

1 - Homology Theory

Clément Maria

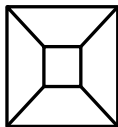
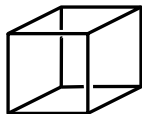
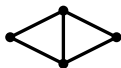
EMAp Summer Program 2023

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Toy Examples

Goal: Define a *topological invariant* of simplicial complex $\mathbf{K} \mapsto \mathbf{H}(\mathbf{K})$ that captures the *connectivity* of the domain $|\mathbf{K}|$, i.e., num. of **connected components**, of **holes**, of **voids**, etc.

simp. cpx.



num. connected components:

1

1

1

1

num. holes:

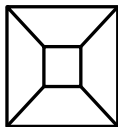
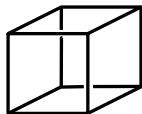
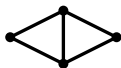
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1

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num. connected components:

1

1

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1

num. holes:

2

1

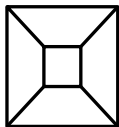
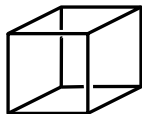
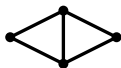
How many holes in the 1-skeleton of the cube?

- A 4
- B 5
- C 6

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num. connected components:

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How many holes in the planar graph?



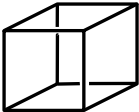
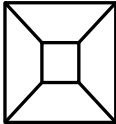
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simp. cpx.				
num. connected components:	1	1	1	1
num. holes:	2	1	5	5

How many holes in the planar graph?

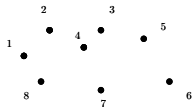
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Simplicial complexes, abstract and geometric

Abstract: Set of *vertices*:

$$V = \{1, \dots, |V|\}$$

Geometric: *Points* in \mathbb{E}^d :



Simplicial complexes, abstract and geometric

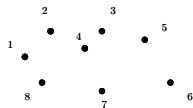
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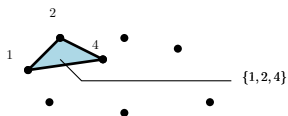
A *simplex* σ is a collection of vertices:

$$\sigma \subseteq V$$

Geometric: Points in \mathbb{E}^d :



Convex hull of points:



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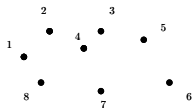
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An *abstract simplicial complex* \mathbf{K} is a family of simplices:

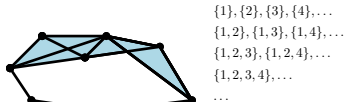
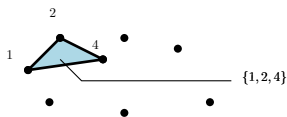
$$\mathbf{K} = \{\sigma_i\}_{i \in I} \text{ s.t.}$$

$$\tau \subseteq \sigma \in \mathbf{K} \Rightarrow \tau \in \mathbf{K}$$

Geometric: Points in \mathbb{E}^d :



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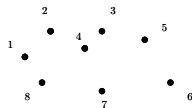
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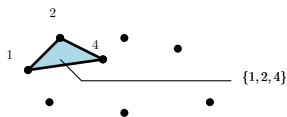
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Dimension of a simplex =
#vertices - 1

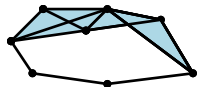
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Convex hull of points:



- {1}, {2}, {3}, {4}, ...
- {1, 2}, {1, 3}, {1, 4}, ...
- {1, 2, 3}, {1, 2, 4}, ...
- {1, 2, 3, 4}, ...
- ...



dim: 0



1



2



3

Simplicial complexes, abstract

Definition (Abstract simplicial complex)

Let $V = \{v_1, \dots, v_n\} \neq \emptyset$ be a finite set of **vertices**.

- A **simplex** σ of **dimension** $d \geq 0$ is a non-empty *subset* $\sigma \subseteq V$ of $d + 1$ vertices,

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- The **d -skeleton** of \mathbf{K} is the subcomplex of \mathbf{K} made of all simplices of dimension at most d .

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Remark

- *A dimension 1 simplicial complex (e.g., the 1-skeleton of a simplicial complex) is combinatorially a graph (with no loop nor multiple edges).*

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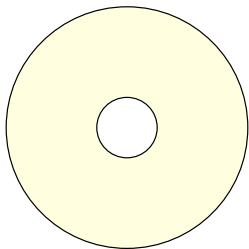
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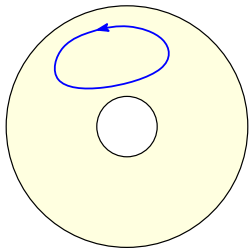
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- The intersection $\tau \cap \sigma$ of distinct simplices is a strict subface of both τ and σ .
- The empty set is not a simplex.

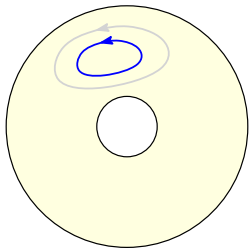
Homology Theory: Topology via linear algebra



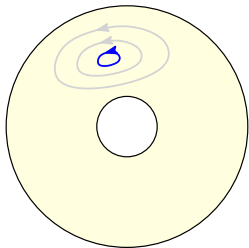
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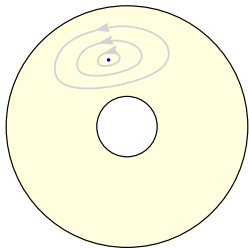
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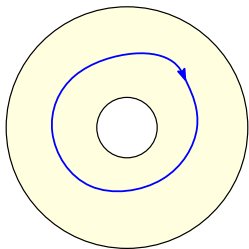
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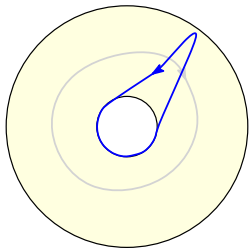
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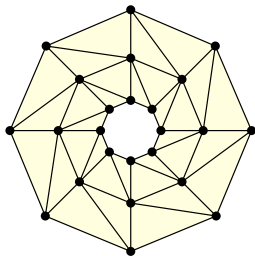
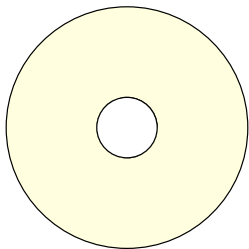
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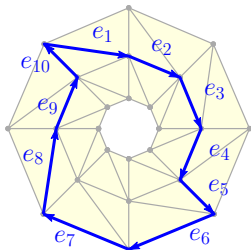
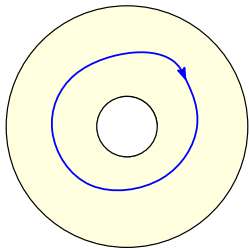


Homology Theory: Topology via linear algebra



\mathbb{F} coefficient field (such as $\mathbb{Z}/2\mathbb{Z}$), and \mathbf{K} a simplicial complex:

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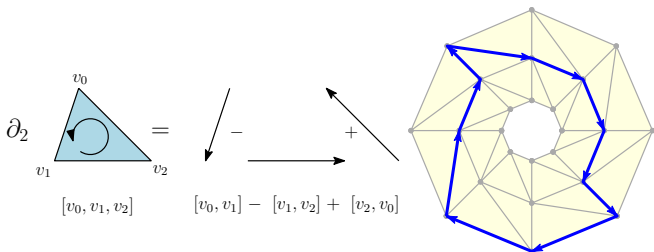
\mathbb{F} coefficient field (such as $\mathbb{Z}/2\mathbb{Z}$), and \mathbf{K} a simplicial complex:

Chain group $\mathbf{C}_d(\mathbf{K})$ of formal sums of d -simplices with \mathbb{F} coefficients:

$$\mathbf{C}_d(\mathbf{K}) = \left\{ \sum_i \gamma_i \sigma_i : \dim \sigma_i = d \text{ and } \gamma_i \in \mathbb{F} \right\} \quad \text{ex: } e_1 + \dots + e_{10} \in \mathbf{C}_1$$

$\longrightarrow \mathbf{C}_d(\mathbf{K})$ is isomorphic to \mathbb{F}^{n_d} the \mathbb{F} -vector space of dimension $n_d :=$ the number of d -dimensional faces of \mathbf{K} .

Homology Theory: Topology via linear algebra



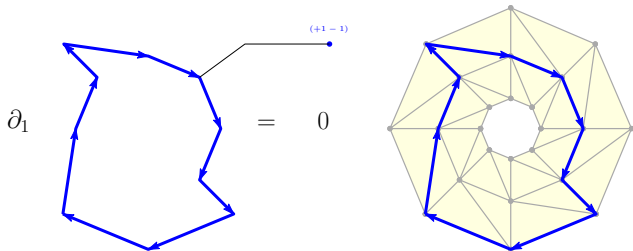
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Boundary operator: linear map $\partial_d : \mathbf{C}_d(\mathbf{K}) \rightarrow \mathbf{C}_{d-1}(\mathbf{K})$ satisfying

$$\partial_d[v_0, \dots, v_d] = \sum_{i=0}^d (-1)^i [v_0, \dots, \widehat{v}_i, \dots, v_d]$$

→ for σ a simplex (seen as a chain), $\partial_d(\sigma)$ equals the alternate sum of **facets** of σ , hence the name “boundary”.

Homology Theory: Topology via linear algebra



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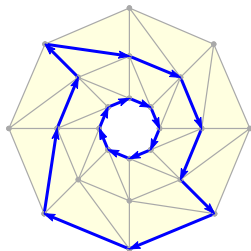
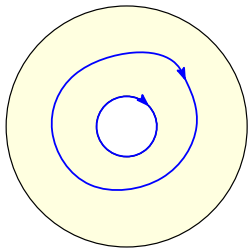
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The kernel of ∂_d , denoted by $\mathbf{Z}_d(\mathbf{K})$, contains the *cycles* of \mathbf{K} .

→ captures the “blue loops”.

Homology Theory: Topology via linear algebra

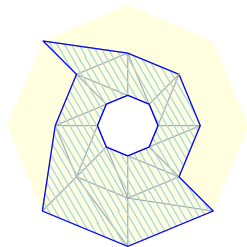
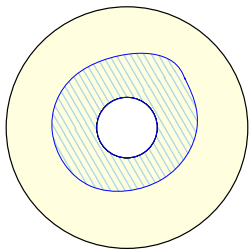


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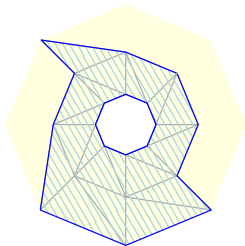
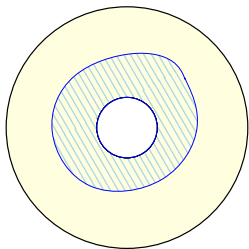


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The image of ∂_{d+1} , denoted by $\mathbf{B}_d(\mathbf{K})$, contains the **boundaries** of \mathbf{K} .
→ captures the “hashed surfaces” separating two equivalent cycles.

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Cycles $\mathbf{Z}_d = \ker \partial_d$ *Boundaries* $\mathbf{B}_d = \text{im } \partial_{d+1}$

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Theorem

For a simplicial complex \mathbf{K} , $\partial_d \circ \partial_{d+1} = 0$ for all $d \geq 0$.

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Theorem

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For $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, and $\sigma = [v_0, \dots, v_d]$ a d -dimensional simplex:

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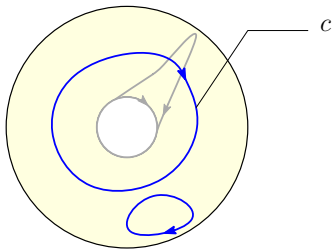
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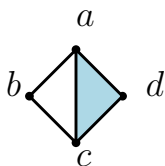
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The dimension $\dim \mathbf{H}_d(\mathbf{K}) = \dim \mathbf{Z}_d - \dim \mathbf{B}_d$ is the d th Betti number β_d of \mathbf{K} (with \mathbb{F} coefficients).



$$\mathbf{H}_1 = \langle [c] \rangle$$

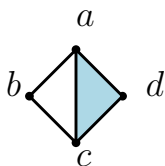
Homology computation



$$\partial_2 \begin{array}{c} acd \\ ac \\ cd \\ ad \\ ab \\ bc \end{array} \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array}$$

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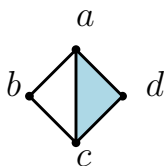
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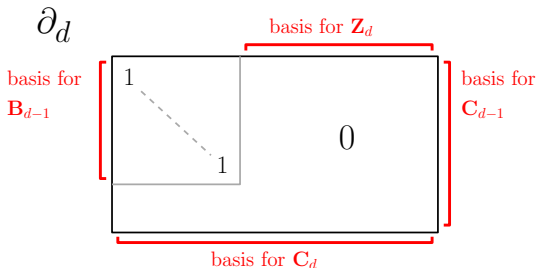
Homology computation



$$\partial_2 \quad \begin{matrix} acd \\ ac \\ cd \\ ad \\ ab \\ bc \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\partial_1 \quad \begin{matrix} ac & cd & ad & ab & bc \end{matrix} \quad \partial_0 = 0$$

a	1	0	1	1	0
b	0	0	0	1	1
c	1	1	0	0	1
d	0	1	1	0	0



Smith normal form
 $O(n^3)$ algorithm

$$\beta_d = \dim \mathbf{Z}_d - \dim \mathbf{B}_d$$

Euler characteristic and Betti numbers

Lemma

Let \mathbf{K} be simplicial complex with n_d d -dimensional simplices, and d th Betti number β_d . Then its Euler characteristic $\chi(\mathbf{K})$ satisfies:

$$\chi(\mathbf{K}) = \sum_d (-1)^d n_d = \sum_d (-1)^d \beta_d$$

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A graph theory exercise:

Prove that β_0 (with $\mathbb{Z}/2\mathbb{Z}$ coefficients) is equal to the number of connected components of the complex.

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- $\partial_0 = 0 \Rightarrow C_0 = \mathbf{Z}_0$ and all vertices are cycles.
- if vertices u and v are in the same connected component, there exists a path $u = v_0, v_1, \dots, v_k = v$ in the 1-skeleton connecting u and v . Hence: $u + \partial_1([v_0v_1] + [v_1v_2] + \dots + [v_{d-1}v_d]) = v$, and u and v belong to the same homology class (they differ by an element of \mathbf{B}_0).

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- if $[u] = [v]$ in \mathbf{H}_0 then there exists $c \in \mathbf{C}_1$ such that $u + v = \partial_1 c$. Considering c as a graph (it is a collection of edges), u and v are the only *odd* degree vertices in c . Because the sum of degrees of a graph is even, u and v must belong to the same connected component.

$$\beta_0 \geq \text{num. of connected components.}$$

Topological invariance

The fundamental property of homology is its topological invariance:

Theorem (Topological invariance)

Let K and K' be simplicial complexes such that their domains $|\mathbf{K}|$ and $|\mathbf{K}'|$ are homotopy equivalent as topological spaces, then

$$\mathbf{H}_d(\mathbf{K}) \cong \mathbf{H}_d(\mathbf{K}'), \quad \forall d.$$

We prove something simpler, and purely combinatorial.

Elementary collapses

Definition

Let \mathbf{K} be a simplicial complex. A *free pair* (τ, σ) is a pair of simplices such that τ has a unique coface σ .

An *elementary collapse* $\mathbf{K} \rightarrow \mathbf{K} - \{\tau, \sigma\}$ is the removal of a free pair.

—→ an elementary collapse is a combinatorial version of a “local deformation retract” in topology.

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For a free pair (τ, σ) in \mathbf{K} , $\mathbf{H}_d(\mathbf{K}) \cong \mathbf{H}_d(\mathbf{K} - \{\sigma, \tau\})$, $\forall d$.

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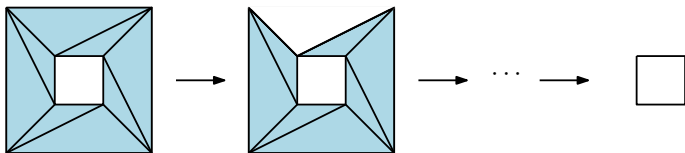
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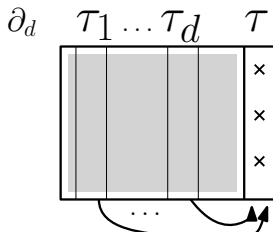
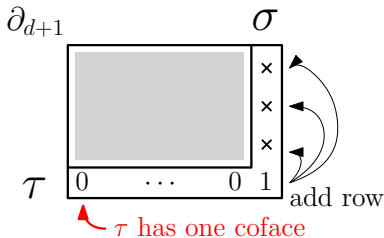
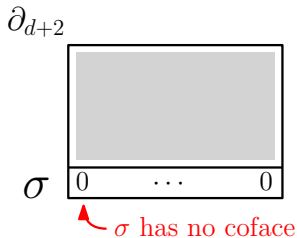
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$$\partial_d \circ \partial_{d+1}(\sigma) = 0 = \partial_d(\tau_1 + \dots + \tau_d) + \partial_d \tau$$

$$\dim \mathbf{B}_d(\mathbf{K}') = \dim \mathbf{B}_d(\mathbf{K}) - 1$$

$$\dim \mathbf{Z}_d(\mathbf{K}') = \dim \mathbf{Z}_d(\mathbf{K}) - 1$$

Chain maps and their homotopy

Definition (Chain map)

A *chain map* $\phi: (\mathbf{C}, \partial) \rightarrow (\mathbf{C}', \partial')$ between two chain complexes is a family of homomorphisms:

$$\phi_p: \mathbf{C}_p \rightarrow \mathbf{C}'_p$$

that commutes with boundary maps.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_{d+2}} & \mathbf{C}_{d+1} & \xrightarrow{\partial_{d+1}} & \mathbf{C}_d & \xrightarrow{\partial_d} & \mathbf{C}_{d-1} & \xrightarrow{\partial_{d-1}} & \cdots \\ & & \downarrow \phi_{d+1} & & \downarrow \phi_d & & \downarrow \phi_{d-1} & & \\ \cdots & \xrightarrow{\partial'_{d+2}} & \mathbf{C}'_{d+1} & \xrightarrow{\partial'_{d+1}} & \mathbf{C}'_d & \xrightarrow{\partial'_d} & \mathbf{C}'_{d-1} & \xrightarrow{\partial'_{d-1}} & \cdots \end{array}$$

Chain maps and their homotopy

Definition (Chain homotopy)

If $\phi, \psi: (\mathbf{C}, \partial) \rightarrow (\mathbf{C}', \partial')$ are chain maps, a *chain homotopy* of ϕ to ψ is a family of homomorphisms

$$D_p: \mathbf{C}_p \rightarrow \mathbf{C}'_{p+1}$$

such that

$$\partial'_{p+1} \circ D_p + D_{p-1} \circ \partial_p = \psi_p - \phi_p$$

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{\partial_{d+2}} & \mathbf{C}_{d+1} & \xrightarrow{\partial_{d+1}} & \mathbf{C}_d & \xrightarrow{\partial_d} & \mathbf{C}_{d-1} & \xrightarrow{\partial_{d-1}} & \dots \\
 & & \downarrow \phi_{d+1} & & \downarrow \phi_d & & \downarrow \phi_{d-1} & & \\
 & & \downarrow \psi_{d+1} & & \downarrow \psi_d & & \downarrow \psi_{d-1} & & \\
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Theorem

If there exists a chain equivalence between two chain complexes, they have same homology.

Proof of invariance by elem. collapse, with homotopy

\mathbf{K} a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\phi} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\psi} \mathbf{C}_\bullet(\mathbf{K})$$

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$$\begin{array}{ccccc}
 & & \mathbf{C}_{d+1} & \xrightarrow{\partial_{d+1}} & \mathbf{C}_d & \xrightarrow{\partial_d} & \mathbf{C}_{d-1} \\
 & \swarrow & \downarrow \psi \circ \phi & & \downarrow \psi \circ \phi & & \swarrow \\
 & & \mathbf{C}_{d+1} & \xrightarrow{\text{id}} & \mathbf{C}_d & & \\
 & \searrow & \downarrow \text{id} & & \downarrow \text{id} & & \searrow \\
 \mathbf{C}_{d+2} & \xrightarrow{0} & \mathbf{C}_{d+1} & \xrightarrow{D_d} & \mathbf{C}_d & &
 \end{array}$$

Proof of invariance by elem. collapse, with homotopy

\mathbf{K} a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\phi} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\psi} \mathbf{C}_\bullet(\mathbf{K})$$

with:

$$\psi(x) = x, \quad \text{i.e. the inclusion } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial\sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \quad D(x) = \begin{cases} \sigma & \text{if } x = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{array}{ccccc}
 & & \mathbf{C}_{d+1} & \xrightarrow{\partial_{d+1}} & \boxed{x} \neq \tau, \sigma & \xrightarrow{\partial_d} & \mathbf{C}_{d-1} \\
 & \swarrow 0 & \downarrow \psi \circ \phi & \downarrow \text{id} & \swarrow D_d & \downarrow \psi \circ \phi & \downarrow \text{id} & \swarrow 0 \\
 & & \mathbf{C}_{d+2} & \xrightarrow{\quad} & \boxed{0} & \xrightarrow{\quad} & \boxed{x-x} & \\
 & & & & & & &
 \end{array}$$

Proof of invariance by elem. collapse, with homotopy

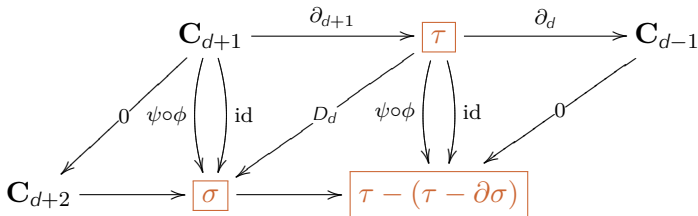
\mathbf{K} a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\phi} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\psi} \mathbf{C}_\bullet(\mathbf{K})$$

with:

$$\psi(x) = x, \quad \text{i.e. the inclusion } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$$

$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial\sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \quad D(x) = \begin{cases} \sigma & \text{if } x = \tau, \\ 0 & \text{otherwise.} \end{cases}$$



Proof of invariance by elem. collapse, with homotopy

\mathbf{K} a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\phi} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\psi} \mathbf{C}_\bullet(\mathbf{K})$$

with:

$$\psi(x) = x, \quad \text{i.e. the inclusion } \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \hookrightarrow \mathbf{C}_\bullet(\mathbf{K})$$

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$$\begin{array}{ccccc}
 & & \boxed{\sigma} & \xrightarrow{\partial_{d+1}} & \boxed{\partial\sigma = \tau + \dots} & \xrightarrow{\partial_d} & \mathbf{C}_{d-1} \\
 & \swarrow 0 & \downarrow \psi \circ \phi & \downarrow \text{id} & \swarrow D_d & \downarrow \psi \circ \phi & \downarrow \text{id} & \swarrow 0 \\
 \mathbf{C}_{d+2} & \xrightarrow{\quad} & \boxed{0 - \sigma} & \xrightarrow{\quad} & \mathbf{C}_d & & &
 \end{array}$$

Proof of invariance by elem. collapse, with homotopy

\mathbf{K} a simplicial complex, (σ, τ) a free pair:

$$\mathbf{C}_\bullet(\mathbf{K}) \xrightarrow{\phi} \mathbf{C}_\bullet(\mathbf{K} - \{\sigma, \tau\}) \xrightarrow{\psi} \mathbf{C}_\bullet(\mathbf{K})$$

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$$\phi(x) = \begin{cases} 0 & \text{if } x = \sigma, \\ \tau - \partial\sigma & \text{if } x = \tau, \text{ and} \\ x & \text{otherwise.} \end{cases} \quad D(x) = \begin{cases} \sigma & \text{if } x = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

The diagram illustrates the relationship between the chain complexes \mathbf{C}_{d+2} , \mathbf{C}_d , and \mathbf{C}_{d-1} under the maps ϕ and ψ . The top row consists of three boxes: $y \neq \sigma$, $\partial y \not\equiv \tau$, and \mathbf{C}_{d-1} . The bottom row consists of two boxes: $y - y$ and \mathbf{C}_d . The leftmost box \mathbf{C}_{d+2} has an arrow pointing to $y - y$. The rightmost box \mathbf{C}_{d-1} has an arrow pointing to \mathbf{C}_d . The top row is connected by ∂_{d+1} and ∂_d . The bottom row is connected by an unlabeled arrow. Vertical arrows connect the top row to the bottom row: 0 from $y \neq \sigma$ to $y - y$, $\psi \circ \phi$ from $y \neq \sigma$ to \mathbf{C}_d , D_d from $\partial y \not\equiv \tau$ to $y - y$, $\psi \circ \phi$ from $\partial y \not\equiv \tau$ to \mathbf{C}_d , and 0 from \mathbf{C}_{d-1} to \mathbf{C}_d . Double-headed vertical arrows labeled id connect $y \neq \sigma$ to $\psi \circ \phi$ and $\partial y \not\equiv \tau$ to $\psi \circ \phi$.

Exercise

Compute the homology of:

- a tetrahedron, and of an empty tetrahedron,
- a circle,
- an annulus.