1 - Homology Theory

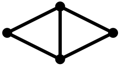
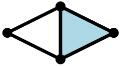
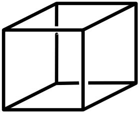
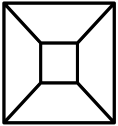
Clément Maria

MPRI 2.14.1 Computational Geometry and Topology

2023–2024

Toy Examples

Goal: Define a *topological invariant* of *simplicial complex* $\mathbf{K} \mapsto \mathbf{H}(\mathbf{K})$ that captures the *connectivity* of the domain $|\mathbf{K}|$, i.e., num. of connected components, of holes, of voids, etc.

simp. cpx.				
num. connected components:	1	1	1	1
num. holes:	2	1		

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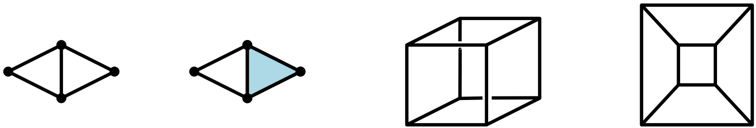

num. connected components:	1	1	1	1
num. holes:	2	1		

How many holes in the 1-skeleton of the cube?

A 4

B 5

C 6

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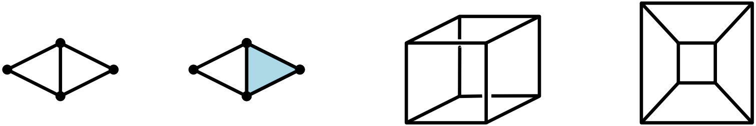

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How many holes in the planar graph?

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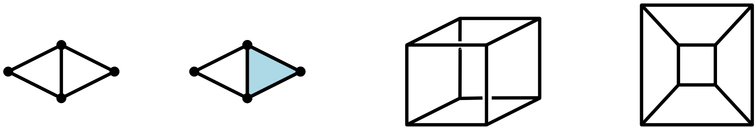

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num. connected components:	1	1	1	1
num. holes:	2	1	5	5

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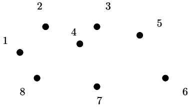
Simplicial complexes, abstract and geometric

Abstract: Set of *vertices*:

$$V = \{1, \ldots, |V|\}$$

Geometric: *Points* in $\mathbb{E}^d$:



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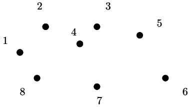
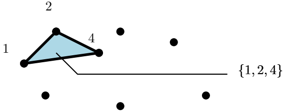


Convex hull of points:



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An *abstract simplicial complex* $\mathbf{K}$ is a family of simplices:
$\mathbf{K} = \{\sigma_i\}_{i \in I}$ s.t.:

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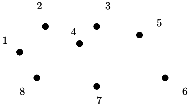

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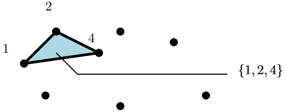

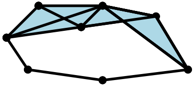

$\{1\}, \{2\}, \{3\}, \{4\}, \ldots$
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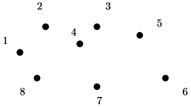

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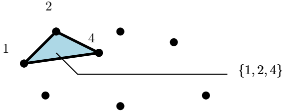

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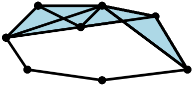
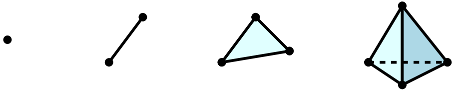


Dimension of a simplex = #vertices -1



Simplicial complexes, abstract

Definition (Abstract simplicial complex)

Let $V = \{v_1, \dots, v_n\} \neq$ be a finite *set* of vertices.

- A simplex σ of dimension $d \geq 0$ is a non-empty *subset* $\sigma \subseteq V$ of $d + 1$ vertices,

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- The *d*-skeleton of $\mathbf{K}$ is the subcomplex of $\mathbf{K}$ made of all simplices of dimension at most d.

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Remark

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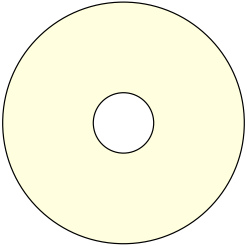
- *The intersection* $\tau \cap \sigma$ *of* distinct *simplices is a* strict *subface of both* τ *and* σ*.*
- *The empty set is not a simplex.*

Homology Theory: Topology via linear algebra



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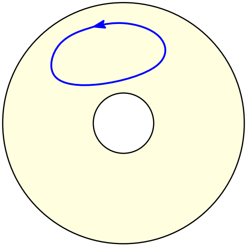

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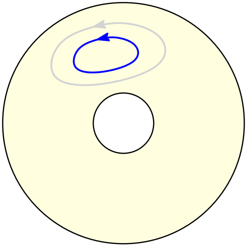

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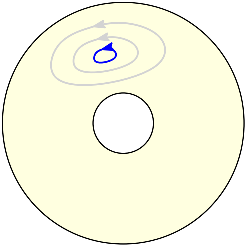

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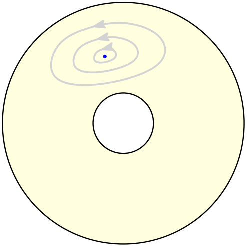

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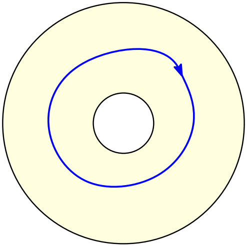

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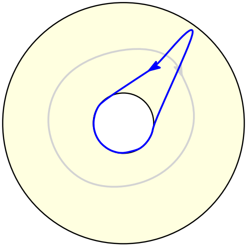

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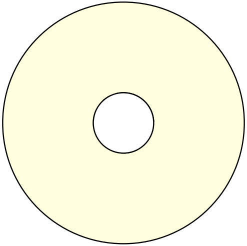

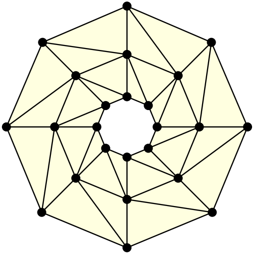

$\mathbb{F}$ coefficient field (such as $\mathbb{Z}/2\mathbb{Z}$), and $\mathbf{K}$ a simplicial complex:

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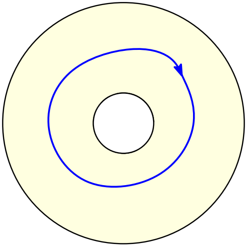

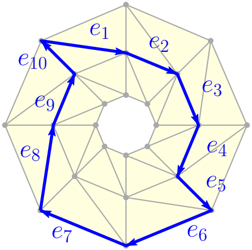

$\mathbb{F}$ coefficient field (such as $\mathbb{Z}/2\mathbb{Z}$), and $\mathbf{K}$ a simplicial complex:

Chain group $\mathbf{C}_d(\mathbf{K})$ of formal sums of d-simplices with $\mathbb{F}$ coefficients:

$$\mathbf{C}_d(\mathbf{K}) = \left\{ \sum_i \gamma_i \sigma_i \ : \ \dim \sigma_i = d \ \text{ and } \ \gamma_i \in \mathbb{F} \right\} \quad \text{ex: } e_1 + \ldots + e_{10} \in \mathbf{C}_1$$

$\longrightarrow$ $\mathbf{C}_d(\mathbf{K})$ is isomorphic to $\mathbb{F}^{n_d}$ the $\mathbb{F}$-vector space of dimension $n_d :=$ the number of d-dimensional faces of $\mathbf{K}$.

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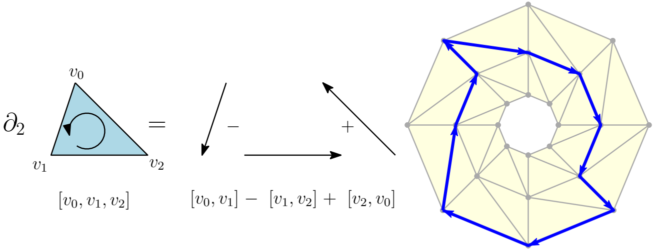

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Boundary operator: linear map $\partial_d : \mathbf{C}_d(\mathbf{K}) \to \mathbf{C}_{d-1}(\mathbf{K})$ satisfying

$$\partial_d[v_0, \ldots, v_d] = \sum_{i=0}^{d} (-1)^i [v_0, \ldots, \widehat{v_i}, \ldots, v_d]$$

$\longrightarrow$ for σ a simplex (seen as a chain), $\partial_d(\sigma)$ equals the alternate sum of facets of σ, hence the name “boundary”.

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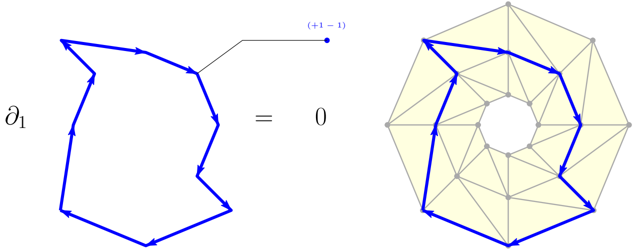

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The kernel of ∂_d, denoted by $\mathbf{Z}_d(\mathbf{K})$, contains the cycles of $\mathbf{K}$.
$\longrightarrow$ captures the “blue loops”.

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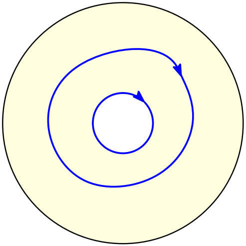

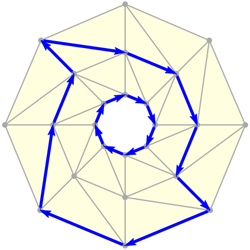

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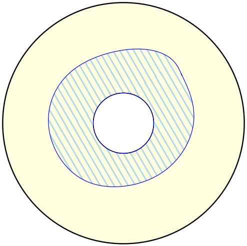

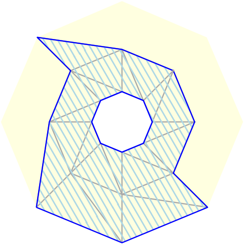

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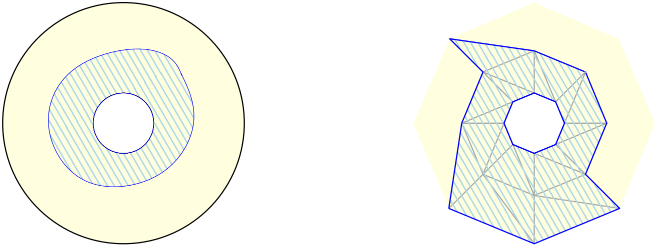

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The image of ∂_{d+1}, denoted by $\mathbf{B}_d(\mathbf{K})$, contains the boundaries of $\mathbf{K}$.
$\longrightarrow$ captures the "hashed surfaces" separating two equivalent cycles.

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Cycles $\mathbf{Z}_d = \ker \partial_d$ *Boundaries* $\mathbf{B}_d = \text{im } \partial_{d+1}$

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Theorem

For a simplicial complex $\mathbf{K}$*,* $\boxed{\partial_d \circ \partial_{d+1} = 0}$ *for all* $d \geq 0$.

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Theorem

For a simplicial complex $\mathbf{K}$*,* $\boxed{\partial_d \circ \partial_{d+1} = 0}$ *for all* $d \geq 0$.

For $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, and $\sigma = [v_0, \ldots, v_d]$ a d-dimensional simplex:

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&= 0
\end{aligned}$$

$\implies \forall d \geq 0, \ \ \mathbf{B}_d \subseteq \mathbf{Z}_d.$

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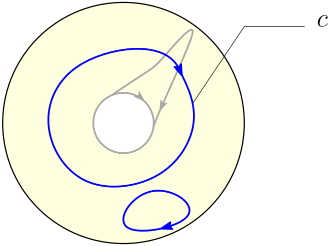
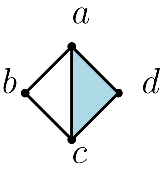
For a simplicial complex $\mathbf{K}$, $\boxed{\partial_d \circ \partial_{d+1} = 0}$ *for all* $d \geq 0$.

The dimension $\dim \mathbf{H}_d(\mathbf{K}) = \dim \mathbf{Z}_d - \dim \mathbf{B}_d$ is the dth Betti number β_d of $\mathbf{K}$ (with $\mathbb{F}$ coefficients).



$\mathbf{H}_1 = \langle\ [c]\ \rangle$

Homology computation



∂_2

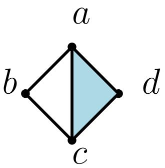
	acd
ac	1
cd	1
ad	1
ab	0
bc	0

∂_1

	ac	cd	ad	ab	bc
a	1	0	1	1	0
b	0	0	0	1	1
c	1	1	0	0	1
d	0	1	1	0	0

$\partial_0 = 0$

Homology computation



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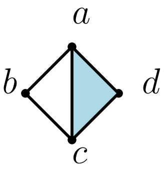
∂_2

	acd
ac	1
$ac+cd$	0
$ac+ad$	0
ab	0
bc	0

∂_1

	ac	cd	ab	$ab+ac+bc$	$ac+cd+ad$
$d+c$	1	0	0	0	0
d	0	1	0	0	0
b	0	0	1	0	0
$a+b+c+d$	0	0	0	0	0

Homology computation



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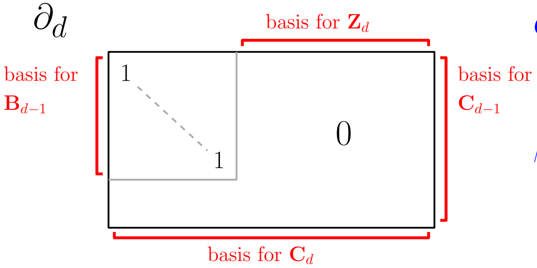
$\partial_0 = 0$

Smith normal form

$O(n^3)$ algorithm



$\beta_d = \dim \mathbf{Z}_d - \dim \mathbf{B}_d$

Euler characteristic and Betti numbers

Lemma

Let $\mathbf{K}$ be simplicial complex with n_d d-dimensional simplices, and dth Betti number β_d. Then its Euler characteristic $\chi(\mathbf{K})$ satisfies:

$$\chi(\mathbf{K}) = \sum_d (-1)^d n_d = \sum_d (-1)^d \beta_d$$

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&= \sum_d (-1)^d \dim \ker \partial_d + \sum_{d+1} (-1)^d \dim \mathrm{im}\ \partial_d
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&= \sum_d (-1)^d \dim \ker \partial_d + \sum_{d+1} (-1)^d \dim \operatorname{im}\ \partial_d \\
&= \sum_d (-1)^d \dim \mathbf{C}_d = \sum_d (-1)^d n_d.
\end{aligned}$$

A graph theory exercise:

Prove that β_0 (with $\mathbb{Z}/2\mathbb{Z}$ coefficients) is equal to the number of connected components of the complex.

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- $\partial_0 = 0 \Rightarrow C_0 = \mathbf{Z}_0$ and all vertices are cycles.
- if vertices u and v are in the same connected component, there exists a path $u = v_0, v_1, \ldots, v_k = v$ in the 1-skeleton connecting u and v. Hence: $u + \partial_1([v_0 v1] + [v_1 v_2] + \ldots [v_{d-1} v_d]) = v$, and u and v belong to the same homology class (they differ by an element of $\mathbf{B}_0$).

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- if $[u] = [v]$ in $\mathbf{H}_0$ then there exists $c \in \mathbf{C}_1$ such that $u + v = \partial_1 c$. Considering c as a graph (it is a collection of edges), u and v are the only *odd* degree vertices in c. Because the sum of degrees of a graph is even, u and v must belong to the same connected component.

$$\beta_0 \geq \text{num. of connected components.}$$

Topological invariance

The fundamental property of homology is its topological invariance:

Theorem (Topological invariance)

Let K and K' be simplicial complexes such that their domains $|\mathbf{K}|$ and $|\mathbf{K}'|$ are homotopy equivalent as topological spaces, then

$$\mathbf{H}_d(\mathbf{K}) \cong \mathbf{H}_d(\mathbf{K}'), \quad \forall d.$$

We prove something simpler, and purely combinatorial.

Elementary collapses

Definition

Let $\mathbf{K}$ be a simplicial complex. A *free pair* (τ, σ) is a pair of simplices such that τ has a unique coface σ.
An *elementary collapse* $\mathbf{K} \to \mathbf{K} - \{\tau, \sigma\}$ is the removal of a free pair.

$\longrightarrow$ an elementary collapse is a combinatorial version of a "local deformation retract" in topology.

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Lemma (Invariance)

For a free pair (τ, σ) *in* $\mathbf{K}$, $\quad \mathbf{H}_d(\mathbf{K}) \cong \mathbf{H}_d(\mathbf{K} - \{\sigma, \tau\}), \quad \forall d.$

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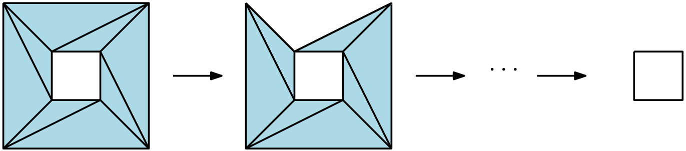

Proof of invariance by elem. collapse, with matrices

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A matrix proof: write $\dim \tau = d$ and $\dim \sigma = d + 1$.

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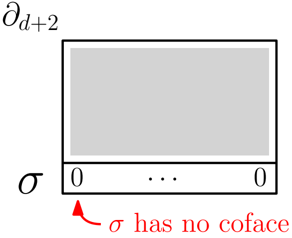
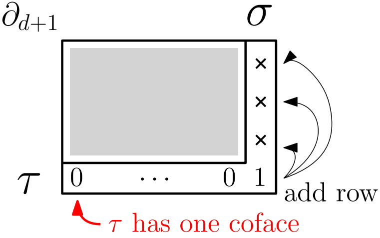
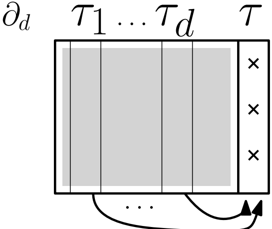






$$\partial_d \circ \partial_{d+1}(\sigma) = 0 = \partial_d(\tau_1 + \ldots + \tau_d) + \partial_d \tau$$

$$\dim \mathbf{B}_d(\mathbf{K}') = \dim \mathbf{B}_d(\mathbf{K}) - 1$$

$$\dim \mathbf{Z}_d(\mathbf{K}') = \dim \mathbf{Z}_d(\mathbf{K}) - 1$$

Chain maps and their homotopy

Definition (Chain map)

A *chain map* $\phi\colon (\mathbf{C}, \partial) \to (\mathbf{C}', \partial')$ between two chain complexes is a family of homomorphisms:

$$\phi_p\colon \mathbf{C}_p \to \mathbf{C}'_p$$

that commutes with boundary maps.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{d+2}} & \mathbf{C}_{d+1} & \xrightarrow{\partial_{d+1}} & \mathbf{C}_d & \xrightarrow{\partial_d} & \mathbf{C}_{d-1} & \xrightarrow{\partial_{d-1}} & \cdots \\
 & \circlearrowleft & \big\downarrow \phi_{d+1} & \circlearrowleft & \big\downarrow \phi_d & \circlearrowleft & \big\downarrow \phi_{d-1} & \circlearrowleft & \\
\cdots & \xrightarrow[\partial'_{d+2}]{} & \mathbf{C}'_{d+1} & \xrightarrow[\partial'_{d+1}]{} & \mathbf{C}'_d & \xrightarrow[\partial'_d]{} & \mathbf{C}'_{d-1} & \xrightarrow[\partial'_{d-1}]{} & \cdots
\end{array}$$

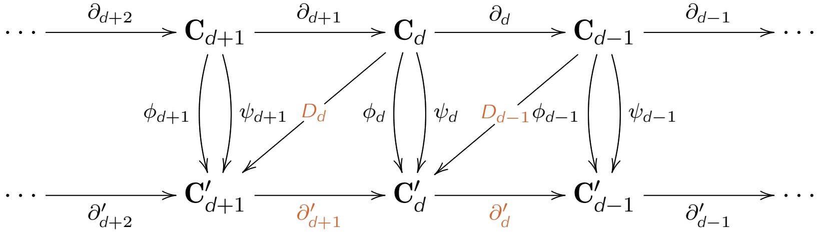
Chain maps and their homotopy

Definition (Chain homotopy)

If $\phi, \psi \colon (\mathbf{C}, \partial) \to (\mathbf{C}', \partial')$ are chain maps, a *chain homotopy* of ϕ to ψ is a family of homomorphisms

$$D_p \colon \mathbf{C}_p \to \mathbf{C}'_{p+1}$$

such that $$\boxed{\partial'_{p+1} \circ D_p + D_{p-1} \circ \partial_p = \psi_p - \phi_p}$$



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A chain map $\phi\colon (\mathbf{C}, \partial) \to (\mathbf{C}', \partial')$ is a *chain equivalence* if there is a chain map $\phi'\colon (\mathbf{C}', \partial') \to (\mathbf{C}, \partial)$ such that $\phi \circ \phi'$ and $\phi' \circ \phi$ are chain homotopic to the identity (of their respective domains).

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$\longrightarrow$ the algebraic version of an homotopy of continuous functions.

Theorem

If there exists a chain equivalence between two chain complexes, they have same homology.

Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\phi} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\psi} \mathbf{C}_\bullet(\mathbf{K})$$

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$$\psi(x) \quad = \quad x, \qquad \text{i.e. the inclusion } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$$

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$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\phi} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\psi} \mathbf{C}_\bullet(\mathbf{K})$$

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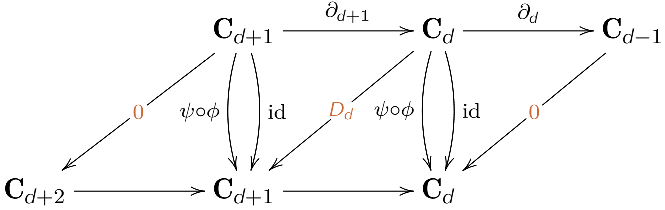
$$\psi(x) = x, \quad \text{i.e. the inclusion } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial\sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \qquad D(x) = \begin{cases} \sigma & \text{if } x = \tau, \\ 0 & \text{otherwise.} \end{cases}$$



Proof of invariance by elem. collapse, with homotopy

$\mathbf{K}$ a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\ \phi\ } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\ \psi\ } \mathbf{C}_\bullet(\mathbf{K})$$

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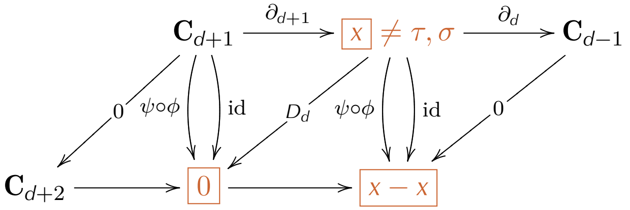

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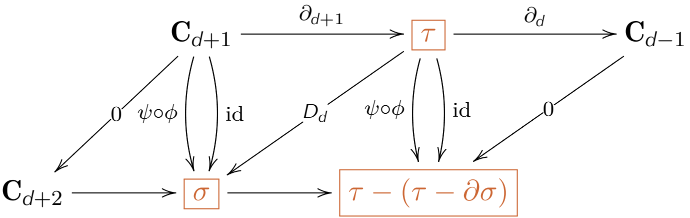

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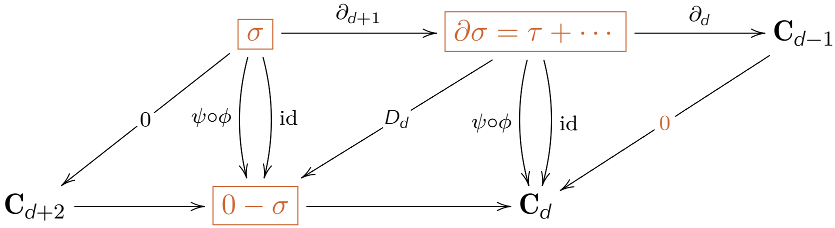

Proof of invariance by elem. collapse, with homotopy

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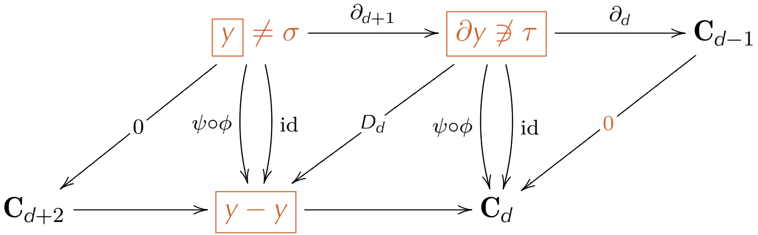

Exercise

Compute the homology of:

- a tetrahedron, and of an empty tetrahedron,
- a circle,
- an annulus.

Prove the other direction ($\phi \circ \psi$) for the chain equivalence proof of the invariance of homology by elementary collapse.