2 - Discrete Morse theory

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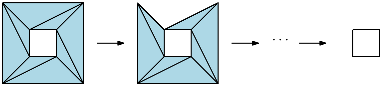
Elementary collapses

Simplification with elementary collapses:



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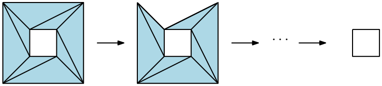
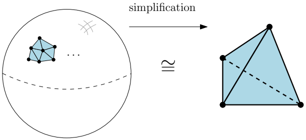


With Discrete Morse theory:



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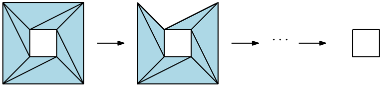

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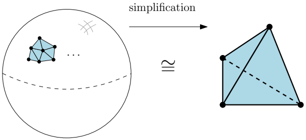
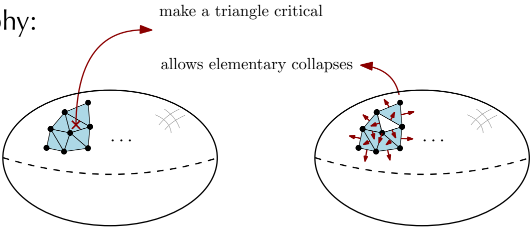


the philosophy:



Morse matching

Fix the coefficient field $\mathbb{Z}/2\mathbb{Z}$.

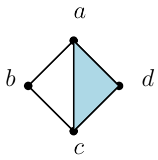
Definition

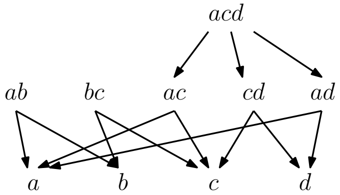
A simplicial complex $\mathbf{K}$ induces a *face partial ordering* $<$ between simplices. It is the transitive closure of the relation $\prec$

$$\tau \prec \sigma \quad \text{iff} \quad \tau \subset \sigma \text{ and } \dim \tau = \dim \sigma - 1.$$

The *Hasse diagram* of a complex is the directed graph (V, E) with:

- $V = \mathbf{K}$,
- $(\sigma \to \tau) \in E$ iff $\tau \prec \sigma$.





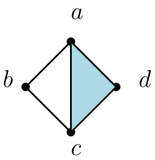
Morse matching

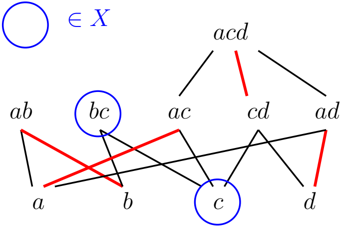
Definition

A *partial matching* of $\mathbf{K}$ is a partition $\mathbf{K} = X \sqcup T \sqcup S$ with a bijective pairing $\omega : T \to S$, where:

$$\tau \prec \omega(\tau), \text{ for all } \tau \in T.$$

$\longrightarrow$ it is a graph matching in the Hasse diagram of $\mathbf{K}$.





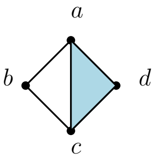
Morse matching

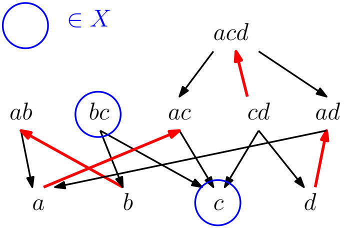
Definition

Let $\mathbf{K}$ be a simplicial complex, $\mathcal{H} = (\mathbf{K}, E)$ its Hasse diagram, and $(X \sqcup T \sqcup S, \omega)$ a partial matching. Define the directed graph $\bar{\mathcal{H}}$ as the graph $(\mathbf{K}, E')$:

- with same underlying undirected graph as $\mathcal{H}$,
- where every edge $(\omega(\tau) \to \tau) \in E$ is reversed in E', i.e., $(\tau \to \omega(\tau)) \in E'$.

If $\bar{\mathcal{H}}$ is acyclic, we call the partial matching a Morse matching.





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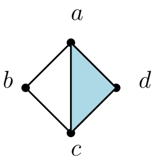

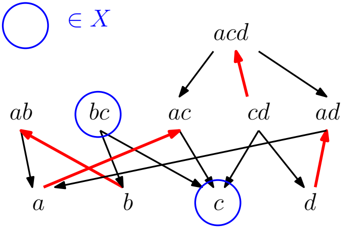

NB: a cycle can only alternate between two dim. d and $d+1$ (ω bij.).

Morse matching

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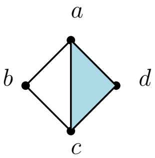
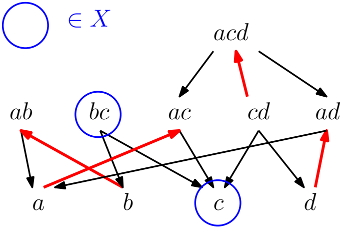
Let $\mathbf{K}$ be a simplicial complex, $\mathcal{H} = (\mathbf{K}, E)$ its Hasse diagram, and $(X \sqcup T \sqcup S, \omega)$ a Morse matching.

- Simplices in X are called critical.
- A Morse matching is optimal if it has the minimal number of critical simplices, over all possible Morse matchings of $\mathbf{K}$.





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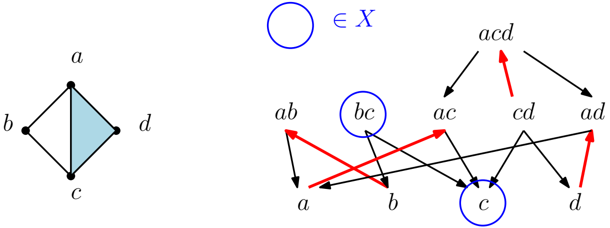
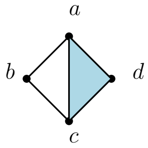
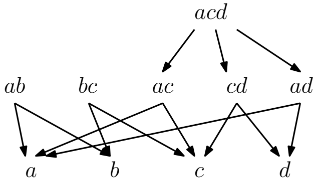
It is **NP-hard** to decide whether a simplicial complex has a Morse matching with less than k critical simplices.



Compute a (good) Morse matching: Heuristic





X :

$T \leftrightarrow S$

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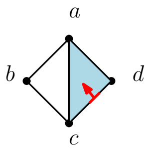

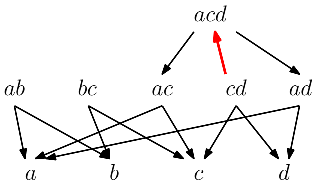

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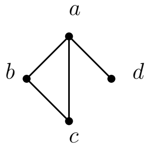

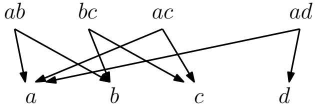

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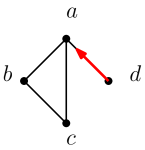

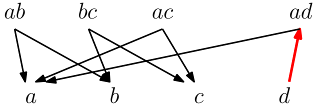

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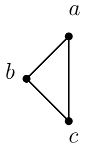

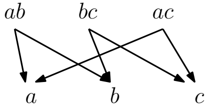

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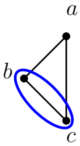

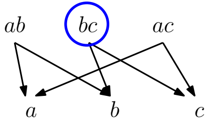

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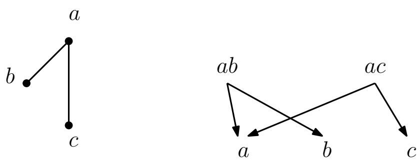

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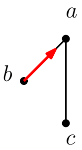

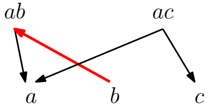

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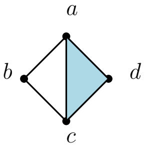

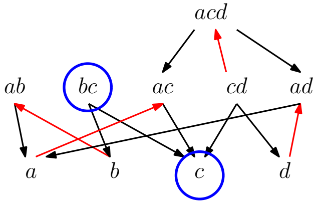

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Lemma (Correction)

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Lemma (Correction)

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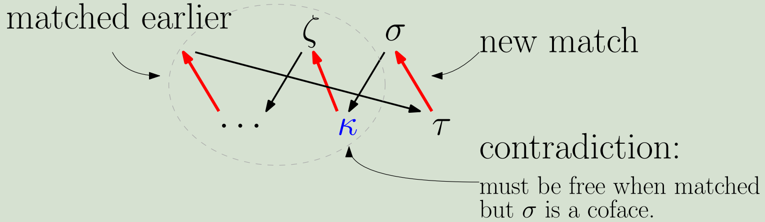
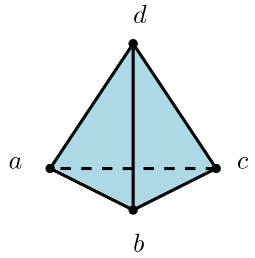
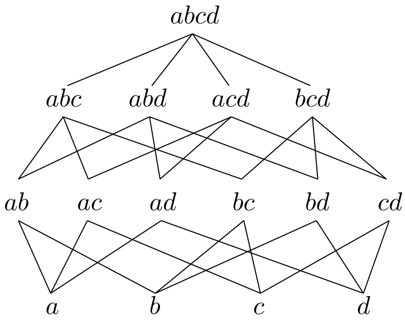
- *The matching is always acyclic: ...*

Suppose that, at some point, reverting an arrow creates a cycle. Pick the first such arrow (τ, σ):



More interesting example





What is the number of critical simplices of an optimal Morse matching?

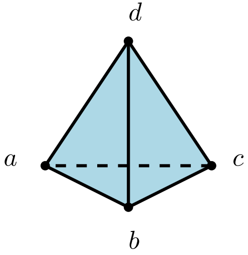
A 0

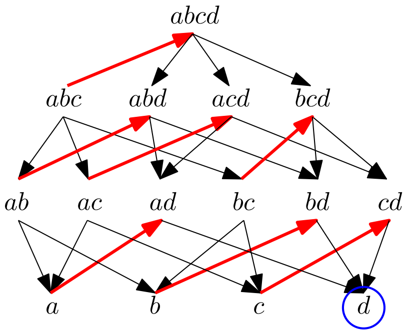
B 1

C 2

D 3

More interesting example





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Fundamental Theorem of Discrete Morse Theory

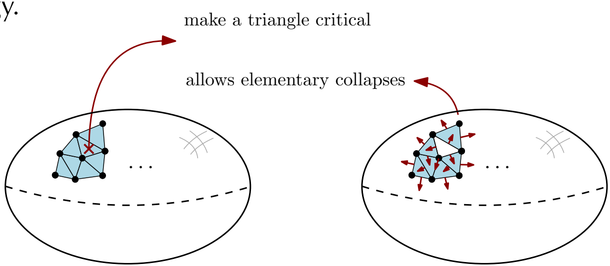
Theorem

Let $\mathbf{K}$ *be a simplicial complex, and* $(X \sqcup T \sqcup S, \omega)$ *a Morse matching. There exists a chain complex* $(\mathbf{C}(X), \partial^X)$ *with cells in* X*, called a Morse complex, whose homology groups are isomorphic to the ones of* $\mathbf{K}$.

More precisely,

- $\mathbf{C}_d(X)$ generated by the d-dimensional simplices of X,
- $\partial^X_d \colon \mathbf{C}_d(X) \to \mathbf{C}_{d-1}(X)$.

Application: reduce a complex drastically when computing its homology.



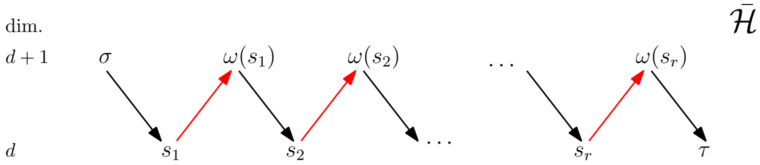
Boundary map ∂^X

Definition

In a complex $\mathbf{K}$ with Morse matching $(X \sqcup T \sqcup S, \omega)$, let $\sigma, \tau \in X$ be critical simplices, with $\dim \sigma = \dim \tau + 1$.
A gradient path from σ to τ is a directed path from σ to τ:

$$\sigma \to s_1 \to \omega(s_1) \to s_2 \to \ldots \to s_r \to \omega(s_r) \to \tau$$

in $\bar{\mathcal{H}}$, alternating between simplices of dimension d and $d-1$:



Boundary map ∂^X

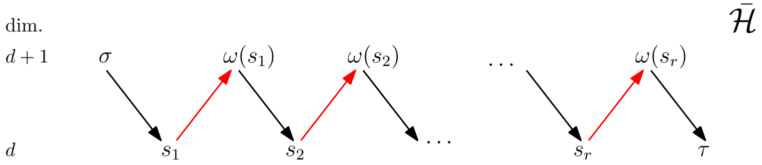
Definition

For $\sigma, \tau \in X$, with $\dim \sigma = \dim \tau + 1$, define

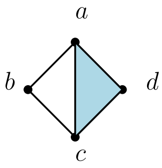
$$[\sigma : \tau] := \# \text{ distinct } (\sigma \to \tau)\text{-gradient path} \in \mathbb{Z}/2\mathbb{Z}.$$

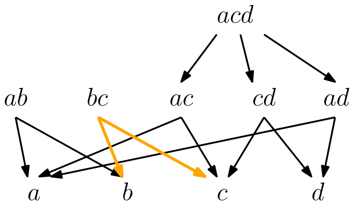
Then the boundary map $\partial^X : \mathbf{C}(X) \to \mathbf{C}(X)$ is defined such that:

$$\partial^X(\sigma) = \sum_{\substack{\tau \in X \\ \dim \sigma = \dim \tau + 1}} [\sigma : \tau] \cdot \tau$$



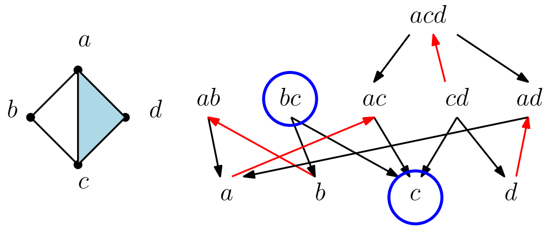
Example: Morse complex





$$\partial(bc) = 1 \cdot b + 1 \cdot c$$

Example: Morse complex



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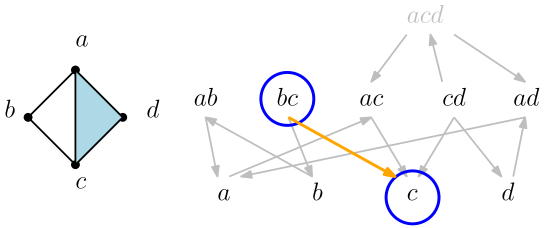

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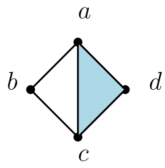

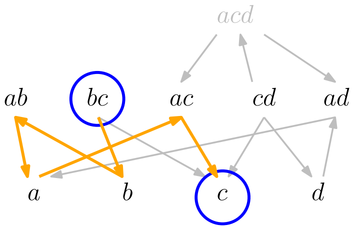

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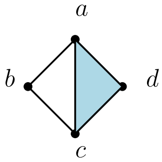

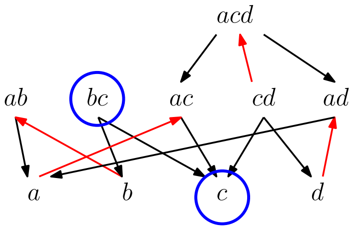

$$\partial^X(bc) = (1+1)\cdot c = 0$$

$$\Rightarrow \mathbf{H}_0 = \mathbf{H}_1 = \mathbf{Z}/2\mathbf{Z}$$

$$\mathbf{H}_2 = 0$$

Example: Morse complex

We need to prove that

- $(\mathbf{C}(X), \partial^X)$ is a chain complex, i.e., $\partial^X \circ \partial^X = 0$,
- $(\mathbf{C}(X), \partial^X)$ and $(\mathbf{C}(\mathbf{K}), \partial)$ have same homology.

The Morse complex is a chain complex

Inductive Proof of: $\partial_d^X \circ \partial_{d+1}^X \sigma = 0 \quad \forall d, \text{ and } \forall \sigma \in X$

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For $X = \mathbf{K},$ and $T = S = \emptyset$, $\partial^X = \partial$ the usual boundary map, and $\partial \circ \partial = 0.$ ✓

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Now, let $X \sqcup T \sqcup S$ be a Morse matching with $\partial^X \circ \partial^X = 0$, and denote by $X_d \subseteq X$ the d-dimensional simplices in X.

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For any $a \in X_{d+1}$, we have:

$$\partial_d^X \circ \partial_{d+1}^X(a) = \partial_d^X\Big(\sum_{b \in X_d}[a:b]b\Big) = \sum_{b \in X_d}\sum_{c \in X_{d-1}}[a:b][b:c]c = 0. \quad (1)$$

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In particular, for any $a \in X_{d+1}$, and $c \in X_{d-1}$, using (1):

$$\sum_{b \neq \tau}[a:b][b:c] + [a:\tau][\tau:c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}.$$

$$\sum_{b \neq \tau} [a : b][b : c] + [a : \tau][\tau : c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \quad (2)$$

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Add (σ, τ) to the Morse matching, denote by $(X', \partial^{X'})$ the new Morse complex, $X' := X \setminus \{\sigma, \tau\}$. For any $a \in X'$: counting $[x : y]$ in (X, ∂^X)

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Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\phi} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\psi} \mathbf{C}_\bullet(\mathbf{K})$$

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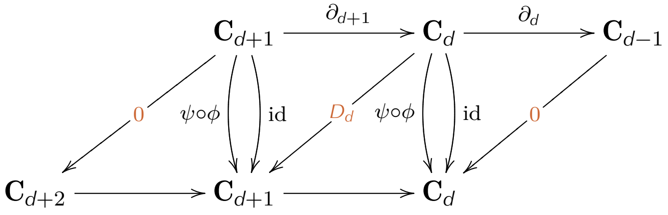

Proof of invariance by elem. collapse, with homotopy

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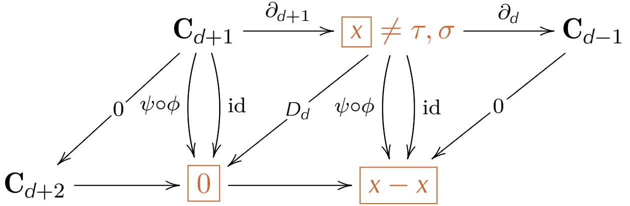

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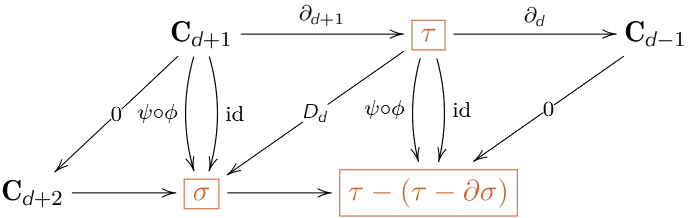

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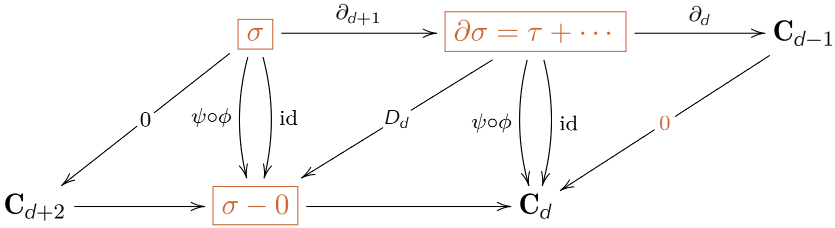

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Commutativity:

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$$\boxed{x} \xrightarrow{\partial^{X'}_{d+1}} \sum_{b \neq \tau} [x : b]b + \sum_{b \neq \tau} [x : \tau][\sigma : b]b + 2[x : \tau]\tau \xrightarrow{\partial^{X'}_d}$$

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Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

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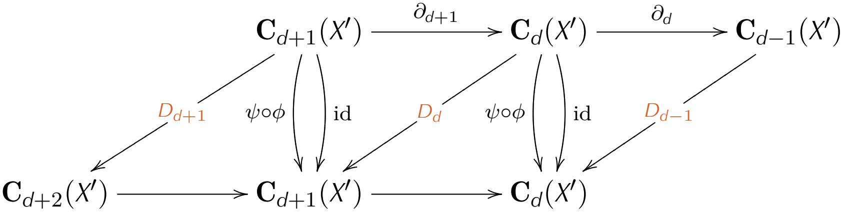
Chain equivalence: Easy side: $\phi \circ \psi = \text{id}$ $\quad x \mapsto x - [x : \tau]\sigma \mapsto x$.



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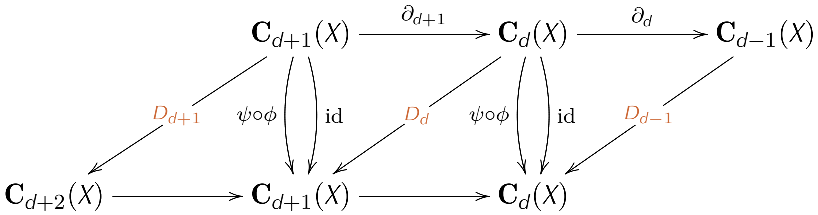
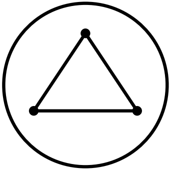


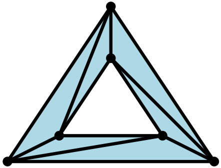
Exercise from last time

Compute the homology of:

- a tetrahedron, and of an empty tetrahedron,

- a circle,
- an annulus.





Exercise from last time

Compute the homology of:

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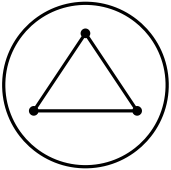
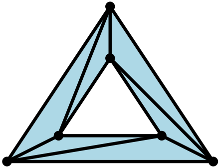
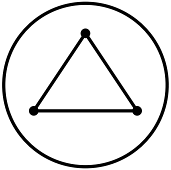


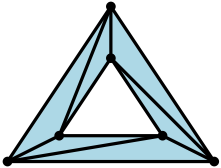


Exercise from last time

Compute the homology of:

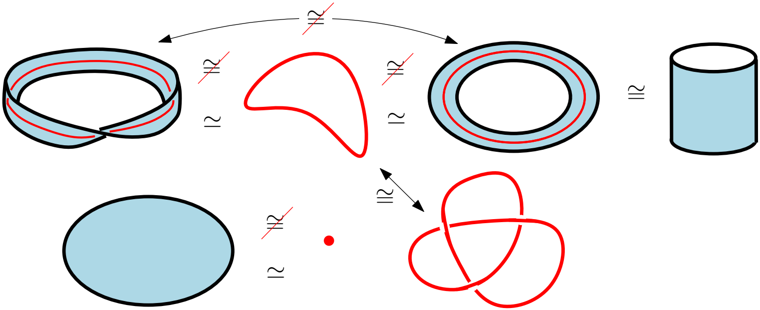
- a tetrahedron, and of an empty tetrahedron, → diagonalise the matrix.
- a circle, → triangulate with 3 edges, diagonalise the matrix.
- an annulus.





Topology

Equivalence of spaces, intuition



Topological and metric spaces

Definition (Topological space "= Set + Continuity")

A *topological space* is a set $\mathbb{X}$, called the *points*, together with a collection of subsets $\mathcal{O}$ of $\mathbb{X}$, called the *open sets*, satisfying:

(i) $\mathbb{X}$ and the empty set $\emptyset$ are open,

(ii) every union of open sets is open,

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Definition (Subspace topology)

Let $\mathbb{A} \subseteq \mathbb{X}$ be a subspace of the topological space $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$. The *subspace topology* induced by $(\mathbb{X}, \mathcal{O})$ on $\mathbb{A}$ is the family of open sets $\mathcal{O}_{\mathbb{A}}$:

$$\mathcal{O}_{\mathbb{A}} := \{\tau \cap \mathbb{A} : \tau \in \mathcal{O}_{\mathbb{X}}\}.$$

Definition (Metric space)

A *metric space* is a set of points $\mathbb{X}$ together with a *distance function* $\mathrm{d} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ satisfying, for every $x, y, z \in \mathbb{X}$,

(i) $\mathrm{d}(x, y) = 0$ *iff* $x = y$,

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Every metric space admits a natural topology, where the open sets are unions of open metric balls.

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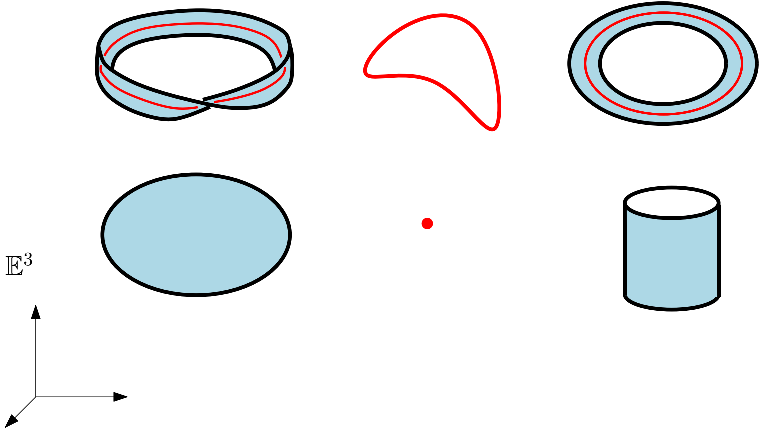

Topological equivalence(s)

Let $(\mathbb{X}, \mathcal{O})$ and $(\mathbb{Y}, \mathcal{O}')$ be two topological spaces.

Definition (Continuous map)

A map $f : \mathbb{X} \to \mathbb{Y}$ is *continuous* if $f^{-1}(U) \in \mathcal{O}$ for any $U \in \mathcal{O}'$, *ie* if the inverse image of any open subset of $\mathbb{Y}$ is open in $\mathbb{X}$.

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A map $f\colon \mathbb{X} \to \mathbb{Y}$ is a *homeomorphism* if f is:

(i) continuous,

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We say that $\mathbb{X}$ and $\mathbb{Y}$ are *homeomorphic* and write $\mathbb{X} \cong \mathbb{Y}$.

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NB: continuous inverse is necessary, as $[0;1)$ and S^1 admits a map $f : [0;1) \ni t \mapsto e^{2\pi i t} \in S^1$ satisfying (i) and (ii).

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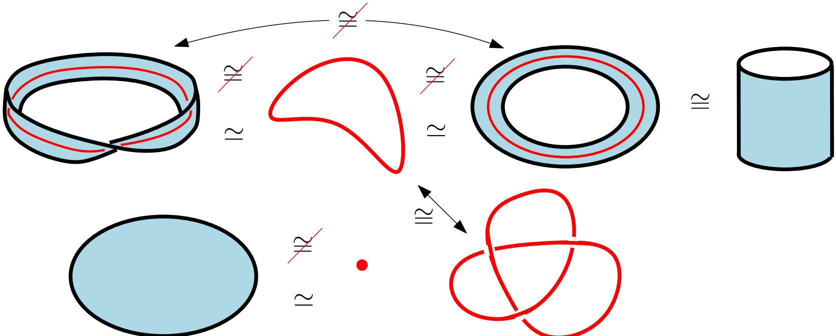

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This is *equality of functions up to continuous deformation*.

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continuous ✓ bijective NO

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- for any t, $f_t|_{\mathbb{A}} = \mathrm{id}_{\mathbb{A}}$. → *fixes subspace* $\mathbb{A}$

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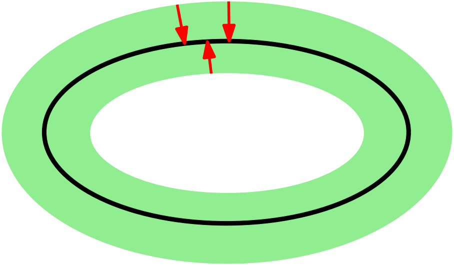

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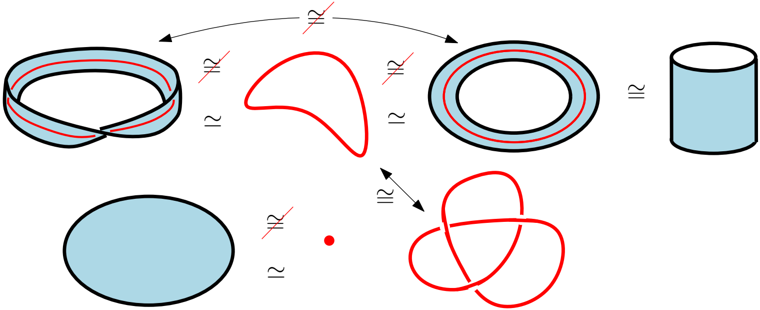
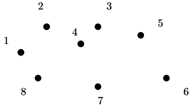


Simplicial complexes, abstract and geometric

Abstract: Set of *vertices*:

$$V = \{1, \ldots, |V|\}$$

Geometric: Points in $\mathbb{E}^d$:



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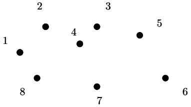

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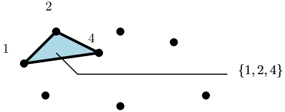

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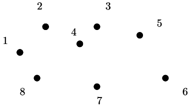

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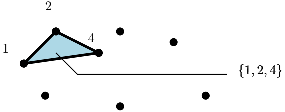

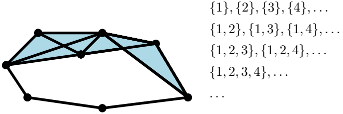

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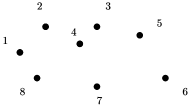

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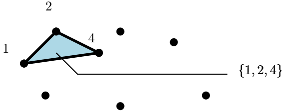

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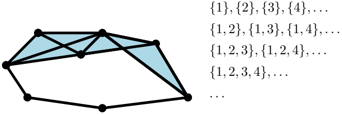
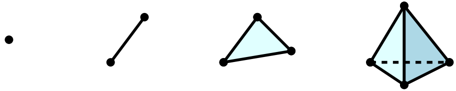


Dimension of a simplex = #vertices −1



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Definition

Realization of an abstract simplex An *abstract* d-dimensional simplex σ admits a canonical realization as the convex hull of the unit points $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)^T$ of $\mathbb{R}^{d+1}$. We denote the image of the embedding $\sigma \to \mathbb{R}^{d+1}$ by $|\sigma|$.

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Geometric realization Let $\mathbf{K}$ be an abstract simplicial complex on n vertices. This is a sub-complex of the n-simplex Δ_n. The geometric realization of $\mathbf{K}$, denoted by $|\mathbf{K}|$, is the image of the embedding $\Delta^n \to \mathbb{R}^n$ restricted to $\mathbf{K} \subseteq \Delta^n$.

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Singular homology

- $\Delta_d = \{x_0, \ldots, x_d\} \subseteq \{1, \ldots, n\}$ for an abstract d simplex.
- $|\Delta_d| \subset \mathbb{R}^{d+1}$ for the image of its canonical embedding as convex hull of the orthonormal basis of $\mathbb{R}^{d+1}$.

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Definition (Singular boundary)

A facet of a singular simplex $\sigma : |\Delta_d| \to \mathbb{X}$ is the restriction of σ (as a function) to a facet of Δ_d:

$$\tau := \sigma : \left| \Delta_d|_{\Delta_{d-1}} \right| \to \mathbb{X}.$$

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Finally, show that if two maps f, g are connected by a homotopy, the induced maps f_*, g_* are equal at the level of homology. □

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