_0 > \varepsilon$, and α, such that $\varepsilon \leq \alpha < \mathrm{reach}(K) - 2\varepsilon_0$, we have:

$$\mathrm{rk}\ (\ \mathbf{H}_d(\check{\mathcal{C}}^{\alpha}(P)) \longrightarrow \mathbf{H}_d(\check{\mathcal{C}}^{\alpha+\varepsilon_0}(P))\ ) = \dim \mathbf{H}_d(K)$$

$\mathrm{rk}\ (\ \mathbf{H}_d(\check{\mathcal{C}}^{\alpha}(P)) \longrightarrow \mathbf{H}_d(\check{\mathcal{C}}^{\alpha+\varepsilon_0}(P))\ )$
counts the number of points (x, y) of the persistence diagram of the filtration:

$$\check{\mathcal{C}}^0(P) \longrightarrow \ldots \longrightarrow \check{\mathcal{C}}^{\mathrm{reach}(K)-\nu}(P)$$

with $x \leq \alpha$ and $y \geq \alpha + \varepsilon_0$.



Theorem

Let P be an ε-sample of a compact K s.t. $0 < \varepsilon < \frac{\mathrm{reach}(K)}{8}$. The homology of K can be read off the persistence diagram of $\mathcal{R}^r(P)$ for $r \geq \varepsilon + 2\varepsilon_0$, for any $\varepsilon < \varepsilon_0 < \frac{\mathrm{reach}(K)}{8}$.

Theorem

Let P be an ε-sample of a compact K s.t. $0 < \varepsilon < \frac{\text{reach}(K)}{8}$. The homology of K can be read off the persistence diagram of $\mathcal{R}^r(P)$ for $r \geq \varepsilon + 2\varepsilon_0$, for any $\varepsilon < \varepsilon_0 < \frac{\text{reach}(K)}{8}$.

Consider

$$\check{\mathcal{C}}^{\varepsilon} \longrightarrow \mathcal{R}^{\varepsilon} \longrightarrow \underbrace{\check{\mathcal{C}}^{\varepsilon+\varepsilon_0}}_{\varepsilon+\varepsilon_0>2\varepsilon} \longrightarrow \check{\mathcal{C}}^{\varepsilon+2\varepsilon_0} \longrightarrow \mathcal{R}^{\varepsilon+2\varepsilon_0} \longrightarrow \check{\mathcal{C}}^{2\varepsilon+4\varepsilon_0}$$

Theorem

Let P be an ε-sample of a compact K s.t. $0 < \varepsilon < \frac{\mathrm{reach}(K)}{8}$. The homology of K can be read off the persistence diagram of $\mathcal{R}^r(P)$ for $r \geq \varepsilon + 2\varepsilon_0$, for any $\varepsilon < \varepsilon_0 < \frac{\mathrm{reach}(K)}{8}$.

Consider

$$\check{\mathcal{C}}^{\varepsilon} \longrightarrow \mathcal{R}^{\varepsilon} \longrightarrow \underbrace{\check{\mathcal{C}}^{\varepsilon+\varepsilon_0}}_{\varepsilon+\varepsilon_0>2\varepsilon} \longrightarrow \check{\mathcal{C}}^{\varepsilon+2\varepsilon_0} \longrightarrow \mathcal{R}^{\varepsilon+2\varepsilon_0} \longrightarrow \check{\mathcal{C}}^{2\varepsilon+4\varepsilon_0}$$

Now, with $2\varepsilon + 4\varepsilon_0 < \mathrm{reach}(K)$,

$$\begin{aligned}\mathrm{rk}\ \left(\mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon}) \to \mathbf{H}_d(\check{\mathcal{C}}^{2\varepsilon+4\varepsilon_0})\right) &= \mathrm{rk}\ (\mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon+\varepsilon_0}) \to \mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon+2\varepsilon_0}) \\ &= \dim \mathbf{H}_d(\mathbf{K}).\end{aligned}$$

Theorem

Let P be an ε-sample of a compact K s.t. $0 < \varepsilon < \frac{\mathrm{reach}(K)}{8}$. The homology of K can be read off the persistence diagram of $\mathcal{R}^r(P)$ for $r \geq \varepsilon + 2\varepsilon_0$, for any $\varepsilon < \varepsilon_0 < \frac{\mathrm{reach}(K)}{8}$.

Consider

$$\check{\mathcal{C}}^{\varepsilon} \longrightarrow \mathcal{R}^{\varepsilon} \longrightarrow \underbrace{\check{\mathcal{C}}^{\varepsilon+\varepsilon_0}}_{\varepsilon+\varepsilon_0>2\varepsilon} \longrightarrow \check{\mathcal{C}}^{\varepsilon+2\varepsilon_0} \longrightarrow \mathcal{R}^{\varepsilon+2\varepsilon_0} \longrightarrow \check{\mathcal{C}}^{2\varepsilon+4\varepsilon_0}$$

Now, with $2\varepsilon + 4\varepsilon_0 < \mathrm{reach}(K)$,

$$\begin{aligned}\mathrm{rk}\ \left(\mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon}) \to \mathbf{H}_d(\check{\mathcal{C}}^{2\varepsilon+4\varepsilon_0})\right) &= \mathrm{rk}\ \left(\mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon+\varepsilon_0}) \to \mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon+2\varepsilon_0}\right) \\ &= \dim \mathbf{H}_d(\mathbf{K}).\end{aligned}$$

$$\Longrightarrow \mathrm{rk}\ \left(\mathbf{H}_d(\mathcal{R}^{\varepsilon}) \to \mathbf{H}_d(\mathcal{R}^{\varepsilon+2\varepsilon_0})\right) = \dim \mathbf{H}_d(\mathbf{K}).$$

Theorem

Let P be an ε-sample of a compact K s.t. $0 < \varepsilon < \frac{\mathrm{reach}(K)}{8}$. The homology of K can be read off the persistence diagram of $\mathcal{R}^r(P)$ for $r \geq \varepsilon + 2\varepsilon_0$, for any $\varepsilon < \varepsilon_0 < \frac{\mathrm{reach}(K)}{8}$.

Consider

$$\check{\mathcal{C}}^{\varepsilon} \longrightarrow \mathcal{R}^{\varepsilon} \longrightarrow \underbrace{\check{\mathcal{C}}^{\varepsilon+\varepsilon_0}}_{\varepsilon+\varepsilon_0 > 2\varepsilon} \longrightarrow \check{\mathcal{C}}^{\varepsilon+2\varepsilon_0} \longrightarrow \mathcal{R}^{\varepsilon+2\varepsilon_0} \longrightarrow \check{\mathcal{C}}^{2\varepsilon+4\varepsilon_0}$$

Now, with $2\varepsilon + 4\varepsilon_0 < \mathrm{reach}(K)$,

$$\begin{aligned} \mathrm{rk}\ \left(\mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon}) \to \mathbf{H}_d(\check{\mathcal{C}}^{2\varepsilon+4\varepsilon_0})\right) &= \mathrm{rk}\ (\mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon+\varepsilon_0}) \to \mathbf{H}_d(\check{\mathcal{C}}^{\varepsilon+2\varepsilon_0}) \\ &= \dim \mathbf{H}_d(\mathbf{K}). \end{aligned}$$

$$\Longrightarrow \mathrm{rk}\ \left(\mathbf{H}_d(\mathcal{R}^{\varepsilon}) \to \mathbf{H}_d(\mathcal{R}^{\varepsilon+2\varepsilon_0})\right) = \dim \mathbf{H}_d(\mathbf{K}).$$

In conclusion, $\dim \mathbf{H}_d(K)$ is exactly the number of points in the persistence diagram of the Rips filtration (up to $r = \varepsilon + 2\varepsilon_0$) in the upper left quadrant based at $(\varepsilon, \varepsilon + 2\varepsilon_0)$.

Morse filtration

Definition

Let $\mathbf{K}$ be a complex, equipped with a filtration,

$$\emptyset := \mathbf{K}_0 \xrightarrow{\mathfrak{S}_1} \mathbf{K}_1 \xrightarrow{\mathfrak{S}_2} \mathbf{K}_2 \xrightarrow{\mathfrak{S}_3} \dots \xrightarrow{\mathfrak{S}_m} \mathbf{K}_m = \mathbf{K}$$

where $\mathfrak{S}_i$ contains at least one simplex.

A Morse filtration for the filtration of $\mathbf{K}$ is a collection of Morse matchings $(X_i \sqcup T_i \sqcup S_i, \omega_i \colon T_i \to S_i)$ for each $\mathbf{K}_i$, $i = 0 \dots m$, satisfying:

$$X_i \subseteq X_{i+1}, \qquad T_i \subseteq T_{i+1}, \qquad S_i \subseteq S_{i+1},$$

$$\omega_{i+1}\big|_{T_i} = \omega_i, \qquad \partial^{X_{i+1}}\big|_{X_i} = \partial^{X_i}, \qquad \forall i = 0 \dots m-1.$$

Morse filtration

Theorem

Let $\mathbf{K}$ *be a complex, equipped with a filtration,*

$$\emptyset := \mathbf{K}_0 \xrightarrow{\mathfrak{S}_1} \mathbf{K}_1 \xrightarrow{\mathfrak{S}_2} \mathbf{K}_2 \xrightarrow{\mathfrak{S}_3} \dots \xrightarrow{\mathfrak{S}_m} \mathbf{K}_m = \mathbf{K}$$

and a Morse filtration $(X_i \sqcup T_i \sqcup S_i, \omega_i \colon T_i \to S_i)$*,* $i = 0 \dots m$*:*

$$X_i \subseteq X_{i+1}, \qquad T_i \subseteq T_{i+1}, \qquad S_i \subseteq S_{i+1},$$
$$\omega_{i+1}\big|_{T_i} = \omega_i, \qquad \partial^{X_{i+1}}\Big|_{X_i} = \partial^{X_i}, \qquad \forall i = 0 \dots m-1.$$

Then the following persistence modules:

$$\mathbf{H}(\mathbf{K}_0, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_1, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_2, \partial) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_m, \partial)$$
$$\mathbf{H}(X_0, \partial^{X_0}) \xrightarrow{\subseteq} \mathbf{H}(X_1, \partial^{X_1}) \xrightarrow{\subseteq} \mathbf{H}(X_2, \partial^{X_2}) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(X_m, \partial^{X_m})$$

are isomorphic.

Morse filtration

Theorem

The following persistence modules are isomorphic:

$$\mathbf{H}(\mathbf{K}_0, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_1, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_2, \partial) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_m, \partial)$$

$$\mathbf{H}(X_0, \partial^{X_0}) \xrightarrow{\subseteq} \mathbf{H}(X_1, \partial^{X_1}) \xrightarrow{\subseteq} \mathbf{H}(X_2, \partial^{X_2}) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(X_m, \partial^{X_m})$$

Morse filtration

Theorem

The following persistence modules are isomorphic:

$$\mathbf{H}(\mathbf{K}_0, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_1, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_2, \partial) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_m, \partial)$$

$$\mathbf{H}(X_0, \partial^{X_0}) \xrightarrow{\subseteq} \mathbf{H}(X_1, \partial^{X_1}) \xrightarrow{\subseteq} \mathbf{H}(X_2, \partial^{X_2}) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(X_m, \partial^{X_m})$$

Sketch of proof:

$$X_i \subseteq X_{i+1}, \quad T_i \subseteq T_{i+1}, \quad S_i \subseteq S_{i+1}, \quad \omega_{i+1}\big|_{T_i} = \omega_i, \quad \partial^{X_{i+1}}\big|_{X_i} = \partial^{X_i},$$

Morse filtration

Theorem

The following persistence modules are isomorphic:

$$\mathbf{H}(\mathbf{K}_0,\partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_1,\partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_2,\partial) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_m,\partial)$$

$$\mathbf{H}(X_0,\partial^{X_0}) \xrightarrow{\subseteq} \mathbf{H}(X_1,\partial^{X_1}) \xrightarrow{\subseteq} \mathbf{H}(X_2,\partial^{X_2}) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(X_m,\partial^{X_m})$$

Sketch of proof:

$$X_i \subseteq X_{i+1}, \quad T_i \subseteq T_{i+1}, \quad S_i \subseteq S_{i+1}, \quad \omega_{i+1}|_{T_i} = \omega_i, \quad \partial^{X_{i+1}}\Big|_{X_i} = \partial^{X_i},$$

implies that:

$$(\mathbf{C}(X_0),\partial^{X_0}) \xrightarrow{\subseteq} (\mathbf{C}(X_1),\partial^{X_1}) \xrightarrow{\subseteq} (\mathbf{C}(X_2),\partial^{X_2}) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} (\mathbf{C}(X_m),\partial^{X_m})$$

is well-defined filtration of complexes

Morse filtration

Theorem

The following persistence modules are isomorphic:

$$\mathbf{H}(\mathbf{K}_0,\partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_1,\partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_2,\partial) \xrightarrow{\subseteq} \ldots \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_m,\partial)$$

$$\mathbf{H}(X_0,\partial^{X_0}) \xrightarrow{\subseteq} \mathbf{H}(X_1,\partial^{X_1}) \xrightarrow{\subseteq} \mathbf{H}(X_2,\partial^{X_2}) \xrightarrow{\subseteq} \ldots \xrightarrow{\subseteq} \mathbf{H}(X_m,\partial^{X_m})$$

Sketch of proof:

$$X_i \subseteq X_{i+1}, \quad T_i \subseteq T_{i+1}, \quad S_i \subseteq S_{i+1}, \quad \omega_{i+1}\big|_{T_i} = \omega_i, \quad \partial^{X_{i+1}}\big|_{X_i} = \partial^{X_i},$$

implies that:

$$(\mathbf{C}(X_0),\partial^{X_0}) \xrightarrow{\subseteq} (\mathbf{C}(X_1),\partial^{X_1}) \xrightarrow{\subseteq} (\mathbf{C}(X_2),\partial^{X_2}) \xrightarrow{\subseteq} \ldots \xrightarrow{\subseteq} (\mathbf{C}(X_m),\partial^{X_m})$$

is well-defined filtration of complexes inducing the persist. module:

$$\mathbf{H}(X_0,\partial^{X_0}) \xrightarrow{\subseteq} \mathbf{H}(X_1,\partial^{X_1}) \xrightarrow{\subseteq} \mathbf{H}(X_2,\partial^{X_2}) \xrightarrow{\subseteq} \ldots \xrightarrow{\subseteq} \mathbf{H}(X_m,\partial^{X_m})$$

Morse filtration

Theorem

The following persistence modules are isomorphic:

$$\mathbf{H}(\mathbf{K}_0, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_1, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_2, \partial) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_m, \partial)$$

$$\mathbf{H}(X_0, \partial^{X_0}) \xrightarrow{\subseteq} \mathbf{H}(X_1, \partial^{X_1}) \xrightarrow{\subseteq} \mathbf{H}(X_2, \partial^{X_2}) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(X_m, \partial^{X_m})$$

Sketch of proof:

For each *i* there is a chain equivalence:

$$(\mathbf{C}(\mathbf{K}_i), \partial) \xrightarrow{\phi_i} (\mathbf{C}(X_i), \partial^{X_i}) \xrightarrow{\psi_i} (\mathbf{C}(\mathbf{K}_i), \partial^{X_i})$$

Morse filtration

Theorem

The following persistence modules are isomorphic:

$$\mathbf{H}(\mathbf{K}_0,\partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_1,\partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_2,\partial) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_m,\partial)$$

$$\mathbf{H}(X_0,\partial^{X_0}) \xrightarrow{\subseteq} \mathbf{H}(X_1,\partial^{X_1}) \xrightarrow{\subseteq} \mathbf{H}(X_2,\partial^{X_2}) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(X_m,\partial^{X_m})$$

Sketch of proof:

For each i there is a chain equivalence:

$$(\mathbf{C}(\mathbf{K}_i),\partial) \xrightarrow{\phi_i} (\mathbf{C}(X_i),\partial^{X_i}) \xrightarrow{\psi_i} (\mathbf{C}(\mathbf{K}_i),\partial^{X_i})$$

where each chain map commutes with inclusion:

$$\begin{tikzcd}
\mathbf{C}(\mathbf{K}_i) \arrow[r, "\subseteq"] \arrow[d, "\phi_i"'] & \mathbf{C}(\mathbf{K}_{i+1}) \arrow[d, "\phi_{i+1}"] \\
\mathbf{C}(X_i) \arrow[r, "\subseteq"] & \mathbf{C}(X_{i+1})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C}(\mathbf{K}_i) \arrow[r, "\subseteq"] & \mathbf{C}(\mathbf{K}_{i+1}) \\
\mathbf{C}(X_i) \arrow[r, "\subseteq"] \arrow[u, "\psi_i"] & \mathbf{C}(X_{i+1}) \arrow[u, "\psi_{i+1}"']
\end{tikzcd}$$

Morse filtration

Theorem

The following persistence modules are isomorphic:

$$\mathbf{H}(\mathbf{K}_0, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_1, \partial) \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_2, \partial) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(\mathbf{K}_m, \partial)$$

$$\mathbf{H}(X_0, \partial^{X_0}) \xrightarrow{\subseteq} \mathbf{H}(X_1, \partial^{X_1}) \xrightarrow{\subseteq} \mathbf{H}(X_2, \partial^{X_2}) \xrightarrow{\subseteq} \dots \xrightarrow{\subseteq} \mathbf{H}(X_m, \partial^{X_m})$$

Sketch of proof:
For each i there is a chain equivalence:

$$(\mathbf{C}(\mathbf{K}_i), \partial) \xrightarrow{\phi_i} (\mathbf{C}(X_i), \partial^{X_i}) \xrightarrow{\psi_i} (\mathbf{C}(\mathbf{K}_i), \partial^{X_i})$$

$\longrightarrow$ enough prove commutativity for addition of a Morse pair (τ, σ):

$$(\mathbf{C}(X), \partial^{X}) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^{X})$$

with $X' = X - \{\sigma, \tau\}$ as used previously (same chain maps).

Topology Inference Problem