Reconstruction de courbes en super-résolution sans-grille

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30th August 2023

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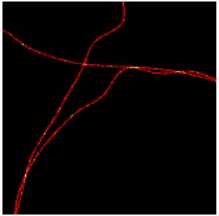

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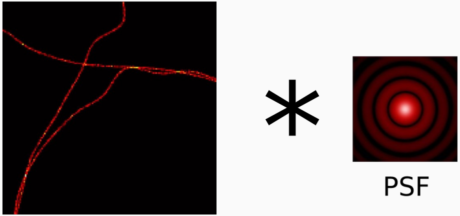

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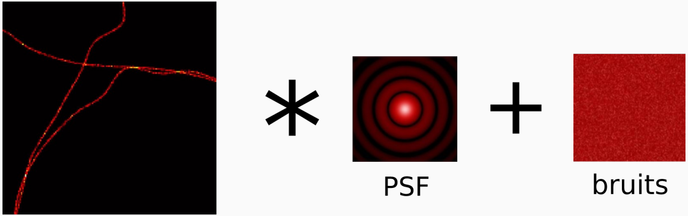

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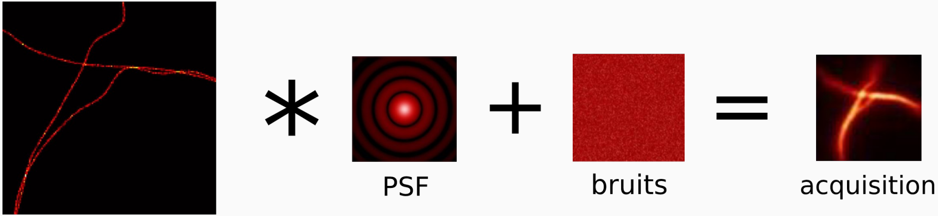

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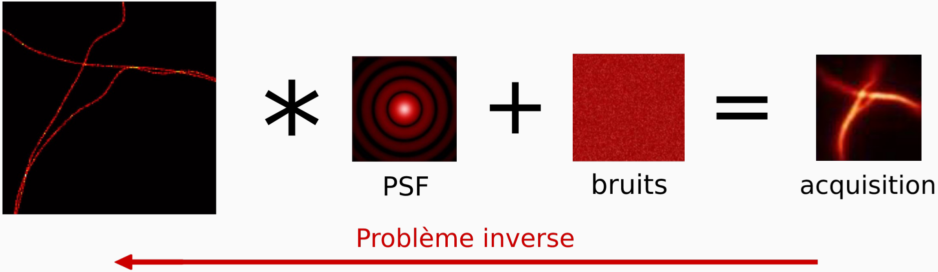
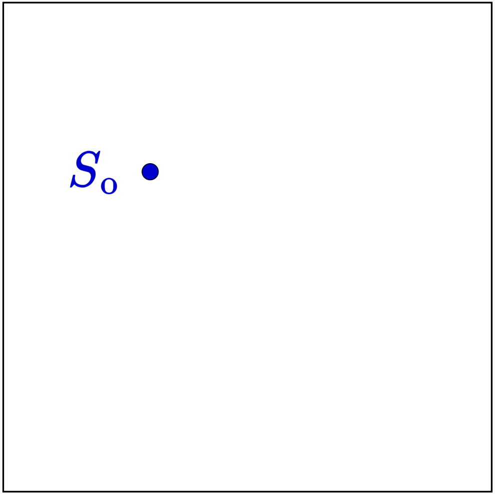


Grid or gridless?



Source to estimate

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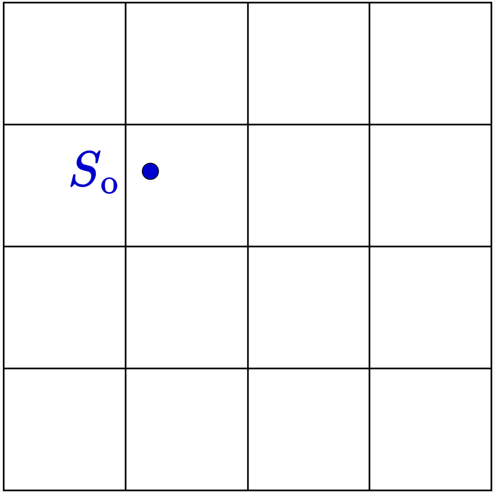
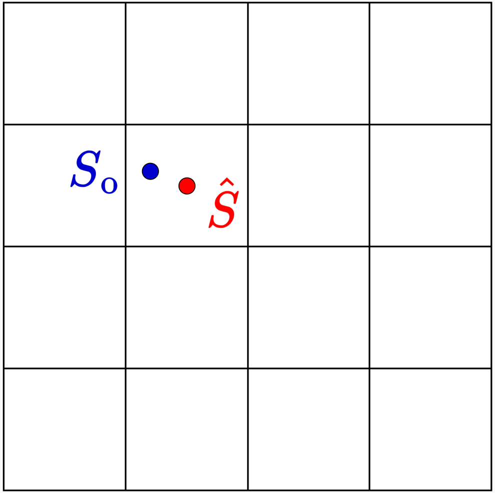


Introducing a grid

Grid or gridless?



Reconstruction $\hat{S}$ on a grid

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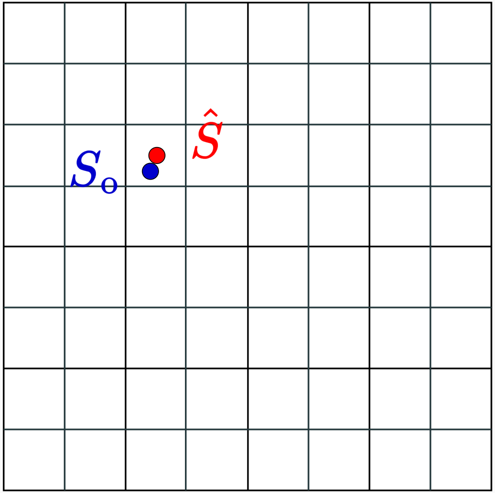

Reconstruction $\hat{S}$ on a finer grid

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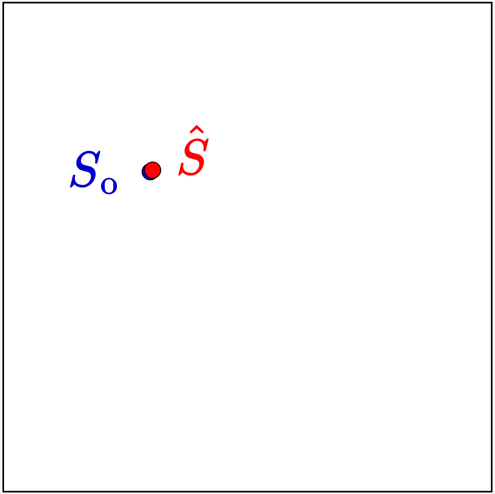

Reconstruction $\hat{S}$ is now **off-the-grid**

Off-the-grid spikes

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Difficult numerical problem: infinite dimensional, non-reflexive. Tackled by greedy algorithm like *Frank-Wolfe* [Frank and Wolfe, 1956] , *etc*.

Some results for spikes reconstruction

Reconstruction by fluorescence microscopy SMLM: acquisition stack with few lit fluorophores per image.

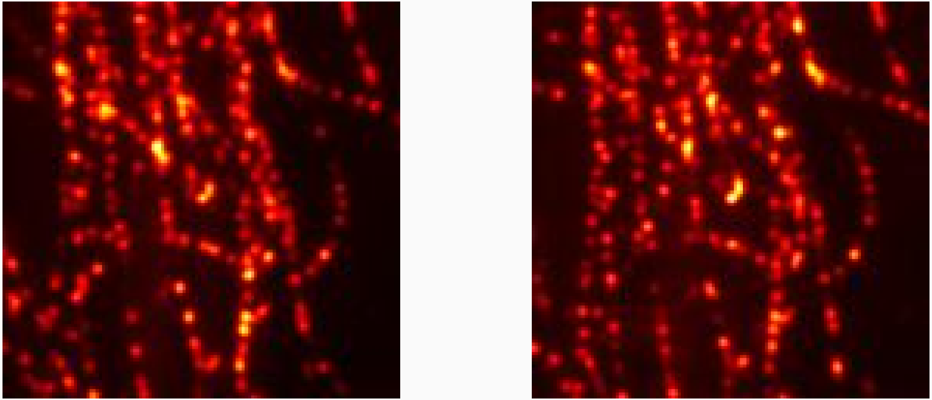
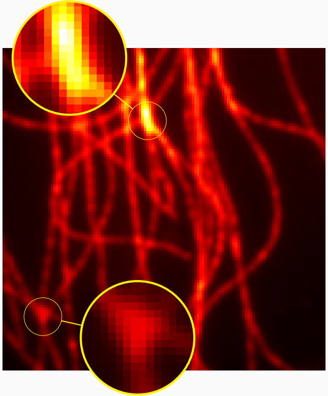


Figure 1: Two excerpts from a SMLM stack

Results on SMLM



Stack mean

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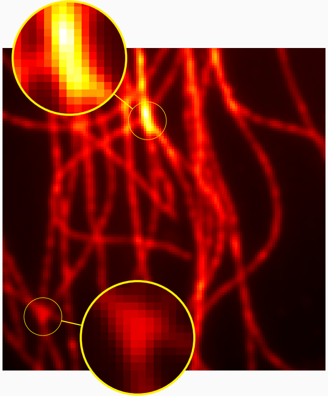

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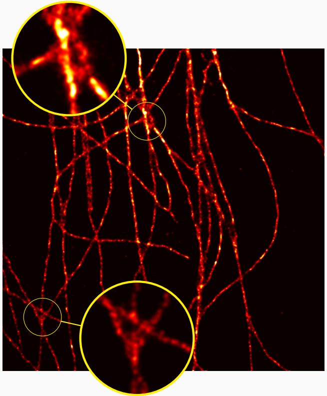

Off-the-grid [Laville et al., 2021]

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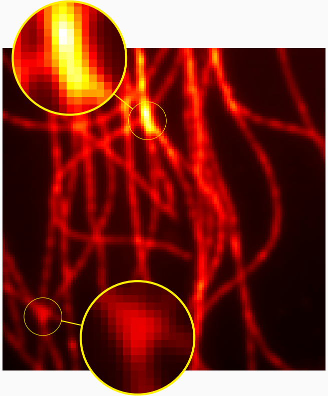

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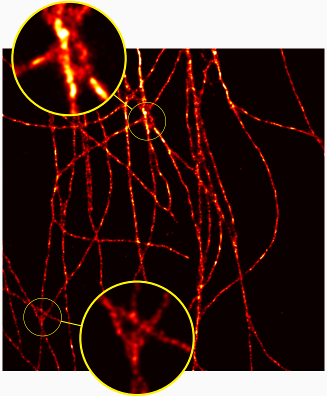

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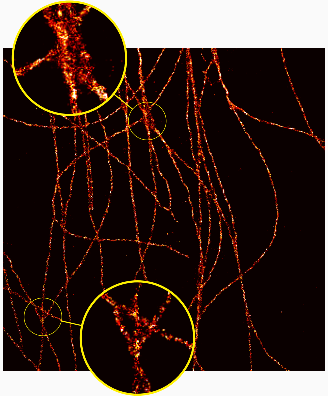

Deep-STORM [Nehme et al., 2018]

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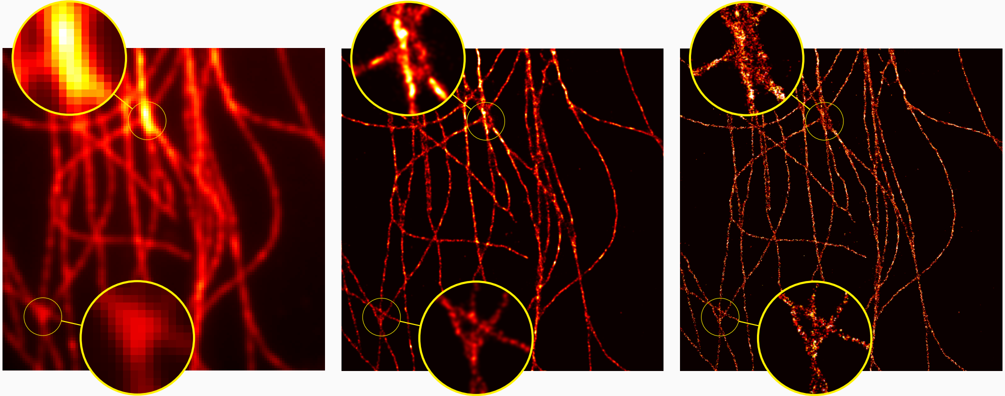

Stack mean — Off-the-grid [Laville et al., 2021] — Deep-STORM [Nehme et al., 2018]

SMLM drawback: a lot of images, no live-cell imaging.

A new divergence regularisation

Geometry encoded in off-the-grid

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	0D
Geometry	Spikes
Space	$\mathcal{M}(\mathcal{X})$
Regulariser	$\|\cdot\|_{\mathrm{TV}}$

$$\delta_x$$

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δ_x

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- $\operatorname{div} \boldsymbol{\mu}_\gamma = \delta_{\gamma(0)} - \delta_{\gamma(1)}$.

CROC energy

Consider the variational problem we coined *Curves Represented On Charges*:

$$\underset{\boldsymbol{m} \in \mathscr{V}}{\operatorname{argmin}} \frac{1}{2} \| y - \Phi \boldsymbol{m} \|_{\mathcal{H}}^2 + \alpha \| \boldsymbol{m} \|_{\mathscr{V}}. \tag{CROC}$$

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Do curve measures minimise (CROC)?

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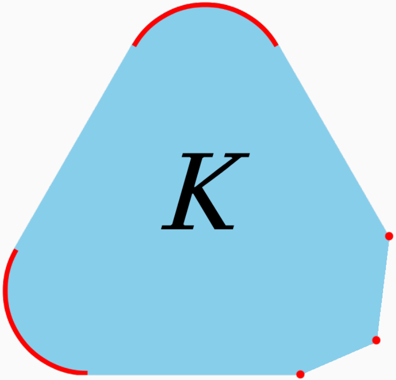
$\operatorname{Ext} K$ is the set of extreme points of K.



$\operatorname{Ext} K$ in red

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Characterise $\operatorname{Ext} \mathcal{B}_E^1$ of the regulariser $\iff$ outline the structure of a *minimum* of F.

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Main result

Let the (non-complete) set of curve measures endowed with weak-$*$ topology:

$$\mathfrak{G} \stackrel{\text{def.}}{=} \left\{ \frac{\boldsymbol{\mu}_{\boldsymbol{\gamma}}}{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathscr{V}}}, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$

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Off-the-grid numerical reconstruction

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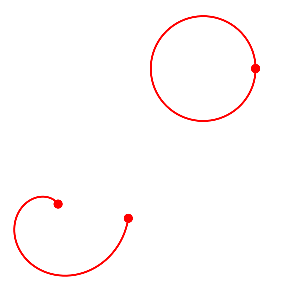
We present the *Charge Sliding Frank-Wolfe* algorithm.

Synthetic problem



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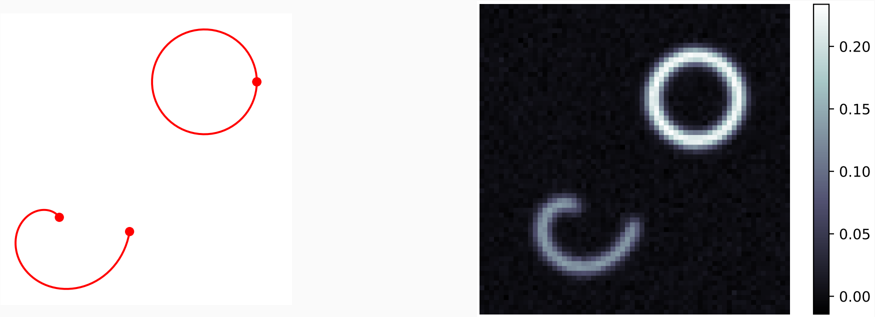

Figure 2: The source and its noisy acquired image I

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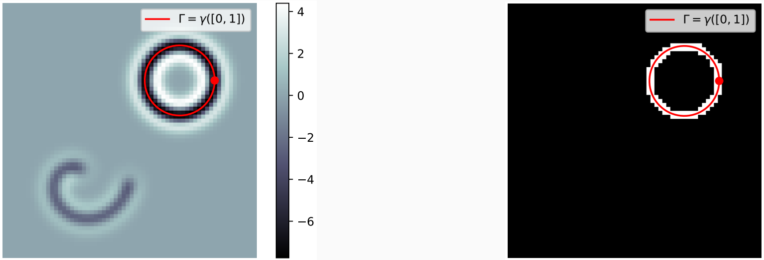

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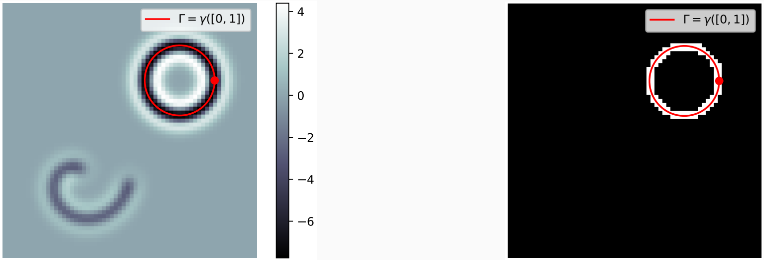

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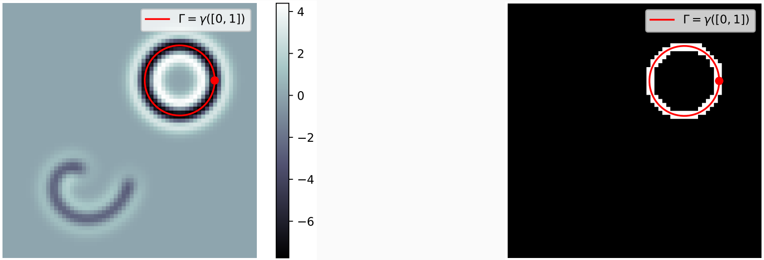

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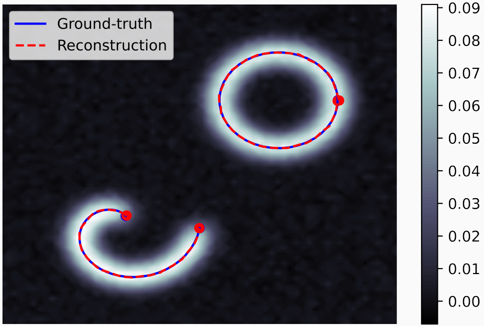

Reconstruction [Laville et al., 2023a].

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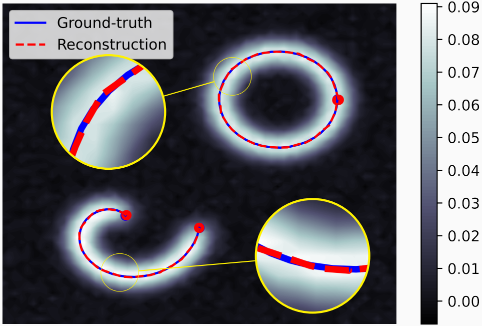

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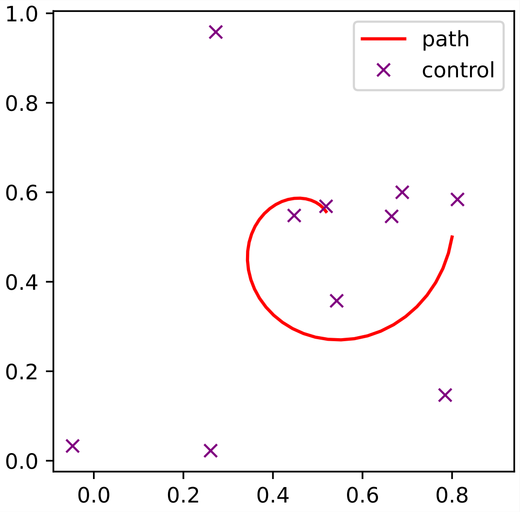

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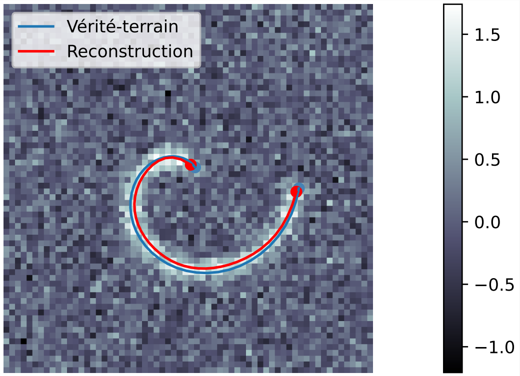
- Bézier curves holds nice regularity properties, encodes a curve with few control points
- Pro: always smooth curves. Cons: prone to shortening.



Conclusion

Partial conclusion

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- struggles with the *vector* operator definition;
- discretisation insights.

Still, there is room for improvements:

- define a *scalar* operator, further enabling curve reconstruction in fluctuation microscopy;
- improve the support estimation step;
- tackle the curve crossing issue.

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Recap: iterate the algorithm

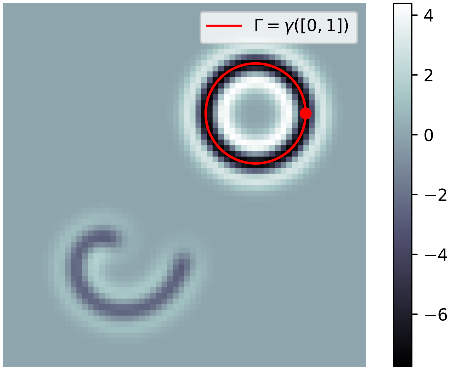

Figure 4: First step of first iteration: certificate and support of new curve estimated

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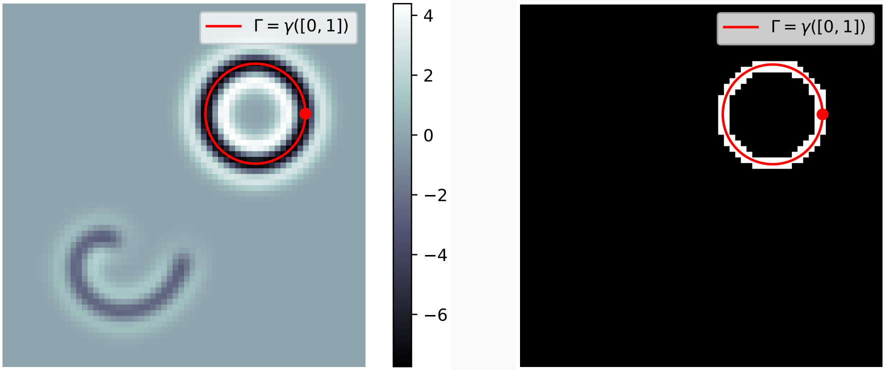

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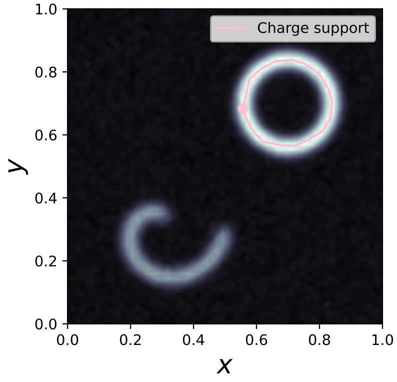

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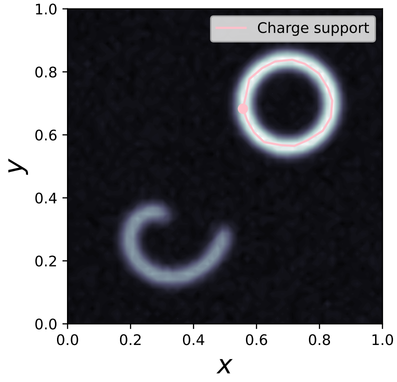

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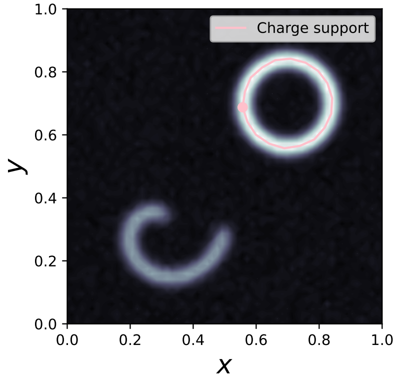

Both amplitude and position optimisation

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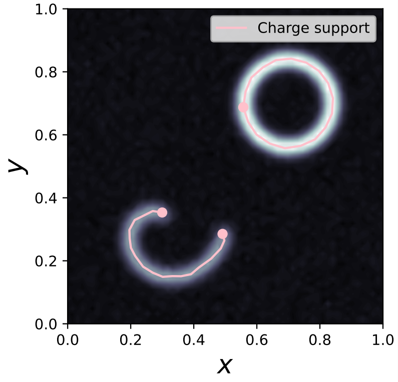

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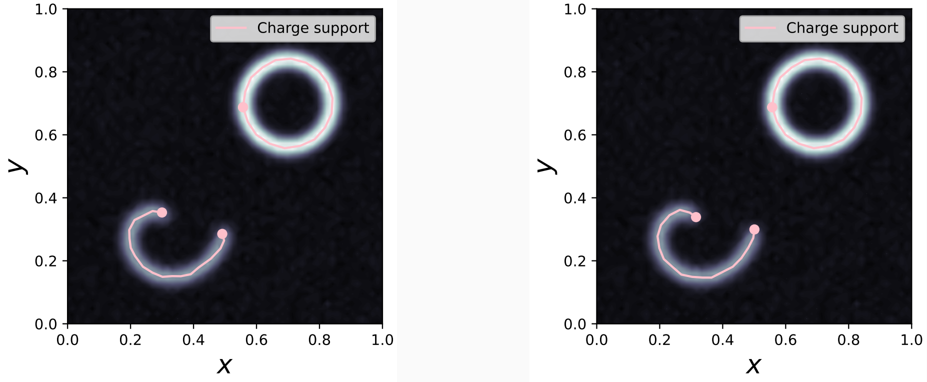

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