

## 3 - Persistent homology

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EMAp Summer Program 2023

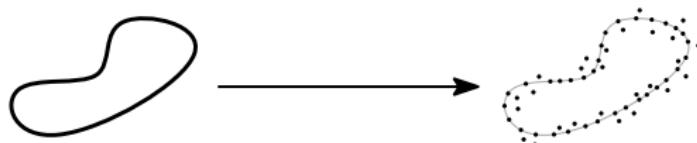
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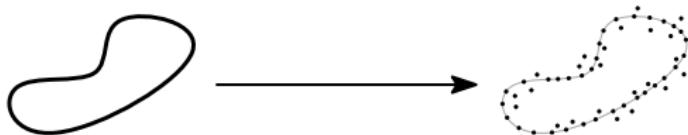
# Topological Inference Problem



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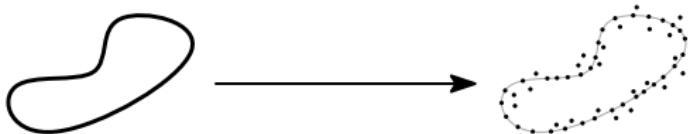
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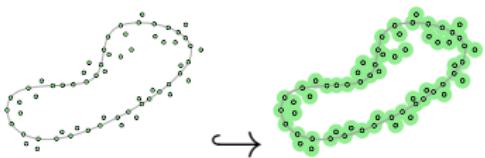
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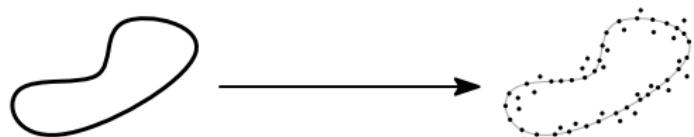
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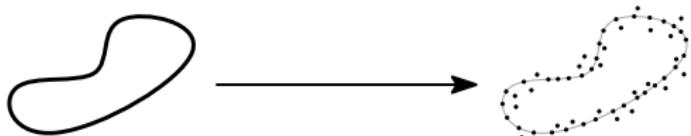
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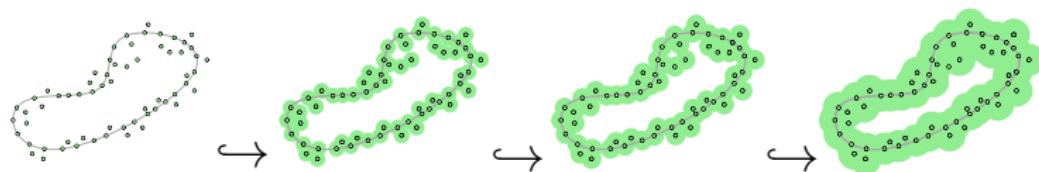
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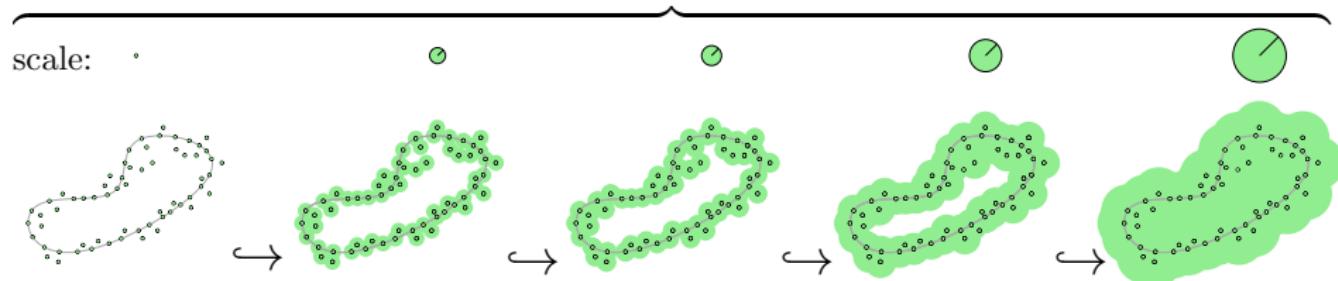
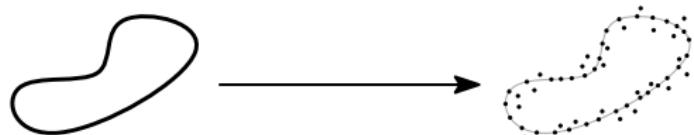
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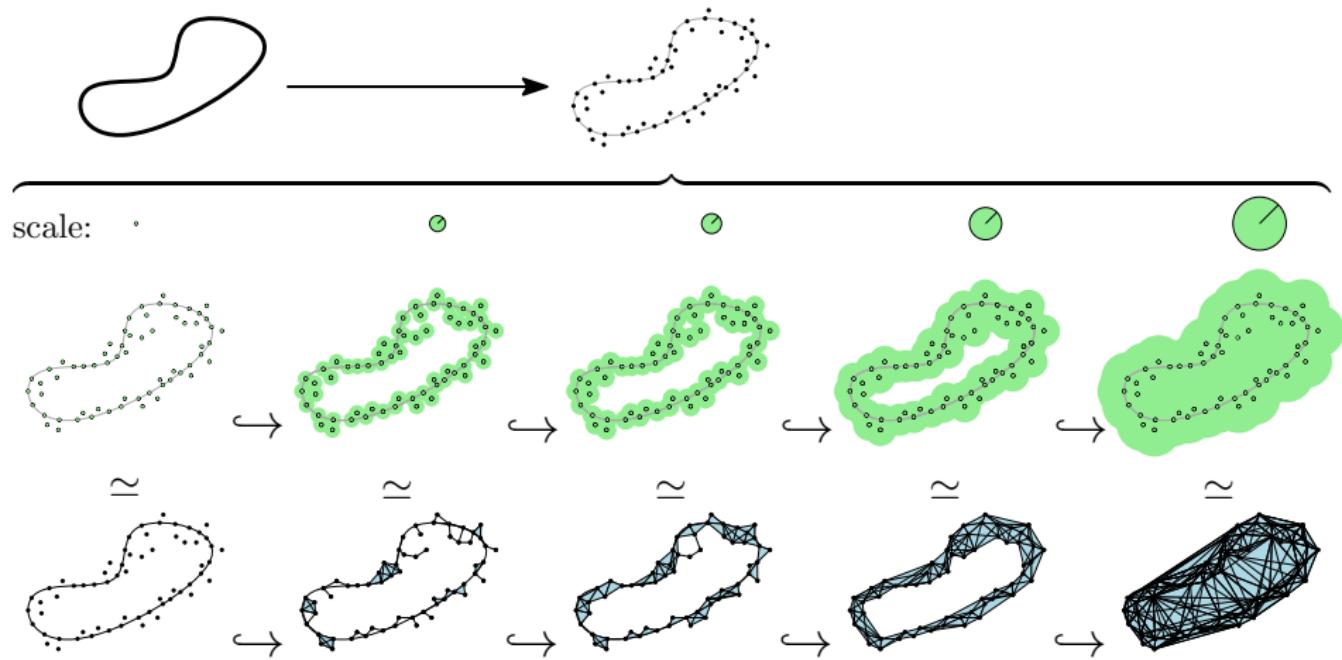
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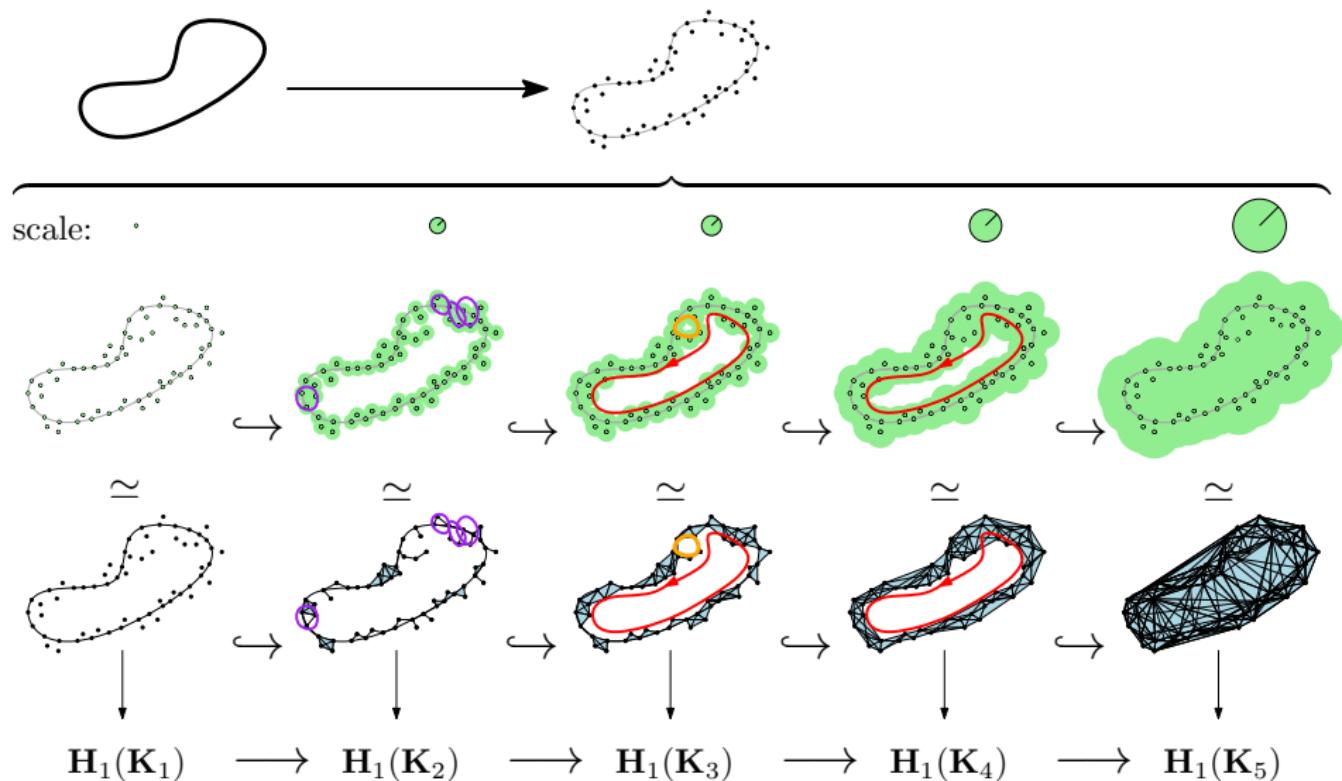
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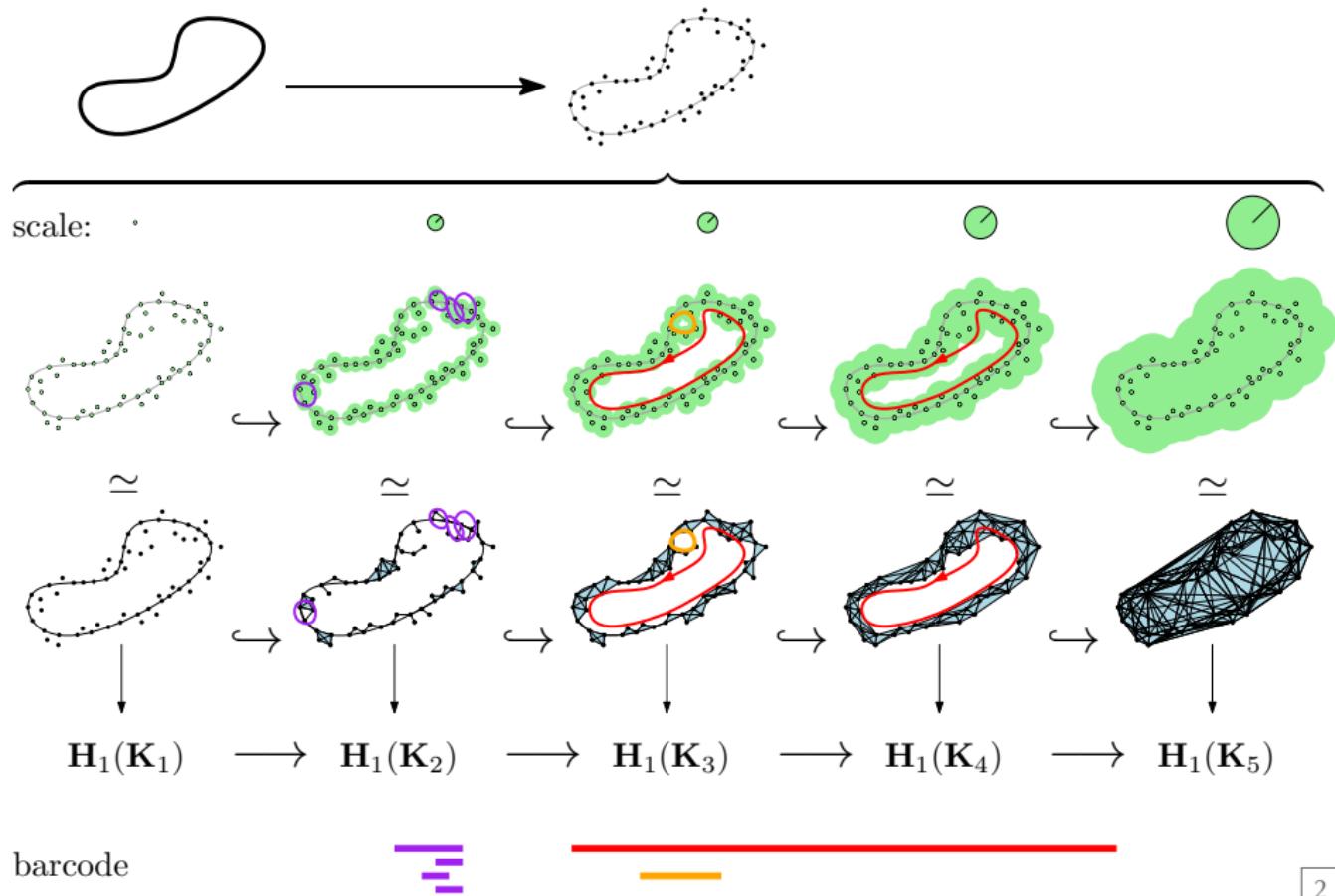
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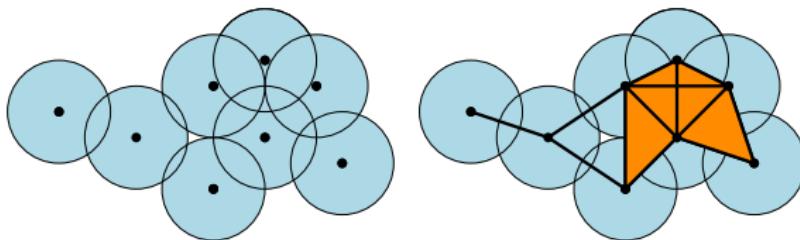


## Cech complex and Nerve

### Definition (Cech complex)

Let  $P = \{p_1, \dots, p_n\}$  be a set of points in  $\mathbb{R}^D$ , and denote by  $\mathcal{B}_r(p)$  the closed ball of radius  $r$  centered at  $p$ . The **Cech complex** of parameter  $r$  on  $P$  is the simplicial complex  $\check{\mathcal{C}}^r(P)$  satisfying:

$$\sigma = \{p_{i_0}, \dots, p_{i_d}\} \in \check{\mathcal{C}}^r(P) \text{ iff } \bigcap_{j=0 \dots d} \mathcal{B}_r(p_{i_j}) \neq \emptyset$$

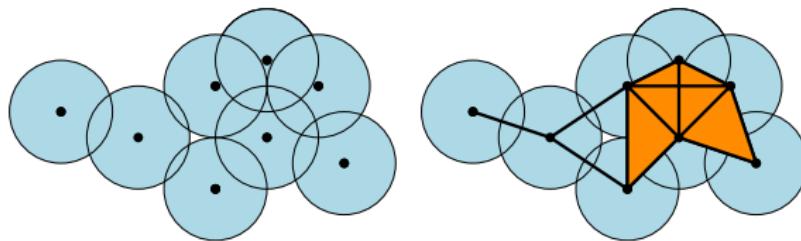


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### Lemma (Nerve)

The domains  $\bigcup_{p_i \in P} \mathcal{B}_r(p_i)$  and  $|\check{\mathcal{C}}^r(P)|$  are homotopy equivalent.

## Cech complex and Nerve

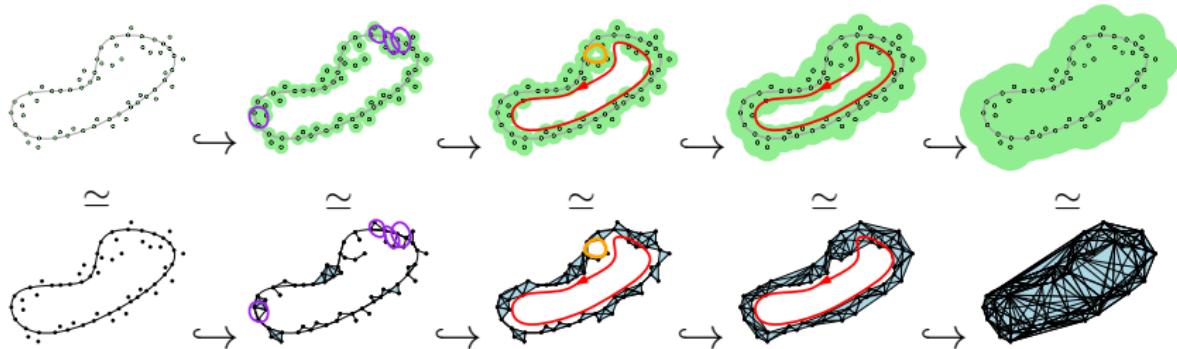
If  $P$  finite,  $|\check{\mathcal{C}}^r(P)|$ ,  $r \geq 0$ , changes homotopy type a finite number of times for  $r = r_1, \dots, r_k$ . Consider  $\alpha_0, \dots, \alpha_k$  such that:

$$0 < \alpha_0 < r_1 < \alpha_1 < \dots < r_k < \alpha_k.$$

We study the sequence:

$$\check{\mathcal{C}}^{\alpha_0}(P) \longrightarrow \check{\mathcal{C}}^{\alpha_1}(P) \longrightarrow \check{\mathcal{C}}^{\alpha_2}(P) \longrightarrow \dots \longrightarrow \check{\mathcal{C}}^{\alpha_k}(P)$$

$$\mathbf{H}_\bullet(\check{\mathcal{C}}^{\alpha_0}(P)) \longrightarrow \mathbf{H}_\bullet(\check{\mathcal{C}}^{\alpha_1}(P)) \longrightarrow \mathbf{H}_\bullet(\check{\mathcal{C}}^{\alpha_2}(P)) \longrightarrow \dots \longrightarrow \mathbf{H}_\bullet(\check{\mathcal{C}}^{\alpha_k}(P))$$



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For a fixed set of indices  $\{1, \dots, n\}$  (or real numbers  $\alpha_1 < \dots < \alpha_n$ ), a **persistence module**, denote by  $\mathbb{V}$ , is a collection  $\{V_1, \dots, V_n\}$  of finite dimensional  $\mathbb{F}$ -vector spaces, and linear maps  $f_i: V_i \rightarrow V_{i+1}$ ,  $1 \leq i < n$ :

$$\mathbb{V} = \quad V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} \dots \xrightarrow{f_{n-1}} V_n$$

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## Definition (Morphism of persistence modules)

Let  $\mathbb{V} = (V_i, f_i)$ ,  $\mathbb{W} = (W_i, g_i)$ ,  $i = 1 \dots n$  be two persistence modules. A **morphism** of persistence modules  $\varphi: \mathbb{V} \rightarrow \mathbb{W}$  is a collection of linear maps  $\{\varphi_i: V_i \rightarrow W_i\}_{i=1\dots n}$  such that each square:

$$\begin{array}{ccc} V_i & \xrightarrow{f_i} & V_{i+1} \\ \varphi_i \downarrow & \circ & \downarrow \varphi_{i+1} \\ W_i & \xrightarrow{g_i} & W_{i+1} \end{array}$$

commutes for  $i = 1 \dots n - 1$ .

$\varphi$  is an **isomorphism** of persistence modules if each  $\varphi_i$  is an isomorphism of vector spaces. We then denote  $\mathbb{V} \cong \mathbb{W}$ .

# Decomposition of persistence modules

Definition (Direct sum of persistence modules)

A **direct sum** of two persistence modules  $\mathbb{V} = (V_i, f_i)$  and  $\mathbb{W} = (W_i, g_i)$ ,  $i = 1 \dots n$ , is the persistence module:

$$\mathbb{V} \oplus \mathbb{W} := (V_i \oplus W_i, f_i \oplus g_i)_{i=1,\dots,n}, \text{ i.e.,:}$$

$$V_1 \oplus W_1 \xrightarrow{f_1 \oplus g_1} V_2 \oplus W_2 \xrightarrow{f_2 \oplus g_2} V_3 \oplus W_3 \xrightarrow{f_3 \oplus g_3} \dots \xrightarrow{f_{n-1} \oplus g_{n-1}} V_n \oplus W_n$$

A persistence module  $\mathbb{V}$  is **indecomposable** iff

$$\mathbb{V} \cong \mathbb{V}_1 \oplus \mathbb{V}_2 \implies \mathbb{V}_1 = 0 \text{ or } \mathbb{V}_2 = 0.$$

# Decomposition

## Theorem (Decomposition)

Every persistence module  $\mathbb{V}$  over the indices  $\{1, \dots, n\}$  admits a unique decomposition into *indecomposable* persistence modules of the form:

$$\mathbb{V} \cong \bigoplus_i \mathbb{I}[b_i; d_i], \quad 1 \leq b_i \leq d_i \leq n,$$

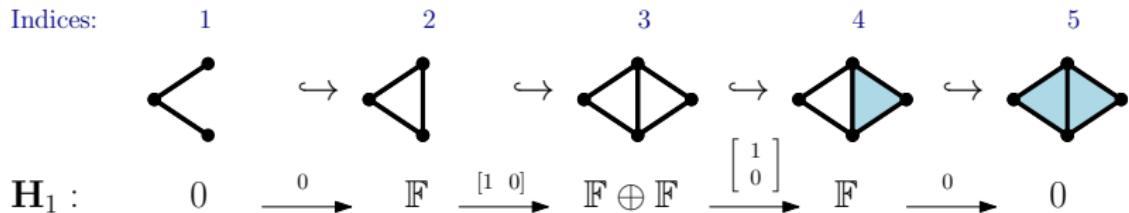
called *interval modules*, such that

$$\mathbb{I}[b; d] :=$$

$$\underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0} 0}_{\text{indices: } [1; b-1]} \xrightarrow{0} \underbrace{\mathbb{F} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \mathbb{F}}_{[b; d]} \xrightarrow{0} \underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0} 0}_{[d+1; n]}$$

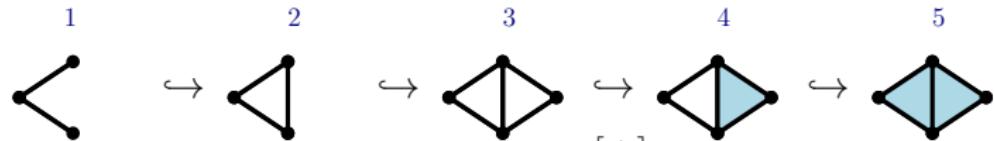
# Representation of interval decompositions

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$H_1 :$

$$\cong \left\{ \begin{array}{ccccccc} 0 & \xrightarrow{0} & \mathbb{F} & \xrightarrow{[1 \ 0]} & \mathbb{F} \oplus \mathbb{F} & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & \mathbb{F} \\ \oplus & & \oplus & & \oplus & & \oplus \\ 0 & \xrightarrow{0} & 0 & \xrightarrow{[0]} & \mathbb{F} & \xrightarrow{[0]} & 0 \end{array} \right.$$

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$$\mathbf{H}_1 : \quad \begin{array}{ccccccc} 1 & \hookrightarrow & 2 & \hookrightarrow & 3 & \hookrightarrow & 4 & \hookrightarrow & 5 \\ 0 & \xrightarrow{0} & \mathbb{F} & \xrightarrow{[1 \ 0]} & \mathbb{F} \oplus \mathbb{F} & \xrightarrow{\left[ \begin{array}{c} 1 \\ 0 \end{array} \right]} & \mathbb{F} & \xrightarrow{0} & 0 \end{array}$$

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barecode:

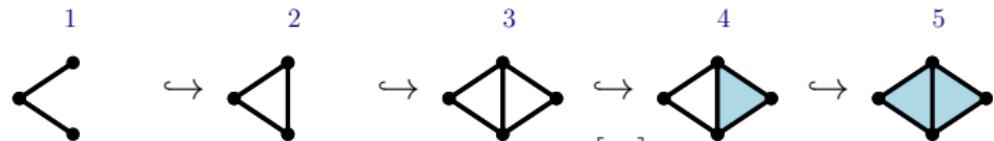
$$I[2; 4] \oplus I[3; 3]$$

## Definition (Representation of decompositions)

Let  $\mathbb{V} \cong \bigoplus_{i \in I} \mathbb{I}[b_i; d_i]$ . The **persistence barcode** of  $\mathbb{V}$  is the drawing of the collection of horizontal intervals  $[b_i; d_i]$ .

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$$\xrightarrow{0} 0$$

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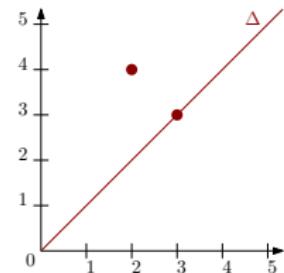
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# Algorithm for persistent homology

## Definition (Filtration)

A **filtration** of a simplicial complex  $\mathbf{K}$  is an ordering  $(\sigma_1, \dots, \sigma_n)$  of its simplices such that  $\mathbf{K}_i := \{\sigma_1, \dots, \sigma_i\}$  is a simplicial complex for all  $i = 1, \dots, n$ . We represent it by the sequence of elementary inclusions:

$$\emptyset := \mathbf{K}_0 \xrightarrow{\sigma_1} \mathbf{K}_1 \xrightarrow{\sigma_2} \mathbf{K}_2 \xrightarrow{\sigma_3} \dots \xrightarrow{\sigma_n} \mathbf{K}_n = \mathbf{K}$$

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Example: Filtration induced by scale in the Čech complex, where each inclusion is *decomposed* further into inclusions of one simplex at a time:

$$\check{\mathcal{C}}^{\alpha_0}(P) \longrightarrow \check{\mathcal{C}}^{\alpha_1}(P) \longrightarrow \check{\mathcal{C}}^{\alpha_2}(P) \longrightarrow \dots \longrightarrow \check{\mathcal{C}}^{\alpha_k}(P)$$

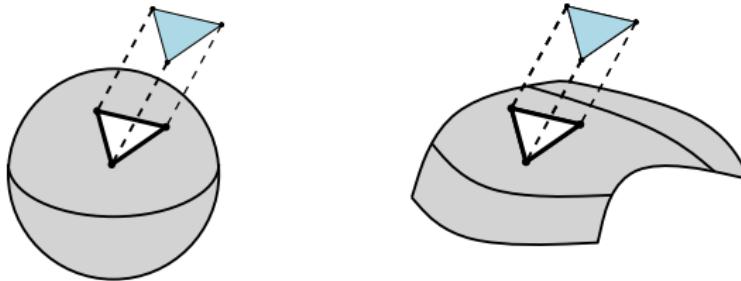


# Algorithm for persistent homology

## Lemma (Elementary inclusion)

Let  $\mathbf{K} \rightarrow \mathbf{K} \cup \{\sigma\}$  be an elementary inclusion, with  $\dim \sigma = d$ . Then:

- either  $\dim \mathbf{H}_{d-1}(\mathbf{K} \cup \{\sigma\}) = \dim \mathbf{H}_{d-1}(\mathbf{K}) - 1$ ,
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Additionally,

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- $\mathbf{B}_d(\mathbf{K}) \subset \mathbf{B}_d(\mathbf{K} \cup \{\sigma\}) \Rightarrow \dim \mathbf{B}_d(\mathbf{K}) \leq \dim \mathbf{B}_d(\mathbf{K} \cup \{\sigma\})$ .

Additionally,

$$\mathbf{C}_d(\mathbf{K} \cup \{\sigma\}) = \mathbf{C}_d(\mathbf{K}) \oplus \langle \sigma \rangle \Rightarrow \dim \mathbf{C}_d(\mathbf{K} \cup \{\sigma\}) = \dim \mathbf{C}_d(\mathbf{K}) + 1.$$

Finally, we have:

$$\dim \mathbf{C}_d(\mathbf{K} \cup \{\sigma\}) = \overbrace{\dim \mathbf{Z}_d(\mathbf{K} \cup \{\sigma\})}^{\dim \ker \partial_d} + \overbrace{\dim \mathbf{B}_{d-1}(\mathbf{K} \cup \{\sigma\})}^{\dim \text{im } \partial_d}, \text{ so:}$$

- either  $\dim \mathbf{B}_{d-1}(\mathbf{K} \cup \{\sigma\}) = \dim \mathbf{B}_{d-1}(\mathbf{K}) + 1$ ,
- or  $\dim \mathbf{Z}_d(\mathbf{K} \cup \{\sigma\}) = \dim \mathbf{Z}_d(\mathbf{K}) + 1$ .

We conclude with  $\dim \mathbf{H}_d = \dim \mathbf{Z}_d - \dim \mathbf{B}_d$ .

# Algorithm for persistent homology

## Lemma

Let  $\mathbf{K} \rightarrow \mathbf{K} \cup \{\sigma\}$  and  $\dim \sigma = d$ . We have:

$$\dim \mathbf{H}_d(\mathbf{K} \cup \{\sigma\}) = \dim \mathbf{H}_d(\mathbf{K}) + 1 \quad \text{iff} \quad [\partial\sigma] = 0 \quad \text{in } \mathbf{H}_d(\mathbf{K}).$$

In which case  $\sigma$  is a *creator*. Otherwise, we have:

$$\dim \mathbf{H}_{d-1}(\mathbf{K} \cup \{\sigma\}) = \dim \mathbf{H}_{d-1}(\mathbf{K}) - 1 \quad \text{and} \quad [\partial\sigma] \neq 0 \quad \text{in } \mathbf{H}_d(\mathbf{K}),$$

and  $\sigma$  is a *destructor*.

Note that  $\partial\sigma \in \mathbf{C}_{d-1}(\mathbf{K})$  because all subfaces of  $\sigma$  are in  $\mathbf{K}$ .

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Denote by  $\mathbf{C}(\mathbf{K}) := \bigoplus_{d \geq 0} \mathbf{C}_d(\mathbf{K})$  and  $\partial := \bigoplus_{d \geq 0} \partial_d: \mathbf{C}(\mathbf{K}) \rightarrow \mathbf{C}(\mathbf{K})$ .

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We call this data a **compatible basis** for the filtration of  $\mathbf{K}$ .

**NB:**  $c_i$  is a chain in  $\mathbf{K}$ . Talking about " $c_i$  in  $\mathbf{K}_i$ " means the *restriction of  $c_i$  in  $\mathbf{K}_i$* .

# Algorithm for persistent homology

## Theorem (Correctness)

Let  $\{c_1, \dots, c_n\}$ ,  $F \sqcup G \sqcup H$  and  $G \leftrightarrow H$  be a compatible basis for  $\mathbf{K}$ . Denote by  $\mathbb{H}(\mathbf{K})$  the persistence module induced by the filtration of  $\mathbf{K}$ :

$$\emptyset \xrightarrow{\sigma_1} \mathbf{K}_1 \xrightarrow{\sigma_2} \mathbf{K}_2 \xrightarrow{\sigma_3} \dots \xrightarrow{\sigma_n} \mathbf{K}_n$$

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then,

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**Claims** (algorithmic proof):

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## Claims (algorithmic proof):

- (a) A compatible basis exists, and  $\{c_1, \dots, c_i\}$  basis for  $\mathbf{C}(\mathbf{K}_i)$ ,

# Algorithm for persistent homology

## Theorem (Correctness)

Let  $\{c_1, \dots, c_n\}$ ,  $F \sqcup G \sqcup H$  and  $G \leftrightarrow H$  be a compatible basis for  $\mathbf{K}$ . Denote by  $\mathbb{H}(\mathbf{K})$  the persistence module induced by the filtration of  $\mathbf{K}$ :

$$\emptyset \xrightarrow{\sigma_1} \mathbf{K}_1 \xrightarrow{\sigma_2} \mathbf{K}_2 \xrightarrow{\sigma_3} \dots \xrightarrow{\sigma_n} \mathbf{K}_n$$
$$\mathbb{H}(\mathbf{K}) := 0 \longrightarrow \mathbf{H}(\mathbf{K}_1) \longrightarrow \mathbf{H}(\mathbf{K}_2) \longrightarrow \dots \longrightarrow \mathbf{H}(\mathbf{K}_n)$$

then,

$$\mathbb{H}(\mathbf{K}) \cong \bigoplus_{g \leftrightarrow h} \mathbb{I}[g; h - 1] \quad \bigoplus_{f \in F} \mathbb{I}[f; n].$$

## Claims (algorithmic proof):

- (a) A compatible basis exists, and  $\{c_1, \dots, c_i\}$  basis for  $\mathbf{C}(\mathbf{K}_i)$ ,  
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Suppose we have a compatible basis for  $\mathbf{K}_i$ , claims (a), (b), (c) are satisfied for the filtration  $\mathbf{K}_0 \rightarrow \dots \rightarrow \mathbf{K}_i$ . Next:  $\mathbf{K}_i \xrightarrow{\sigma_{i+1}} \mathbf{K}_{i+1}$

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This is a compatible basis for  $\mathbf{K}_0 \rightarrow \dots \rightarrow \mathbf{K}_{i+1}$ , satisfying (a),(b),(c).

exercise

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**if**  $[\partial\sigma_{i+1}] \neq 0$  in  $\mathbf{H}(\mathbf{K}_i)$ , then [ $\partial\sigma_{i+1}$  is a cycle]

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Let  $f_0 = \max\{f \in F : \varepsilon_f \neq 0\}$ .

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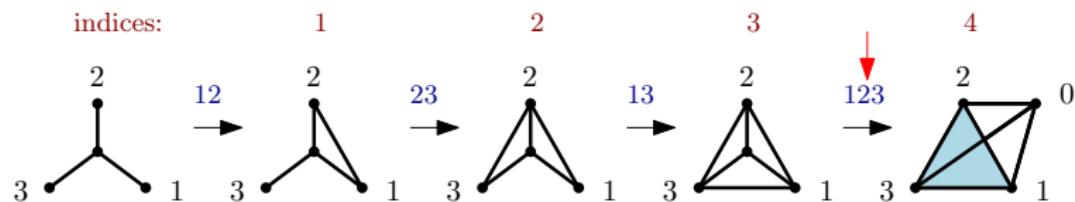
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Note that (a) is maintained by the choice of  $f_0$  as max.

**exercise**

# Algorithm for persistent homology

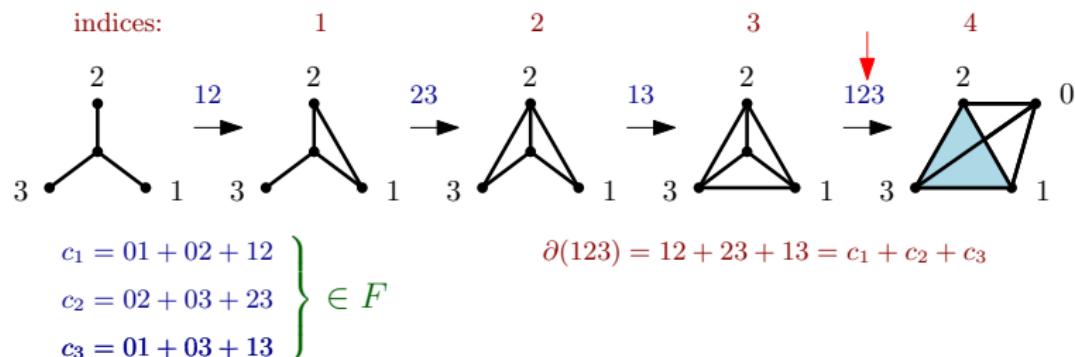
Example:



$$\left. \begin{array}{l} c_1 = 01 + 02 + 12 \\ c_2 = 02 + 03 + 23 \\ c_3 = 01 + 03 + 13 \end{array} \right\} \in F$$

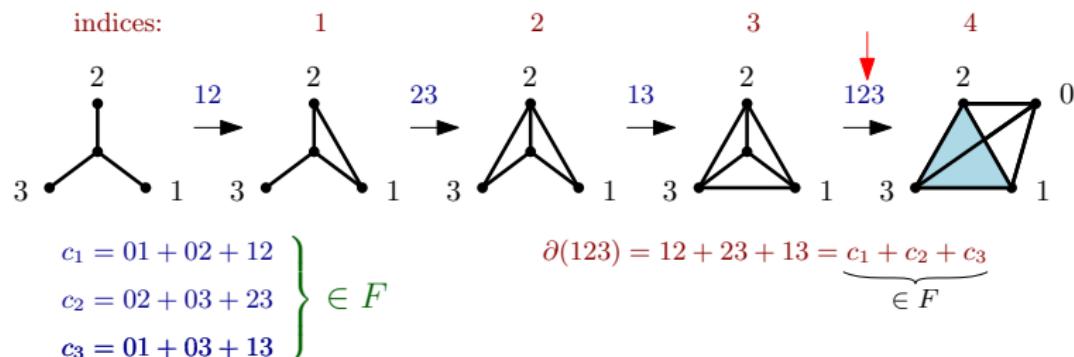
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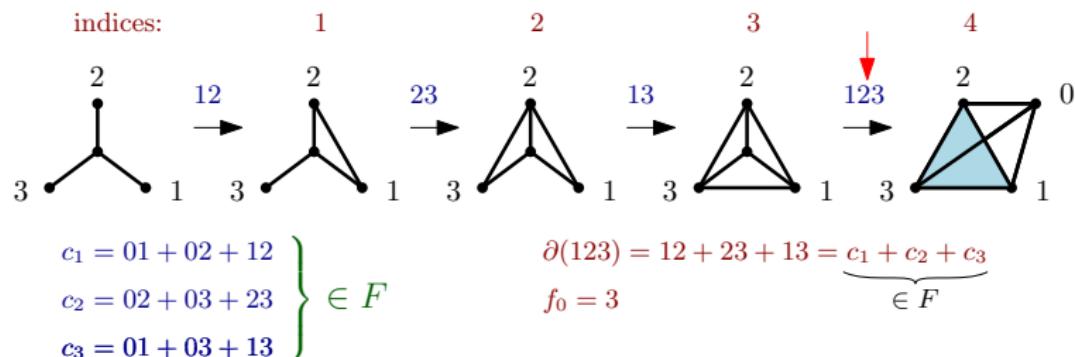
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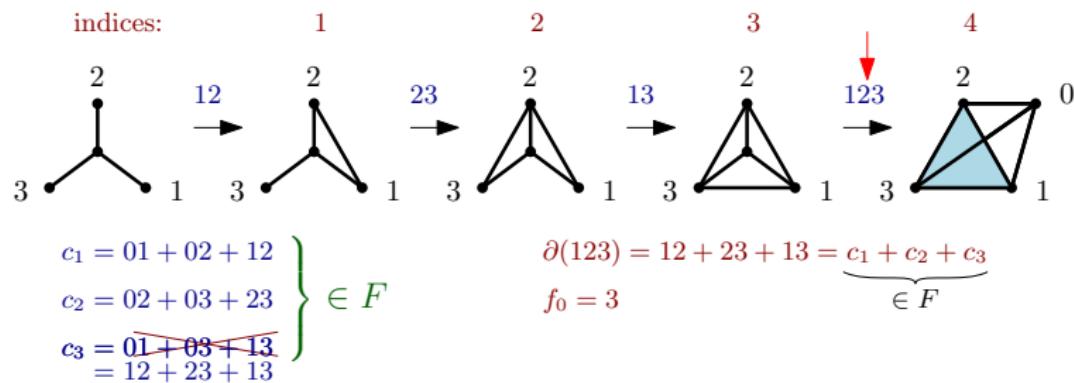
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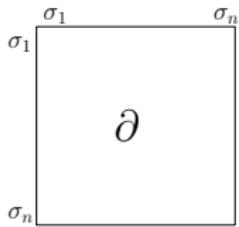
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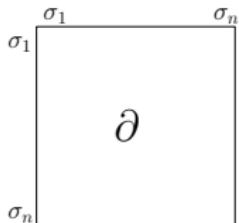
## Matrix implementation

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Undefined if column = 0.



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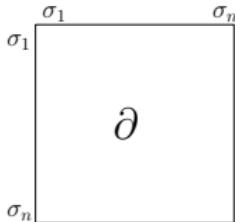
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- **For**  $i = 1 \dots n,$ 
  - ▶ **While**  $\exists j < i$  s.t.  $\text{low}_j = \text{low}_i,$ 
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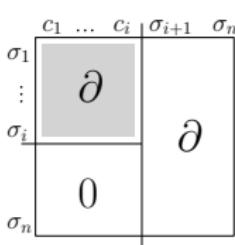
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**Invariant:** at the end of step  $i$ , we have:

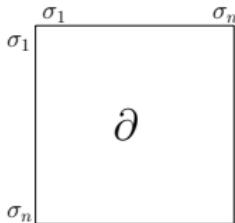


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i.e., there is a compatible basis  $c_1, \dots, c_i$  for  $\mathbf{K}_i$  such that  $\text{col}_k = \partial c_k, 1 \leq k \leq i.$

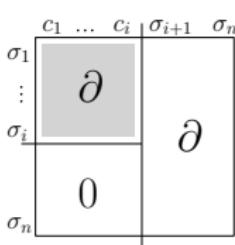
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$O(n^3)$

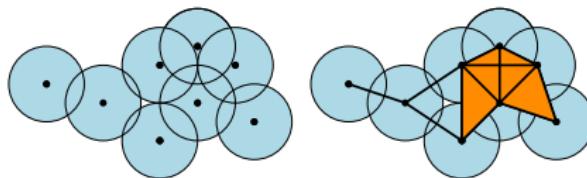
## Comparing geometric diagrams

If  $P$  finite,  $|\check{\mathcal{C}}^r(P)|$ ,  $r \geq 0$ , changes homotopy type a finite number of times for  $r = r_1, \dots, r_k$ . Consider  $\alpha_0, \dots, \alpha_k$  such that:

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Geometric filtration:

$$\check{\mathcal{C}}^{\alpha_0}(P) \longrightarrow \check{\mathcal{C}}^{\alpha_1}(P) \longrightarrow \check{\mathcal{C}}^{\alpha_2}(P) \longrightarrow \dots \longrightarrow \check{\mathcal{C}}^{\alpha_k}(P)$$



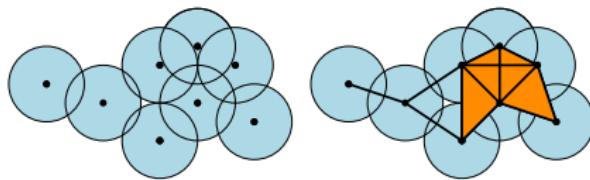
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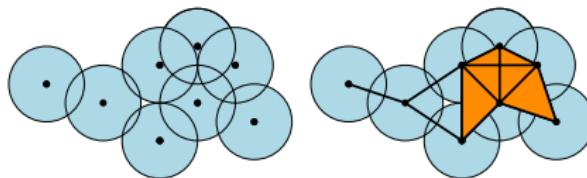
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→ use points  $(\alpha_b, \alpha_d)$  instead of  $(b, d)$  to get the **geometric persistence diagram**.

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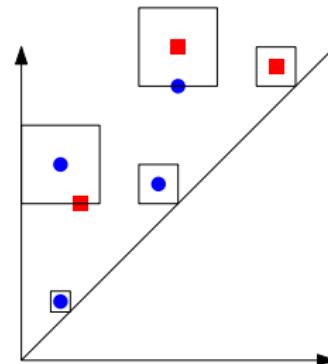
Let  $D = \{(b_i, d_i)\}_{i \in I} \cup \{x = y\}$  and  $D' = \{(b'_j, d'_j)\}_{j \in J} \cup \{x = y\}$  be two persistence diagrams,  $b, d \in \mathbb{R}$ .

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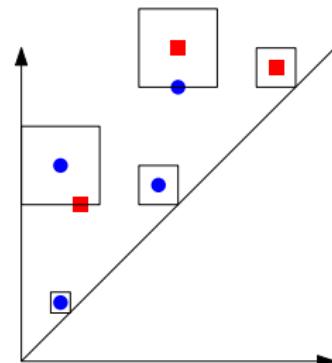
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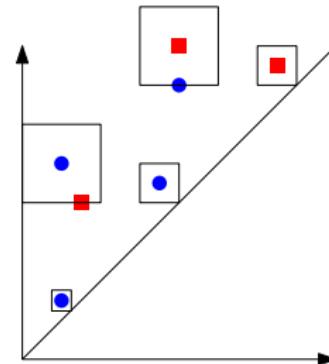
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**Theorem (Stability on a simplicial complex)**

Let  $f, g: \mathbf{K} \rightarrow \mathbb{R}$  be two functions on a same simplicial complex  $\mathbf{K}$ , inducing filtrations  $\mathbf{K}_\alpha = f^{-1}((-\infty; \alpha])$  and  $\mathbf{K}'_\gamma = g^{-1}((-\infty; \gamma])$ . Then

$$d_B(D(f), D(g)) \leq \|f - g\|_\infty.$$

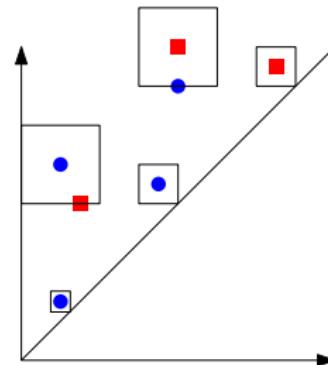
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## Theorem (Stability on general space)

Let  $f, g: M \rightarrow \mathbb{R}$  be two functions on a same metric space  $M$ , satisfying some “tameness” conditions. If  $\exists \varepsilon \geq 0$  s.t.  $\forall r \in \mathbb{R}$   $f^{-1}(-\infty; r] \subseteq g^{-1}(-\infty; r + \varepsilon]$  and  $g^{-1}(-\infty; r] \subseteq f^{-1}(-\infty; r + \varepsilon]$ , then  $d_B(D(f), D(g)) \leq \varepsilon$ .