2 - Discrete Morse theory

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EMAp Summer Program 2023

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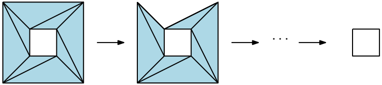
Elementary collapses

Simplification with elementary collapses:



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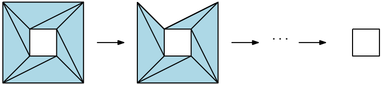
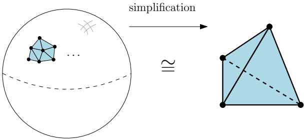


With Discrete Morse theory:



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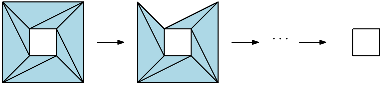

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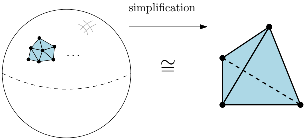
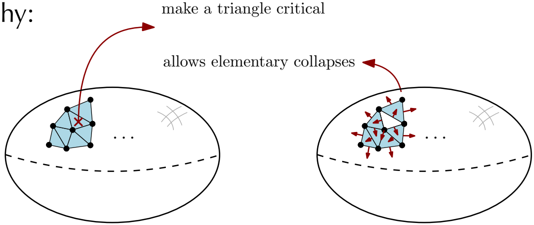


the philosophy:



Morse matching

Fix the coefficient field $\mathbb{Z}/2\mathbb{Z}$.

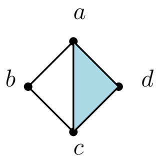
Definition

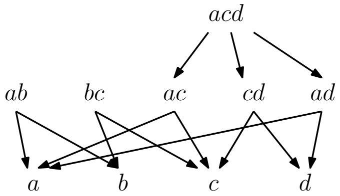
A simplicial complex $\mathbf{K}$ induces a *face partial ordering* $<$ between simplices. It is the transitive closure of the relation $\prec$

$$\tau \prec \sigma \quad \text{iff} \quad \tau \subset \sigma \text{ and } \dim \tau = \dim \sigma - 1.$$

The *Hasse diagram* of a complex is the directed graph (V, E) with:

- $V = \mathbf{K} - \{\emptyset\}$,
- $(\sigma \to \tau) \in E$ iff $\tau \prec \sigma$.





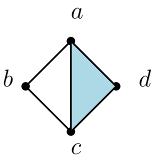
Morse matching

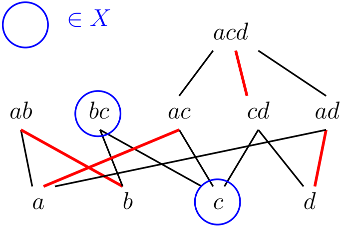
Definition

A *partial matching* of $\mathbf{K}$ is a partition $\mathbf{K} = X \sqcup T \sqcup S$ with a bijective pairing $\omega : T \to S$, where:

$$\tau \prec \omega(\tau), \text{ for all } \tau \in T.$$

$\longrightarrow$ it is a graph matching in the Hasse diagram of $\mathbf{K}$.





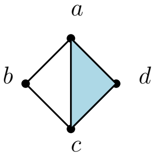
Morse matching

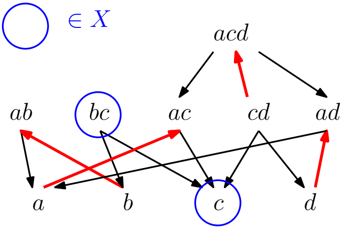
Definition

Let $\mathbf{K}$ be a simplicial complex, $\mathcal{H} = (\mathbf{K}, E)$ its Hasse diagram, and $(X \sqcup T \sqcup S, \omega)$ a partial matching. Define the directed graph $\bar{\mathcal{H}}$ as the graph $(\mathbf{K}, E')$:

- with same underlying undirected graph as $\mathcal{H}$,
- where every edge $(\omega(\tau) \to \tau) \in E$ is reversed in E', i.e., $(\tau \to \omega(\tau)) \in E'$.

If $\bar{\mathcal{H}}$ is acyclic, we call the partial matching a Morse matching.





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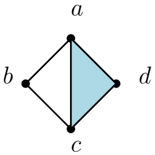
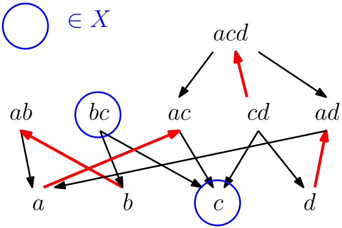




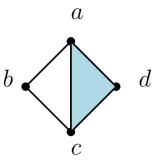
NB: a cycle can only alternate between two dim. d and $d+1$ (ω bij.).

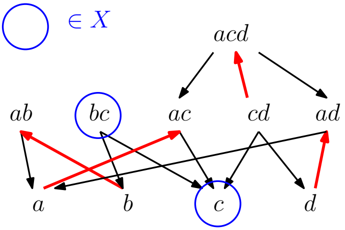
Morse matching

Definition

Let $\mathbf{K}$ be a simplicial complex, $\mathcal{H} = (\mathbf{K}, E)$ its Hasse diagram, and $(X \sqcup T \sqcup S, \omega)$ a Morse matching.

- Simplices in X are called critical.
- A Morse matching is optimal if it has the minimal number of critical simplices, over all possible Morse matchings of $\mathbf{K}$.





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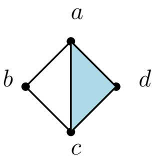
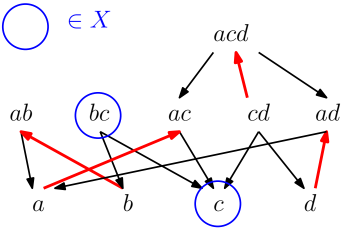
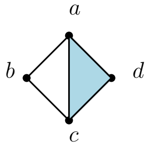
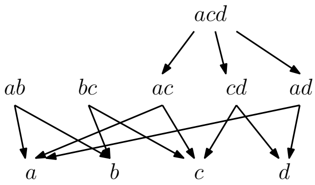
It is **NP-hard** to decide whether a simplicial complex has a Morse matching with less than k critical simplices.





Compute a (good) Morse matching: Heuristic





X :

$T \leftrightarrow S$

- *A* set of *available simplices*; $A \leftarrow \mathbf{K}$, $X, T, S \leftarrow \emptyset$
- **While** $A \neq \emptyset$:
  - **if** there is a free pair (τ, σ) in A, match τ with σ:
    - $A \leftarrow A \setminus \{\tau, \sigma\}$, ***perform elementary collapse***
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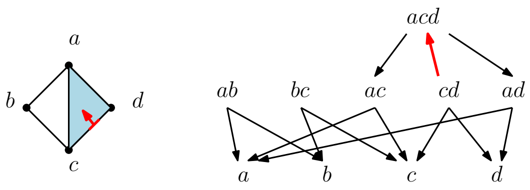

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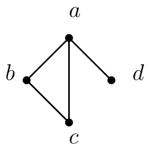

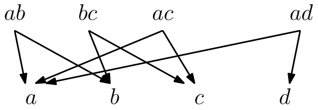

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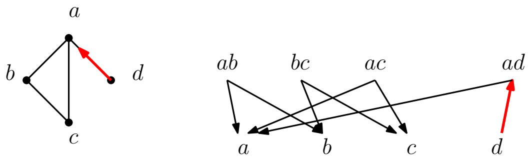

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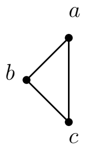

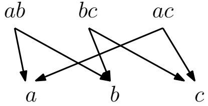

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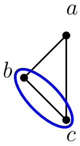

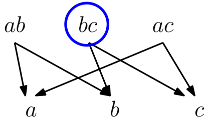

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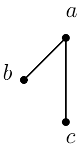

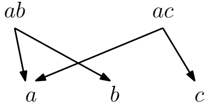

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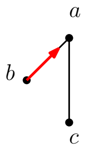

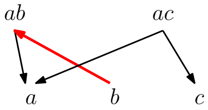

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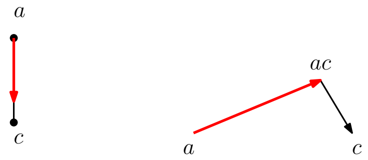

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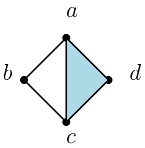

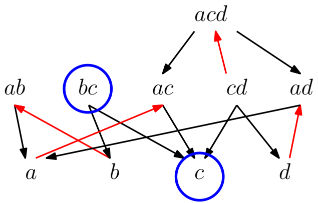

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Lemma (Correction)

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Compute a (good) Morse matching: Heuristic

Lemma (Correction)

- *A always contains a simplicial complex: only perform elementary collapses and maximal simplex removals.*

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Compute a (good) Morse matching: Heuristic

Lemma (Correction)

- *A always contains a simplicial complex: only perform elementary collapses and maximal simplex removals.*
- *$T \times S$ always contains a* matching*: after being matched, a simplex is removed from the available simplices.*

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Lemma (Correction)

- *A always contains a simplicial complex: only perform elementary collapses and maximal simplex removals.*
- *$T \times S$ always contains a* matching*: after being matched, a simplex is removed from the available simplices.*
- *The matching is always* acyclic*: ...*

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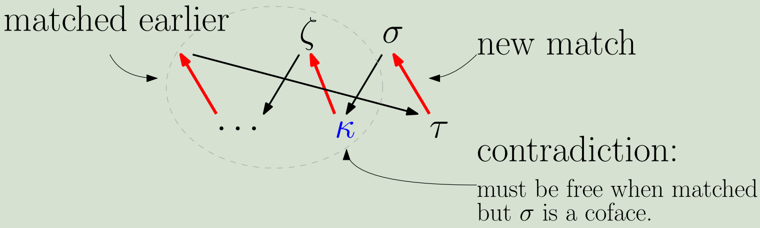
Compute a (good) Morse matching: Heuristic

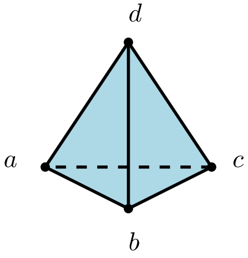
Lemma (Correction)

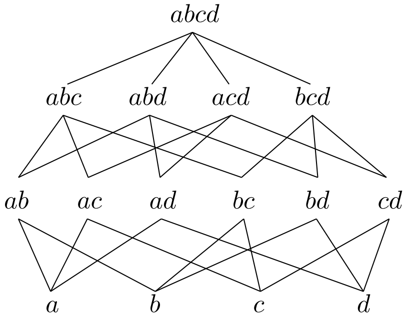
- *A always contains a simplicial complex: only perform elementary collapses and maximal simplex removals.*
- *$T \times S$ always contains a* matching*: after being matched, a simplex is removed from the available simplices.*
- *The matching is always* acyclic*: ...*

Suppose that, at some point, reverting an arrow creates a cycle. Pick the first such arrow (τ, σ):



More interesting example





What is the number of critical simplices of an optimal Morse matching?

A 0

B 1

C 2

D 3

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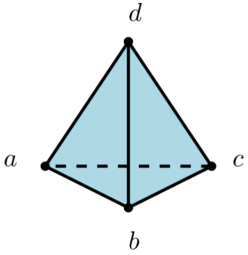

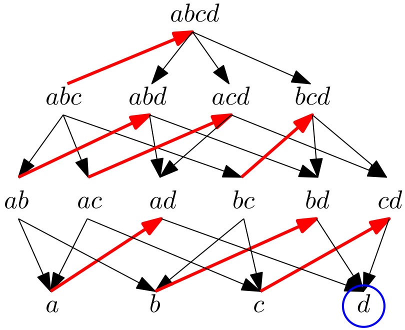

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Fundamental Theorem of Discrete Morse Theory

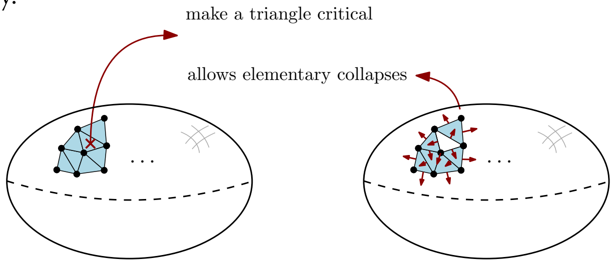
Theorem

Let $\mathbf{K}$ be a simplicial complex, and $(X \sqcup T \sqcup S, \omega)$ a Morse matching. There exists a chain complex $(\mathbf{C}(X), \partial^X)$ with cells in X, called a Morse complex, whose homology groups are isomorphic to the ones of $\mathbf{K}$.

More precisely,

- $\mathbf{C}_d(X)$ generated by the d-dimensional simplices of X,
- $\partial_d^X \colon \mathbf{C}_d(X) \to \mathbf{C}_{d-1}(X)$.

Application: reduce a complex drastically when computing its homology.



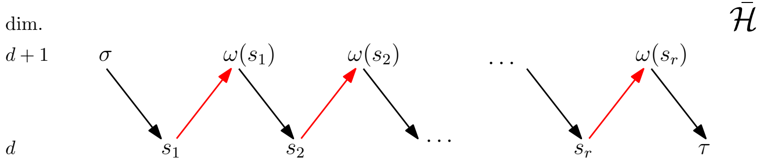
Boundary map ∂^X

Definition

In a complex $\mathbf{K}$ with Morse matching $(X \sqcup T \sqcup S, \omega)$, let $\sigma, \tau \in X$ be critical simplices, with $\dim \sigma = \dim \tau + 1$.
A gradient path from σ to τ is a directed path from σ to τ:

$$\sigma \to s_1 \to \omega(s_1) \to s_2 \to \ldots \to s_r \to \omega(s_r) \to \tau$$

in $\bar{\mathcal{H}}$, alternating between simplices of dimension d and $d-1$:



Boundary map ∂^X

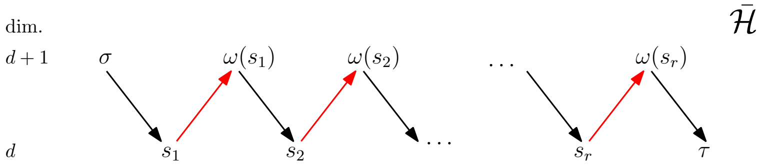
Definition

For $\sigma, \tau \in X$, with $\dim \sigma = \dim \tau + 1$, define

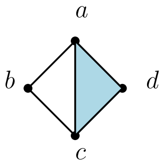
$$[\sigma : \tau] := \#\text{ distinct } (\sigma \to \tau)\text{-gradient path} \in \mathbb{Z}/2\mathbb{Z}.$$

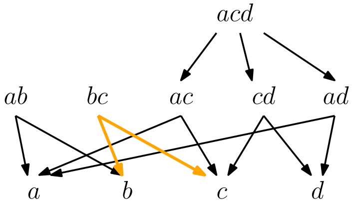
Then the boundary map $\partial^X : \mathbf{C}(X) \to \mathbf{C}(X)$ is defined such that:

$$\partial^X(\sigma) = \sum_{\substack{\tau \in X \\ \dim \sigma = \dim \tau + 1}} [\sigma : \tau] \cdot \tau$$



Example: Morse complex





$$\partial(bc) = 1 \cdot b + 1 \cdot c$$

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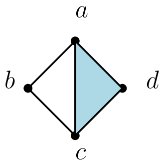

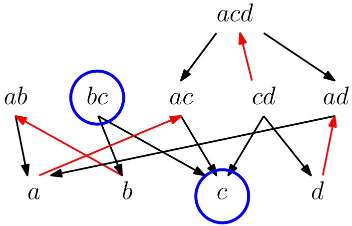

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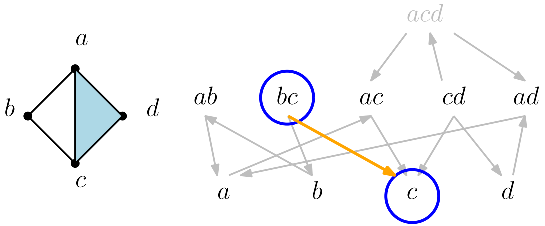

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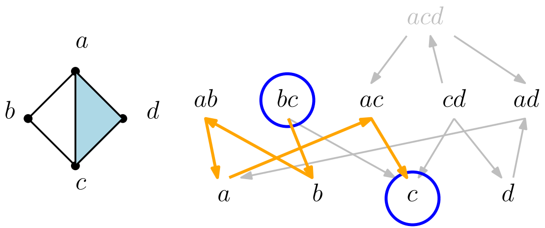

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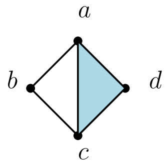

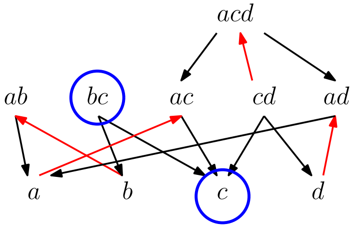

$$\partial^X(bc) = (1+1)\cdot c = 0$$

$$\Rightarrow \mathbf{H}_0 = \mathbf{H}_1 = \mathbf{Z}/2\mathbf{Z}$$

$$\mathbf{H}_2 = 0$$

Example: Morse complex

We need to prove that

- $(\mathbf{C}(X), \partial^X)$ is a chain complex, i.e., $\partial^X \circ \partial^X = 0$,
- $(\mathbf{C}(X), \partial^X)$ and $(\mathbf{C}(\mathbf{K}), \partial)$ have same homology.

The Morse complex is a chain complex

Inductive Proof of: $\partial_d^X \circ \partial_{d+1}^X \sigma = 0 \quad \forall d, \text{ and } \forall \sigma \in X$

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Now, let $X \sqcup T \sqcup S$ be a Morse matching with $\partial^X \circ \partial^X = 0$, and denote by $X_d \subseteq X$ the d-dimensional simplices in X.

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For any $a \in X_{d+1}$, we have:

$$\partial_d^X \circ \partial_{d+1}^X(a) = \partial_d^X(\sum_{b \in X_d}[a:b]b) = \sum_{b \in X_d}\sum_{c \in X_{d-1}}[a:b][b:c]c = 0. \quad (1)$$

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In particular, for any $a \in X_{d+1}$, and $c \in X_{d-1}$, using (1):

$$\sum_{b \neq \tau}[a:b][b:c] + [a:\tau][\tau:c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}.$$

$$\sum_{b \neq \tau} [a : b][b : c] + [a : \tau][\tau : c] = 0 \qquad \forall a \in X_{d+1}, c \in X_{d-1}. \tag{2}$$

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Add (σ, τ) to the Morse matching, denote by $(X', \partial^{X'})$ the new Morse complex, $X' := X \setminus \{\sigma, \tau\}$. For any $a \in X'$: counting $[x : y]$ in (X, ∂^X)

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Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\quad\phi\quad} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\quad\psi\quad} \mathbf{C}_\bullet(\mathbf{K})$$

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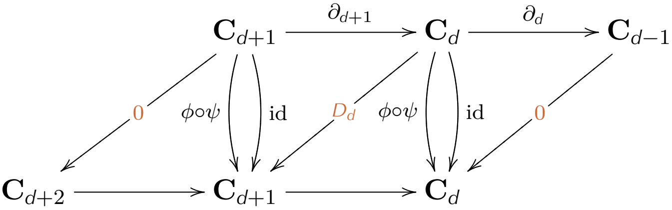

Proof of invariance by elem. collapse, with homotopy

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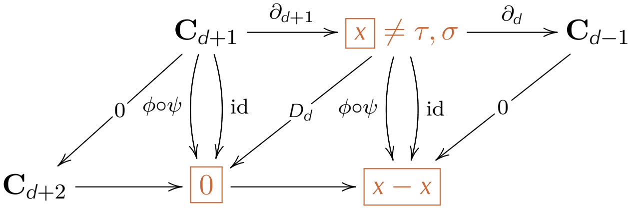

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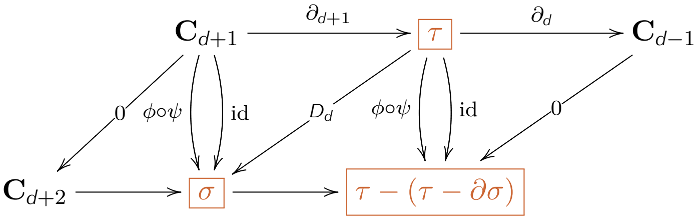

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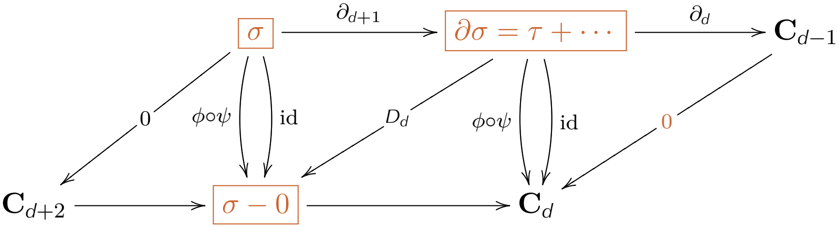

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Inductively: Base case trivial. ✓

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\boxed{x} & \xrightarrow{\ \partial^{X'}_{d+1}\ } & \sum_{b \neq \tau}[x : b]b + \sum_{b \neq \tau}[x : \tau][\sigma : b]b + 2[x : \tau]\tau & \xrightarrow{\ \partial^{X'}_d\ } &
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$$\begin{array}{ccccc} \mathbf{C}_{d+1}(X) & \xrightarrow{\partial^X_{d+1}} & \boxed{\tau} & \xrightarrow{\partial^X_d} & \mathbf{C}_{d-1}(X) \\ \downarrow{\scriptstyle \phi_{d+1}} & & \downarrow{\scriptstyle \phi_d} & & \downarrow{\scriptstyle \phi_{d-1}} \\ \mathbf{C}_{d+1}(X') & \xrightarrow{\partial^{X'}_{d+1}} & \mathbf{C}_d(X') & \xrightarrow{\partial^{X'}_d} & \mathbf{C}_{d-1}(X') \end{array}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad \psi(x) = x - [x : \tau]\sigma$$

Commutativity:

$$\begin{array}{ccccc}
\mathbf{C}_{d+1}(X) & \xrightarrow{\partial^X_{d+1}} & \boxed{\tau} & \xrightarrow{\partial^X_d} & \partial^X \tau \\
\downarrow{\scriptstyle \phi_{d+1}} & & \downarrow{\scriptstyle \phi_d} & & \downarrow{\scriptstyle \phi_{d-1}} \\
\mathbf{C}_{d+1}(X') & \xrightarrow{\partial^{X'}_{d+1}} & \mathbf{C}_d(X') & \xrightarrow{\partial^{X'}_d} & \mathbf{C}_{d-1}(X')
\end{array}$$

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Commutativity:

$$\begin{array}{ccccc}
\mathbf{C}_{d+1}(X) & \xrightarrow{\ \partial^X_{d+1}\ } & \boxed{\tau} & \xrightarrow{\ \partial^X_d\ } & \partial^X \tau \\
\downarrow \phi_{d+1} & & \downarrow \phi_d & & \downarrow \phi_{d-1} \\
\mathbf{C}_{d+1}(X') & \xrightarrow{\ \partial^{X'}_{d+1}\ } & \tau - \partial^X \sigma = \sum_{b \neq \tau} [\sigma : b] b & \xrightarrow{\ \partial^{X'}_d\ } & \partial^X \tau
\end{array}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓

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Commutativity: $\partial^X \partial^X \sigma = 0 = \partial^X (\sum_{b \neq \tau} [\sigma : b] b) + \partial^X \tau$

$$\begin{array}{ccccc} \mathbf{C}_{d+1}(X) & \xrightarrow{\partial^X_{d+1}} & \boxed{\tau} & \xrightarrow{\partial^X_d} & \partial^X \tau \\ \downarrow{\scriptstyle \phi_{d+1}} & & \downarrow{\scriptstyle \phi_d} & & \downarrow{\scriptstyle \phi_{d-1}} \\ \mathbf{C}_{d+1}(X') & \xrightarrow{\partial^{X'}_{d+1}} & \sum_{b \neq \tau} [\sigma : b] b & \xrightarrow{\partial^{X'}_d} & \sum_{b \neq \tau} [\sigma : b] \sum_c [b : c] c = \partial^X \tau \end{array}$$

Proof of invariance by adding a Morse pair

Inductively: Base case trivial. ✓
X a Morse complex, $(\sigma, \tau) \in X \times X$ such that $X' := X \setminus \{\sigma, \tau\}$ a valid Morse complex. $\dim \sigma = \dim \tau + 1 = d + 1$

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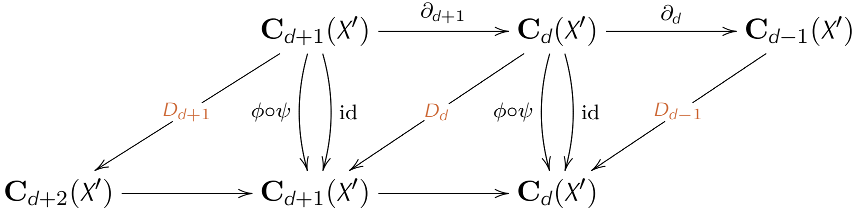
Chain equivalence: Easy side: $\phi \circ \psi = \text{id} \quad x \mapsto x - [x : \tau]\sigma \mapsto x.$



Proof of invariance by adding a Morse pair

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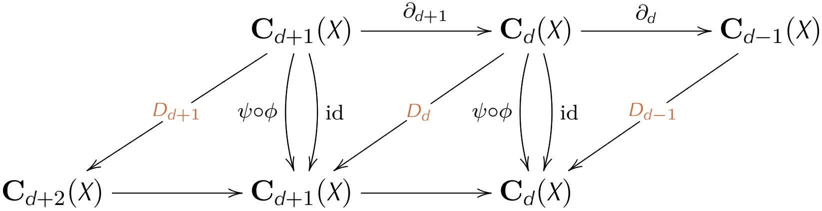
$$(\mathbf{C}(X), \partial^X) \xrightarrow{\phi} (\mathbf{C}(X'), \partial^{X'}) \xrightarrow{\psi} (\mathbf{C}(X), \partial^X) \quad \text{with}$$

$$\psi \circ \phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial^X \sigma & \text{if } x = \tau, \\ x - [x : \tau]\sigma & \text{otherwise.} \end{cases} \qquad D(x) = \begin{cases} \sigma & \text{if } x = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Chain equivalence:

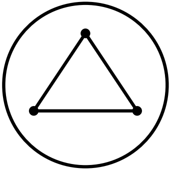


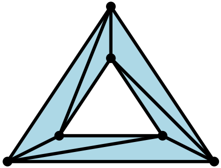
Exercise from last time

Compute the homology of:

- a tetrahedron, and of an empty tetrahedron,

- a circle,
- an annulus.

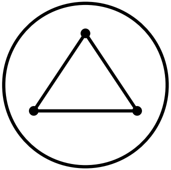


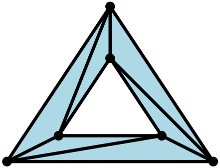


Exercise from last time

Compute the homology of:

- a tetrahedron, and of an empty tetrahedron, $\rightarrow$ diagonalise the matrix.
- a circle,
- an annulus.





Exercise from last time

Compute the homology of:

- a tetrahedron, and of an empty tetrahedron, $\rightarrow$ diagonalise the matrix.
- a circle, $\rightarrow$ triangulate with 3 edges, diagonalise the matrix.
- an annulus.