Off-the-grid covariance-based super-resolution.

Bastien Laville, Laure Blanc-Féraud, Gilles Aubert

Team Morpheme: Inria SAM, CNRS, UCA (France)

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Introduction

Biomedical context

Aim

Image biological structures at small scales

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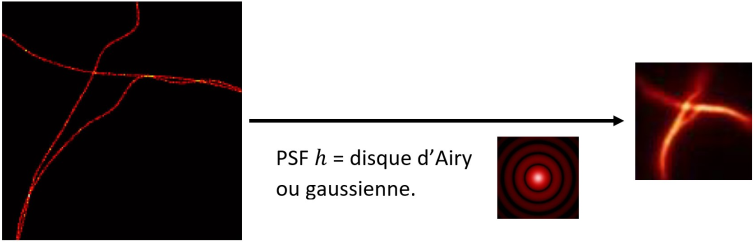
Physical limitation due to diffraction for bodies $< 200\ \mathrm{nm}$: convolution by the microscope's *point spread function*.



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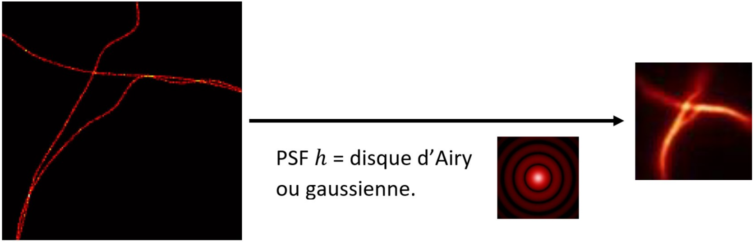
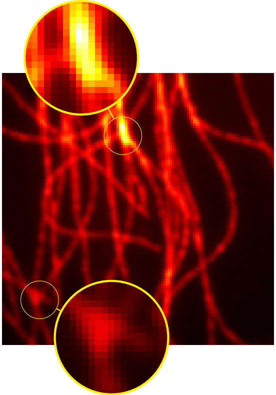


Reconstruction e.g. by fluorescence microscopy SMLM: acquisition stack with few lit fluorophores per image. Drawback: many images ($\approx 1 \times 10^4$, does not allow imaging of living cells).

On an EPFL SMLM Challenge stack (10000 images, high density):



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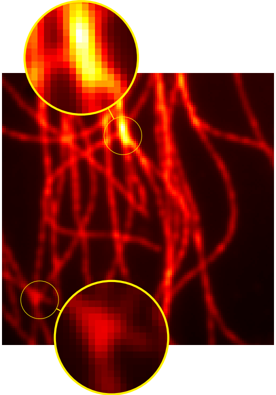
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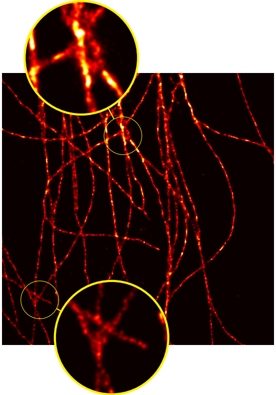
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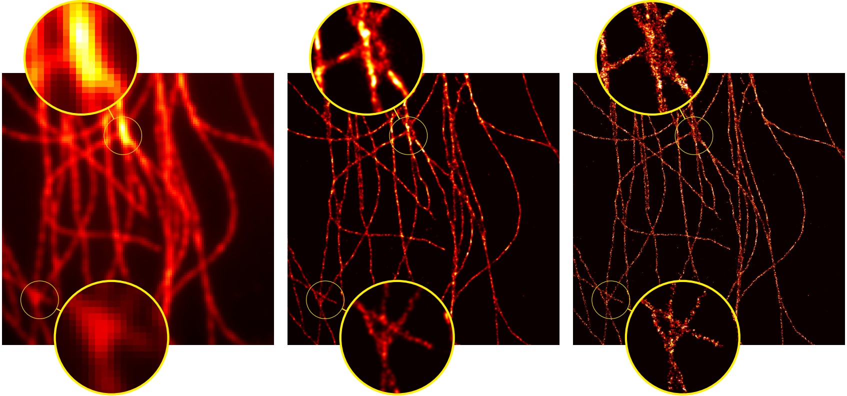

Mean of stack Off-the-grid Deep-STORM

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SOFI imaging (Super-resolution optical fluctuation imaging).
Applications: imaging for localisation in fluorescence microscopy, etc. [Dertinger10].

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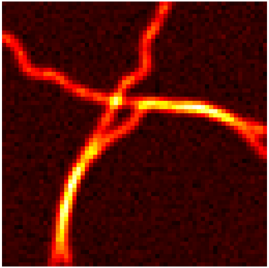

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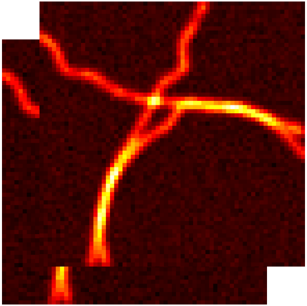

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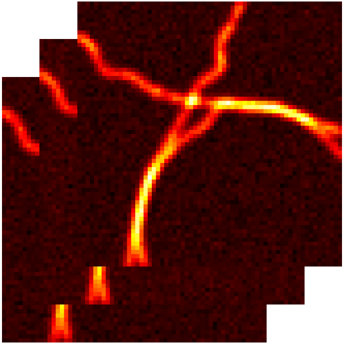
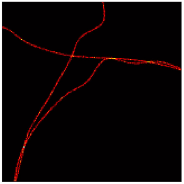


Off-the-grid digest

Inverse problem: from an acquisition, we reconstruct spike positions and amplitudes. Super-resolution grid problem:

Acquisition



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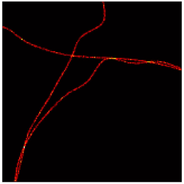

Reconstruction

- off-the-grid super-resolution can be understood as the 'limit' of an increasingly fine grid;
- not limited by a (fine) grid, but still limited by the noise.

Discrete case

- the reconstructed peaks are necessarily on the fine grid;
- (Non-)convex combinatorial optimisation;
- fast numerical computation;
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Off-the-grid case

- not limited by the grid;
- convexity of the functional on an infinite dimensional space;
- existence and uniqueness guarantees;
- recent field of research.

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- topological dual of $\mathscr{C}(\mathcal{X})$ for $\langle f, m \rangle = \int_{\mathcal{X}} f \, \mathrm{d}m$. Generalisation of $\mathrm{L}^1(\mathcal{X})$ since $\mathrm{L}^1(\mathcal{X}) \hookrightarrow \mathcal{M}(\mathcal{X})$;

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- Banach for the TV-norm: $m \in \mathcal{M}(\mathcal{X})$,

$$|m|(\mathcal{X}) \stackrel{\text{def.}}{=} \sup \left(\int_{\mathcal{X}} f \, \mathrm{d}m \,\middle|\, f \in \mathscr{C}(\mathcal{X}), \|f\|_{\infty,\mathcal{X}} \leq 1 \right).$$

If $m = \sum_{i=1}^N a_i \delta_{x_i}$ is a discrete measure then $|m|(\mathcal{X}) = \sum_{i=1}^N |a_i|$.

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We call **BLASSO** the optimisation problem [Castro12, Duval15] for $\lambda > 0$:

$$\underset{m \in \mathcal{M}(\mathcal{X})}{\operatorname{argmin}} \frac{1}{2} \|y - \Phi m\|^2_{\mathrm{L}^2(\mathcal{X})} + \lambda |m|(\mathcal{X}) \qquad (\mathcal{P}_\lambda(y))$$

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The optimisation space $\mathcal{M}(\mathcal{X})$ is an infinite dimensional space, reflexive only for weak-* topology: a difficult problem.

Dynamic off-the-grid

Quantities at stake

- acquisition stack (images in $\mathrm{L}^2(\mathcal{X})$) during $[0, T]$;

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- we define $y : [0, T] \to \mathrm{L}^2(\mathcal{X})$ the SOFI acquisition stack ;
- we aim to reconstruct the *dynamic* measure:

$$t \mapsto \mu(t) \stackrel{\text{def.}}{=} \sum_{i=1}^{N} a_i(t) \delta_{x_i} \in \mathrm{L}^2(0, T; \mathcal{M}(\mathcal{X}))$$

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Cumulants are a tool to reconstruct the positions x_i. Example : temporal mean $\bar{y} \stackrel{\text{def.}}{=} \frac{1}{T}\int_0^T y(\cdot,t)\,\mathrm{d}t$. One have $\Phi m_{a,x} = \bar{y}$ where $m_{a,x} \stackrel{\text{def.}}{=} \sum_{i=1}^N \bar{a}_i \delta_{x_i}$ and $\bar{a}_i$ is the mean of $a_i(\cdot)$.

Build the variational problem

Let R_y be the temporal covariance, $\forall u, v \in \mathcal{X}$ we get:

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\begin{aligned}
R_y(u,v) &\stackrel{\text{def.}}{=} \frac{1}{T}\int_0^T \left(y(u,t) - \bar{y}(u)\right)\left(y(v,t) - \bar{y}(v)\right)\,\mathrm{d}t \\
&= \dots \quad \text{(independence of fluctuations [Dertinger10])} \\
&= \sum_{i=1}^{N} \underbrace{M_i}_{a_i \text{ variance}} \; h(u - x_i)h(v - x_i) \\
&= \int_{\mathcal{X}} h(u-x)h(v-x)\,\mathrm{d}m_{M,x}(x) \\
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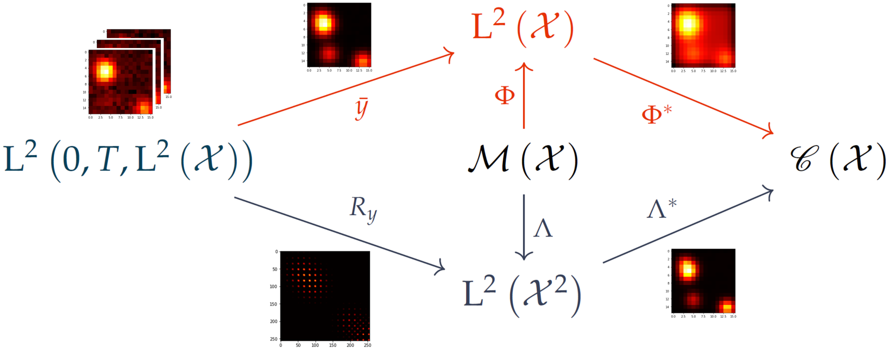
$m_{M,x} \overset{\text{def.}}{=} \sum_{i=1}^N M_i \delta_{x_i}$ shares the same positions as $\mu = \sum_{i=1}^N a_i(t)\delta_{x_i}$, we call $\Lambda : \mathcal{M}(\mathcal{X}) \to \mathrm{L}^2(\mathcal{X}^2)$ this « covariance operator ».

Quantities digest



Legend: dynamic part, temporal mean part $\bar{y}$ and covariance R_y.

BLASSO on cumulants

Let $\lambda > 0$,
covariance problem writes down:

$$\underset{m \in \mathcal{M}(\mathcal{X})}{\operatorname{argmin}} T_\lambda(m) \overset{\text{def.}}{=} \frac{1}{2} \| R_y - \Lambda(m) \|^2_{\mathrm{L}^2(\mathcal{X}^2)} + \lambda |m|(\mathcal{X}) \qquad (\mathcal{Q}_\lambda(y))$$

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Let $\Delta \stackrel{\text{def.}}{=} \min_{i \neq j} |x_i - x_j|$ be the *minimum* separation distance

Proposition

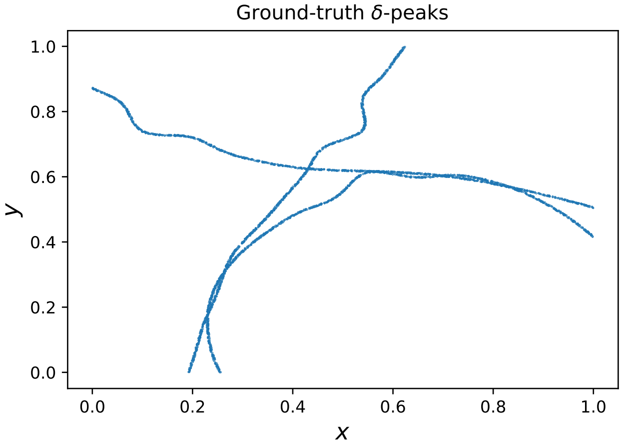
Support of a ***real*** *Radon measure in noiseless setting is reconstructed:*

- *if* $\Delta \gtrsim 1,1\sigma$ *[Bendory16] for mean-based reconstruction;*
- *if* $\Delta \gtrsim 1,1\sigma/\sqrt{2}$ *for covariance-based reconstruction* $(\mathcal{Q}_\lambda(y))$*:* ***better!****.*

Numerical 2D results `SOFItool`

Test on 2D tubulins from ISBI challenge 2016:

- stack of 1000 acquisitions 64×64 simulated by `SOFItool`;
- 8700 emitters scattered along the tubulins; **high** background noise + Poisson noise at 4 + Gaussian noise at 1×10^{-2}. SNR ≈ 10 db.



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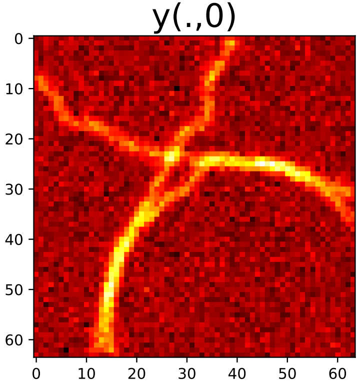

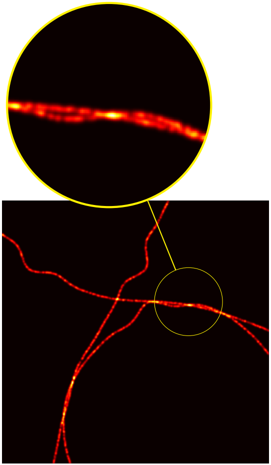

Figure 1: Ground-truth

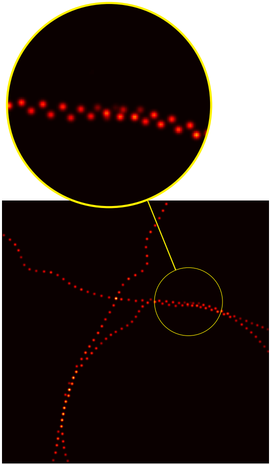

Figure 2: $(\mathcal{Q}_\lambda(y))$

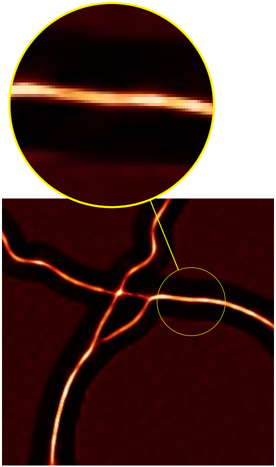

Figure 3: SRRF [Culley18]

Conclusion

Take home statements:

- off-the-grid methods squeeze all the 'information' out of the acquisition y: no discretisation drawback;
- allow performing structural assumption on the minimiser;
- strong results for existence and uniqueness of BLASSO solution;
- one efficient numerical algorithm: Sliding Frank-Wolfe.

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Outlook:

- spikes moving in position w.r.t time: *tracking* problem;
- theory only suited for spikes and sets: what about other source structures (e.g. curves)?

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Sliding Frank-Wolfe

Algorithm 1: *Sliding Frank-Wolfe.*

Entrées: Acquisition $y \in \mathcal{H}$, nombre d'itérations K, $\lambda > 0$

Initialisation : $m^{[0]} = 0$ $N^{[k]} = 0$

for *Récurrence pour l'étape* k, $0 \leq k \leq K$ **do**

Pour $m^{[k]} = \sum_{i=1}^{N^{[k]}} a_i^{[k]} \delta_{x_i^{[k]}}$ telle que $a_i^{[k]} \in \mathbb{R}$, $x_i^{[k]} \in \mathcal{X}$, trouver $x_*^{[k]} \in \mathcal{X}$ tel que :

$$x_*^{[k]} \in \underset{x \in \mathcal{X}}{\operatorname{argmax}} \left| \eta^{[k]}(x) \right| \qquad \text{où} \quad \eta^{[k]}(x) \overset{\text{def.}}{=} \frac{1}{\lambda} \Phi^*(\Phi m^{[k]} - y),$$

if $\left| \eta^{[k]}(x_*^{[k]}) \right| < 1$ **then**

$m^{[k]}$ est la solution du BLASSO. Stop.

else

Calculer $m^{[k+1/2]} = \sum_{i=1}^{N^{[k]}} a_i^{[k+1/2]} \delta_{x_i^{[k+1/2]}} + a_{N^{[k]}+1}^{[k+1/2]} \delta_*^{[k+1/2]}$ telle que :

$$a_i^{[k+1/2]} \in \underset{a \in \mathbb{R}^{N^{[k]}+1}}{\operatorname{argmin}} \frac{1}{2} \|y - \Phi_{x^{[k+1/2]}}(a)\|_{\mathcal{H}}^2 + \lambda \|a\|_1$$

$$\text{pour} \quad x^{[k+1/2]} \overset{\text{def.}}{=} \left(x_1^{[k]}, \ldots, x_{N^{[k]}}^{[k]}, x_*^{[k]} \right).$$

Calculer $m^{[k+1]} = \sum_{i=1}^{N^{[k+1]}} a_i^{[k+1]} \delta_{x_i^{[k+1]}}$ telle que :

$$(a_i^{[k+1]}, x_i^{[k+1]}) \in \underset{(a,x) \in R}{\operatorname{argmax}} \frac{1}{2} \|y - \Phi_{x^{[k+1/2]}}(a)\|_{\mathcal{H}}^2 + \lambda \|a\|_1$$

end

end

Sortie: Mesure discrète $m^{[k]}$ pour k l'itération d'arrêt.