Reconstruction de courbes en super-résolution sans-grille

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Introduction
Biomedical imaging

Objective
To image live biological structures at small scales.
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**Physical** limitation due to **diffraction** for bodies < 200 nm: convolution by the microscope’s point spread function (PSF).
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Problème inverse
Grid or gridless?

\[ S_0 \]

Source to estimate
Grid or gridless?

Introducing a grid

\[ S_0 \]
Grid or gridless?

Reconstruction $\hat{S}$ on a grid
Grid or gridless?

Reconstruction  \( \hat{S} \) on a finer grid
Grid or gridless?

Reconstruction $\hat{S}$ is now **off-the-grid**
Off-the-grid spikes
• $\mathcal{X}$ is a compact of $\mathbb{R}^d$;
Quantities

- $\mathcal{X}$ is a compact of $\mathbb{R}^d$;
- how to model spikes? Through Dirac measure $\delta_x$, element of the set of Radon measures $\mathcal{M}(\mathcal{X})$;
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- $\mathcal{X}$ is a compact of $\mathbb{R}^d$;
- how to model spikes? Through Dirac measure $\delta_x$, element of the set of Radon measures $\mathcal{M}(\mathcal{X})$;
- topological dual of $\mathcal{C}_0(\mathcal{X})$ equipped with $\langle f, m \rangle = \int_{\mathcal{X}} f \, dm$. Generalises $L^1(\mathcal{X})$; $L^1(\mathcal{X}) \hookrightarrow \mathcal{M}(\mathcal{X})$;
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• Banach endowed with TV-norm: $m \in \mathcal{M}(\mathcal{X})$,

$$|m|(\mathcal{X}) \overset{\text{def}}{=} \sup \left( \int_{\mathcal{X}} f \, dm \left| \begin{array}{c} f \in C_0(\mathcal{X}), \|f\|_{\infty,\mathcal{X}} \leq 1 \end{array} \right. \right).$$

If $m = \sum_{i=1}^{N} a_i \delta_{x_i}$ a discrete measure
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If $m = \sum_{i=1}^{N} a_i \delta_{x_i}$ a discrete measure, then $|m|(\mathcal{X}) = \sum_{i=1}^{N} |a_i|$. 
A LASSO equivalent for measures

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• $\Phi : \mathcal{M}(\mathcal{X}) \to \mathbb{R}^p$ the acquisition operator, e.g. $\Phi m_{a_0,x_0} \overset{\text{def.}}{=} \sum_{i=1}^{N} a_i h(x - x_i)$;
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We call **BLASSO** the problem

[de Castro and Gamboa, 2012, Bredies and Pikkarainen, 2012] for $\lambda > 0$:

$$\arg\min_{m \in \mathcal{M}(\mathcal{X})} \frac{1}{2} \|y - \Phi m\|_{\mathbb{R}^p}^2 + \lambda |m|(\mathcal{X})$$

($\mathcal{P}_\lambda(y)$)
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\[
\begin{aligned}
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& \quad (\mathcal{P}_\lambda(y))
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One of its minimisers is a sum of Dirac, close to \( m_{a_0, x_0} \) [Duval and Peyré, 2014].
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One of its minimisers is a sum of Dirac, close to $m_{a_0,x_0}$ [Duval and Peyré, 2014].

Difficult numerical problem: infinite dimensional, non-reflexive. Tackled by greedy algorithm like **Frank-Wolfe** [Frank and Wolfe, 1956] , etc.
Some results for spikes reconstruction

Reconstruction by fluorescence microscopy SMLM: acquisition stack with few lit fluorophores per image.

**Figure 1**: Two excerpts from a SMLM stack
Results on SMLM

Stack mean
Results on SMLM

Stack mean

Off-the-grid [Laville et al., 2021]
Results on SMLM

Stack mean

Results on SMLM

Stack mean  
Off-the-grid [Laville et al., 2021]  
Deep-STORM [Nehme et al., 2018]

SMLM drawback: a lot of images, no live-cell imaging.
A new divergence regularisation
Geometry encoded in off-the-grid
<table>
<thead>
<tr>
<th>Geometry</th>
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<tbody>
<tr>
<td>Space</td>
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Geometry encoded in off-the-grid
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$\delta_x$

$\chi_E$

<sup>1</sup>[de Castro et al., 2021]
Geometry encoded in off-the-grid

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\(\delta_x\) \hspace{1cm} ? \hspace{1cm} \chi_E

\(^1\)[de Castro et al., 2021]
Desperate times call for desperate measures

- let $\mathcal{M}(\mathcal{X})^2$ be the space of vector Radon measures;
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• let $\mathcal{M}(\mathcal{X})^2$ be the space of vector Radon measures;

• let $\mathcal{V} \overset{\text{def.}}{=} \left\{ m \in \mathcal{M}(\mathcal{X})^2, \, \text{div}(m) \in \mathcal{M}(\mathcal{X}) \right\}$ the space of charges, or divergence vector fields. It is a Banach equipped with $\|\cdot\|_\mathcal{V} \overset{\text{def.}}{=} \|\cdot\|_{\text{TV}}^2 + \|\text{div}(\cdot)\|_{\text{TV}}$;
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- let $\gamma : [0, 1] \to \mathbb{R}^d$ a 1-rectifiable parametrised Lipschitz curve, we say that $\mu_\gamma \in \mathcal{V}$ is a measure supported on a curve $\gamma$ if:

$$\forall g \in C_0(\mathcal{X})^2, \quad \langle \mu_\gamma, g \rangle_{\mathcal{M}^2} \overset{\text{def.}}{=} \int_0^1 g(\gamma(t)) \cdot \dot{\gamma}(t) \, dt.$$
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• $\text{div } \mu_{\gamma} = \delta_{\gamma(0)} - \delta_{\gamma(1)}$. 

Consider the variational problem we coined *Curves Represented On Charges*: 

$$\text{argmin}_{m \in \mathcal{Y}} \frac{1}{2} \| y - \Phi m \|_H^2 + \alpha \| m \|_Y. \quad (1)$$

- $\| y - \Phi m \|_H^2$ is the data-term;
- $\| m \|_TV$ weights down the curve length, i.e. $\| \mu_{\gamma} \|_TV = H_1(\gamma((0;1)))$;
- $\| \text{div} m \|_TV$ is the (open) curve counting term.
CROC energy

Consider the variational problem we coined *Curves Represented On Charges*:

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\underset{m \in V}{\text{argmin}} \, \frac{1}{2} \| y - \Phi m \|^2_H + \alpha (\| m \|_{TV^2} + \| \text{div} \, m \|_{TV})
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\]

Do curve measures minimise (CROC)?
Extremepoints

Definition
Let $X$ be a topological vector space and $K \subset X$. An extremepoint $x$ of $K$ is a point such that $\forall y, z \in K$:

$\forall \lambda \in (0; 1); x = \lambda y + (1 - \lambda) z \Rightarrow x = y = z$

$\text{Ext}_K$ is the set of extremepoints of $K$. 

12
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Extreme points

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$\text{Ext } K$ is the set of extreme points of $K$. 

Ext $K$ in red
Let $F : E \to \mathbb{R}^m$, $G$ the data-term, $R$ the regulariser, $\alpha > 0$. 

$$F = G + \alpha R$$
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$B_1^E$ is the unit-ball of $R$: $B_1^E \overset{\text{def.}}{=} \{ u \in E \mid R(u) \leq 1 \}$. 
Link with extreme points: the representer theorem

Let $F : E \rightarrow \mathbb{R}^m$, $G$ the data-term, $R$ the regulariser, $\alpha > 0$.

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**Theorem (from [Boyer et al., 2019, Bredies and Carioni, 2019])**

There exists a minimiser of $F$ which is a linear sum of extreme points of $\text{Ext } B^1_E$. 
Link with extreme points: the representer theorem

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**Theorem (from [Boyer et al., 2019, Bredies and Carioni, 2019])**

*There exists a minimiser of $F$ which is a linear sum of extreme points of $\text{Ext } B^1_E$*

Characterise $\text{Ext } B^1_E$ of the regulariser $\iff$ outline the structure of a *minimum* of $F$. 
• If $E = \mathcal{M}(\mathcal{X})$ and $R = \|\cdot\|_{TV}$, then:

\[ \text{Ext}(\mathcal{B}(\mathcal{M}(\mathcal{X}))) = \{x; x \in \mathcal{X}\} \]

• If $E = \mathcal{BV}(\mathcal{X})$ and $R = \|\cdot\|_{BV}$, then:

\[ \text{Ext}(\mathcal{B}(\mathcal{BV}(\mathcal{X}))) = \text{Per}(E); E \subset \mathcal{X} \text{ is simple} \]

• If $E = \mathcal{V}$ and $R = \|\cdot\|_{V}$, then:

\[ \text{Ext}(\mathcal{B}(\mathcal{V})) = ? \]
Extreme points in measure spaces

- If $E = \mathcal{M}(\mathcal{X})$ and $R = \|\cdot\|_{TV}$, then:

$$\text{Ext}(\mathcal{B}_\mathcal{M}) = \{\delta_x, x \in \mathcal{X}\}.$$
Extreme points in measure spaces

- If $E = \mathcal{M}(\mathcal{X})$ and $R = \|\cdot\|_{TV}$, then:
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  \]

- If $E = \mathcal{BV}(\mathcal{X})$ and $R = \|\cdot\|_{BV}$, then:
  \[
  \text{Ext}(\mathcal{B}_\mathcal{BV}) = \left\{\frac{1}{\text{Per}(E)} \chi_E, E \subset \mathcal{X} \text{ is simple}\right\}.
  \]
Extreme points in measure spaces

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  \]

- If $E = \mathcal{V}$ and $R = \|\cdot\|_\mathcal{V}$, then:
  \[
  \text{Ext}(\mathcal{B}_\mathcal{V}) = ?
  \]
Main result

Let the (non-complete) set of curve measures endowed with weak-* topology:

$$\mathcal{G} \overset{\text{def.}}{=} \left\{ \frac{\mu_\gamma}{\|\mu_\gamma\|_\gamma}, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$
Main result

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**Theorem (Main result of [Laville et al., 2023b])**

Let $B^1_\mathcal{V} \overset{\text{def.}}{=} \{ m \in \mathcal{V}, \| m \|_\mathcal{V} \leq 1 \}$ the unit ball of the $\mathcal{V}$-norm.
Let the (non-complete) set of curve measures endowed with weak-\(^\ast\) topology:

\[
\mathcal{G} \overset{\text{def.}}{=} \left\{ \frac{\mu_{\gamma}}{\| \mu_{\gamma} \|_{\mathcal{Y}}} \right\}, \; \gamma \text{ Lipschitz } 1\text{-rectifiable simple curve}
\]

**Theorem (Main result of [Laville et al., 2023b])**

Let \( B_{\mathcal{Y}}^1 \overset{\text{def.}}{=} \{ m \in \mathcal{Y}, \| m \|_{\mathcal{Y}} \leq 1 \} \) the unit ball of the \( \mathcal{Y} \)-norm. Then,

\[
\text{Ext}(B_{\mathcal{Y}}^1) = \mathcal{G}.
\]
Off-the-grid numerical reconstruction
• No Hilbertian structure on measure spaces: no proximal algorithm;

General setup in off-the-grid
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- we use the Frank-Wolfe algorithm, designed to minimise a differentiable functional on a weakly compact set;
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⇒ perfect with our latter results!
General setup in off-the-grid

- No Hilbertian structure on measure spaces: no proximal algorithm;
- we use the Frank-Wolfe algorithm, designed to minimise a differentiable functional on a weakly compact set;
- it recovers the solution by iteratively adding and optimising extreme points of the regulariser.

→ perfect with our latter results!

We present the Charge Sliding Frank-Wolfe algorithm.
Synthetic problem

Figure 2: The source and its noisy acquired image
Figure 2: The source and its noisy acquired image $I$
Acquisition process and certificate

- a possible choice consists in setting $\Phi = \ast \nabla h$ since:

![Figure 3: The certificate $\mid$ on the left, $u$ on the right.](image-url)
• a possible choice consists in setting $\Phi = \ast \nabla h$ since:
  • $\mu_\gamma$ is vector, hence we need vector datum $y = \text{like the gradient}$;
• a possible choice consists in setting $\Phi = \star \nabla h$ since:
  • $\mu_\gamma$ is vector, hence we need vector datum $y = \text{like the gradient}$;
  • let $u$ be the support of the curve, then we feel that:

$$\eta = \Phi^*(\Phi m - y) \approx \Delta u$$

*Figure 3:* The certificate $|\eta|$ on the left, $u$ on the right.
Acquisition process and certificate

- A possible choice consists in setting $\Phi = \ast \nabla h$ since:
  - $\mu_\gamma$ is vector, hence we need vector datum $y = \nabla I$ like the gradient;
  - let $u$ be the support of the curve, then we feel that:

$$\eta = \Phi^* (\Phi m - y) \simeq \Delta u$$

![Figure 3](image_url)

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Figure 3: The certificate $|\eta|$ on the left, $u$ on the right.
Final results

Reconstruction [Laville et al., 2023a].
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Another discretisation

- polygonal works well, **under peculiar circumstances**;
Another discretisation

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Another discretisation

- polygonal works well, **under peculiar circumstances**;
- Bézier curves holds nice regularity properties, encodes a curve with few control points
- Pro: always smooth curves. Cons: prone to shortening.
Conclusion
<table>
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<th>Recap</th>
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<td>• Charge Sliding Frank-Wofe, an algorithm designed to recover off-the-grid curves in inverse problem;</td>
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Partial conclusion

Recap

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### Recap

- Charge Sliding Frank-Woge, an algorithm designed to recover off-the-grid curves in inverse problem;
- struggles with the *vector* operator definition;
- discretisation insights.
Partial conclusion

Recap

- Charge Sliding Frank-Wofe, an algorithm designed to recover off-the-grid curves in inverse problem;
- struggles with the vector operator definition;
- discretisation insights.

Still, there is room for improvements:

- define a scalar operator, further enabling curve reconstruction in fluctuation microscopy;
- improve the support estimation step;
- tackle the curve crossing issue.


Towards off-the-grid algorithms for total variation regularized inverse problems.
In Lecture Notes in Computer Science, pages 553–564. Springer International Publishing.


Exact reconstruction using beurling minimal extrapolation.


Exact support recovery for sparse spikes deconvolution.
**An algorithm for quadratic programming.**  

Laville, B., Blanc-Féraud, L., and Aubert, G. (2023a).  
**Off-the-grid charge algorithm for curve reconstruction in inverse problems.**  

Laville, B., Blanc-Féraud, L., and Aubert, G. (2023b).  
**Off-the-grid curve reconstruction through divergence regularization: An extreme point result.**  
*SIAM Journal on Imaging Sciences*, 16(2):867–885.

Recap: iterate the algorithm

Figure 4: First step of first iteration: certificate and support of new curve estimated
Recap: iterate the algorithm

\[ \Gamma = \gamma([0, 1]) \]

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Figure 4: First step of first iteration: certificate and support of new curve estimated
Recap: iterate the algorithm

Amplitude optimisation

Figure 4: First iteration: second and third steps
Recap: iterate the algorithm

**Amplitude optimisation**

**Both amplitude and position optimisation**

**Figure 4:** First iteration: second and third steps
Recap: iterate the algorithm

**Figure 4:** Second iteration: another curve is found
Recap: iterate the algorithm

**Figure 4:** Second iteration: another curve is found