Off-the-grid curve reconstruction: bridge the gap between 0 and 2-rectifiable measures

Bastien Laville, Laure Blanc-Féraud, Gilles Aubert

15th May 2023

Morpheme team, Inria SAM,
CNRS, Université Côte d'Azur

Table of contents

Introduction

Biomedical imaging

Objective

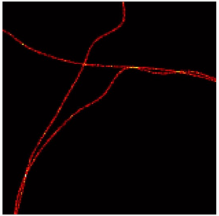
To image **live** biological structures at **small scales**.

Biomedical imaging

Objective

To image **live** biological structures at **small scales**.

Physical limitation due to diffraction for bodies < 200 nm: convolution by the microscope's point spread function (PSF).

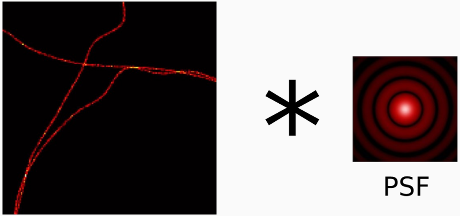


Biomedical imaging

Objective

To image **live** biological structures at **small scales**.

Physical limitation due to diffraction for bodies < 200 nm: convolution by the microscope's point spread function (PSF).

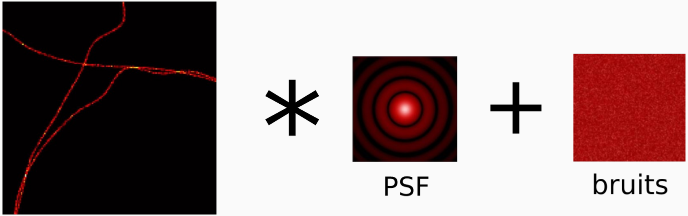


Biomedical imaging

Objective

To image **live** biological structures at **small scales**.

Physical limitation due to diffraction for bodies < 200 nm: convolution by the microscope's point spread function (PSF).

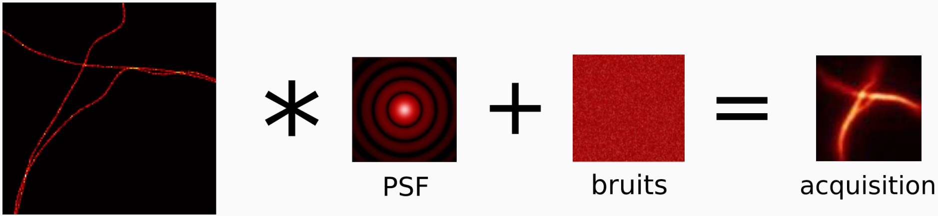


Biomedical imaging

Objective

To image **live** biological structures at **small scales**.

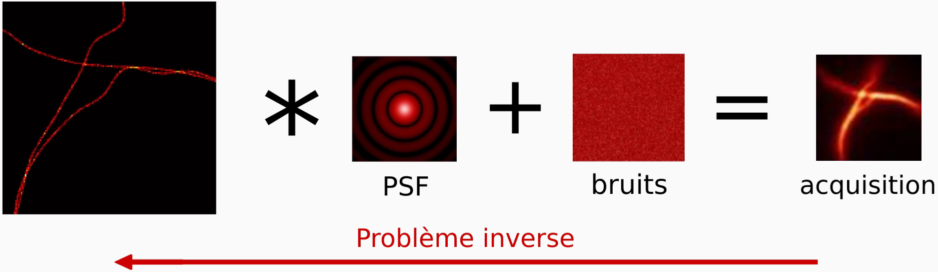
Physical limitation due to diffraction for bodies < 200 nm: convolution by the microscope's point spread function (PSF).



Objective

To image **live** biological structures at **small scales**.

Physical limitation due to diffraction for bodies < 200 nm: convolution by the microscope's point spread function (PSF).



Solve inverse problem through variational approach

- S_o is the source;

Solve inverse problem through variational approach

- S_o is the source;
- it is observed through y: blur Φ, noise…;

Solve inverse problem through variational approach

- S_{o} is the source;
- it is observed through y: blur Φ, noise…;
- how to build $\hat{S}$ from y?

Solve inverse problem through variational approach

- S_{o} is the source;
- it is observed through y: blur Φ, noise…;
- how to build $\hat{S}$ from y?

Solve inverse problem through variational approach

- S_{o} is the source;
- it is observed through y: blur Φ, noise…;
- how to build $\hat{S}$ from y?

Variational optimisation

- use a *prior* on S_{o};

Solve inverse problem through variational approach

- S_o is the source;
- it is observed through y: blur Φ, noise…;
- how to build $\hat{S}$ from y?

Variational optimisation

- use a *prior* on S_o;
- among all sources S, penalise the ones fulfiling the prior;

Solve inverse problem through variational approach

- S_o is the source;
- it is observed through y: blur Φ, noise…;
- how to build $\hat{S}$ from y?

Variational optimisation

- use a *prior* on S_o;
- among all sources S, penalise the ones fulfiling the prior;
- $\hat{S}$ minimises $S \mapsto \|y - \Phi S\|_2^2 + \alpha R(S)$;

Solve inverse problem through variational approach

- S_o is the source;
- it is observed through y: blur Φ, noise…;
- how to build $\hat{S}$ from y?

Variational optimisation

- use a *prior* on S_o;
- among all sources S, penalise the ones fulfiling the prior;
- $\hat{S}$ minimises $S \mapsto \|y - \Phi S\|_2^2 + \alpha R(S)$;
- $\|y - \Phi S\|_2^2$ penalises the closeness of y and the source S;

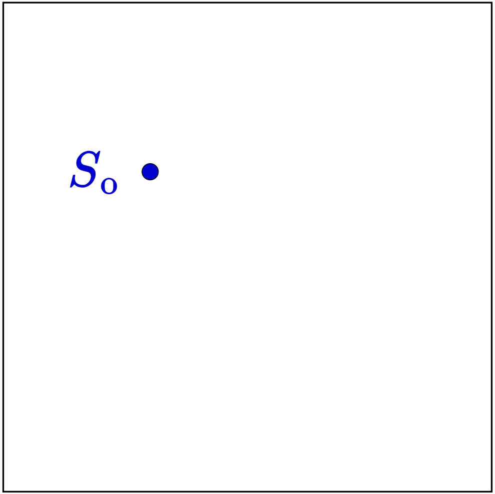
Solve inverse problem through variational approach

- S_{o} is the source;
- it is observed through y: blur Φ, noise…;
- how to build $\hat{S}$ from y?

Variational optimisation

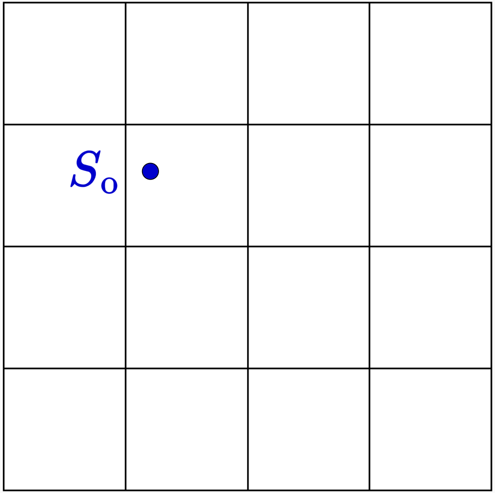
- use a *prior* on S_{o};
- among all sources S, penalise the ones fulfiling the prior;
- $\hat{S}$ minimises $S \mapsto \|y - \Phi S\|_2^2 + \alpha R(S)$;
- $\|y - \Phi S\|_2^2$ penalises the closeness of y and the source S;
- $R(S)$ regularises the problem (well-posed) and enforces more or less the prior on S w $\alpha > 0$.

Grid or gridless?



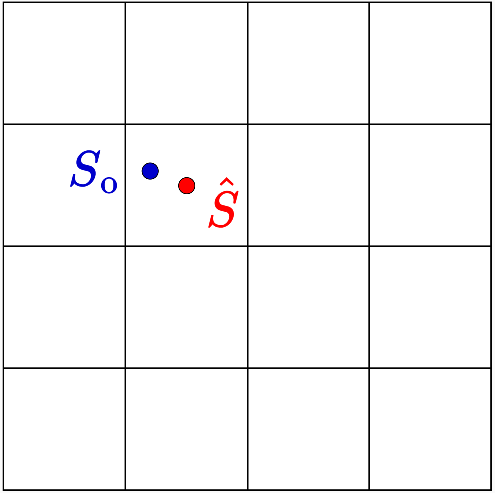
Source to estimate

Grid or gridless?



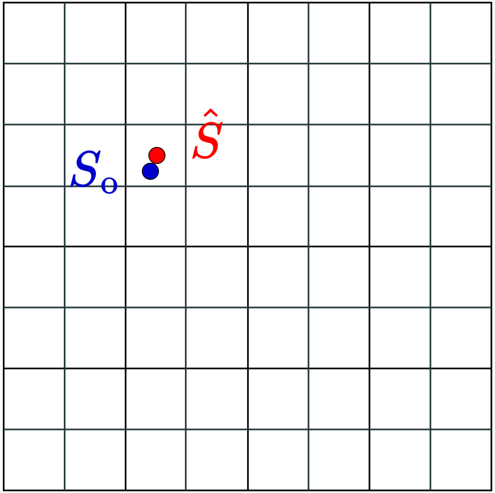
Introducing a grid

Grid or gridless?



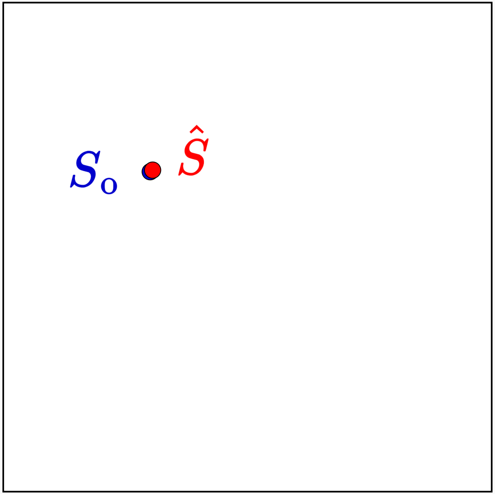
Reconstruction $\hat{S}$ on a grid

Grid or gridless?



Reconstruction $\hat{S}$ on a finer grid

Grid or gridless?



Reconstruction $\hat{S}$ is now **off-the-grid**

Off-the-grid review: spikes and sets

Quantities

- $\mathcal{X}$ is a compact of $\mathbb{R}^d$;

Quantities

- $\mathcal{X}$ is a compact of $\mathbb{R}^d$;
- how to model spikes ? Through Dirac measure δ_x, element of the *set of Radon measures* $\mathcal{M}(\mathcal{X})$;

Quantities

- $\mathcal{X}$ is a compact of $\mathbb{R}^d$;
- how to model spikes ? Through Dirac measure δ_x, element of the *set of Radon measures* $\mathcal{M}(\mathcal{X})$;
- topological dual of $\mathscr{C}_0(\mathcal{X})$ equipped with $\langle f, m\rangle = \int_{\mathcal{X}} f \, \mathrm{d}m$. Generalises $\mathrm{L}^1(\mathcal{X})$; $\mathrm{L}^1(\mathcal{X}) \hookrightarrow \mathcal{M}(\mathcal{X})$;

Quantities

- $\mathcal{X}$ is a compact of $\mathbb{R}^d$;
- how to model spikes ? Through Dirac measure δ_x, element of the *set of Radon measures* $\mathcal{M}(\mathcal{X})$;
- topological dual of $\mathscr{C}_0(\mathcal{X})$ equipped with $\langle f, m \rangle = \int_{\mathcal{X}} f \, \mathrm{d}m$. Generalises $\mathrm{L}^1(\mathcal{X})$; $\mathrm{L}^1(\mathcal{X}) \hookrightarrow \mathcal{M}(\mathcal{X})$;
- Banach endowed with TV-norm : $m \in \mathcal{M}(\mathcal{X})$,

$$|m|(\mathcal{X}) \stackrel{\text{def.}}{=} \sup \left(\int_{\mathcal{X}} f \, \mathrm{d}m \,\middle|\, f \in \mathscr{C}_0(\mathcal{X}), \|f\|_{\infty,\mathcal{X}} \leq 1 \right).$$

If $m = \sum_{i=1}^N a_i \delta_{x_i}$ a discrete measure, then $|m|(\mathcal{X}) = \sum_{i=1}^N |a_i|$.

A LASSO equivalent for measures

- Let the source $m_{a_0,x_0} \stackrel{\text{def.}}{=} \sum_{i=1}^{N} a_i \delta_{x_i} \in \mathcal{M}(\mathcal{X})$ a discrete measure;

A LASSO equivalent for measures

- Let the source $m_{a_0,x_0} \stackrel{\text{def.}}{=} \sum_{i=1}^{N} a_i \delta_{x_i} \in \mathcal{M}(\mathcal{X})$ a discrete measure;
- $\Phi : \mathcal{M}(\mathcal{X}) \to \mathbb{R}^p$ the acquisition operator, e.g. $\Phi m_{a_0,x_0} \stackrel{\text{def.}}{=} \sum_{i=1}^{N} a_i h(x - x_i)$;

A LASSO equivalent for measures

- Let the source $m_{a_0,x_0} \stackrel{\text{def.}}{=} \sum_{i=1}^N a_i \delta_{x_i} \in \mathcal{M}(\mathcal{X})$ a discrete measure;
- $\Phi : \mathcal{M}(\mathcal{X}) \to \mathbb{R}^p$ the acquisition operator, e.g. $\Phi m_{a_0,x_0} \stackrel{\text{def.}}{=} \sum_{i=1}^N a_i h(x - x_i)$;
- $w \in \mathrm{L}^2(\mathcal{X})$ additive noise;

A LASSO equivalent for measures

- Let the source $m_{a_0,x_0} \overset{\text{def.}}{=} \sum_{i=1}^N a_i \delta_{x_i} \in \mathcal{M}(\mathcal{X})$ a discrete measure;
- $\Phi : \mathcal{M}(\mathcal{X}) \to \mathbb{R}^p$ the acquisition operator, e.g. $\Phi m_{a_0,x_0} \overset{\text{def.}}{=} \sum_{i=1}^N a_i h(x - x_i)$;
- $w \in \mathrm{L}^2(\mathcal{X})$ additive noise;
- $y \overset{\text{def.}}{=} \Phi m_{a_0,x_0} + w$.

A LASSO equivalent for measures

- Let the source $m_{a_0,x_0} \stackrel{\text{def.}}{=} \sum_{i=1}^{N} a_i \delta_{x_i} \in \mathcal{M}(\mathcal{X})$ a discrete measure;
- $\Phi : \mathcal{M}(\mathcal{X}) \to \mathbb{R}^p$ the acquisition operator, e.g. $\Phi m_{a_0,x_0} \stackrel{\text{def.}}{=} \sum_{i=1}^{N} a_i h(x - x_i)$;
- $w \in \mathrm{L}^2(\mathcal{X})$ additive noise;
- $y \stackrel{\text{def.}}{=} \Phi m_{a_0,x_0} + w$.

We call **BLASSO** the problem
[Candès and Fernandez-Granda, 2013, Duval and Peyré, 2014] for $\lambda > 0$:

$$\underset{m \in \mathcal{M}(\mathcal{X})}{\operatorname{argmin}} \frac{1}{2} \|y - \Phi m\|_{\mathbb{R}^p}^2 + \lambda |m|(\mathcal{X}) \tag{$\mathcal{P}_\lambda(y)$}$$

A LASSO equivalent for measures

- Let the source $m_{a_0,x_0} \overset{\text{def.}}{=} \sum_{i=1}^N a_i \delta_{x_i} \in \mathcal{M}(\mathcal{X})$ a discrete measure;
- $\Phi : \mathcal{M}(\mathcal{X}) \to \mathbb{R}^p$ the acquisition operator, e.g. $\Phi m_{a_0,x_0} \overset{\text{def.}}{=} \sum_{i=1}^N a_i h(x - x_i)$;
- $w \in \mathrm{L}^2(\mathcal{X})$ additive noise;
- $y \overset{\text{def.}}{=} \Phi m_{a_0,x_0} + w$.

We call **BLASSO** the problem
[Candès and Fernandez-Granda, 2013, Duval and Peyré, 2014] for $\lambda > 0$:

$$\operatorname*{argmin}_{m \in \mathcal{M}(\mathcal{X})} \frac{1}{2} \|y - \Phi m\|_{\mathbb{R}^p}^2 + \lambda |m|(\mathcal{X}) \qquad (\mathcal{P}_\lambda(y))$$

One of its minimisers is a sum of Dirac, close to m_{a_0,x_0}.

A LASSO equivalent for measures

- Let the source $m_{a_0,x_0} \stackrel{\text{def.}}{=} \sum_{i=1}^N a_i \delta_{x_i} \in \mathcal{M}(\mathcal{X})$ a discrete measure;
- $\Phi : \mathcal{M}(\mathcal{X}) \to \mathbb{R}^p$ the acquisition operator, e.g. $\Phi m_{a_0,x_0} \stackrel{\text{def.}}{=} \sum_{i=1}^N a_i h(x - x_i)$;
- $w \in \mathrm{L}^2(\mathcal{X})$ additive noise;
- $y \stackrel{\text{def.}}{=} \Phi m_{a_0,x_0} + w$.

We call **BLASSO** the problem
[Candès and Fernandez-Granda, 2013, Duval and Peyré, 2014] for $\lambda > 0$:

$$\underset{m \in \mathcal{M}(\mathcal{X})}{\operatorname{argmin}} \frac{1}{2} \|y - \Phi m\|_{\mathbb{R}^p}^2 + \lambda |m|(\mathcal{X}) \qquad (\mathcal{P}_\lambda(y))$$

One of its minimisers is a sum of Dirac, close to m_{a_0,x_0}.

Difficult numerical problem: infinite dimensional, non-reflexive. Tackled by greedy algorithm like *Frank-Wolfe* [Denoyelle et al., 2019] , *etc*.

Some results for spikes reconstruction

Reconstruction by fluorescence microscopy SMLM: acquisition stack with few lit fluorophores per image.

Reconstruction by fluorescence microscopy SMLM: acquisition stack with few lit fluorophores per image.

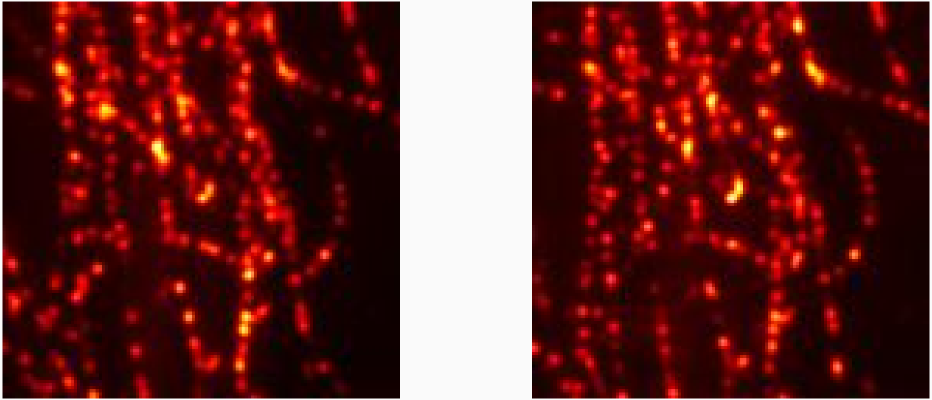
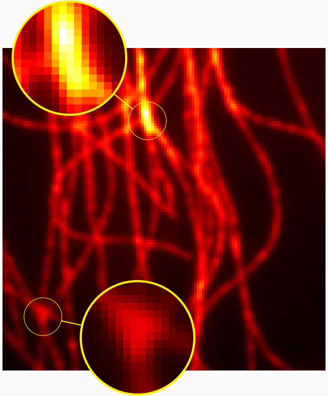


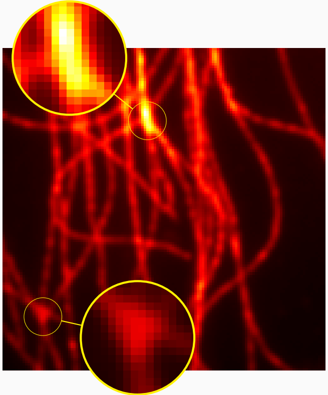
Figure 1: Two excerpts from a SMLM stack

Results on SMLM

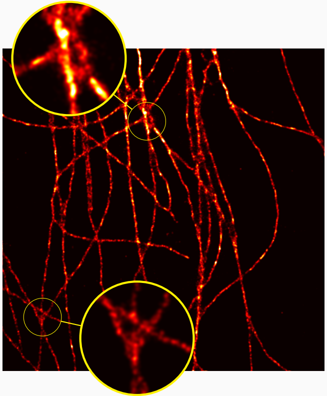


Stack mean

Results on SMLM

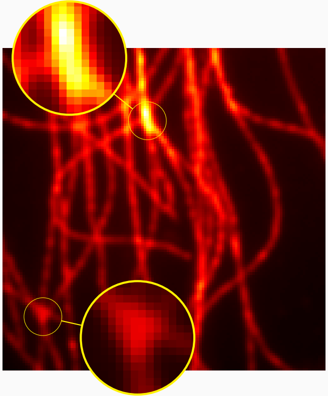


Stack mean

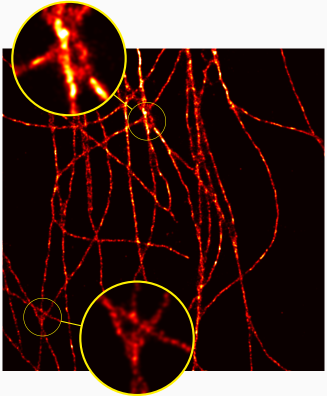


Off-the-grid [Laville et al., 2021]

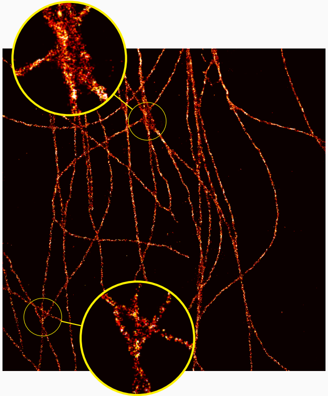
Results on SMLM



Stack mean

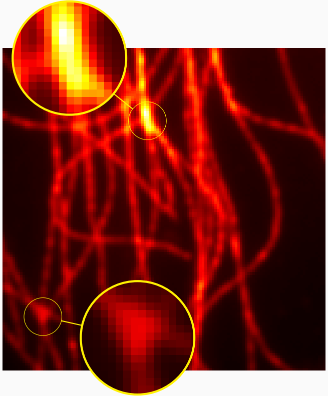


Off-the-grid [Laville et al., 2021]

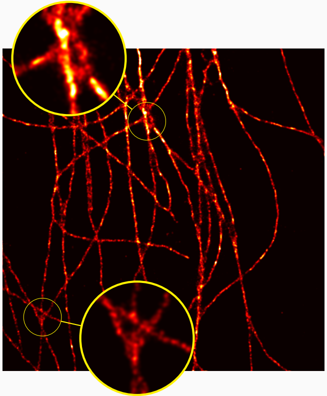


Deep-STORM [Nehme et al., 2018]

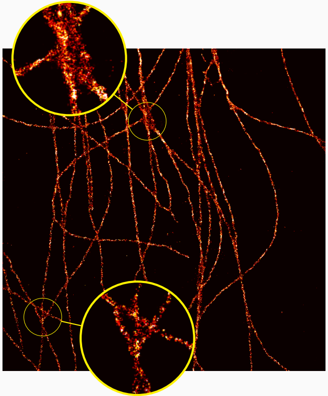
Results on SMLM



Stack mean



Off-the-grid [Laville et al., 2021]



Deep-STORM [Nehme et al., 2018]

SMLM drawback: a lot of images, no live-cell imaging.

2-rectifiable measures reconstruction

- how to model sets measures? Through χ_E where E is a **simple set**, belonging to $\mathrm{BV}(\mathcal{X})$ the set of function of *bounded variation*;

2-rectifiable measures reconstruction

- how to model sets measures? Through χ_E where E is a **simple set**, belonging to $\mathrm{BV}(\mathcal{X})$ the set of function of *bounded variation*;
- $\mathrm{BV}(\mathcal{X}) = \left\{ u \in \mathrm{L}^2(\mathcal{X}) \mid \text{"}\nabla u\text{"} \in \boldsymbol{\mathcal{M}(\mathcal{X})^2} \right\}$;

2-rectifiable measures reconstruction

- how to model sets measures? Through χ_E where E is a **simple set**, belonging to $\mathrm{BV}(\mathcal{X})$ the set of function of *bounded variation*;
- $\mathrm{BV}(\mathcal{X}) = \left\{ u \in \mathrm{L}^2(\mathcal{X}) \mid \mathrm{D}u \in \boldsymbol{\mathcal{M}(\mathcal{X})^2} \right\}$;
- Banach endowed with BV-norm : $u \in \mathrm{BV}(\mathcal{X})$,

$$\|u\|_{\mathrm{BV}} \stackrel{\text{def.}}{=} \|u\|_1 + \|\mathrm{D}u\|_{\mathrm{TV}}.$$

2-rectifiable measures reconstruction

- how to model sets measures? Through χ_E where E is a **simple set**, belonging to $\mathrm{BV}(\mathcal{X})$ the set of function of *bounded variation*;
- $\mathrm{BV}(\mathcal{X}) = \left\{u \in \mathrm{L}^2(\mathcal{X}) \mid \mathrm{D}u \in \boldsymbol{\mathcal{M}(\mathcal{X})^2}\right\}$;
- Banach endowed with BV-norm : $u \in \mathrm{BV}(\mathcal{X})$,

$$\|u\|_{\mathrm{BV}} \overset{\text{def.}}{=} \|u\|_1 + \|\mathrm{D}u\|_{\mathrm{TV}}.$$

If $u = \chi_E$,

2-rectifiable measures reconstruction

- how to model sets measures? Through χ_E where E is a **simple set**, belonging to $\mathrm{BV}(\mathcal{X})$ the set of function of *bounded variation*;
- $\mathrm{BV}(\mathcal{X}) = \left\{u \in \mathrm{L}^2(\mathcal{X}) \mid \mathrm{D}u \in \boldsymbol{\mathcal{M}(\mathcal{X})^2}\right\}$;
- Banach endowed with BV-norm : $u \in \mathrm{BV}(\mathcal{X})$,

$$\|u\|_{\mathrm{BV}} \stackrel{\text{def.}}{=} \|u\|_1 + \|\mathrm{D}u\|_{\mathrm{TV}}.$$

If $u = \chi_E$, then $\|\mathrm{D}u\|_{\mathrm{TV}} = \mathrm{Per}(E)$;

2-rectifiable measures reconstruction

- how to model sets measures? Through χ_E where E is a **simple set**, belonging to $\mathrm{BV}(\mathcal{X})$ the set of function of *bounded variation*;
- $\mathrm{BV}(\mathcal{X}) = \left\{ u \in \mathrm{L}^2(\mathcal{X}) \mid \mathrm{D}u \in \boldsymbol{\mathcal{M}(\mathcal{X})^2} \right\}$;
- Banach endowed with BV-norm : $u \in \mathrm{BV}(\mathcal{X})$,

$$\|u\|_{\mathrm{BV}} \stackrel{\mathrm{def.}}{=} \|u\|_1 + \|\mathrm{D}u\|_{\mathrm{TV}}.$$

  If $u = \chi_E$, then $\|\mathrm{D}u\|_{\mathrm{TV}} = \mathrm{Per}(E)$;
- Let $\lambda > 0$, the adaptation of BLASSO [de Castro et al., 2021] writes down:

$$\underset{u \in \mathrm{BV}(\mathcal{X})}{\operatorname{argmin}} \frac{1}{2} \|y - \Phi u\|^2_{\mathrm{L}^2(\mathcal{X})} + \lambda \|u\|_{\mathrm{BV}} \qquad (\mathcal{S}_\lambda(y))$$

2-rectifiable measures reconstruction

- how to model sets measures? Through χ_E where E is a **simple set**, belonging to $\mathrm{BV}(\mathcal{X})$ the set of function of *bounded variation*;
- $\mathrm{BV}(\mathcal{X}) = \left\{ u \in \mathrm{L}^2(\mathcal{X}) \mid \mathrm{D}u \in \boldsymbol{\mathcal{M}(\mathcal{X})^2} \right\}$;
- Banach endowed with BV-norm : $u \in \mathrm{BV}(\mathcal{X})$,

$$\|u\|_{\mathrm{BV}} \stackrel{\text{def.}}{=} \|u\|_1 + \|\mathrm{D}u\|_{\mathrm{TV}}.$$

  If $u = \chi_E$, then $\|\mathrm{D}u\|_{\mathrm{TV}} = \mathrm{Per}(E)$;
- Let $\lambda > 0$, the adaptation of BLASSO [de Castro et al., 2021] writes down:

$$\underset{u \in \mathrm{BV}(\mathcal{X})}{\operatorname{argmin}} \frac{1}{2}\|y - \Phi u\|^2_{\mathrm{L}^2(\mathcal{X})} + \lambda \|u\|_{\mathrm{BV}} \qquad (\mathcal{S}_\lambda(y))$$

  One of its minimisers is a sum of level sets χ_E!

Geometry encoded in off-the-grid

Geometry encoded in off-the-grid

	TV	BV
Geometry	Spikes	Sets
Space	$\mathcal{M}(\mathcal{X})$	$\mathrm{BV}(\mathcal{X})$
Regulariser	$\lVert\cdot\rVert_{\mathrm{TV}}$	$\lVert\cdot\rVert_1 + \lVert \mathrm{D}\cdot\rVert_{\mathrm{TV}}$

$$\delta_x$$

Geometry encoded in off-the-grid

	TV	BV
Geometry	Spikes	Sets
Space	$\mathcal{M}(\mathcal{X})$	$\mathrm{BV}(\mathcal{X})$
Regulariser	$\|\cdot\|_{\mathrm{TV}}$	$\|\cdot\|_1 + \|\mathrm{D}\cdot\|_{\mathrm{TV}}$

δ_x

χ_E

Geometry encoded in off-the-grid

	TV	?	BV
Geometry	Spikes	Curves	Sets
Space	$\mathcal{M}(\mathcal{X})$	?	$\mathrm{BV}(\mathcal{X})$
Regulariser	$\lVert\cdot\rVert_{\mathrm{TV}}$	?	$\lVert\cdot\rVert_1 + \lVert \mathrm{D}\cdot\rVert_{\mathrm{TV}}$

δ_x ? χ_E

A new divergence regularisation

A new functional space

- let $\mathcal{M}(\mathcal{X})^2$ be the space of vector Radon measures;

A new functional space

- let $\boldsymbol{\mathcal{M}(\mathcal{X})^2}$ be the space of vector Radon measures;
- let $\mathscr{V} \stackrel{\text{def.}}{=} \left\{ \boldsymbol{m} \in \boldsymbol{\mathcal{M}(\mathcal{X})^2}, \ \operatorname{div}(\boldsymbol{m}) \in \mathcal{M}(\mathcal{X}) \right\}$ the space of *charges*, or *divergence vector fields*. It is a Banach equipped with $\|\cdot\|_{\mathscr{V}} \stackrel{\text{def.}}{=} \|\cdot\|_{\mathrm{TV}^2} + \|\operatorname{div}(\cdot)\|_{\mathrm{TV}}$;

A new functional space

- let $\boldsymbol{\mathcal{M}}\left(\boldsymbol{\mathcal{X}}\right)^{\mathbf{2}}$ be the space of vector Radon measures;
- let $\mathscr{V} \stackrel{\text{def.}}{=} \left\{\boldsymbol{m} \in \boldsymbol{\mathcal{M}}\left(\boldsymbol{\mathcal{X}}\right)^{\mathbf{2}},\, \mathrm{div}(\boldsymbol{m}) \in \mathcal{M}\left(\mathcal{X}\right)\right\}$ the space of *charges*, or *divergence vector fields*. It is a Banach equipped with $\|\cdot\|_{\mathscr{V}} \stackrel{\text{def.}}{=} \|\cdot\|_{\mathrm{TV}^2} + \|\mathrm{div}(\cdot)\|_{\mathrm{TV}}$;
- let $\gamma : [0,1] \to \mathbb{R}^d$ a 1-rectifiable parametrised Lipschitz curve,

A new functional space

- let $\boldsymbol{\mathcal{M}}\left(\mathcal{X}\right)^{\mathbf{2}}$ be the space of vector Radon measures;
- let $\mathscr{V} \overset{\text{def.}}{=} \left\{\boldsymbol{m} \in \boldsymbol{\mathcal{M}}\left(\mathcal{X}\right)^{\mathbf{2}},\ \operatorname{div}(\boldsymbol{m}) \in \mathcal{M}\left(\mathcal{X}\right)\right\}$ the space of *charges*, or *divergence vector fields*. It is a Banach equipped with $\|\cdot\|_{\mathscr{V}} \overset{\text{def.}}{=} \|\cdot\|_{\mathrm{TV}^2} + \|\operatorname{div}(\cdot)\|_{\mathrm{TV}}$;
- let $\gamma : [0,1] \to \mathbb{R}^d$ a 1-rectifiable parametrised Lipschitz curve, we say that $\boldsymbol{\mu}_{\boldsymbol{\gamma}} \in \mathscr{V}$ *is a measure* ***supported on a curve*** γ if:

$$\forall \boldsymbol{g} \in \mathbf{C_0}(\mathcal{X})^{\mathbf{2}}, \quad \langle \boldsymbol{\mu}_{\gamma}, \boldsymbol{g} \rangle_{\boldsymbol{\mathcal{M}}^{\mathbf{2}}} \overset{\text{def.}}{=} \int_0^1 \boldsymbol{g}(\gamma(t)) \cdot \dot{\gamma}(t)\,\mathrm{d}t.$$

- a curve is closed is $\gamma(0) = \gamma(1)$, open otherwise;

A new functional space

- let $\boldsymbol{\mathcal{M}}\left(\mathcal{X}\right)^{\mathbf{2}}$ be the space of vector Radon measures;
- let $\mathscr{V} \stackrel{\text{def.}}{=} \left\{\boldsymbol{m} \in \boldsymbol{\mathcal{M}}\left(\mathcal{X}\right)^{\mathbf{2}}, \operatorname{div}(\boldsymbol{m}) \in \mathcal{M}\left(\mathcal{X}\right)\right\}$ the space of *charges*, or *divergence vector fields*. It is a Banach equipped with $\|\cdot\|_{\mathscr{V}} \stackrel{\text{def.}}{=} \|\cdot\|_{\mathrm{TV}^2} + \|\operatorname{div}(\cdot)\|_{\mathrm{TV}}$;
- let $\gamma : [0,1] \to \mathbb{R}^d$ a 1-rectifiable parametrised Lipschitz curve, we say that $\boldsymbol{\mu}_\gamma \in \mathscr{V}$ *is a measure* ***supported on a curve*** γ if:

$$\forall \boldsymbol{g} \in \mathbf{C_0}(\mathcal{X})^{\mathbf{2}}, \quad \langle \boldsymbol{\mu}_\gamma, \boldsymbol{g} \rangle_{\boldsymbol{\mathcal{M}}^{\mathbf{2}}} \stackrel{\text{def.}}{=} \int_0^1 \boldsymbol{g}(\gamma(t)) \cdot \dot{\gamma}(t) \,\mathrm{d}t.$$

- a curve is closed is $\gamma(0) = \gamma(1)$, open otherwise;
- simple if γ is an injective mapping

A new functional space

- let $\boldsymbol{\mathcal{M}}(\mathcal{X})^{\mathbf{2}}$ be the space of vector Radon measures;
- let $\mathscr{V} \stackrel{\text{def.}}{=} \left\{ \boldsymbol{m} \in \boldsymbol{\mathcal{M}}(\mathcal{X})^{\mathbf{2}},\ \operatorname{div}(\boldsymbol{m}) \in \mathcal{M}(\mathcal{X}) \right\}$ the space of *charges*, or *divergence vector fields*. It is a Banach equipped with $\|\cdot\|_{\mathscr{V}} \stackrel{\text{def.}}{=} \|\cdot\|_{\mathrm{TV}^2} + \|\operatorname{div}(\cdot)\|_{\mathrm{TV}}$;
- let $\gamma : [0,1] \to \mathbb{R}^d$ a 1-rectifiable parametrised Lipschitz curve, we say that $\boldsymbol{\mu}_\gamma \in \mathscr{V}$ *is a measure **supported on a curve*** γ if:

$$\forall \boldsymbol{g} \in \mathbf{C}_{\mathbf{0}}(\mathcal{X})^{\mathbf{2}}, \quad \langle \boldsymbol{\mu}_\gamma, \boldsymbol{g} \rangle_{\boldsymbol{\mathcal{M}}^{\mathbf{2}}} \stackrel{\text{def.}}{=} \int_0^1 \boldsymbol{g}(\gamma(t)) \cdot \dot{\gamma}(t) \, \mathrm{d}t.$$

- a curve is closed is $\gamma(0) = \gamma(1)$, open otherwise;
- simple if γ is an injective mapping
- $\operatorname{div} \boldsymbol{\mu}_\gamma = \delta_{\gamma(0)} - \delta_{\gamma(1)}$.

A convex decomposition

Let the (non-complete) subset of $\mathscr{V}$ endowed with weak-$*$ topology:

$$\mathfrak{G} \stackrel{\text{def.}}{=} \left\{ \frac{\boldsymbol{\mu}_{\boldsymbol{\gamma}}}{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathscr{V}}}, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$

A convex decomposition

Let the (non-complete) subset of $\mathscr{V}$ endowed with weak-$*$ topology:

$$\mathfrak{G} \stackrel{\text{def.}}{=} \left\{ \frac{\boldsymbol{\mu}_{\boldsymbol{\gamma}}}{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathscr{V}}}, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$

Theorem (Smirnov Theorem A. [Smirnov, 1993])

Let $\boldsymbol{T} \in \mathscr{V}$. *Then* $\boldsymbol{T} = \boldsymbol{P} + \boldsymbol{Q}$ *where* $\boldsymbol{P}$ *is a solenoid i.e.* $\operatorname{div} \boldsymbol{P} = 0$

A convex decomposition

Let the (non-complete) subset of $\mathscr{V}$ endowed with weak-$*$ topology:

$$\mathfrak{G} \stackrel{\text{def.}}{=} \left\{ \frac{\boldsymbol{\mu_\gamma}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{V}}}, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$

Theorem (Smirnov Theorem A. [Smirnov, 1993])

Let $\boldsymbol{T} \in \mathscr{V}$. Then $\boldsymbol{T} = \boldsymbol{P} + \boldsymbol{Q}$ where $\boldsymbol{P}$ is a solenoid i.e. $\operatorname{div} \boldsymbol{P} = 0$ and $\boldsymbol{Q}$ is completely decomposable on $\mathfrak{G}$, i.e. there exists a Borel measure ρ s.t.:

$$\boldsymbol{Q} = \int_{\mathfrak{G}} \boldsymbol{R} \, \mathrm{d}\rho(\boldsymbol{R}),$$

A convex decomposition

Let the (non-complete) subset of $\mathscr{V}$ endowed with weak-$*$ topology:

$$\mathfrak{G} \overset{\text{def.}}{=} \left\{ \frac{\boldsymbol{\mu}_\gamma}{\|\boldsymbol{\mu}_\gamma\|_{\mathscr{V}}}, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$

Theorem (Smirnov Theorem A. [Smirnov, 1993])

Let $\boldsymbol{T} \in \mathscr{V}$. Then $\boldsymbol{T} = \boldsymbol{P} + \boldsymbol{Q}$ where $\boldsymbol{P}$ is a solenoid i.e. $\operatorname{div} \boldsymbol{P} = 0$ and $\boldsymbol{Q}$ is completely decomposable on $\mathfrak{G}$, i.e. there exists a Borel measure ρ s.t.:

$$\boldsymbol{Q} = \int_{\mathfrak{G}} \boldsymbol{R} \, \mathrm{d}\rho(\boldsymbol{R}), \quad \|\boldsymbol{Q}\|_{\mathrm{TV}^2} = \int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}^2} \, \mathrm{d}\rho(\boldsymbol{R}),$$

A convex decomposition

Let the (non-complete) subset of $\mathscr{V}$ endowed with weak-$*$ topology:

$$\mathfrak{G} \stackrel{\text{def.}}{=} \left\{ \frac{\boldsymbol{\mu}_\gamma}{\|\boldsymbol{\mu}_\gamma\|_{\mathscr{V}}}, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$

Theorem (Smirnov Theorem A. [Smirnov, 1993])

Let $\boldsymbol{T} \in \mathscr{V}$. Then $\boldsymbol{T} = \boldsymbol{P} + \boldsymbol{Q}$ where $\boldsymbol{P}$ is a solenoid i.e. div $\boldsymbol{P} = 0$ *and $\boldsymbol{Q}$ is completely decomposable on $\mathfrak{G}$, i.e. there exists a Borel measure ρ s.t.:*

$$\boldsymbol{Q} = \int_{\mathfrak{G}} \boldsymbol{R} \,\mathrm{d}\rho(\boldsymbol{R}), \quad \|\boldsymbol{Q}\|_{\mathrm{TV}^2} = \int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}^2} \,\mathrm{d}\rho(\boldsymbol{R}), \quad \dots \tag{1}$$

A convex decomposition

Theorem (Smirnov Theorem A. [Smirnov, 1993])

Let $\boldsymbol{T} \in \mathscr{V}$. *Then* $\boldsymbol{T} = \boldsymbol{P} + \boldsymbol{Q}$ *where* $\boldsymbol{P}$ *is a solenoid i.e.* $\operatorname{div} \boldsymbol{P} = 0$ *and* $\boldsymbol{Q}$ *is completely decomposable on* $\mathfrak{G}$*, i.e. there exists a Borel measure* ρ *s.t.:*

$$\boldsymbol{Q} = \int_{\mathfrak{G}} \boldsymbol{R} \, \mathrm{d}\rho(\boldsymbol{R}), \quad \|\boldsymbol{Q}\|_{\mathrm{TV}^2} = \int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}^2} \, \mathrm{d}\rho(\boldsymbol{R}), \quad \ldots \tag{1}$$

The following only holds for $d = 2$:

A convex decomposition

Theorem (Smirnov Theorem A. [Smirnov, 1993])

Let $\boldsymbol{T} \in \mathscr{V}$. Then $\boldsymbol{T} = \boldsymbol{P} + \boldsymbol{Q}$ where $\boldsymbol{P}$ is a solenoid i.e. $\operatorname{div} \boldsymbol{P} = 0$ and $\boldsymbol{Q}$ is completely decomposable on $\mathfrak{G}$, i.e. there exists a Borel measure ρ s.t.:

$$\boldsymbol{Q} = \int_{\mathfrak{G}} \boldsymbol{R} \,\mathrm{d}\rho(\boldsymbol{R}), \quad \|\boldsymbol{Q}\|_{\mathrm{TV}^2} = \int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}^2} \,\mathrm{d}\rho(\boldsymbol{R}), \quad \dots \tag{1}$$

The following only holds for $d = 2$:

Theorem (Khavin & Smirnov [Khavin and Smirnov, 1998])

Let $\boldsymbol{P}$ a solenoid of $\mathscr{V}$. It is completely decomposable on $\mathfrak{G}$.

A convex decomposition

Theorem (Smirnov Theorem A. [Smirnov, 1993])

Let $\boldsymbol{T} \in \mathscr{V}$. Then $\boldsymbol{T} = \boldsymbol{P} + \boldsymbol{Q}$ where $\boldsymbol{P}$ is a solenoid i.e. $\operatorname{div} \boldsymbol{P} = 0$ *and $\boldsymbol{Q}$ is completely decomposable on $\mathfrak{G}$, i.e. there exists a Borel measure ρ s.t.:*

$$\boldsymbol{Q} = \int_{\mathfrak{G}} \boldsymbol{R} \, \mathrm{d}\rho(\boldsymbol{R}), \quad \|\boldsymbol{Q}\|_{\mathrm{TV}^2} = \int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}^2} \, \mathrm{d}\rho(\boldsymbol{R}), \quad \ldots \tag{1}$$

The following only holds for $d = 2$:

Theorem (Khavin & Smirnov [Khavin and Smirnov, 1998])

Let $\boldsymbol{P}$ a solenoid of $\mathscr{V}$. It is completely decomposable on $\mathfrak{G}$.

Then any charge of $\mathscr{V}$ for $d = 2$ is completely decomposable on $\mathfrak{G}$.

Consider the variational problem we coined *Curves Represented On Charges*:

$$\underset{\boldsymbol{m}\in\mathscr{V}}{\operatorname{argmin}}\ \frac{1}{2}\|y - \Phi\,\boldsymbol{m}\|_{\mathcal{H}}^{2} + \alpha\|\boldsymbol{m}\|_{\mathscr{V}}\,. \tag{CROC}$$

CROC energy

Consider the variational problem we coined *Curves Represented On Charges*:

$$\operatorname*{argmin}_{\boldsymbol{m}\in\mathscr{V}} \frac{1}{2}\|y - \Phi\,\boldsymbol{m}\|_{\mathcal{H}}^2 + \alpha\|\boldsymbol{m}\|_{\mathscr{V}}. \tag{CROC}$$

- $\frac{1}{2}\|y - \Phi\,\boldsymbol{m}\|_{\mathcal{H}}^2$ is the data-term;

CROC energy

Consider the variational problem we coined *Curves Represented On Charges*:

$$\operatorname*{argmin}_{\boldsymbol{m}\in\mathscr{V}} \frac{1}{2}\|y-\Phi\,\boldsymbol{m}\|_{\mathcal{H}}^2+\alpha(\|\boldsymbol{m}\|_{\mathrm{TV}^2}+\|\operatorname{div}\boldsymbol{m}\|_{\mathrm{TV}}) \tag{CROC}$$

- $\frac{1}{2}\|y-\Phi\,\boldsymbol{m}\|_{\mathcal{H}}^2$ is the data-term;
- $\|\boldsymbol{m}\|_{\mathrm{TV}^2}$ weights down the curve length, *i.e.* $\|\boldsymbol{\mu_\gamma}\|_{\mathrm{TV}^2}=\mathscr{H}_1(\gamma((0,1)))$;

CROC energy

Consider the variational problem we coined *Curves Represented On Charges*:

$$\underset{\boldsymbol{m}\in\mathscr{V}}{\operatorname{argmin}} \frac{1}{2}\|y - \Phi\,\boldsymbol{m}\|_{\mathcal{H}}^2 + \alpha(\|\boldsymbol{m}\|_{\mathrm{TV}^2} + \|\operatorname{div}\boldsymbol{m}\|_{\mathrm{TV}}) \tag{CROC}$$

- $\frac{1}{2}\|y - \Phi\,\boldsymbol{m}\|_{\mathcal{H}}^2$ is the data-term;
- $\|\boldsymbol{m}\|_{\mathrm{TV}^2}$ weights down the curve length, *i.e.* $\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathrm{TV}^2} = \mathscr{H}_1(\boldsymbol{\gamma}((0,1)))$;
- $\|\operatorname{div}\boldsymbol{m}\|_{\mathrm{TV}}$ is the (open) curve counting term.

CROC energy

Consider the variational problem we coined *Curves Represented On Charges*:

$$\operatorname*{argmin}_{\boldsymbol{m}\in\mathscr{V}} \frac{1}{2}\|y - \Phi\,\boldsymbol{m}\|_{\mathcal{H}}^2 + \alpha(\|\boldsymbol{m}\|_{\mathrm{TV}^2} + \|\operatorname{div}\boldsymbol{m}\|_{\mathrm{TV}}) \tag{CROC}$$

- $\frac{1}{2}\|y - \Phi\,\boldsymbol{m}\|_{\mathcal{H}}^2$ is the data-term;
- $\|\boldsymbol{m}\|_{\mathrm{TV}^2}$ weights down the curve length, *i.e.* $\|\boldsymbol{\mu_\gamma}\|_{\mathrm{TV}^2} = \mathscr{H}_1(\gamma((0,1)))$;
- $\|\operatorname{div}\boldsymbol{m}\|_{\mathrm{TV}}$ is the (open) curve counting term.

Do curve measures minimise (CROC)?

Extreme points

Extreme points

Definition

Let X be a topological vector space and $K \subset X$. An *extreme point* x of K is a point such that $\forall y, z \in K$:

Extreme points

Definition

Let X be a topological vector space and $K \subset X$. An *extreme point* x of K is a point such that $\forall y, z \in K$:

$$\forall \lambda \in (0,1),\ x = \lambda y + (1-\lambda) z \implies x = y = z$$

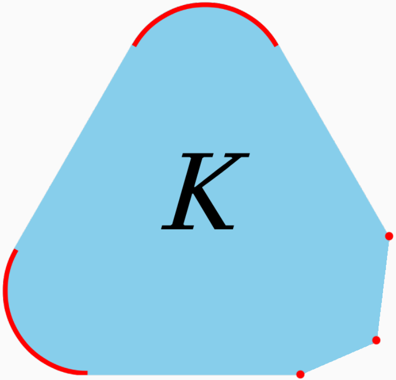
Extreme points

Definition

Let X be a topological vector space and $K \subset X$. An *extreme point* x of K is a point such that $\forall y, z \in K$:

$$\forall \lambda \in (0,1),\ x = \lambda y + (1-\lambda) z \implies x = y = z$$

$\operatorname{Ext} K$ is the set of extreme points of K.



$\operatorname{Ext} K$ in red

Link with extreme points: general setup

Let $F : E \to \mathbb{R}^m$, G the data-term, R the regulariser, $\alpha > 0$.

$$F = G + \alpha R$$

Link with extreme points: general setup

Let $F : E \to \mathbb{R}^m$, G the data-term, R the regulariser, $\alpha > 0$.

$$F = G + \alpha R$$

Theorem (Representer theorem [Bredies and Fanzon, 2019, Duval and Peyré, 2014])

There exists a minimiser of F which is a linear sum of extreme points of the unit-ball of R, $\operatorname{Ext} \mathcal{B}_E^1 \stackrel{\text{def.}}{=} \{u \in E \,|\, R(E) \leq 1\}$.

Link with extreme points: general setup

Let $F : E \to \mathbb{R}^m$, G the data-term, R the regulariser, $\alpha > 0$.

$$F = G + \alpha R$$

Theorem (Representer theorem [Bredies and Fanzon, 2019, Duval and Peyré, 2014])

There exists a minimiser of F which is a linear sum of extreme points of the unit-ball of R, $\mathrm{Ext}\, \mathcal{B}_E^1 \stackrel{\text{def.}}{=} \{u \in E \,|\, R(E) \leq 1\}$.

Characterise $\mathrm{Ext}\, \mathcal{B}_E^1$ of the regulariser $\iff$ outline the structure of a *minimum* of F.

Extreme points in measure spaces

- If $E = \mathcal{M}(\mathcal{X})$ and $R = \|\cdot\|_{\mathrm{TV}}$, then:

$$\mathrm{Ext}(\mathcal{B}_{\mathcal{M}}) = \{\delta_x, x \in \mathcal{X}\}\,.$$

Extreme points in measure spaces

- If $E = \mathcal{M}(\mathcal{X})$ and $R = \|\cdot\|_{\mathrm{TV}}$, then:

$$\mathrm{Ext}(\mathcal{B}_{\mathcal{M}}) = \{\delta_x, x \in \mathcal{X}\}.$$

- If $E = \mathrm{BV}(\mathcal{X})$ and $R = \|\cdot\|_{\mathrm{BV}}$, then:

$$\mathrm{Ext}(\mathcal{B}_{\mathrm{BV}}) = \left\{ \frac{1}{\mathrm{Per}(E)} \chi_E,\ E \subset \mathcal{X} \text{ is simple} \right\}.$$

Extreme points in measure spaces

- If $E = \mathcal{M}(\mathcal{X})$ and $R = \|\cdot\|_{\mathrm{TV}}$, then:

$$\mathrm{Ext}(\mathcal{B}_{\mathcal{M}}) = \{\delta_x, x \in \mathcal{X}\}.$$

- If $E = \mathrm{BV}(\mathcal{X})$ and $R = \|\cdot\|_{\mathrm{BV}}$, then:

$$\mathrm{Ext}(\mathcal{B}_{\mathrm{BV}}) = \left\{\frac{1}{\mathrm{Per}(E)}\chi_E,\ E \subset \mathcal{X} \text{ is simple}\right\}.$$

- If $E = \mathscr{V}$ and $R = \|\cdot\|_{\mathscr{V}}$, then:

$$\mathrm{Ext}(\mathcal{B}_{\mathscr{V}}) = ?$$

Main result

Let the (non-complete) set of curve measures endowed with weak-$*$ topology:

$$\mathfrak{G} \stackrel{\text{def.}}{=} \left\{ \frac{\boldsymbol{\mu}_{\boldsymbol{\gamma}}}{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathscr{V}}}, \, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$

Main result

Let the (non-complete) set of curve measures endowed with weak-$*$ topology:

$$\mathfrak{G} \stackrel{\text{def.}}{=} \left\{ \frac{\boldsymbol{\mu}_{\boldsymbol{\gamma}}}{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathscr{V}}}, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$

Theorem (Main result of [Laville et al., 2023])

Let $\mathcal{B}_{\mathscr{V}}^1 \stackrel{\text{def.}}{=} \{\boldsymbol{m} \in \mathscr{V}, \|\boldsymbol{m}\|_{\mathscr{V}} \leq 1\}$ *the unit ball of the* $\mathscr{V}$*-norm.*

Main result

Let the (non-complete) set of curve measures endowed with weak-$*$ topology:

$$\mathfrak{G} \stackrel{\text{def.}}{=} \left\{ \frac{\boldsymbol{\mu_\gamma}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{V}}}, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$

Theorem (Main result of [Laville et al., 2023])

Let $\mathcal{B}^1_{\mathscr{V}} \stackrel{\text{def.}}{=} \{\boldsymbol{m} \in \mathscr{V}, \|\boldsymbol{m}\|_{\mathscr{V}} \leq 1\}$ *the unit ball of the* $\mathscr{V}$*-norm. Then,*

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) = \mathfrak{G}.$$

Off-the-grid numerical reconstruction

General setup in off-the-grid

- No Hilbertian structure on measure spaces: no proximal algorithm;

General setup in off-the-grid

- No Hilbertian structure on measure spaces: no proximal algorithm;
- we use the Frank-Wolfe algorithm, designed to minimise a differentiable functional on a weakly compact set;

General setup in off-the-grid

- No Hilbertian structure on measure spaces: no proximal algorithm;
- we use the Frank-Wolfe algorithm, designed to minimise a differentiable functional on a weakly compact set;
- it recovers the solution by iteratively adding and optimising extreme points of the regulariser.

General setup in off-the-grid

- No Hilbertian structure on measure spaces: no proximal algorithm;
- we use the Frank-Wolfe algorithm, designed to minimise a differentiable functional on a weakly compact set;
- it recovers the solution by iteratively adding and optimising extreme points of the regulariser.

General setup in off-the-grid

- No Hilbertian structure on measure spaces: no proximal algorithm;
- we use the Frank-Wolfe algorithm, designed to minimise a differentiable functional on a weakly compact set;
- it recovers the solution by iteratively adding and optimising extreme points of the regulariser.

$\hookrightarrow$ perfect with our latter results!

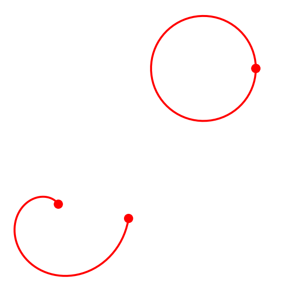
General setup in off-the-grid

- No Hilbertian structure on measure spaces: no proximal algorithm;
- we use the Frank-Wolfe algorithm, designed to minimise a differentiable functional on a weakly compact set;
- it recovers the solution by iteratively adding and optimising extreme points of the regulariser.

$\hookrightarrow$ perfect with our latter results!

We present the *Charge Sliding Frank-Wolfe* algorithm.

Synthetic problem



Synthetic problem

Figure 2: The source and its noisy acquired image

Acquisition process and certificate

- a possible choice consists in setting $\Phi = *\nabla h$ since:

Acquisition process and certificate

- a possible choice consists in setting $\Phi = *\nabla h$ since:
  - μ_γ is vector, hence we need vector *data* like the gradient;

Acquisition process and certificate

- a possible choice consists in setting $\Phi = *\nabla h$ since:
    - μ_γ is vector, hence we need vector *data* like the gradient;
    - let u be the support of the curve, then $\mu_\gamma \simeq \nabla u$ and:

$$\eta = \Phi^*(\Phi m - y) \simeq \Delta u$$

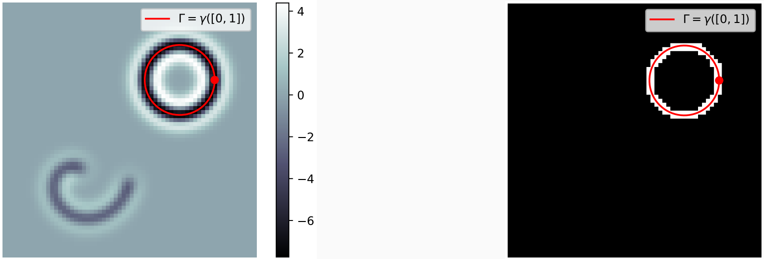


Figure 3: The certificate η on the left, u on the right.

Acquisition process and certificate

- a possible choice consists in setting $\Phi = *\nabla h$ since:
  - $\boldsymbol{\mu}_\gamma$ is vector, hence we need vector *data* like the gradient;
  - let u be the support of the curve, then $\boldsymbol{\mu}_\gamma \simeq \nabla u$ and:

$$\eta = \Phi^*(\Phi m - y) \simeq \Delta u$$

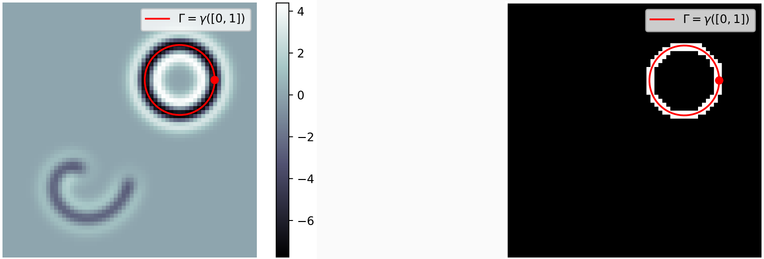
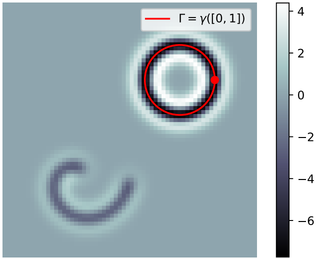


Figure 3: The certificate η on the left, u on the right.

Acquisition process and certificate

- a possible choice consists in setting $\Phi = *\nabla h$ since:
  - $\boldsymbol{\mu}_\gamma$ is vector, hence we need vector *data* like the gradient;
  - let u be the support of the curve, then $\boldsymbol{\mu}_\gamma \simeq \nabla u$ and:

$$\eta = \Phi^*(\Phi m - y) \simeq \Delta u$$



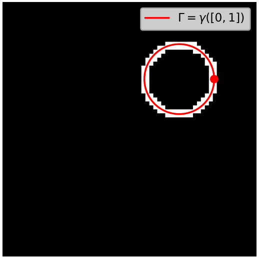
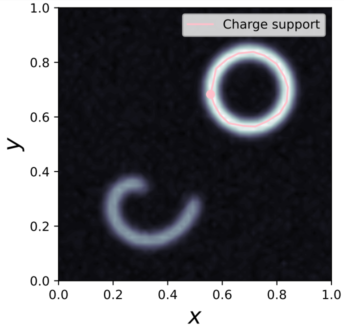


Figure 3: The certificate η on the left, u on the right.

Amplitude and sliding steps

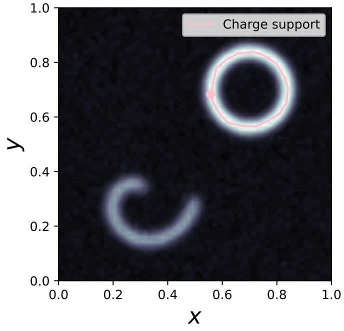
Amplitude and sliding steps



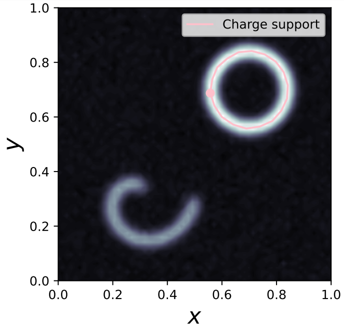
Amplitude optimisation

- we optimise the amplitude a of the new estimated curve;

Amplitude and sliding steps



Amplitude optimisation



Both amplitude and position optimisation

- we optimise the amplitude a of the new estimated curve;
- we perform a *sliding*: we optimise on both amplitudes a and positions γ.

Recap: iterate the algorithm

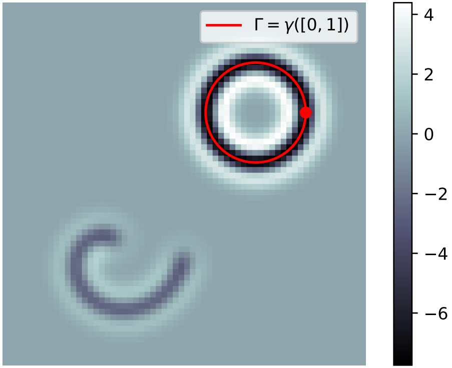


Figure 4: First step of first iteration: certificate and support of new curve estimated

Recap: iterate the algorithm

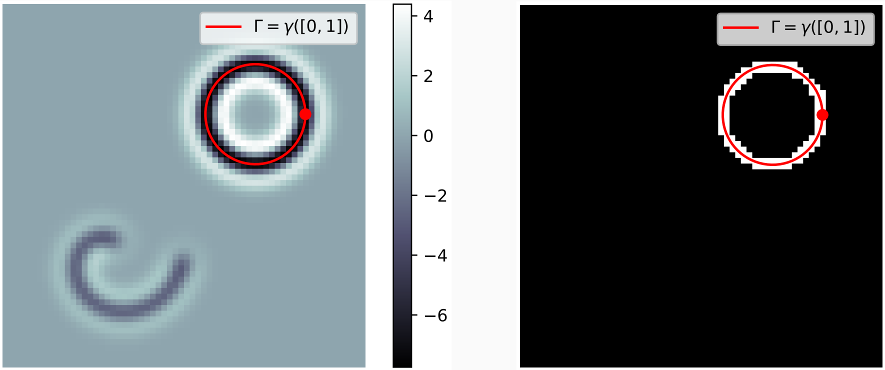
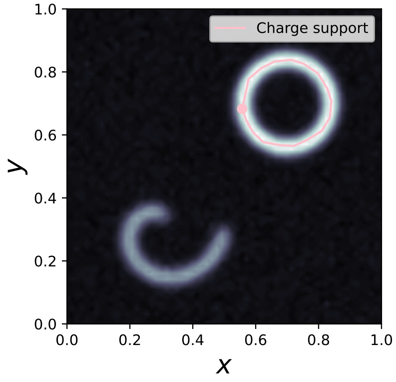


Figure 4: First step of first iteration: certificate and support of new curve estimated

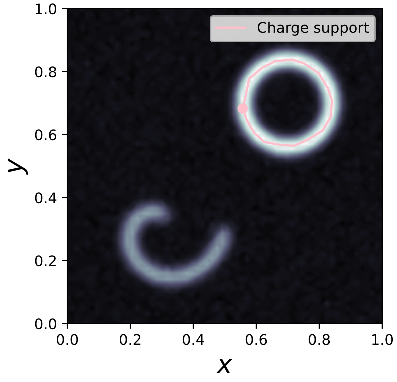
Recap: iterate the algorithm



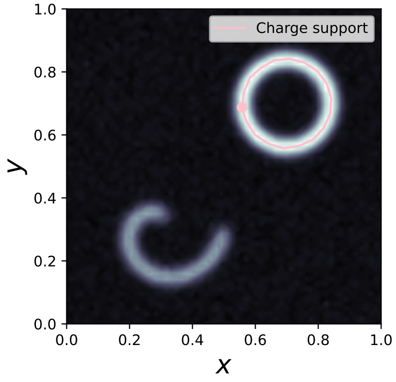
Amplitude optimisation

Figure 4: First iteration: second and third steps

Recap: iterate the algorithm



Amplitude optimisation



Both amplitude and position optimisation

Figure 4: First iteration: second and third steps

Recap: iterate the algorithm

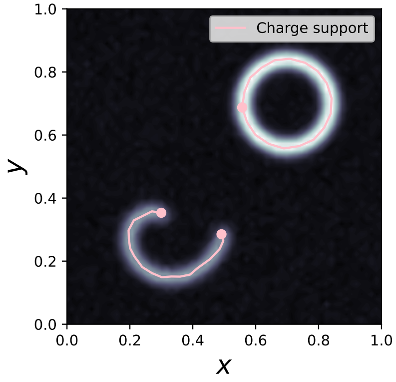


Figure 4: Second iteration: another curve is found

Recap: iterate the algorithm

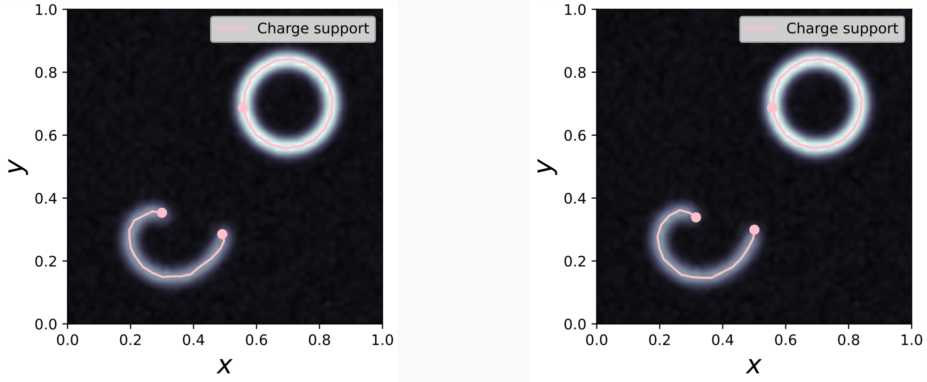
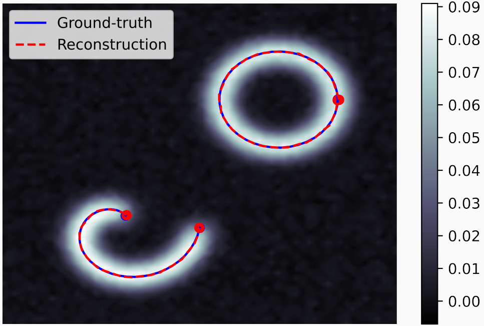


Figure 4: Second iteration: another curve is found

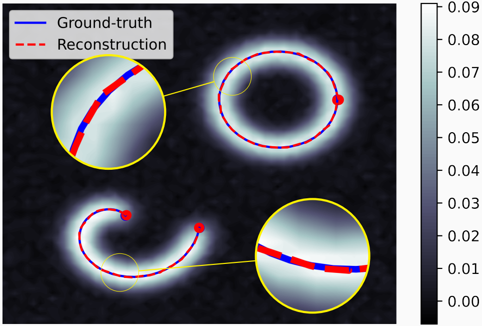
Final results

Final results



Reconstruction

Final results



Reconstruction

Conclusion

Take home messages

- off-the-grid methods yields compelling results (yet scarcely used by applicative researchers);

Take home messages

- off-the-grid methods yields compelling results (yet scarcely used by applicative researchers);
- we brought up to date the space of charges $\mathscr{V}$;

Take home messages

- off-the-grid methods yields compelling results (yet scarcely used by applicative researchers);
- we brought up to date the space of charges $\mathscr{V}$;
- we introduced the functional (CROC), we proved $\mathrm{Ext}(\mathcal{B}_{\mathscr{V}}) = \mathfrak{G}$;

Take home messages

- off-the-grid methods yields compelling results (yet scarcely used by applicative researchers);
- we brought up to date the space of charges $\mathscr{V}$;
- we introduced the functional (CROC), we proved $\mathrm{Ext}(\mathcal{B}_{\mathscr{V}}) = \mathfrak{G}$;
- we proposed a Charge Sliding Frank-Wolfe for curve reconstruction.

Take home messages

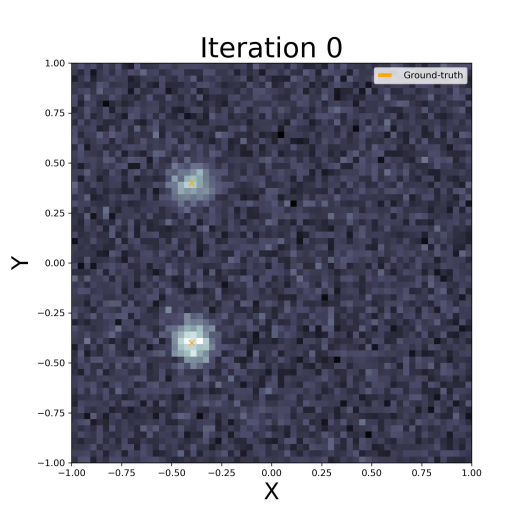
- off-the-grid methods yields compelling results (yet scarcely used by applicative researchers);
- we brought up to date the space of charges $\mathscr{V}$;
- we introduced the functional (CROC), we proved $\mathrm{Ext}(\mathcal{B}_{\mathscr{V}}) = \mathfrak{G}$;
- we proposed a Charge Sliding Frank-Wolfe for curve reconstruction.

Take home messages

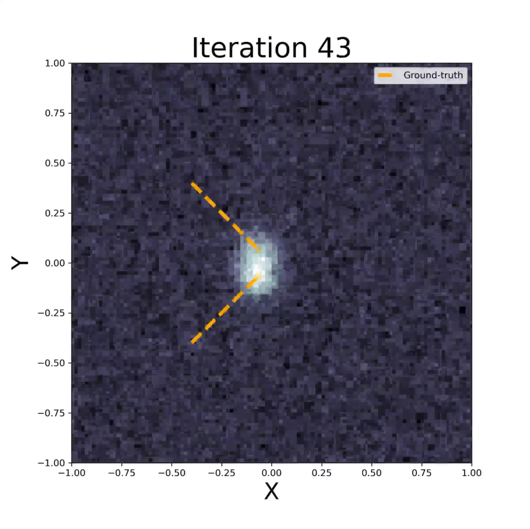
- off-the-grid methods yields compelling results (yet scarcely used by applicative researchers);
- we brought up to date the space of charges $\mathscr{V}$;
- we introduced the functional (CROC), we proved $\mathrm{Ext}(\mathcal{B}_{\mathscr{V}}) = \mathfrak{G}$;
- we proposed a Charge Sliding Frank-Wolfe for curve reconstruction.

In the future: better curve estimation support, test on experimental data, curves untangling, *etc*.

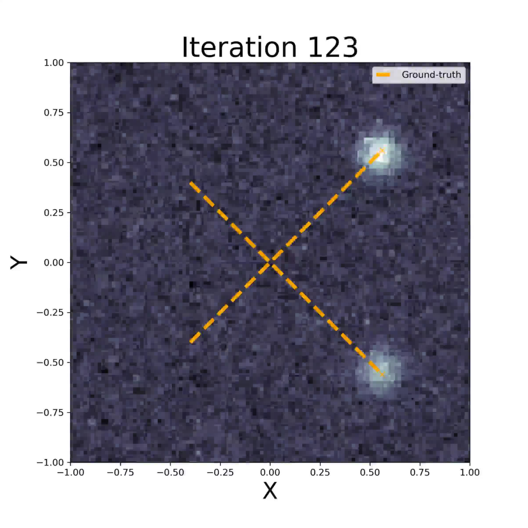
Perspectives: curves untangling, inspired by dynamic off-the-grid



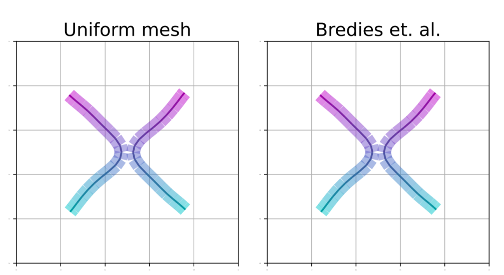
Perspectives: curves untangling, inspired by dynamic off-the-grid



Perspectives: curves untangling, inspired by dynamic off-the-grid

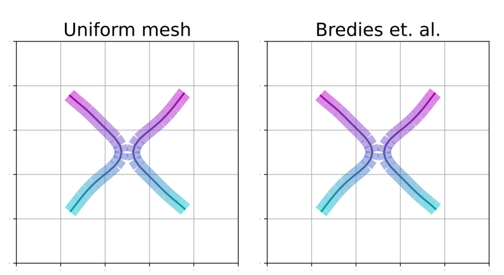


Perspectives: curves untangling, inspired by dynamic off-the-grid

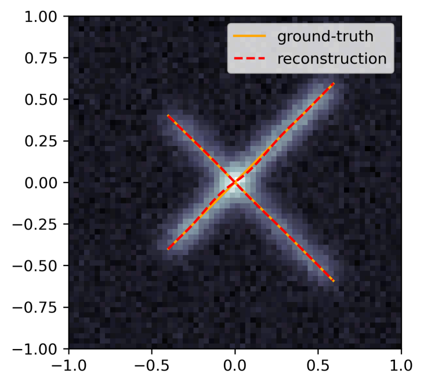


Reconstruction in literature (from Duval and Tovey, 2022)

Perspectives: curves untangling, inspired by dynamic off-the-grid



Reconstruction in literature (from Duval and Tovey, 2022)



Reconstruction (WIP Laville and Théo Bertrand from CEREMADE)

References i

Bredies, K. and Fanzon, S. (2019).
An optimal transport approach for solving dynamic inverse problems in spaces of measures.

Candès, E. J. and Fernandez-Granda, C. (2013).
Towards a mathematical theory of super-resolution.
Communications on Pure and Applied Mathematics, 67(6):906–956.

de Castro, Y., Duval, V., and Petit, R. (2021).
Towards off-the-grid algorithms for total variation regularized inverse problems.
In *Lecture Notes in Computer Science*, pages 553–564. Springer International Publishing.

References ii

Denoyelle, Q., Duval, V., Peyré, G., and Soubies, E. (2019).
The sliding frank–wolfe algorithm and its application to super-resolution microscopy.
Inverse Problems, 36(1):014001.

Duval, V. and Peyré, G. (2014).
Exact support recovery for sparse spikes deconvolution.
Foundations of Computational Mathematics, 15(5):1315–1355.

Khavin, V. P. and Smirnov, S. K. (1998).
Approximation and extension problems for some classes of vector fields.
St. Petersburg Department of Steklov Institute of Mathematics, Russian Academy of Sciences, 10(3):507–528.

References iii

- Laville, B., Blanc-Féraud, L., and Aubert, G. (2021).
  Off-The-Grid Variational Sparse Spike Recovery: Methods and Algorithms.
  Journal of Imaging, 7(12):266.
- Laville, B., Blanc-Féraud, L., and Aubert, G. (2023).
  Off-the-grid curve reconstruction through divergence regularisation: an extreme point result.
  Preprint.
- Nehme, E., Weiss, L. E., Michaeli, T., and Shechtman, Y. (2018).
  Deep-STORM: super-resolution single-molecule microscopy by deep learning.
  5(4):458.

References iv

Smirnov, S. K. (1993).
Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows.
St. Petersburg Department of Steklov Institute of Mathematics, Russian Academy of Sciences, 5(4):206–238.

See our work on

https://www-sop.inria.fr/members/Bastien.Laville/

Proof recipe I

First inclusion:

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) \supset \mathfrak{G}$$

Proof recipe I

First inclusion:

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) \supset \mathfrak{G}$$

Let γ a simple Lipschitz curve and μ_γ the measure supported on this curve. By contradiction, let $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{B}^1_{\mathscr{V}}$ and for $\lambda \in (0,1)$:

$$\frac{\boldsymbol{\mu_\gamma}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{V}}} = \lambda \boldsymbol{u_1} + (1-\lambda)\boldsymbol{u_2}.$$

Proof recipe I

First inclusion:

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) \supset \mathfrak{G}$$

Let γ a simple Lipschitz curve and μ_γ the measure supported on this curve. By contradiction, let $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{B}^1_{\mathscr{V}}$ and for $\lambda \in (0,1)$:

$$\frac{\boldsymbol{\mu_\gamma}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{V}}} = \lambda \boldsymbol{u_1} + (1-\lambda)\boldsymbol{u_2}.$$

By Smirnov's decomposition, $\boldsymbol{u_i} = \int_{\mathfrak{G}} \boldsymbol{R} \,\mathrm{d}\rho_i(\boldsymbol{R})$ where ρ_i is a Borel measure.

Proof recipe I

First inclusion:

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) \supset \mathfrak{G}$$

Let γ a simple Lipschitz curve and μ_γ the measure supported on this curve. By contradiction, let $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{B}^1_{\mathscr{V}}$ and for $\lambda \in (0,1)$:

$$\frac{\boldsymbol{\mu_\gamma}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{V}}} = \lambda \boldsymbol{u_1} + (1-\lambda)\boldsymbol{u_2}.$$

By Smirnov's decomposition, $\boldsymbol{u_i} = \int_{\mathfrak{G}} \boldsymbol{R} \,\mathrm{d}\rho_i(\boldsymbol{R})$ where ρ_i is a Borel measure. Also:

- **$\boldsymbol{u_1}, \boldsymbol{u_2}$ has support included in μ_γ support, ditto for $\operatorname{spt} \boldsymbol{R} \subset \operatorname{spt} \boldsymbol{\mu_\gamma}$ [Smirnov, 1993];**

Proof recipe I

First inclusion:

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) \supset \mathfrak{G}$$

Let γ a simple Lipschitz curve and μ_γ the measure supported on this curve. By contradiction, let $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{B}^1_{\mathscr{V}}$ and for $\lambda \in (0,1)$:

$$\frac{\boldsymbol{\mu_\gamma}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{V}}} = \lambda \boldsymbol{u_1} + (1-\lambda)\boldsymbol{u_2}.$$

By Smirnov's decomposition, $\boldsymbol{u_i} = \int_{\mathfrak{G}} \boldsymbol{R} \,\mathrm{d}\rho_i(\boldsymbol{R})$ where ρ_i is a Borel measure. Also:

$\boldsymbol{u_1}, \boldsymbol{u_2}$ has support included in μ_γ support, ditto for $\mathrm{spt}\, \boldsymbol{R} \subset \mathrm{spt}\, \boldsymbol{\mu_\gamma}$ [Smirnov, 1993];

moreover, each $\boldsymbol{R}$ has *maximal* length implying $\mathrm{spt}\, \boldsymbol{R} = \mathrm{spt}\, \boldsymbol{\mu_\gamma}$.

Proof recipe II

$$\operatorname{spt} \boldsymbol{R} = \operatorname{spt} \boldsymbol{\mu}_{\gamma}.$$

Proof recipe II

spt $\boldsymbol{R} = $ spt $\boldsymbol{\mu}_{\boldsymbol{\gamma}}$. **Otherwise** spt $\boldsymbol{R} \subsetneq$ spt $\boldsymbol{\mu}_{\gamma}$ $\|\boldsymbol{R}\|_{\mathrm{TV}} < \dfrac{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathrm{TV}}}{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathscr{V}}}$,

Proof recipe II

$\operatorname{spt} \boldsymbol{R} = \operatorname{spt} \boldsymbol{\mu_\gamma}$. **Otherwise** $\operatorname{spt} \boldsymbol{R} \subsetneq \operatorname{spt} \boldsymbol{\mu_\gamma}$ $\|\boldsymbol{R}\|_{\mathrm{TV}} < \frac{\|\boldsymbol{\mu_\gamma}\|_{\mathrm{TV}}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{V}}}$**, therefore,**

$$\int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}} \, \mathrm{d}\rho(\boldsymbol{R}) < \frac{\|\boldsymbol{\mu_\gamma}\|_{\mathrm{TV}}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{V}}} \underbrace{\rho(\mathfrak{G})}_{=1} = \int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}} \, \mathrm{d}\rho(\boldsymbol{R}),$$

thus $\operatorname{spt} \boldsymbol{R} = \operatorname{spt} \boldsymbol{\mu_\gamma}$**,**

Proof recipe II

spt $\boldsymbol{R} = \operatorname{spt} \boldsymbol{\mu_\gamma}$. **Otherwise** spt $\boldsymbol{R} \subsetneq \operatorname{spt} \boldsymbol{\mu_\gamma}$ $\|\boldsymbol{R}\|_{\mathrm{TV}} < \dfrac{\|\boldsymbol{\mu_\gamma}\|_{\mathrm{TV}}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{V}}}$**, therefore,**

$$\int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}} \,\mathrm{d}\rho(\boldsymbol{R}) < \frac{\|\boldsymbol{\mu_\gamma}\|_{\mathrm{TV}}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{V}}} \underbrace{\rho(\mathfrak{G})}_{=1} = \int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}} \,\mathrm{d}\rho(\boldsymbol{R}),$$

thus spt $\boldsymbol{R} = \operatorname{spt} \boldsymbol{\mu_\gamma}$**,**

each $\boldsymbol{R}$ is supported on a simple Lipschitz curve $\gamma_{\boldsymbol{R}}$.

Proof recipe II

$\operatorname{spt} \boldsymbol{R} = \operatorname{spt} \boldsymbol{\mu}_{\boldsymbol{\gamma}}$. **Otherwise** $\operatorname{spt} \boldsymbol{R} \subsetneq \operatorname{spt} \boldsymbol{\mu}_{\boldsymbol{\gamma}}$ $\|\boldsymbol{R}\|_{\mathrm{TV}} < \frac{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathrm{TV}}}{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathscr{V}}}$**, therefore,**

$$\int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}} \, \mathrm{d}\rho(\boldsymbol{R}) < \frac{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathrm{TV}}}{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathscr{V}}} \underbrace{\rho(\mathfrak{G})}_{=1} = \int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}} \, \mathrm{d}\rho(\boldsymbol{R}),$$

thus $\operatorname{spt} \boldsymbol{R} = \operatorname{spt} \boldsymbol{\mu}_{\boldsymbol{\gamma}}$**,**

each $\boldsymbol{R}$ is supported on a simple Lipschitz curve $\gamma_{\boldsymbol{R}}$.

Hence, each $\gamma_{\boldsymbol{R}}$ is a reparametrisation of γ yielding $\boldsymbol{R} = \frac{\boldsymbol{\mu}_{\boldsymbol{\gamma}}}{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathscr{V}}}$

Proof recipe II

$\operatorname{spt} \boldsymbol{R} = \operatorname{spt} \boldsymbol{\mu}_\gamma$. **Otherwise** $\operatorname{spt} \boldsymbol{R} \subsetneq \operatorname{spt} \boldsymbol{\mu}_\gamma$ $\|\boldsymbol{R}\|_{\mathrm{TV}} < \dfrac{\|\boldsymbol{\mu}_\gamma\|_{\mathrm{TV}}}{\|\boldsymbol{\mu}_\gamma\|_{\mathscr{V}}}$**, therefore,**

$$\int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}} \,\mathrm{d}\rho(\boldsymbol{R}) < \frac{\|\boldsymbol{\mu}_\gamma\|_{\mathrm{TV}}}{\|\boldsymbol{\mu}_\gamma\|_{\mathscr{V}}} \underbrace{\rho(\mathfrak{G})}_{=1} = \int_{\mathfrak{G}} \|\boldsymbol{R}\|_{\mathrm{TV}} \,\mathrm{d}\rho(\boldsymbol{R}),$$

thus $\operatorname{spt} \boldsymbol{R} = \operatorname{spt} \boldsymbol{\mu}_\gamma$**,**

each $\boldsymbol{R}$ **is supported on a simple Lipschitz curve** $\gamma_{\boldsymbol{R}}$**.**

Hence, each $\gamma_{\boldsymbol{R}}$ **is a reparametrisation of** γ **yielding** $\boldsymbol{R} = \frac{\boldsymbol{\mu}_\gamma}{\|\boldsymbol{\mu}_\gamma\|_{\mathscr{V}}}$**, eventually:**

$$\boldsymbol{u}_i = \int_{\mathfrak{G}} \boldsymbol{R} \,\mathrm{d}\rho_i = \int_{\mathfrak{G}} \frac{\boldsymbol{\mu}_\gamma}{\|\boldsymbol{\mu}_\gamma\|_{\mathscr{V}}} \,\mathrm{d}\rho_i = \frac{\boldsymbol{\mu}_\gamma}{\|\boldsymbol{\mu}_\gamma\|_{\mathscr{V}}} \underbrace{\rho_i(\mathfrak{G})}_{=1} = \frac{\boldsymbol{\mu}_\gamma}{\|\boldsymbol{\mu}_\gamma\|_{\mathscr{V}}}.$$

Contradiction, then $\boldsymbol{\mu}_\gamma$ **is an extreme point.**

$\square$

Proof recipe III

Second inclusion:

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) \subset \mathfrak{G}$$

Proof recipe III

Second inclusion:

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) \subset \mathfrak{G}$$

Let $\boldsymbol{T} \in \mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}})$, then there exists a finite (probability) Borel measure ρ s.t.:

$$\boldsymbol{T} = \int_{\mathfrak{G}} \boldsymbol{R} \, \mathrm{d}\rho(\boldsymbol{R}),$$

Proof recipe III

Second inclusion:

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{Y}}) \subset \mathfrak{G}$$

Let $\boldsymbol{T} \in \mathrm{Ext}(\mathcal{B}^1_{\mathscr{Y}})$, then there exists a finite (probability) Borel measure ρ s.t.:

$$\boldsymbol{T} = \int_{\mathfrak{G}} \boldsymbol{R} \, \mathrm{d}\rho(\boldsymbol{R}),$$

either ρ is supported on a singleton of $\mathfrak{G}$, then there exists μ_γ s.t. $\boldsymbol{T} = \dfrac{\boldsymbol{\mu_\gamma}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{Y}}}$

Proof recipe III

Second inclusion:

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{Y}}) \subset \mathfrak{G}$$

Let $\boldsymbol{T} \in \mathrm{Ext}(\mathcal{B}^1_{\mathscr{Y}})$, then there exists a finite (probability) Borel measure ρ s.t.:

$$\boldsymbol{T} = \int_{\mathfrak{G}} \boldsymbol{R} \, \mathrm{d}\rho(\boldsymbol{R}),$$

either ρ is supported on a singleton of $\mathfrak{G}$, then there exists μ_γ s.t. $\boldsymbol{T} = \dfrac{\boldsymbol{\mu_\gamma}}{\|\boldsymbol{\mu_\gamma}\|_{\mathscr{Y}}}$

or there exists a Borel set $A \subset \mathfrak{G}$ with arbitrary $0 < \rho(A) < 1$ and:

$$\rho = |\rho|(A) \left(\frac{1}{|\rho|(A)} \rho \llcorner A \right) + |\rho|(A^c) \left(\frac{1}{|\rho|(A^c)} \rho \llcorner A^c \right).$$

Proof recipe IV

Then,

$$\boldsymbol{T} = |\rho|(A)\underbrace{\left[\int_{\mathfrak{G}} \frac{1}{|\rho|(A)}\boldsymbol{R}\,\mathrm{d}(\rho \llcorner A)(\boldsymbol{R})\right]}_{\stackrel{\text{def.}}{=}\boldsymbol{u_1}} + |\rho|(A^c)\underbrace{\left[\int_{\mathfrak{G}} \frac{1}{|\rho|(A^c)}\boldsymbol{R}\,\mathrm{d}(\rho \llcorner A^c)(\boldsymbol{R})\right]}_{\stackrel{\text{def.}}{=}\boldsymbol{u_2}}$$

Proof recipe IV

Then,

$$\boldsymbol{T} = |\rho|(A)\underbrace{\left[\int_{\mathfrak{G}} \frac{1}{|\rho|(A)} \boldsymbol{R}\,\mathrm{d}(\rho \llcorner A)(\boldsymbol{R})\right]}_{\stackrel{\text{def.}}{=} \boldsymbol{u_1}} + |\rho|(A^c)\underbrace{\left[\int_{\mathfrak{G}} \frac{1}{|\rho|(A^c)} \boldsymbol{R}\,\mathrm{d}(\rho \llcorner A^c)(\boldsymbol{R})\right]}_{\stackrel{\text{def.}}{=} \boldsymbol{u_2}}$$

A **is chosen (up to a neighbourhood) as a convex set, hence** $\boldsymbol{u_1} = \int_A \boldsymbol{R}\,\mathrm{d}\rho(\boldsymbol{R})$ **belongs to** A**, while conversely** $\boldsymbol{u_2} \in A^c$**, thus** $\boldsymbol{u_1} \neq \boldsymbol{u_2}$**.**

Proof recipe IV

Then,

$$\boldsymbol{T} = |\rho|(A)\underbrace{\left[\int_{\mathfrak{G}} \frac{1}{|\rho|(A)} \boldsymbol{R}\,\mathrm{d}(\rho \llcorner A)(\boldsymbol{R})\right]}_{\stackrel{\text{def.}}{=} \boldsymbol{u_1}} + |\rho|(A^c)\underbrace{\left[\int_{\mathfrak{G}} \frac{1}{|\rho|(A^c)} \boldsymbol{R}\,\mathrm{d}(\rho \llcorner A^c)(\boldsymbol{R})\right]}_{\stackrel{\text{def.}}{=} \boldsymbol{u_2}}$$

A **is chosen (up to a neighbourhood) as a convex set, hence** $\boldsymbol{u_1} = \int_A \boldsymbol{R}\,\mathrm{d}\rho(\boldsymbol{R})$ **belongs to** A**, while conversely** $\boldsymbol{u_2} \in A^c$**, thus** $\boldsymbol{u_1} \neq \boldsymbol{u_2}$**. Eventually, thanks to Smirnov's decomposition:**

$$\begin{aligned} \|\boldsymbol{u_1}\|_{\mathscr{V}} &\leq \int_{\mathfrak{G}} \frac{1}{|\rho|(A)} \underbrace{\|\boldsymbol{R}\|_{\mathscr{V}}}_{=1} \mathrm{d}(\rho \llcorner A)(\boldsymbol{R}) \\ &\leq \frac{|\rho|(A)}{|\rho|(A)} = 1. \end{aligned}$$

Proof recipe V

Then $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{B}^1_{\mathcal{Y}}$ while $\boldsymbol{u_1} \neq \boldsymbol{u_2}$, thus reaching a non-trivial convex combination:

$$\boldsymbol{T} = \lambda \boldsymbol{u_1} + (1 - \lambda)\boldsymbol{u_2},$$

Proof recipe V

Then $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{B}^1_{\mathcal{Y}}$ while $\boldsymbol{u_1} \neq \boldsymbol{u_2}$, thus reaching a non-trivial convex combination:

$$\boldsymbol{T} = \lambda \boldsymbol{u_1} + (1 - \lambda) \boldsymbol{u_2},$$

thereby reaching a contradiction, and therefore concluding the proof.

$\square$