# Off-the-grid charge algorithm for curve reconstruction in inverse problems.

9th International Conference SSVM

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# Introduction

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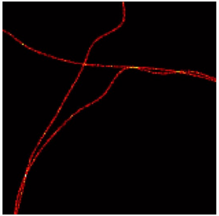

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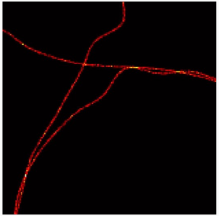

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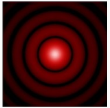

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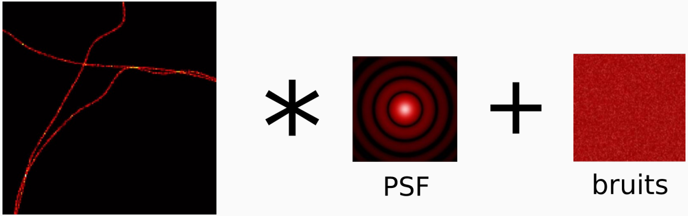

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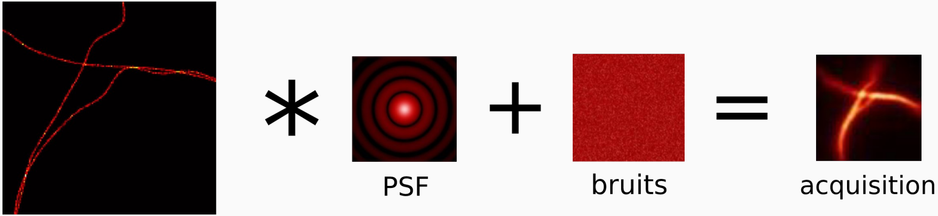

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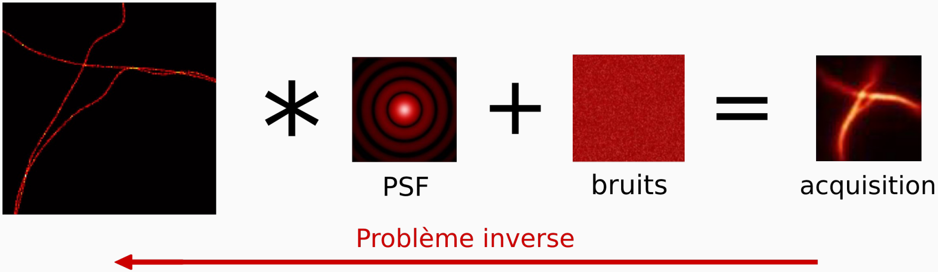

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- $R(S)$ regularises the problem (well-posed) and enforces more or less the prior on $S$ w $\alpha > 0$.

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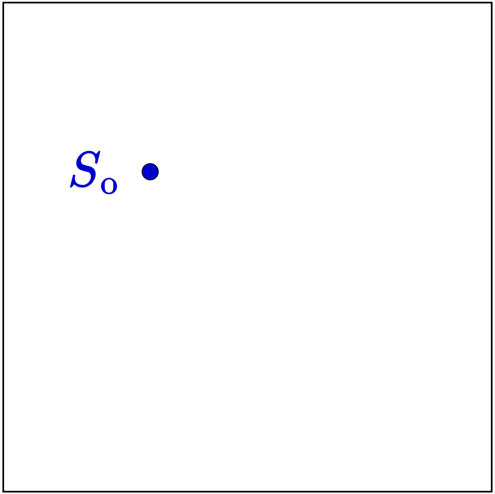


Source to estimate

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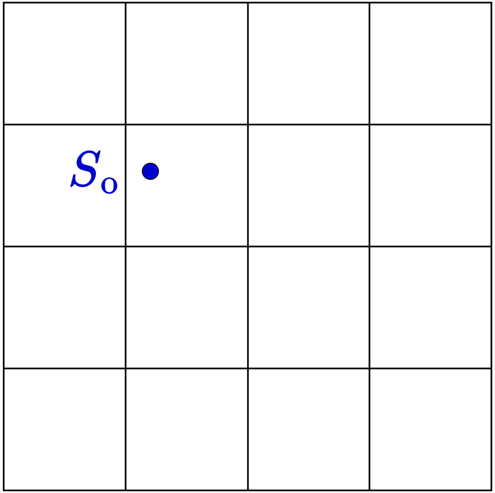


Introducing a grid

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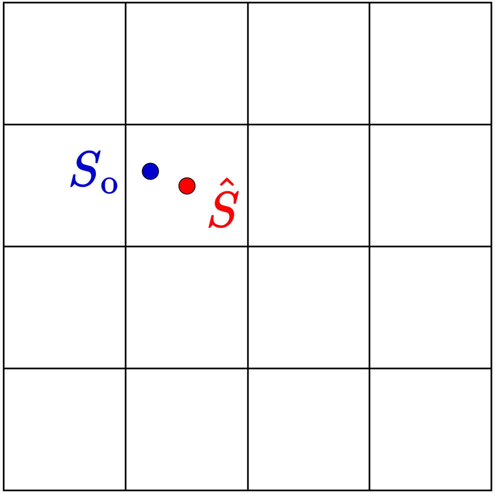


Reconstruction $\hat{S}$ on a grid

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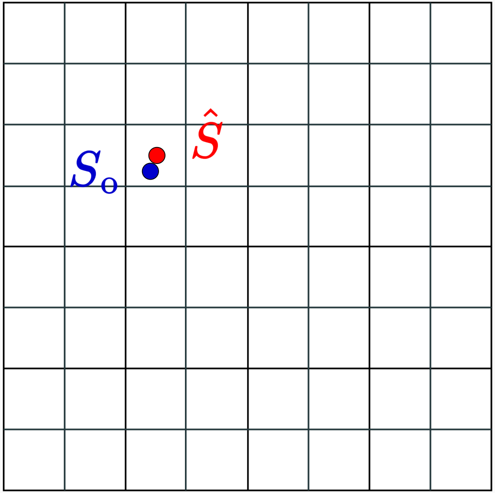


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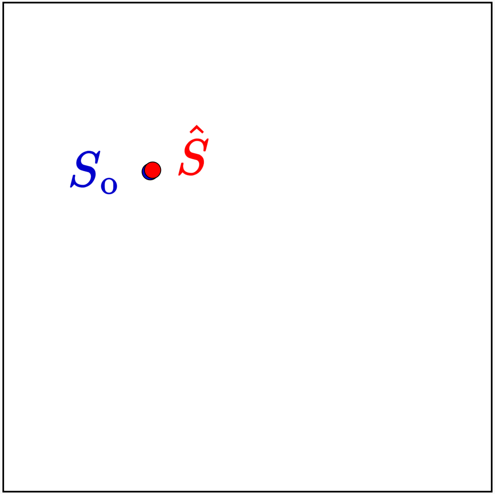


Reconstruction $\hat{S}$ is now **off-the-grid**

# Off-the-grid 101: spikes and sets

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If $m = \sum_{i=1}^N a_i \delta_{x_i}$ a discrete measure, then $|m|(\mathcal{X}) = \sum_{i=1}^N |a_i|$.

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Difficult numerical problem: infinite dimensional, non-reflexive. Tackled by greedy algorithm like *Frank-Wolfe* [Denoyelle et al., 2019] , *etc*.

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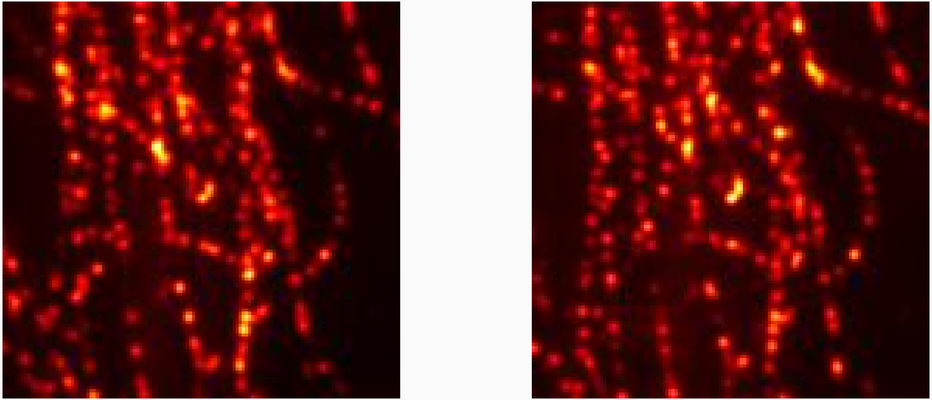
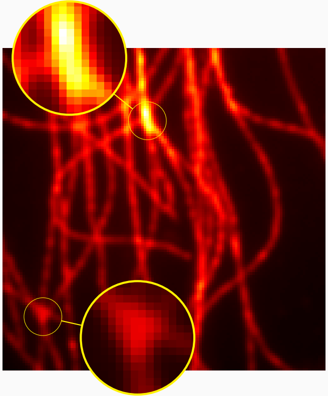


**Figure 1:** Two excerpts from a SMLM stack

# Results on SMLM



Stack mean

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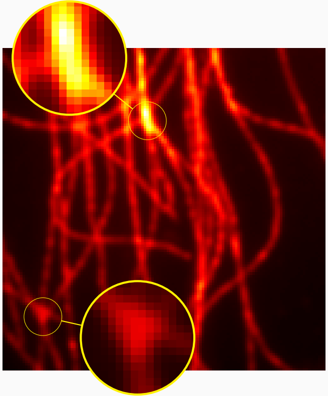

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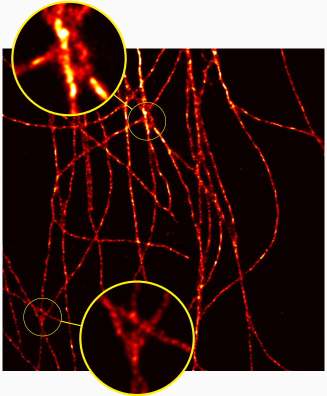

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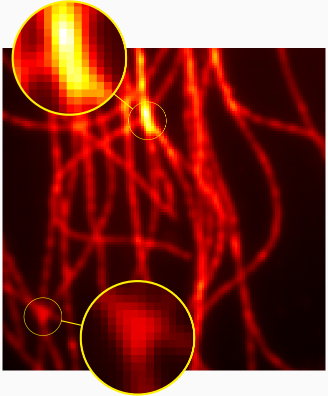

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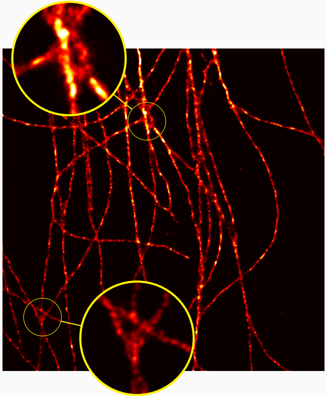

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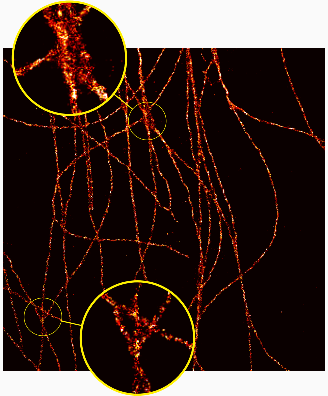

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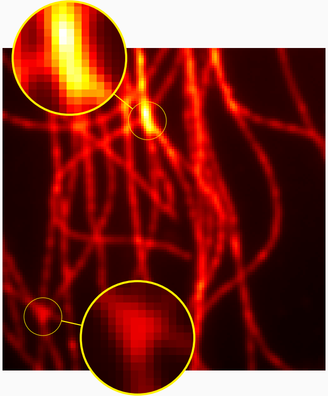

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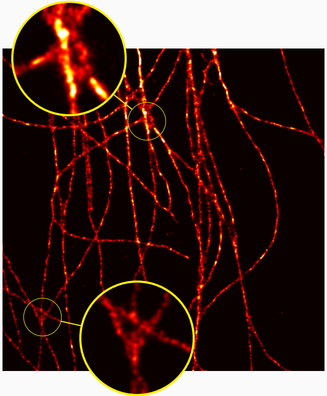

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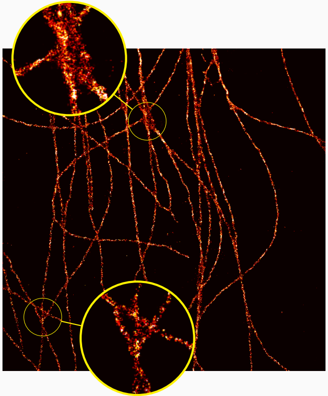

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SMLM drawback: a lot of images, no live-cell imaging.

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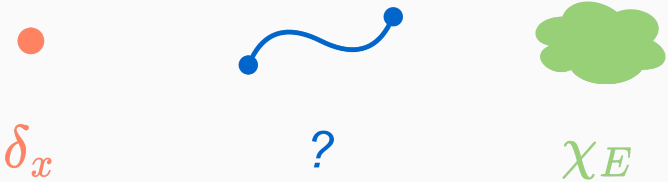

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# CROC energy

Consider the variational problem we coined *Curves Represented On Charges*:

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Do curve measures minimise (CROC)?

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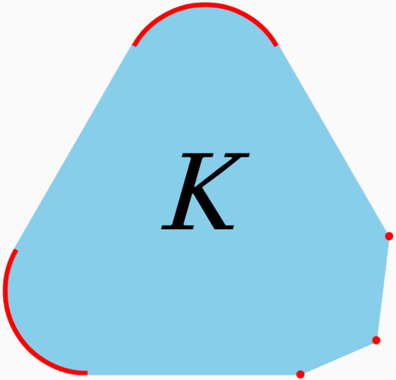
Ext $K$ is the set of extreme points of $K$.




Ext $K$ in red

# Link with extreme points: general setup

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*There exists a minimiser of F which is a linear sum of extreme points of the unit-ball of R,* $\operatorname{Ext} \mathcal{B}_E^1 \overset{\text{def.}}{=} \{u \in E \,|\, R(u) \leq 1\}$ *[Bredies and Fanzon, 2019, Duval and Peyré, 2014].*

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Characterise $\operatorname{Ext} \mathcal{B}_E^1$ of the regulariser $\iff$ outline the structure of a *minimum* of $F$.

# Extreme points in measure spaces

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- If $E = \mathscr{V}$ and $R = \|\cdot\|_{\mathscr{V}}$, then:

$$\mathrm{Ext}(\mathcal{B}_{\mathscr{V}}) = ?$$

# Main result

Let the (non-complete) set of curve measures endowed with weak-$*$ topology:

$$\mathfrak{G} \stackrel{\text{def.}}{=} \left\{ \frac{\boldsymbol{\mu}_\gamma}{\left\|\boldsymbol{\mu}_\gamma\right\|_{\mathscr{V}}}, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$

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# Numerical OG curve reconstruction

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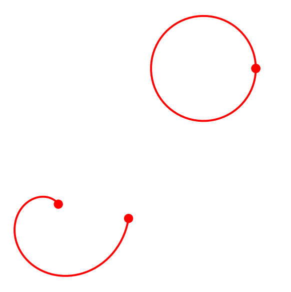
We present the *Charge Sliding Frank-Wolfe* algorithm.

# Synthetic problem



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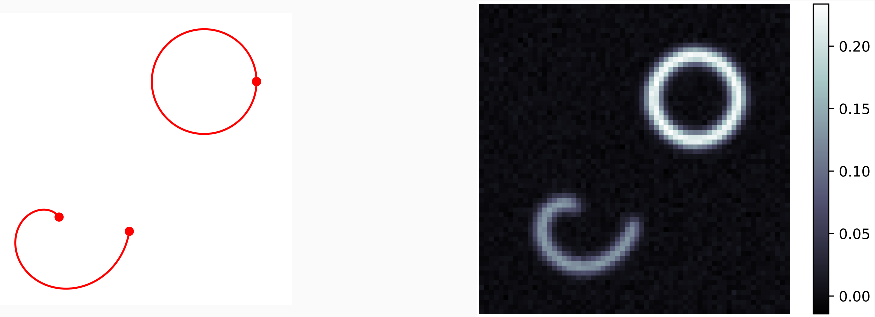


**Figure 2:** The source and its noisy acquired image

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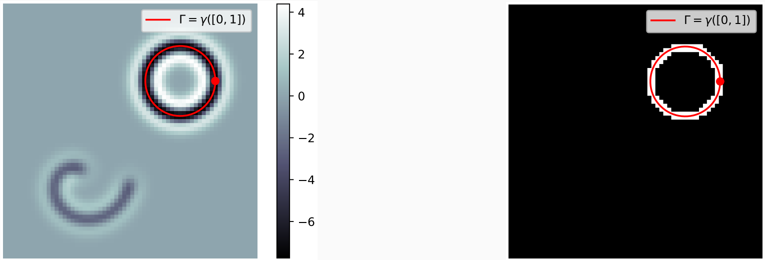


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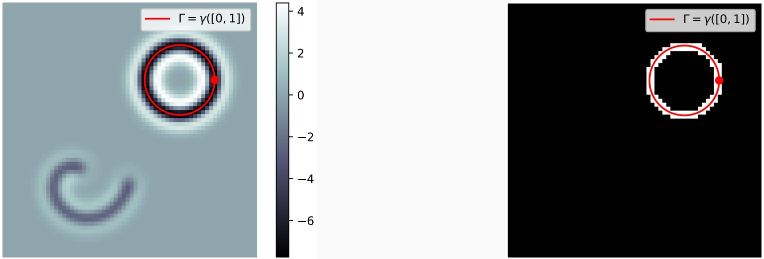


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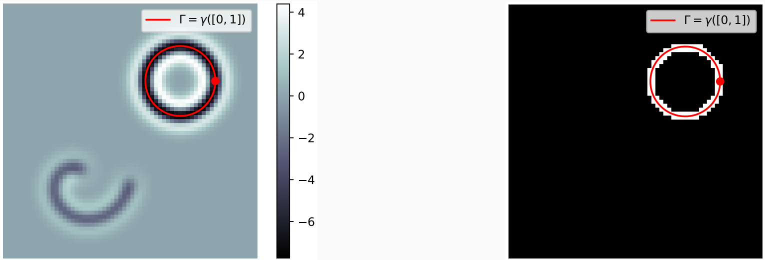


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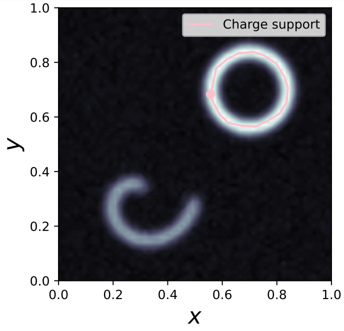


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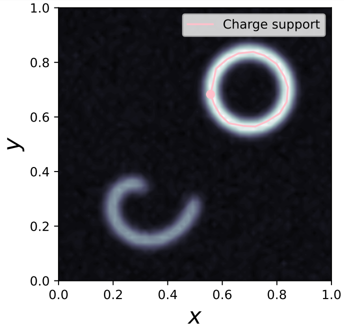


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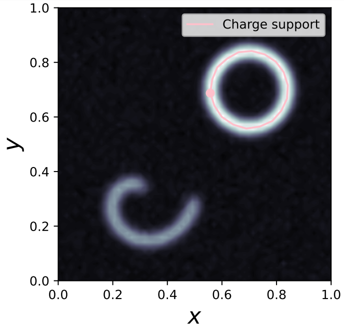


Both amplitude and position optimisation

- we optimise the amplitude $a$ of the new estimated curve;
- we perform a *sliding*: we optimise on both amplitudes $a$ and positions $\gamma$.

# Recap: iterate the algorithm

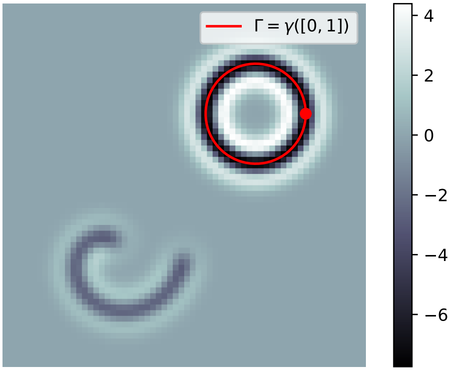


**Figure 4:** First step of first iteration: certificate and support of new curve estimated

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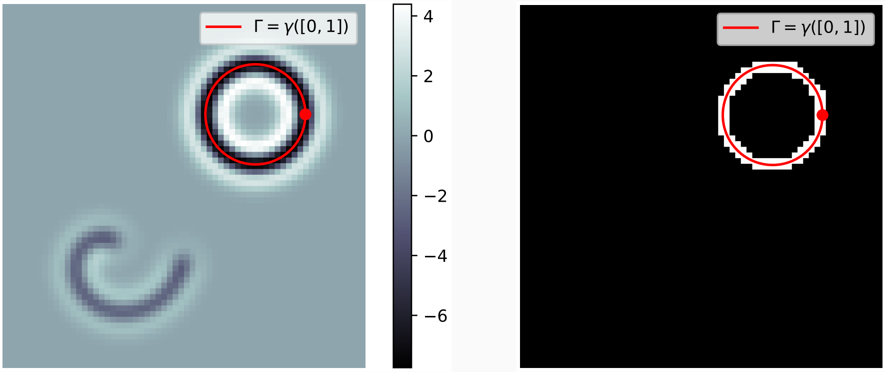


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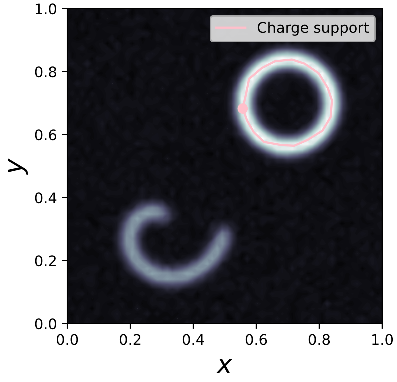


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**Figure 4:** First iteration: second and third steps

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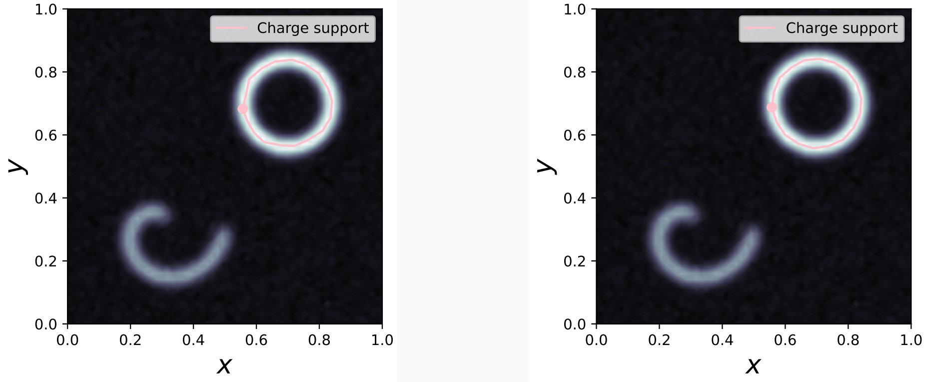


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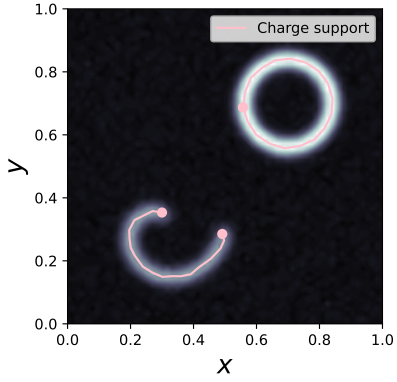
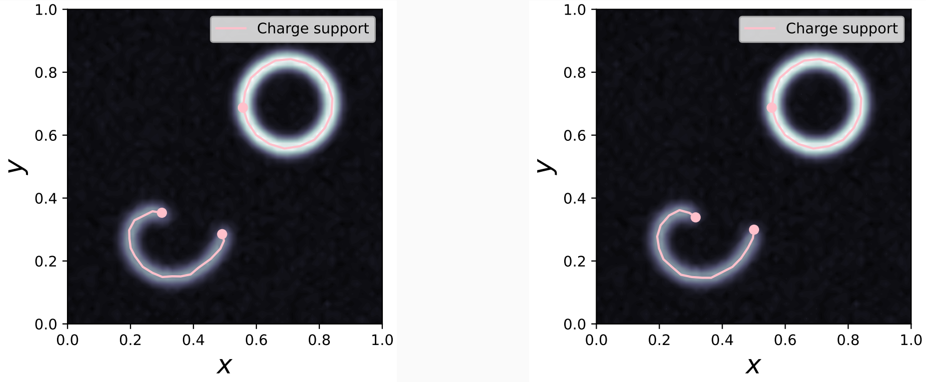



**Figure 4:** Second iteration: another curve is found

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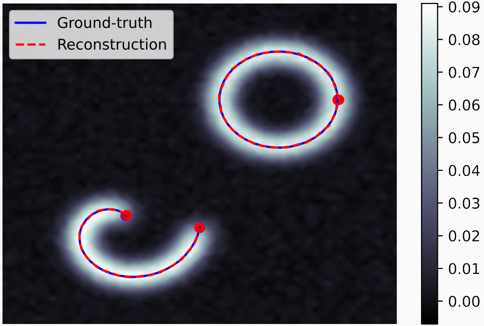


Reconstruction

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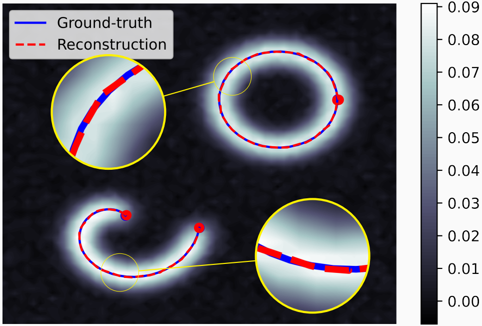


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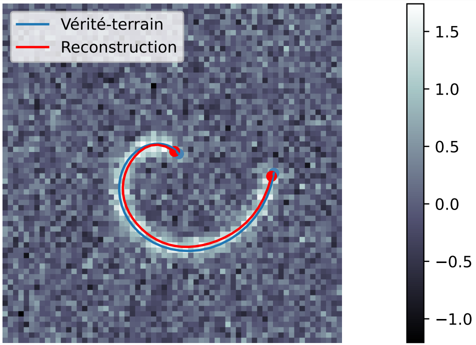


Reconstruction (splines discretisation)

# Conclusion

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In the future: better curve estimation support, test on experimental data, curves untangling, *etc*.

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**See our work on**

**https://www-sop.inria.fr/members/Bastien.Laville/**

# Proof recipe I

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**moreover, each $\boldsymbol{R}$ has *maximal* length implying $\operatorname{spt} \boldsymbol{R} = \operatorname{spt} \mu_\gamma$.**

# Proof recipe II

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**thus** $\text{spt}\,\boldsymbol{R} = \text{spt}\,\mu_\gamma$**,**

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**Hence, each $\gamma_{\boldsymbol{R}}$ is a reparametrisation of $\gamma$ yielding $\boldsymbol{R} = \dfrac{\mu_\gamma}{\|\mu_\gamma\|_{\mathscr{V}}}$, eventually:**

$$\boldsymbol{u}_i = \int_{\mathfrak{G}} \boldsymbol{R} \,\mathrm{d}\rho_i = \int_{\mathfrak{G}} \frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathscr{V}}} \,\mathrm{d}\rho_i = \frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathscr{V}}} \underbrace{\rho_i(\mathfrak{G})}_{=1} = \frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathscr{V}}}.$$

**Contradiction, then $\mu_\gamma$ is an extreme point.**

$\square$

# Proof recipe III

**Second inclusion:**

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) \subset \mathfrak{G}$$

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**either** $\rho$ **is supported on a singleton of** $\mathfrak{G}$**, then there exists** $\mu_\gamma$ **s.t.** $\boldsymbol{T} = \dfrac{\boldsymbol{\mu}_\gamma}{\|\boldsymbol{\mu}_\gamma\|_{\mathscr{V}}}$

**or there exists a Borel set** $A \subset \mathfrak{G}$ **with arbitrary** $0 < \rho(A) < 1$ **and:**

$$\rho = |\rho|(A)\left(\frac{1}{|\rho|(A)}\rho \llcorner A\right) + |\rho|(A^c)\left(\frac{1}{|\rho|(A^c)}\rho \llcorner A^c\right).$$

# Proof recipe IV

**Then,**

$$\boldsymbol{T} = |\rho|(A)\underbrace{\left[\int_{\mathfrak{G}} \frac{1}{|\rho|(A)}\boldsymbol{R}\,\mathrm{d}(\rho \llcorner A)(\boldsymbol{R})\right]}_{\overset{\text{def.}}{=}\boldsymbol{u_1}} + |\rho|(A^c)\underbrace{\left[\int_{\mathfrak{G}} \frac{1}{|\rho|(A^c)}\boldsymbol{R}\,\mathrm{d}(\rho \llcorner A^c)(\boldsymbol{R})\right]}_{\overset{\text{def.}}{=}\boldsymbol{u_2}}$$

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$A$ **is chosen (up to a neighbourhood) as a convex set, hence** $\boldsymbol{u_1} = \int_A \boldsymbol{R}\,\mathrm{d}\rho(\boldsymbol{R})$ **belongs to** $A$**, while conversely** $\boldsymbol{u_2} \in A^c$**, thus** $\boldsymbol{u_1} \neq \boldsymbol{u_2}$**.**

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$$\begin{aligned} \|\boldsymbol{u_1}\|_{\mathscr{V}} &\leq \int_{\mathfrak{G}} \frac{1}{|\rho|(A)} \underbrace{\|\boldsymbol{R}\|_{\mathscr{V}}}_{=1} \mathrm{d}(\rho \llcorner A)(\boldsymbol{R}) \\ &\leq \frac{|\rho|(A)}{|\rho|(A)} = 1. \end{aligned}$$

# Proof recipe V

**Then $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{B}^1_{\mathscr{Y}}$ while $\boldsymbol{u_1} \neq \boldsymbol{u_2}$, thus reaching a non-trivial convex combination:**

$$\boldsymbol{T} = \lambda \boldsymbol{u_1} + (1 - \lambda) \boldsymbol{u_2},$$

## Proof recipe V

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**thereby reaching a contradiction, and therefore concluding the proof.**

□