CGL 2 - Discrete Morse theory

Clément Maria

INRIA Sophia Antipolis-Méditerranée

MPRI 2020-2021

Morse matching

Fix the coefficient field $\mathbb{Z}/2\mathbb{Z}$.

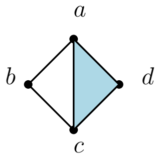
Definition

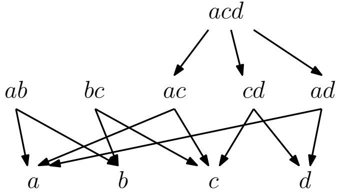
A simplicial complex $\mathbf{K}$ induces a *face partial ordering* $<$ between simplices. It is the transitive closure of the relation $\prec$

$$\tau \prec \sigma \quad \text{iff} \quad \tau \subset \sigma \text{ and } \dim \tau = \dim \sigma - 1.$$

The *Hasse diagram* of a complex is the directed graph (V, E) with:

- $V = \mathbf{K} - \{\emptyset\}$,
- $(\sigma \to \tau) \in E$ iff $\tau \prec \sigma$.



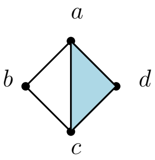


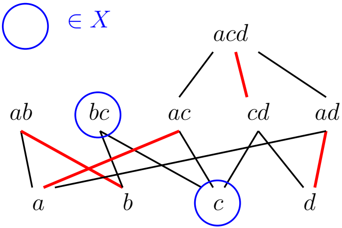
Definition

A *partial matching* of $\mathbf{K}$ is a partition $\mathbf{K} = X \sqcup T \sqcup S$ with a bijective pairing $\omega : T \to S$, where:

$$\tau \prec \omega(\tau), \text{ for all } \tau \in T.$$

$\longrightarrow$ it is a graph matching in the Hasse diargam of $\mathbf{K}$.





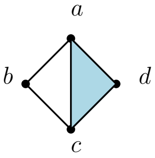
Morse matching

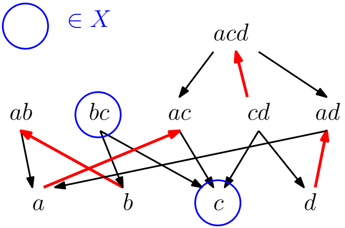
Definition

Let $\mathbf{K}$ be a simplicial complex, $\mathcal{H} = (\mathbf{K}, E)$ its Hasse diagram, and $(X \sqcup T \sqcup S, \omega)$ a partial matching. Define the directed graph $\bar{\mathcal{H}}$ as the graph $(\mathbf{K}, E')$:

- with same underlying undirected graph as $\mathcal{H}$,
- where every edge $(\omega(\tau) \to \tau) \in E$ is reversed in E', i.e., $(\tau \to \omega(\tau)) \in E'$.

If $\bar{\mathcal{H}}$ is acyclic, we call the partial matching a Morse matching.





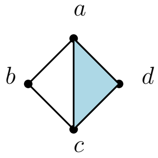
Morse matching

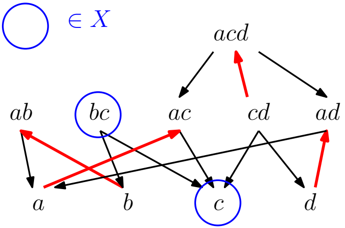
Definition

Let $\mathbf{K}$ be a simplicial complex, $\mathcal{H} = (\mathbf{K}, E)$ its Hasse diagram, and $(X \sqcup T \sqcup S, \omega)$ a partial matching. Define the directed graph $\bar{\mathcal{H}}$ as the graph $(\mathbf{K}, E')$:

- with same underlying undirected graph as $\mathcal{H}$,
- where every edge $(\omega(\tau) \to \tau) \in E$ is reversed in E', i.e., $(\tau \to \omega(\tau)) \in E'$.

If $\bar{\mathcal{H}}$ is acyclic, we call the partial matching a Morse matching.





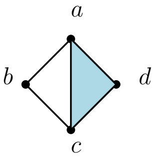
NB: a cycle can only alternate between two dim. d and $d+1$ (ω bij.).

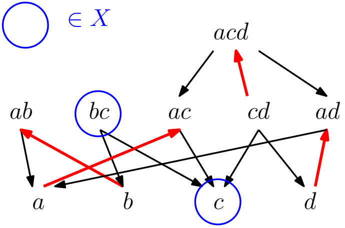
Morse matching

Definition

Let $\mathbf{K}$ be a simplicial complex, $\mathcal{H} = (\mathbf{K}, E)$ its Hasse diagram, and $(X \sqcup T \sqcup S, \omega)$ a Morse matching.

- Simplices in X are called critical.
- A Morse matching is optimal if it has the minimal number of critical simplices, over all possible Morse matchings of $\mathbf{K}$.





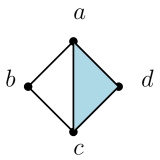
Morse matching

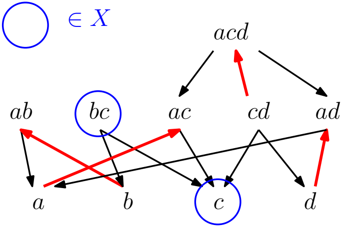
Definition

Let $\mathbf{K}$ be a simplicial complex, $\mathcal{H} = (\mathbf{K}, E)$ its Hasse diagram, and $(X \sqcup T \sqcup S, \omega)$ a Morse matching.

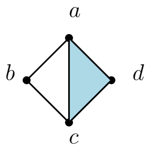
- Simplices in X are called critical.
- A Morse matching is optimal if it has the minimal number of critical simplices, over all possible Morse matchings of $\mathbf{K}$.

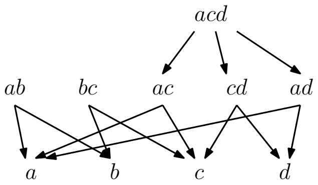
It is **NP-hard** to decide whether a simplicial complex has a Morse matching with less than k critical simplices.





Compute a (good) Morse matching: Heuristic



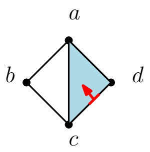


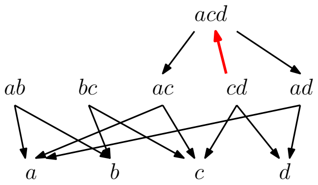
X :

$T \leftrightarrow S$

- A set of *available simplices*; $A \mapsfrom \mathbf{K}$, $X, T, S \mapsfrom \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, match τ with σ:
    - $A \mapsfrom A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \mapsfrom T \cup \{\tau\}$,
    - $S \mapsfrom S \cup \{\sigma\}$ $\Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \mapsfrom X \cup \{\zeta\}$,
    - $A \mapsfrom A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic





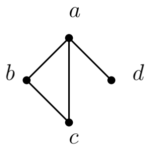
X :

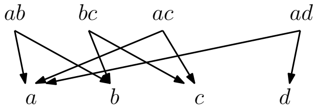
$T \leftrightarrow S$

$cd \;,\; acd$

- A set of *available simplices*; $A \leftarrow \mathbf{K}$, $X, T, S \leftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, match τ with σ:
    - $A \leftarrow A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \leftarrow T \cup \{\tau\}$,
    - $S \leftarrow S \cup \{\sigma\}$ $\Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \leftarrow X \cup \{\zeta\}$,
    - $A \leftarrow A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic





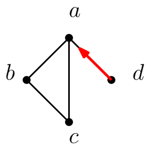
$X:$

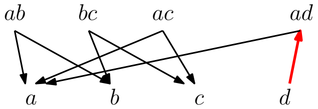
$T \leftrightarrow S$

$cd \;,\; acd$

- *A* set of *available simplices*; $A \leftarrow \mathbf{K}$, $X, T, S \leftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, match τ with σ:
    - $A \leftarrow A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \leftarrow T \cup \{\tau\}$,
    - $S \leftarrow S \cup \{\sigma\} \quad \Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \leftarrow X \cup \{\zeta\}$,
    - $A \leftarrow A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic



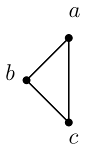


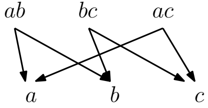
X :

T	$\leftrightarrow$	S
cd	,	acd
d	,	ad

- *A* set of *available simplices*; $A \mapsfrom \mathbf{K}$, $X, T, S \mapsfrom \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, match τ with σ:
    - $A \mapsfrom A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \mapsfrom T \cup \{\tau\}$,
    - $S \mapsfrom S \cup \{\sigma\}$ $\Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \mapsfrom X \cup \{\zeta\}$,
    - $A \mapsfrom A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic



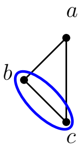


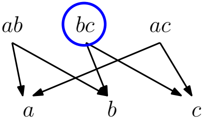
$X :$

$$\begin{array}{ccc} T & \leftrightarrow & S \\ cd & , & acd \\ d & , & ad \end{array}$$

- *A* set of *available simplices*; $\quad A \mapsfrom \mathbf{K}, \quad X, T, S \mapsfrom \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, $\quad$ match τ with σ:
    - $A \mapsfrom A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \mapsfrom T \cup \{\tau\}$,
    - $S \mapsfrom S \cup \{\sigma\} \quad \Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \mapsfrom X \cup \{\zeta\}$,
    - $A \mapsfrom A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic



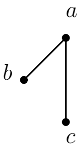


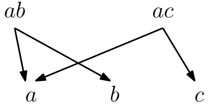
$X : \; bc$

$$\begin{array}{ccc} T & \leftrightarrow & S \\ cd & , & acd \\ d & , & ad \end{array}$$

- A set of *available simplices*; $\quad A \leftarrow \mathbf{K}, \quad X, T, S \leftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, $\quad$ match τ with σ:
    - $A \leftarrow A \setminus \{\tau, \sigma\}$, $\qquad$ ***perform elementary collapse***
    - $T \leftarrow T \cup \{\tau\}$,
    - $S \leftarrow S \cup \{\sigma\} \quad \Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \leftarrow X \cup \{\zeta\}$,
    - $A \leftarrow A \setminus \{\zeta\}$ $\qquad$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic



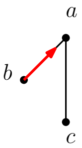


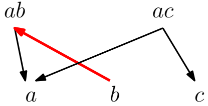
$X: \quad bc$

$$\begin{array}{ccc} T & \leftrightarrow & S \\ cd & , & acd \\ d & , & ad \end{array}$$

- A set of *available simplices*; $\quad A \leftarrow \mathbf{K}, \quad X, T, S \leftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, $\quad$ match τ with σ:
    - $A \leftarrow A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \leftarrow T \cup \{\tau\}$,
    - $S \leftarrow S \cup \{\sigma\} \quad \Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \leftarrow X \cup \{\zeta\}$,
    - $A \leftarrow A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic



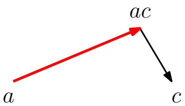


$X: \quad bc$

T	$\leftrightarrow$	S
cd	,	acd
d	,	ad
b	,	ab

- A set of *available simplices*; $\quad A \leftarrow \mathbf{K}, \quad X, T, S \leftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, $\quad$ match τ with σ:
    - $A \leftarrow A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \leftarrow T \cup \{\tau\}$,
    - $S \leftarrow S \cup \{\sigma\} \quad \Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \leftarrow X \cup \{\zeta\}$,
    - $A \leftarrow A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic



$X: \quad bc$

T	$\leftrightarrow$	S
cd	,	acd
d	,	ad
b	,	ab
a	,	ac

- A set of *available simplices*; $\quad A \leftarrow \mathbf{K}, \quad X, T, S \leftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, $\quad$ match τ with σ:
    - $A \leftarrow A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \leftarrow T \cup \{\tau\}$,
    - $S \leftarrow S \cup \{\sigma\} \quad \Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \leftarrow X \cup \{\zeta\}$,
    - $A \leftarrow A \setminus \{\zeta\}$ ***remove maximal simplex***

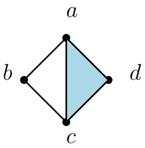
Compute a (good) Morse matching: Heuristic

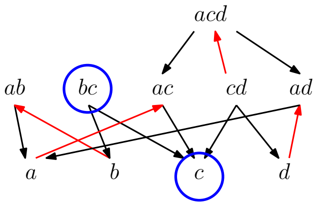
$X: \quad bc \ , \ c$

T	$\leftrightarrow$	S
cd	,	acd
d	,	ad
b	,	ab
a	,	ac

- A set of *available simplices*; $\quad A \leftarrow \mathbf{K}, \quad X, T, S \leftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, $\quad$ match τ with σ:
    - $A \leftarrow A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \leftarrow T \cup \{\tau\}$,
    - $S \leftarrow S \cup \{\sigma\} \quad \Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \leftarrow X \cup \{\zeta\}$,
    - $A \leftarrow A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic





$X: \quad bc \;,\; c$

$T \leftrightarrow S$

cd	,	acd
d	,	ad
a	,	ac
b	,	ab

- A set of *available simplices*; $\quad A \leftarrow \mathbf{K}, \quad X, T, S \leftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, $\quad$ match τ with σ:
    - $A \leftarrow A \setminus \{\tau, \sigma\}$, $\qquad$ ***perform elementary collapse***
    - $T \leftarrow T \cup \{\tau\}$,
    - $S \leftarrow S \cup \{\sigma\} \quad \Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \leftarrow X \cup \{\zeta\}$,
    - $A \leftarrow A \setminus \{\zeta\}$ $\qquad$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic

Lemma (Correction)

- A set of *available simplices*; $\quad A \leftarrow \mathbf{K}, \quad X, T, S \leftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, match τ with σ:
    - $A \leftarrow A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \leftarrow T \cup \{\tau\}$,
    - $S \leftarrow S \cup \{\sigma\} \quad \Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \leftarrow X \cup \{\zeta\}$,
    - $A \leftarrow A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic

Lemma (Correction)

- *A always contains a simplicial complex: only perform elementary collapses and maximal simplex removals.*

- A set of *available simplices*; $A \hookleftarrow \mathbf{K}$, $X, T, S \hookleftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, match τ with σ:
    - $A \hookleftarrow A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \hookleftarrow T \cup \{\tau\}$,
    - $S \hookleftarrow S \cup \{\sigma\}$ $\Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \hookleftarrow X \cup \{\zeta\}$,
    - $A \hookleftarrow A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic

Lemma (Correction)

- *A always contains a simplicial complex: only perform elementary collapses and maximal simplex removals.*
- *$T \times S$ always contains a* matching*: after being matched, a simplex is removed form the available simplices.*

- A set of *available simplices*; $A \hookleftarrow \mathbf{K}$, $X, T, S \hookleftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, match τ with σ:
    - $A \hookleftarrow A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \hookleftarrow T \cup \{\tau\}$,
    - $S \hookleftarrow S \cup \{\sigma\}$ $\Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \hookleftarrow X \cup \{\zeta\}$,
    - $A \hookleftarrow A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic

Lemma (Correction)

- *A always contains a simplicial complex: only perform elementary collapses and maximal simplex removals.*
- *$T \times S$ always contains a* matching*: after being matched, a simplex is removed form the available simplices.*
- *The matching is always* acyclic*: ...*

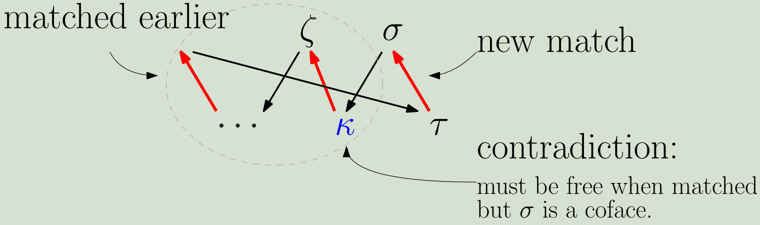
- A set of *available simplices*; $A \leftarrow \mathbf{K}$, $X, T, S \leftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, match τ with σ:
    - $A \leftarrow A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
    - $T \leftarrow T \cup \{\tau\}$,
    - $S \leftarrow S \cup \{\sigma\}$ $\Rightarrow \omega(\tau) = \sigma$
  - **else** make any available *maximal* simplex $\zeta \in A$ critical:
    - set $X \leftarrow X \cup \{\zeta\}$,
    - $A \leftarrow A \setminus \{\zeta\}$ ***remove maximal simplex***

Compute a (good) Morse matching: Heuristic

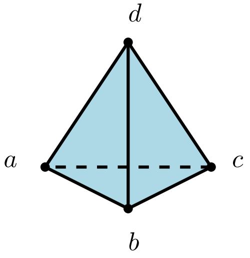
Lemma (Correction)

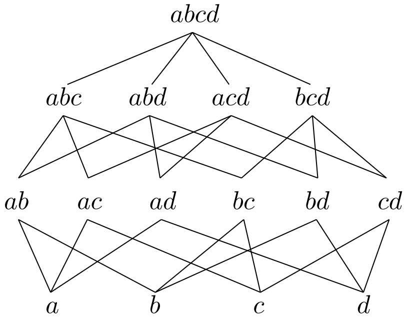
- *A always contains a simplicial complex: only perform elementary collapses and maximal simplex removals.*
- *$T \times S$ always contains a* matching*: after being matched, a simplex is removed form the available simplices.*
- *The matching is always* acyclic*: ...*

Suppose that, at some point, reverting an arrow creates a cycle. Pick the first such arrow (τ, σ):



More interesting example





What is the number of critical simplices of an optimal Morse matching?

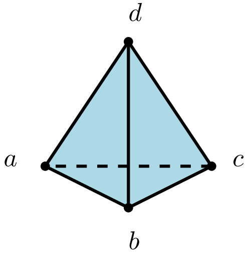
A 0

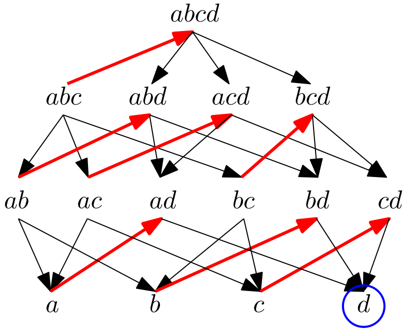
B 1

C 2

D 3

More interesting example





What is the number of critical simplices of an optimal Morse matching?

A 0

B 1

C 2

D 3

Fundamental Theorem of Discrete Morse Theory

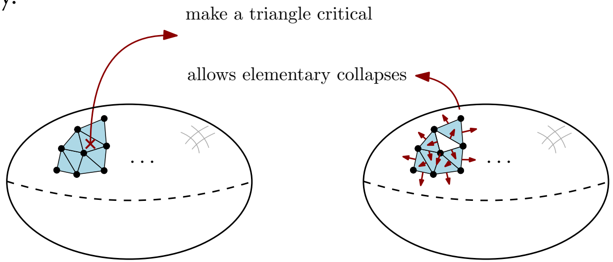
Theorem

Let $\mathbf{K}$ be a simplicial complex, and $(X \sqcup T \sqcup S, \omega)$ a Morse matching. There exists a chain complex $(\mathbf{C}(X), \partial^X)$ with cells in X, called a Morse complex, whose homology groups are isomorphic to the ones of $\mathbf{K}$.

More precisely,

- $\mathbf{C}_d(X)$ generated by the d-dimensional simplices of X,
- $\partial_d^X \colon \mathbf{C}_d(X) \to \mathbf{C}_{d-1}(X)$.

Application: reduce a complex drastically when computing its homology.



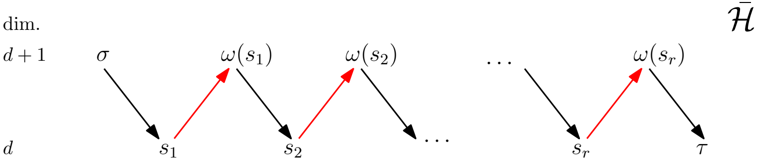
Boundary map ∂^X

Definition

In a complex $\mathbf{K}$ with Morse matching $(X \sqcup T \sqcup S, \omega)$, let $\sigma, \tau \in X$ be critical simplices, with $\dim \sigma = \dim \tau + 1$.
A gradient path from σ to τ is a directed path from σ to τ:

$$\sigma \to s_1 \to \omega(s_1) \to s_2 \to \ldots \to s_r \to \omega(s_r) \to \tau$$

in $\bar{\mathcal{H}}$, alternating between simplices of dimension d and $d-1$:



Boundary map ∂^X

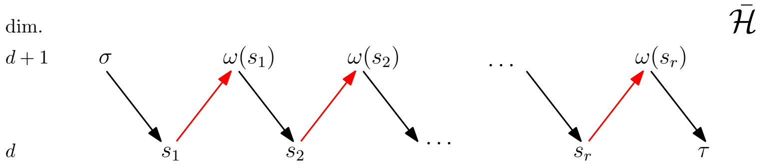
Definition

For $\sigma, \tau \in X$, with $\dim \sigma = \dim \tau + 1$, define

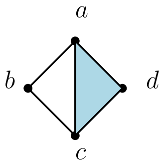
$$[\sigma : \tau] := \#\text{ distinct } (\sigma \to \tau)\text{-gradient path} \in \mathbb{Z}/2\mathbb{Z}.$$

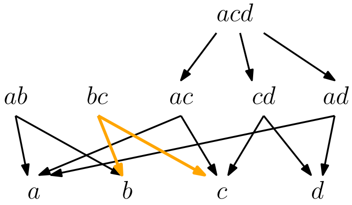
Then the boundary map $\partial^X : \mathbf{C}(X) \to \mathbf{C}(X)$ is defined such that:

$$\partial^X(\sigma) = \sum_{\substack{\tau \in X \\ \dim \sigma = \dim \tau + 1}} [\sigma : \tau] \cdot \tau$$



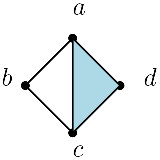
Example: Morse complex

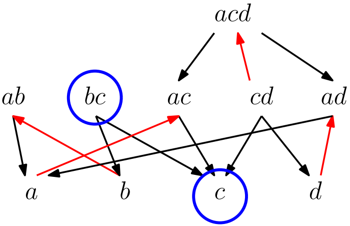




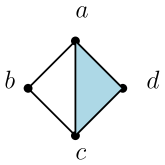
$$\partial(bc) = 1 \cdot b + 1 \cdot c$$

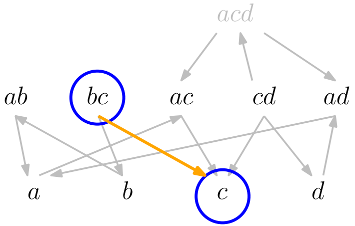
Example: Morse complex



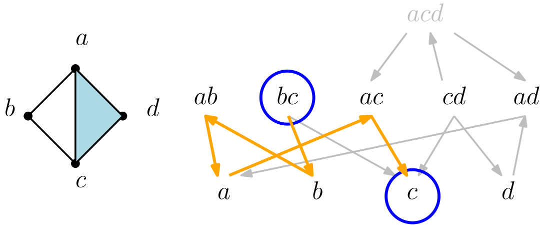


Example: Morse complex

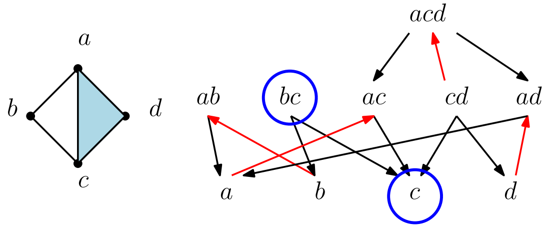




Example: Morse complex



Example: Morse complex



$$\partial^X(bc) = (1+1)\cdot c = 0$$

$$\Rightarrow \mathbf{H}_0 = \mathbf{H}_1 = \mathbf{Z}/2\mathbf{Z}$$

$$\mathbf{H}_2 = 0$$

Example: Morse complex

We need to prove that

- $(\mathbf{C}(X), \partial^X)$ is a chain complex, i.e., $\partial^X \circ \partial^X = 0$,
- $(\mathbf{C}(X), \partial^X)$ and $(\mathbf{C}(\mathbf{K}), \partial)$ have same homology.

The Morse complex is a chain complex

Inductive Proof of: $\partial_d^X \circ \partial_{d+1}^X \sigma = 0 \quad \forall d,$ and $\forall \sigma \in X$

The Morse complex is a chain complex

Inductive Proof of: $\partial_d^X \circ \partial_{d+1}^X \sigma = 0 \quad \forall d,$ and $\forall \sigma \in X$

For $X = \mathbf{K},$ and $T = S = \emptyset$, $\partial^X = \partial$ the usual boundary map, and $\partial \circ \partial = 0.$ ✓

The Morse complex is a chain complex

Inductive Proof of: $\partial_d^X \circ \partial_{d+1}^X \sigma = 0 \quad \forall d,$ and $\forall \sigma \in X$

For $X = \mathbf{K},$ and $T = S = \emptyset$, $\partial^X = \partial$ the usual boundary map, and $\partial \circ \partial = 0.$ ✓

Now, let $X \sqcup T \sqcup S$ be a Morse matching with $\partial^X \circ \partial^X = 0$, and denote by $X_d \subseteq X$ the d-dimensional simplices in X.

The Morse complex is a chain complex

Inductive Proof of: $\partial^X_d \circ \partial^X_{d+1}\sigma = 0 \quad \forall d, \text{ and } \forall \sigma \in X$

For $X = \mathbf{K}$, and $T = S = \emptyset$, $\partial^X = \partial$ the usual boundary map, and $\partial \circ \partial = 0$. ✓

Now, let $X \sqcup T \sqcup S$ be a Morse matching with $\partial^X \circ \partial^X = 0$, and denote by $X_d \subseteq X$ the d-dimensional simplices in X.

For any $a \in X_{d+1}$, we have:

$$\partial^X_d \circ \partial^X_{d+1}(a) = \partial^X_d(\sum_{b \in X_d}[a:b]b) = \sum_{b \in X_d}\sum_{c \in X_{d-1}}[a:b][b:c]c = 0. \quad (1)$$

The Morse complex is a chain complex

Inductive Proof of: $\partial_d^X \circ \partial_{d+1}^X \sigma = 0 \quad \forall d,$ and $\forall \sigma \in X$

For $X = \mathbf{K},$ and $T = S = \emptyset$, $\partial^X = \partial$ the usual boundary map, and $\partial \circ \partial = 0.$ ✓

Now, let $X \sqcup T \sqcup S$ be a Morse matching with $\partial^X \circ \partial^X = 0$, and denote by $X_d \subseteq X$ the d-dimensional simplices in X.

For any $a \in X_{d+1}$, we have:

$$\partial_d^X \circ \partial_{d+1}^X(a) = \partial_d^X(\sum_{b \in X_d}[a:b]b) = \sum_{b \in X_d}\sum_{c \in X_{d-1}}[a:b][b:c]c = 0. \quad (1)$$

Let $\sigma, \tau \in X$ be such that pairing σ and τ leads to a valid Morse matching $(X \setminus \{\sigma,\tau\}) \sqcup (T \cup \{\tau\}) \sqcup (S \cup \{\sigma\})$.

The Morse complex is a chain complex

Inductive Proof of: $\partial_d^X \circ \partial_{d+1}^X \sigma = 0 \quad \forall d, \text{ and } \forall \sigma \in X$

For $X = \mathbf{K}$, and $T = S = \emptyset$, $\partial^X = \partial$ the usual boundary map, and $\partial \circ \partial = 0$. ✓

Now, let $X \sqcup T \sqcup S$ be a Morse matching with $\partial^X \circ \partial^X = 0$, and denote by $X_d \subseteq X$ the d-dimensional simplices in X.

For any $a \in X_{d+1}$, we have:

$$\partial_d^X \circ \partial_{d+1}^X(a) = \partial_d^X\Big(\sum_{b \in X_d} [a:b]b\Big) = \sum_{b \in X_d} \sum_{c \in X_{d-1}} [a:b][b:c]c = 0. \quad (1)$$

Let $\sigma, \tau \in X$ be such that pairing σ and τ leads to a valid Morse matching $(X \setminus \{\sigma, \tau\}) \sqcup (T \cup \{\tau\}) \sqcup (S \cup \{\sigma\})$.

In particular, for any $a \in X_{d+1}$, and $c \in X_{d-1}$, using (1):

$$\sum_{b \neq \tau} [a:b][b:c] + [a:\tau][\tau:c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}.$$

$$\sum_{b \neq \tau} [a : b][b : c] + [a : \tau][\tau : c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \tag{2}$$

$$\sum_{b \neq \tau} [a : b][b : c] + [a : \tau][\tau : c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \tag{2}$$

Add (σ, τ) to the Morse matching, denote by $(X', \partial^{X'})$ the new Morse complex, $X' := X \setminus \{\sigma, \tau\}$. For any $a \in X'$: counting $[x : y]$ in (X, ∂^X)

$$\sum_{b \neq \tau} [a : b][b : c] + [a : \tau][\tau : c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \qquad (2)$$

Add (σ, τ) to the Morse matching, denote by $(X', \partial^{X'})$ the new Morse complex, $X' := X \setminus \{\sigma, \tau\}$. For any $a \in X'$: $\boxed{\text{counting } [x : y] \text{ in } (X, \partial^X)}$

$$\partial_d^{X'} \circ \partial_{d+1}^{X'}(a) \;=\; \partial_d^{X'} \sum_{\substack{b \neq \tau \\ b \in X_d}} \Big(\underbrace{[a : b]}_{\substack{\text{paths not traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{already in } (X, \partial^X))}} + \underbrace{[a : \tau][\sigma : b]}_{\substack{\text{paths traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{new in } (X', \partial^{X'}))}} \Big) \cdot b$$

$$\sum_{b \neq \tau} [a:b][b:c] + [a:\tau][\tau:c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \qquad (2)$$

Add (σ, τ) to the Morse matching, denote by $(X', \partial^{X'})$ the new Morse complex, $X' := X \setminus \{\sigma, \tau\}$. For any $a \in X'$: $\boxed{\text{counting } [x:y] \text{ in } (X, \partial^X)}$

$$\partial_d^{X'} \circ \partial_{d+1}^{X'}(a) = \partial_d^{X'} \sum_{\substack{b \neq \tau \\ b \in X_d}} \Big(\underbrace{[a:b]}_{\substack{\text{paths not traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{already in } (X, \partial^X))}} + \underbrace{[a:\tau][\sigma:b]}_{\substack{\text{paths traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{new in } (X', \partial^{X'}))}} \Big) \cdot b$$

$$\sum_{b \neq \tau} [a:b][b:c] + [a:\tau][\tau:c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \qquad (2)$$

Add (σ, τ) to the Morse matching, denote by $(X', \partial^{X'})$ the new Morse complex, $X' := X \setminus \{\sigma, \tau\}$. For any $a \in X'$: $\boxed{\text{counting } [x:y] \text{ in } (X, \partial^X)}$

$$\partial_d^{X'} \circ \partial_{d+1}^{X'}(a) = \partial_d^{X'} \sum_{\substack{b \neq \tau \\ b \in X_d}} \Big(\underbrace{[a:b]}_{\substack{\text{paths not traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{already in } (X, \partial^X))}} + \underbrace{[a:\tau][\sigma:b]}_{\substack{\text{paths traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{new in } (X', \partial^{X'}))}} \Big) \cdot b$$

$$\sum_{b \neq \tau} [a:b][b:c] + [a:\tau][\tau:c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \qquad (2)$$

Add (σ, τ) to the Morse matching, denote by $(X', \partial^{X'})$ the new Morse complex, $X' := X \setminus \{\sigma, \tau\}$. For any $a \in X'$: $\boxed{\text{counting } [x:y] \text{ in } (X, \partial^X)}$

$$\partial_d^{X'} \circ \partial_{d+1}^{X'}(a) = \partial_d^{X'} \sum_{\substack{b \neq \tau \\ b \in X_d}} \Big(\underbrace{[a:b]}_{\substack{\text{paths not traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{already in } (X, \partial^X))}} + \underbrace{[a:\tau][\sigma:b]}_{\substack{\text{paths traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{new in } (X', \partial^{X'}))}} \Big) \cdot b$$

$$\text{by linearity} \quad = \sum_{b \neq \tau} ([a:b] + [a:\tau][\sigma:b]) \sum_{c \in X_{d-1}} [b:c] \cdot c$$

$$\sum_{b \neq \tau} [a:b][b:c] + [a:\tau][\tau:c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \tag{2}$$

Add (σ, τ) to the Morse matching, denote by $(X', \partial^{X'})$ the new Morse complex, $X' := X \setminus \{\sigma, \tau\}$. For any $a \in X'$: $\boxed{\text{counting } [x:y] \text{ in } (X, \partial^X)}$

$$\partial_d^{X'} \circ \partial_{d+1}^{X'}(a) = \partial_d^{X'} \sum_{\substack{b \neq \tau \\ b \in X_d}} \Big(\underbrace{[a:b]}_{\substack{\text{paths not traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{already in } (X, \partial^X))}} + \underbrace{[a:\tau][\sigma:b]}_{\substack{\text{paths traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{new in } (X', \partial^{X'}))}} \Big) \cdot b$$

$$\text{by linearity} \quad = \sum_{b \neq \tau} ([a:b] + [a:\tau][\sigma:b]) \sum_{c \in X_{d-1}} [b:c] \cdot c$$

$$= \sum_{c \in X_{d-1}} \underbrace{\sum_{b \neq \tau} [a:b][b:c]}_{=[a:\tau][\tau:c] \text{ by } (2)} \cdot c + [a:\tau] \cdot \sum_{c} \underbrace{\sum_{b \neq \tau} [\sigma:b][b:c]}_{=[\sigma:\tau][\tau:c] = [\tau:c]} \cdot c$$

$$\sum_{b\neq\tau}[a:b][b:c]+[a:\tau][\tau:c]=0 \qquad \forall a\in X_{d+1}, c\in X_{d-1}. \tag{2}$$

Add (σ,τ) to the Morse matching, denote by $(X',\partial^{X'})$ the new Morse complex, $X':=X\setminus\{\sigma,\tau\}$. For any $a\in X'$: $\boxed{\text{counting } [x:y] \text{ in } (X,\partial^X)}$

$$\partial_d^{X'}\circ\partial_{d+1}^{X'}(a) = \partial_d^{X'}\sum_{\substack{b\neq\tau\\ b\in X_d}}\Big(\underbrace{[a:b]}_{\substack{\text{paths not traversing}\\ \text{edge } \tau\to\sigma\\ (\textbf{already in } (X,\partial^X))}} + \underbrace{[a:\tau][\sigma:b]}_{\substack{\text{paths traversing}\\ \text{edge } \tau\to\sigma\\ (\textbf{new in } (X',\partial^{X'}))}}\Big)\cdot b$$

$$\text{by linearity} \quad = \sum_{b\neq\tau}([a:b]+[a:\tau][\sigma:b])\sum_{c\in X_{d-1}}[b:c]\cdot c$$

$$= \sum_{c\in X_{d-1}}\underbrace{\sum_{b\neq\tau}[a:b][b:c]}_{=[a:\tau][\tau:c]\text{ by (2)}}\cdot c + [a:\tau]\cdot\sum_{c}\underbrace{\sum_{b\neq\tau}[\sigma:b][b:c]}_{=[\sigma:\tau][\tau:c]=[\tau:c]}\cdot c$$

$$\sum_{b \neq \tau} [a:b][b:c] + [a:\tau][\tau:c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \qquad (2)$$

Add (σ, τ) to the Morse matching, denote by $(X', \partial^{X'})$ the new Morse complex, $X' := X \setminus \{\sigma, \tau\}$. For any $a \in X'$: $\boxed{\text{counting } [x:y] \text{ in } (X, \partial^X)}$

$$\partial_d^{X'} \circ \partial_{d+1}^{X'}(a) = \partial_d^{X'} \sum_{\substack{b \neq \tau \\ b \in X_d}} \Big(\underbrace{[a:b]}_{\substack{\text{paths not traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{already in } (X, \partial^X))}} + \underbrace{[a:\tau][\sigma:b]}_{\substack{\text{paths traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{new in } (X', \partial^{X'}))}} \Big) \cdot b$$

$$\text{by linearity} \quad = \sum_{b \neq \tau} ([a:b] + [a:\tau][\sigma:b]) \sum_{c \in X_{d-1}} [b:c] \cdot c$$

$$= \sum_{c \in X_{d-1}} \underbrace{\sum_{b \neq \tau} [a:b][b:c]}_{=[a:\tau][\tau:c] \text{ by (2)}} \cdot c + [a:\tau] \cdot \sum_{c} \underbrace{\sum_{b \neq \tau} [\sigma:b][b:c]}_{=[\sigma:\tau][\tau:c]=[\tau:c]} \cdot c$$

$$\sum_{b \neq \tau} [a:b][b:c] + [a:\tau][\tau:c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \tag{2}$$

Add (σ, τ) to the Morse matching, denote by $(X', \partial^{X'})$ the new Morse complex, $X' := X \setminus \{\sigma, \tau\}$. For any $a \in X'$: $\boxed{\text{counting } [x:y] \text{ in } (X, \partial^X)}$

$$\partial_d^{X'} \circ \partial_{d+1}^{X'}(a) = \partial_d^{X'} \sum_{\substack{b \neq \tau \\ b \in X_d}} \Big(\underbrace{[a:b]}_{\substack{\text{paths not traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{already in } (X, \partial^X))}} + \underbrace{[a:\tau][\sigma:b]}_{\substack{\text{paths traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{new in } (X', \partial^{X'}))}} \Big) \cdot b$$

$$\text{by linearity} \quad = \sum_{b \neq \tau} ([a:b] + [a:\tau][\sigma:b]) \sum_{c \in X_{d-1}} [b:c] \cdot c$$

$$= \sum_{c \in X_{d-1}} \underbrace{\sum_{b \neq \tau} [a:b][b:c]}_{=[a:\tau][\tau:c] \text{ by (2)}} \cdot c \;+\; [a:\tau] \cdot \sum_{c} \underbrace{\sum_{b \neq \tau} [\sigma:b][b:c]}_{=[\sigma:\tau][\tau:c] = [\tau:c]} \cdot c$$

NB: $[\sigma:\tau]$ is 1 as otherwise we get a cycle adding (σ, τ) to the matching.

$$\sum_{b \neq \tau} [a:b][b:c] + [a:\tau][\tau:c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \qquad (2)$$

Add (σ, τ) to the Morse matching, denote by $(X', \partial^{X'})$ the new Morse complex, $X' := X \setminus \{\sigma, \tau\}$. For any $a \in X'$: $\boxed{\text{counting } [x:y] \text{ in } (X, \partial^X)}$

$$\partial_d^{X'} \circ \partial_{d+1}^{X'}(a) = \partial_d^{X'} \sum_{\substack{b \neq \tau \\ b \in X_d}} \Big(\underbrace{[a:b]}_{\substack{\text{paths not traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{already in } (X, \partial^X))}} + \underbrace{[a:\tau][\sigma:b]}_{\substack{\text{paths traversing} \\ \text{edge } \tau \to \sigma \\ (\textbf{new in } (X', \partial^{X'}))}} \Big) \cdot b$$

$$\text{by linearity} \quad = \sum_{b \neq \tau} ([a:b] + [a:\tau][\sigma:b]) \sum_{c \in X_{d-1}} [b:c] \cdot c$$

$$= \sum_{c \in X_{d-1}} \underbrace{\sum_{b \neq \tau} [a:b][b:c]}_{=[a:\tau][\tau:c] \text{ by } (2)} \cdot c + [a:\tau] \cdot \sum_{c} \underbrace{\sum_{b \neq \tau} [\sigma:b][b:c]}_{=[\sigma:\tau][\tau:c] = [\tau:c]} \cdot c = 0$$

NB: $[\sigma:\tau]$ is 1 as otherwise we get a cycle adding (σ, τ) to the matching.

Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\quad\phi\quad} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\quad\psi\quad} \mathbf{C}_\bullet(\mathbf{K})$$

Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\ \phi\ } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\ \psi\ } \mathbf{C}_\bullet(\mathbf{K})$$

with:

$\psi(x) \quad = \quad x, \quad$ i.e. the inclusion $\mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$

Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\quad\phi\quad} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\quad\psi\quad} \mathbf{C}_\bullet(\mathbf{K})$$

with:

$$\psi(x) \quad = \quad x, \quad \text{i.e. the inclusion } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$$

$$\phi(x) \quad = \quad \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial\sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases}$$

Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\phi} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\psi} \mathbf{C}_\bullet(\mathbf{K})$$

with:

$$\psi(x) \quad = \quad x, \quad \text{i.e. the inclusion } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$$

$$\phi(x) \quad = \quad \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial\sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad D(x) = \begin{cases} \sigma & \text{if } x = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\ \phi\ } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\ \psi\ } \mathbf{C}_\bullet(\mathbf{K})$$

with:

$$\psi(x) \;=\; x, \quad \text{i.e. the inclusion } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$$

$$\phi(x) \;=\; \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial\sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad D(x) = \begin{cases} \sigma & \text{if } x = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}
 & \mathbf{C}_{d+1} \arrow[r, "\partial_{d+1}"] \arrow[dl, "0"'] \arrow[d, bend right, "\phi\circ\psi"'] \arrow[d, bend left, "\mathrm{id}"] & \mathbf{C}_d \arrow[r, "\partial_d"] \arrow[dl, "D_d"'] \arrow[d, bend right, "\phi\circ\psi"'] \arrow[d, bend left, "\mathrm{id}"] & \mathbf{C}_{d-1} \arrow[dl, "0"'] \\
\mathbf{C}_{d+2} \arrow[r] & \mathbf{C}_{d+1} \arrow[r] & \mathbf{C}_d &
\end{tikzcd}$$

Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\quad\phi\quad} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\quad\psi\quad} \mathbf{C}_\bullet(\mathbf{K})$$

with:

$$\psi(x) \quad = \quad x, \quad \text{i.e. the inclusion } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$$

$$\phi(x) \quad = \quad \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial\sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad\qquad D(x) = \begin{cases} \sigma & \text{if } x = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}
 & \mathbf{C}_{d+1} \arrow[r, "\partial_{d+1}"] \arrow[dl, "0"'] \arrow[d, bend right, "\phi\circ\psi"'] \arrow[d, bend left, "\mathrm{id}"] & \boxed{x} \neq \tau, \sigma \arrow[r, "\partial_d"] \arrow[dl, "D_d"'] \arrow[d, bend right, "\phi\circ\psi"'] \arrow[d, bend left, "\mathrm{id}"] & \mathbf{C}_{d-1} \arrow[dl, "0"'] \\
\mathbf{C}_{d+2} \arrow[r] & \boxed{0} \arrow[r] & \boxed{x - x} &
\end{tikzcd}$$

Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\quad\phi\quad} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\quad\psi\quad} \mathbf{C}_\bullet(\mathbf{K})$$

with:

$$\psi(x) = x, \quad \text{i.e. the inclusion } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial\sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad D(x) = \begin{cases} \sigma & \text{if } x = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}
 & \mathbf{C}_{d+1} \arrow[r, "\partial_{d+1}"] \arrow[dl, "0"'] \arrow[d, bend right, "\phi\circ\psi"'] \arrow[d, bend left, "\mathrm{id}"] & \boxed{\tau} \arrow[r, "\partial_d"] \arrow[dl, "D_d"'] \arrow[d, bend right, "\phi\circ\psi"'] \arrow[d, bend left, "\mathrm{id}"] & \mathbf{C}_{d-1} \arrow[dl, "0"'] \\
\mathbf{C}_{d+2} \arrow[r] & \boxed{\sigma} \arrow[r] & \boxed{\tau - (\tau - \partial\sigma)} &
\end{tikzcd}$$

Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\phi} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\psi} \mathbf{C}_\bullet(\mathbf{K})$$

with:

$$\psi(x) \quad = \quad x, \quad \text{i.e. the inclusion } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$$

$$\phi(x) \quad = \quad \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial\sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad D(x) = \begin{cases} \sigma & \text{if } x = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}
 & \boxed{\sigma} \arrow[r, "\partial_{d+1}"] \arrow[dl, "0"'] \arrow[d, bend right, "\phi\circ\psi"'] \arrow[d, bend left, "\mathrm{id}"] & \boxed{\partial\sigma = \tau + \cdots} \arrow[r, "\partial_d"] \arrow[dl, "D_d"'] \arrow[d, bend right, "\phi\circ\psi"'] \arrow[d, bend left, "\mathrm{id}"] & \mathbf{C}_{d-1} \arrow[dl, "0"'] \\
\mathbf{C}_{d+2} \arrow[r] & \boxed{\sigma - 0} \arrow[r] & \mathbf{C}_d &
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓
X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\mathbf{C}_{d+1}(X) \arrow[r, "\partial^X_{d+1}"] & \mathbf{C}_d(X) \arrow[r, "\partial^X_d"] & \mathbf{C}_{d-1}(X) \\
\mathbf{C}_{d+1}(X') \arrow[u, "\psi_{d+1}"] \arrow[r, "\partial^{X'}_{d+1}"] & \mathbf{C}_d(X') \arrow[u, "\psi_d"] \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X') \arrow[u, "\psi_{d-1}"]
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. $\checkmark$
X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\mathbf{C}_{d+1}(X) \arrow[r, "\partial^X_{d+1}"] & \mathbf{C}_d(X) \arrow[r, "\partial^X_d"] & \mathbf{C}_{d-1}(X) \\
\boxed{x} \arrow[u, "\psi_{d+1}"] \arrow[r, "\partial^{X'}_{d+1}"] & \mathbf{C}_d(X') \arrow[u, "\psi_d"] \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X') \arrow[u, "\psi_{d-1}"]
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd} x - [x : \tau]\sigma \arrow[r, "\partial^X_{d+1}"] & \partial^X x - [x : \tau]\partial^x \sigma \arrow[r, "\partial^X_d"] & \mathbf{C}_{d-1}(X) \\ \boxed{x} \arrow[u, "\psi_{d+1}"] \arrow[r, "\partial^{X'}_{d+1}"] & \mathbf{C}_d(X') \arrow[u, "\psi_d"] \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X') \arrow[u, "\psi_{d-1}"] \end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓
X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\quad \phi \quad} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\quad \psi \quad} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
x - [x : \tau]\sigma \arrow[r, "\partial^X_{d+1}"] & \partial^X x - [x : \tau]\partial^x\sigma \arrow[r, "\partial^X_d"] & {} \\
\boxed{x} \arrow[u, "\psi_{d+1}"] \arrow[r, "\partial^{X'}_{d+1}"] & \sum_{b \neq \tau}[x : b]b + \sum_{b \neq \tau}[x : \tau][\sigma : b]b + 2[x : \tau]\tau \arrow[u, "\psi_d"] \arrow[r, "\partial^{X'}_d"] & {}
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. $\checkmark$
X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\mathbf{C}_{d+1}(X) \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & \mathbf{C}_d(X) \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\
\mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \mathbf{C}_d(X') \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X')
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd} \boxed{\sigma} \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & \mathbf{C}_d(X) \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\ \mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \mathbf{C}_d(X') \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X') \end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. $\checkmark$

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd} \boxed{\sigma} \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & (\partial^X \sigma - \tau) + \tau \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\ \mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \mathbf{C}_d(X') \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X') \end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd} \boxed{\sigma} \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & (\partial^X \sigma - \tau) + \tau \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\ \mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & (\partial^X \sigma - \tau) + \tau - \partial^X \sigma = 0 \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X') \end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\ \phi\ } (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\ \psi\ } (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd} \boxed{\sigma} \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & (\partial^X \sigma - \tau) + \tau \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X') \arrow[d, "\phi_{d-1}"'] \\ 0 \arrow[r, "\partial^{X'}_{d+1}"] & 0 \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X') \end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\boxed{a \neq \sigma} \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & \mathbf{C}_d(X) \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\
\mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \mathbf{C}_d(X') \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X')
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\boxed{a \neq \sigma} \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & (\partial^X a - [a : \tau]\tau) + [a : \tau]\tau \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\
\mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \mathbf{C}_d(X') \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X')
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\boxed{a \neq \sigma} \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & (\partial^X a - [a : \tau]\tau) + [a : \tau]\tau \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\
\mathbf{C}_{d+1}(X) \arrow[r, "\partial^{X'}_{d+1}"] & (\partial^X a - [a : \tau]\tau) + [a : \tau](\tau - \partial^X \sigma) \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X')
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓
X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\boxed{a \neq \sigma} \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & (\partial^X a - [a:\tau]\tau) + [a:\tau]\tau \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\
\mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \sum_{b \neq \tau} [a:b]b + [a:\tau](\tau - \partial^X \sigma) \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X')
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓
X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\ \phi\ } (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\ \psi\ } (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\boxed{a \neq \sigma} \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & (\partial^X a - [a : \tau]\tau) + [a : \tau]\tau \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\
\mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \sum_{b \neq \tau}[a : b]b + [a : \tau] \sum_{b \neq \tau}[\sigma : b]b \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X')
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓
X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\boxed{a \neq \sigma} \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & (\partial^X a - [a : \tau]\tau) + [a : \tau]\tau \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\
\mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \partial^{X'} a \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X')
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓
X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\boxed{a \neq \sigma} \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & (\partial^X a - [a : \tau]\tau) + [a : \tau]\tau \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\
a \arrow[r, "\partial^{X'}_{d+1}"] & \partial^{X'} a \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X')
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. $\checkmark$

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\mathbf{C}_{d+1}(X) \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & \boxed{\tau} \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \mathbf{C}_{d-1}(X) \arrow[d, "\phi_{d-1}"'] \\
\mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \mathbf{C}_d(X') \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X')
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓
X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\mathbf{C}_{d+1}(X) \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & \boxed{\tau} \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \partial^X \tau \arrow[d, "\phi_{d-1}"'] \\
\mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \mathbf{C}_d(X') \arrow[r, "\partial^{X'}_d"] & \mathbf{C}_{d-1}(X')
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd} \mathbf{C}_{d+1}(X) \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & \boxed{\tau} \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \partial^X \tau \arrow[d, "\phi_{d-1}"'] \\ \mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \mathbf{C}_d(X') \arrow[r, "\partial^{X'}_d"] & \partial^X \tau \end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓
X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{tikzcd}
\mathbf{C}_{d+1}(X) \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & \boxed{\tau} \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \partial^X \tau \arrow[d, "\phi_{d-1}"'] \\
\mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \tau - \partial^X \sigma = \sum_{b \neq \tau} [\sigma : b] b \arrow[r, "\partial^{X'}_d"] & \partial^X \tau
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity: $\partial^X \partial^X \sigma = 0 = \partial^X (\sum_{b \neq \tau} [\sigma : b] b) + \partial^X \tau$

$$\begin{tikzcd}
\mathbf{C}_{d+1}(X) \arrow[r, "\partial^X_{d+1}"] \arrow[d, "\phi_{d+1}"'] & \boxed{\tau} \arrow[r, "\partial^X_d"] \arrow[d, "\phi_d"'] & \partial^X \tau \arrow[d, "\phi_{d-1}"'] \\
\mathbf{C}_{d+1}(X') \arrow[r, "\partial^{X'}_{d+1}"] & \sum_{b \neq \tau} [\sigma : b] b \arrow[r, "\partial^{X'}_d"] & \sum_{b \neq \tau} [\sigma : b] \sum_c [b : c] c = \partial^X \tau
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. $\checkmark$

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\ \phi\ } (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\ \psi\ } (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Chain equivalence: Easy side: $\phi \circ \psi = \mathrm{id}$ $\quad x \mapsto x - [x : \tau]\sigma \mapsto x.$

$$\begin{tikzcd}
 & \mathbf{C}_{d+1}(X') \arrow[r, "\partial_{d+1}"] \arrow[dl, "D_{d+1}"'] \arrow[d, bend right, "\phi\circ\psi"'] \arrow[d, bend left, "\mathrm{id}"] & \mathbf{C}_d(X') \arrow[r, "\partial_d"] \arrow[dl, "D_d"'] \arrow[d, bend right, "\phi\circ\psi"'] \arrow[d, bend left, "\mathrm{id}"] & \mathbf{C}_{d-1}(X') \arrow[dl, "D_{d-1}"'] \\
\mathbf{C}_{d+2}(X') \arrow[r] & \mathbf{C}_{d+1}(X') \arrow[r] & \mathbf{C}_d(X') &
\end{tikzcd}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. $\checkmark$

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\psi \circ \phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \\ x - [x : \tau]\sigma & \text{otherwise.} \end{cases} \qquad D(x) = \begin{cases} \sigma & \text{if } x = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Chain equivalence:

$$\begin{tikzcd}
 & \mathbf{C}_{d+1}(X) \arrow[r, "\partial_{d+1}"] \arrow[dl, "D_{d+1}"'] \arrow[d, bend right, "\psi\circ\phi"'] \arrow[d, bend left, "\mathrm{id}"] & \mathbf{C}_d(X) \arrow[r, "\partial_d"] \arrow[dl, "D_d"'] \arrow[d, bend right, "\psi\circ\phi"'] \arrow[d, bend left, "\mathrm{id}"] & \mathbf{C}_{d-1}(X) \arrow[dl, "D_{d-1}"'] \\
\mathbf{C}_{d+2}(X) \arrow[r] & \mathbf{C}_{d+1}(X) \arrow[r] & \mathbf{C}_d(X) &
\end{tikzcd}$$