Some developments on off-the-grid curve reconstruction: divergence regularisation, untangling by Riemannian metric

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Institut Mathématiques de Bordeaux

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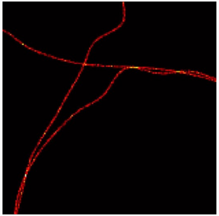

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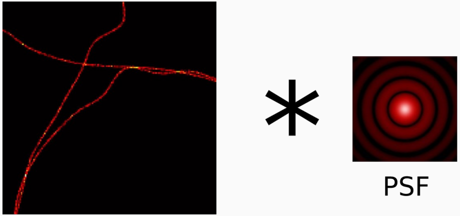

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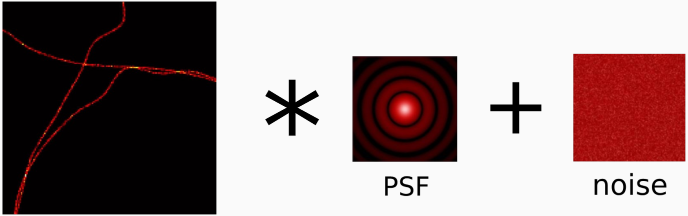

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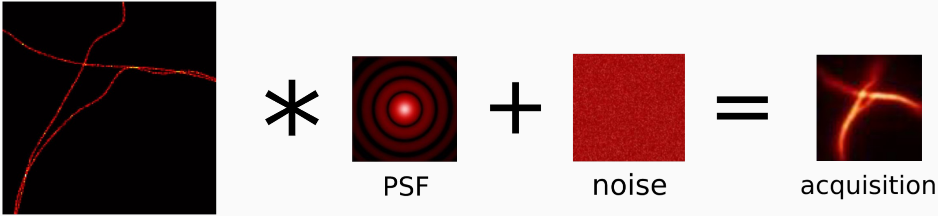

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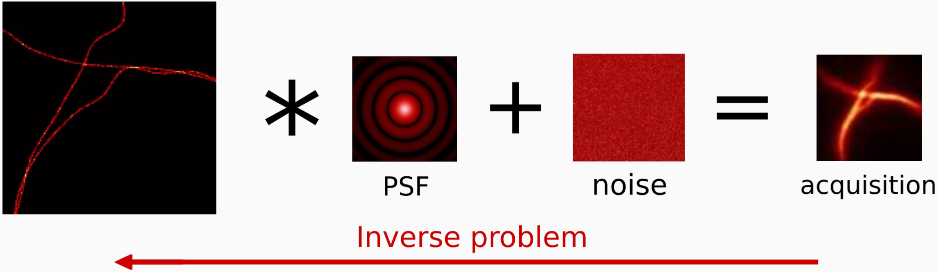

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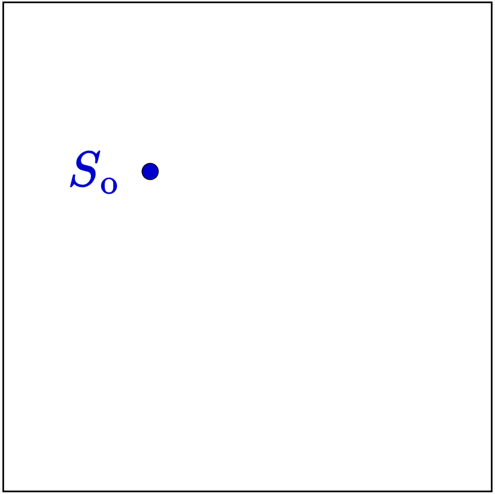

Source to estimate

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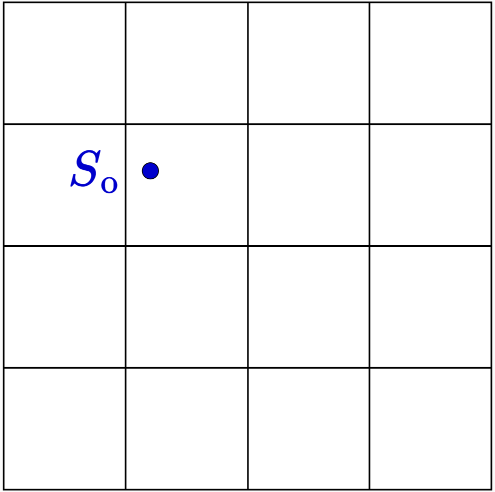

Introducing a grid

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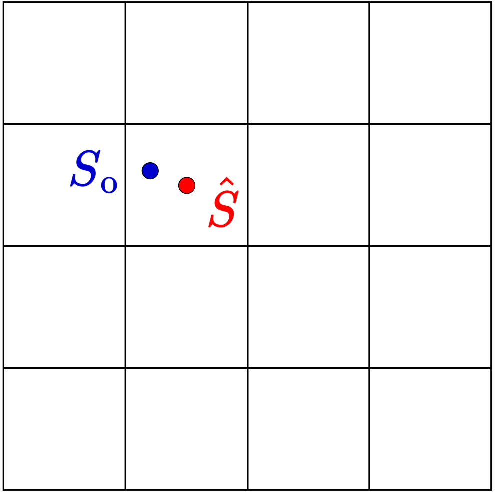

Reconstruction $\hat{S}$ on a grid

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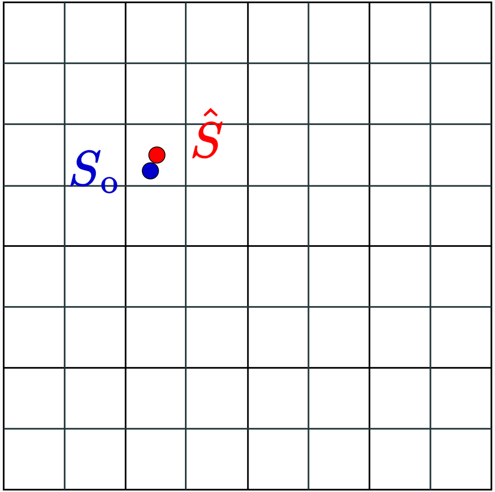

Reconstruction $\hat{S}$ on a finer grid

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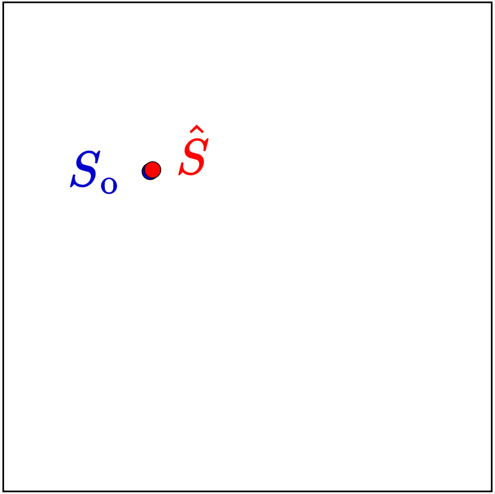

Reconstruction $\hat{S}$ is now **off-the-grid**

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- $R(S)$ regularises the problem (well-posed) and enforces more or less the prior on S w. $\alpha > 0$.

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Difficult numerical problem: infinite dimensional, non-reflexive. Tackled by greedy algorithm like *Frank-Wolfe* [Frank and Wolfe, 1956] , *etc*.

Some results for spikes reconstruction

Reconstruction by fluorescence microscopy SMLM: acquisition stack with few lit fluorophores per image.

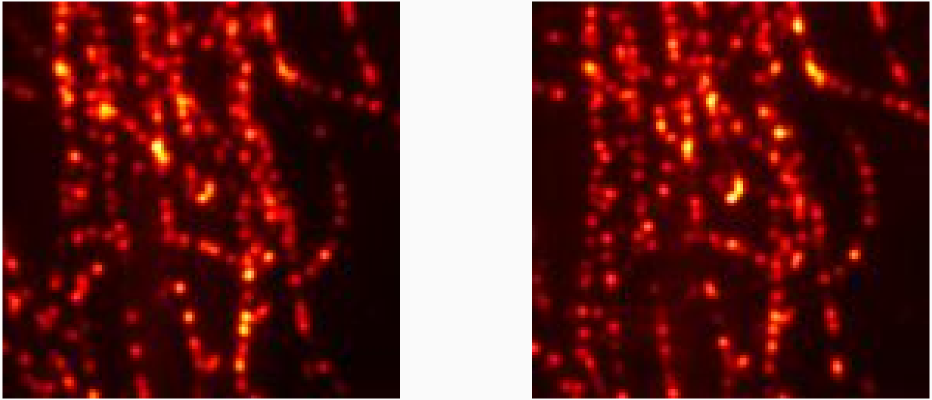
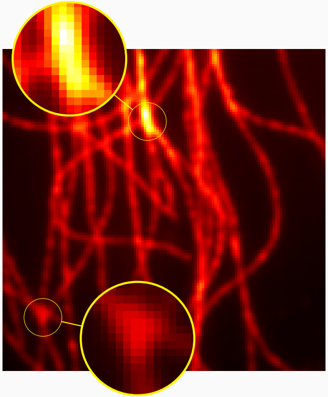


Figure 1: Two excerpts from a SMLM stack

Results on SMLM



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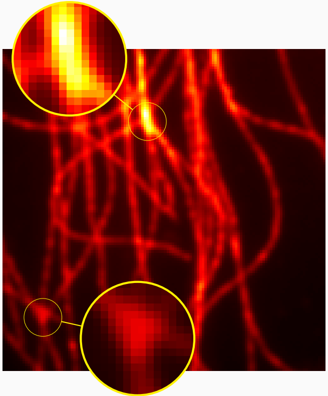

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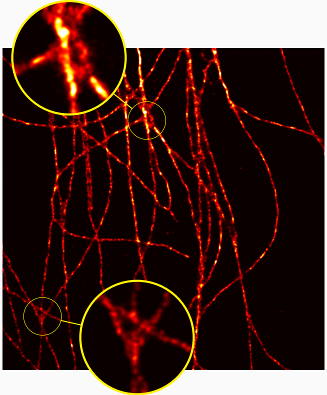

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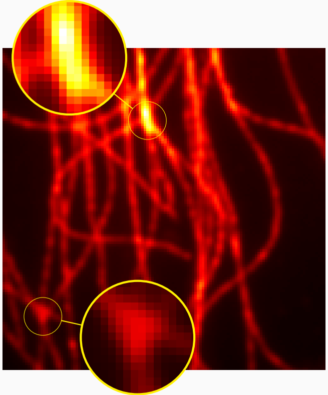

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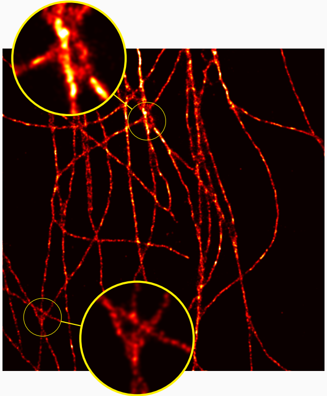

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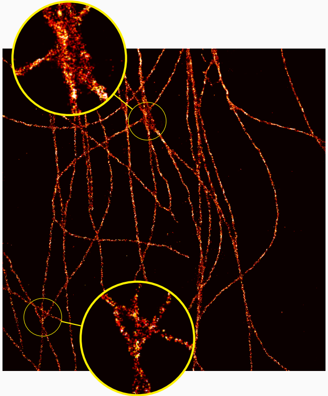

Deep-STORM [Nehme et al., 2018]

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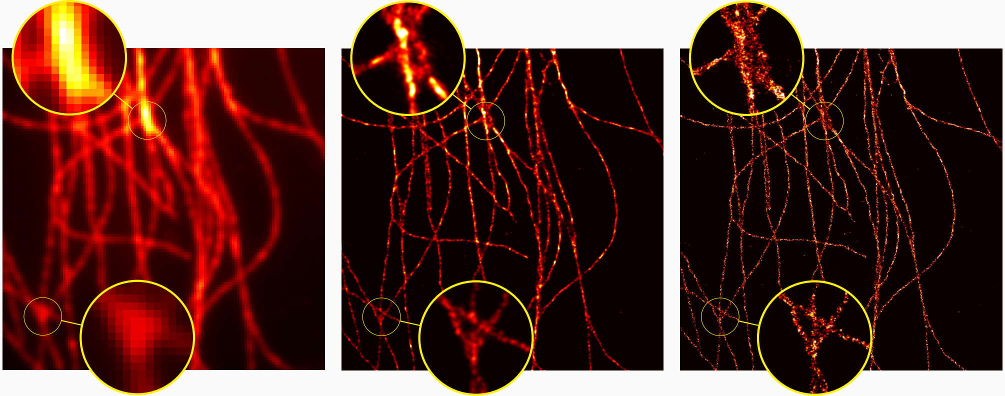

Stack mean | Off-the-grid [Laville et al., 2021] | Deep-STORM [Nehme et al., 2018]

SMLM drawback: a lot of images, no live-cell imaging.

Static off-the-grid curve in $\mathscr{V}$

Bastien Laville

Laure Blanc-Féraud

Gilles Aubert

Related papers

- *Off-the-grid curve reconstruction through divergence regularisation: an extreme point result.* **SIAM Journal on Imaging Sciences (SIIMS), June 2023.**
- *Off-the-grid charge algorithm for curve reconstruction in inverse problems.* **In Springer Lecture Notes in Computer Science 14009, May 2023.**

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$$\delta_x$$

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δ_x

χ_E

Geometry encoded in off-the-grid

	0D	1D	2D
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δ_x ? χ_E

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- let $\gamma : [0,1] \to \mathbb{R}^2$ a 1-rectifiable parametrised Lipschitz curve, we say that $\boldsymbol{\mu}_\gamma \in \mathscr{V}$ *is a measure* ***supported on a curve*** γ if:

$$\forall \boldsymbol{g} \in \mathbf{C_0}(\mathcal{X})^{\mathbf{2}}, \quad \langle \boldsymbol{\mu}_\gamma, \boldsymbol{g} \rangle_{\boldsymbol{\mathcal{M}}^{\mathbf{2}}} \overset{\text{def.}}{=} \int_0^1 \boldsymbol{g}(\gamma(t)) \cdot \dot{\gamma}(t)\, \mathrm{d}t.$$

- a curve is closed is $\gamma(0) = \gamma(1)$, open otherwise;

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Do curve measures minimise (CROC)?

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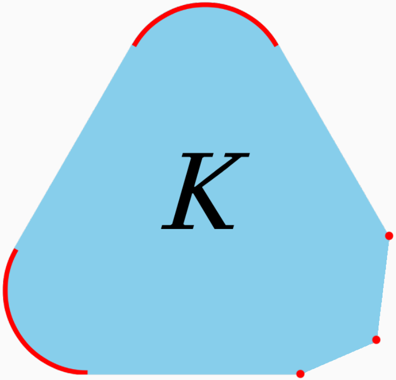
$\operatorname{Ext} K$ is the set of extreme points of K.



$\operatorname{Ext} K$ in red

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Let $F : E \to \mathbb{R}^p$, G the data-term, R the regulariser, $\alpha > 0$.

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Characterise $\mathrm{Ext}\, \mathcal{B}_E^1$ of the regulariser $\iff$ outline the structure of a *minimum* of F.

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Main result

Let the (non-complete) set of curve measures endowed with weak-$*$ topology:

$$\mathfrak{G} \stackrel{\text{def.}}{=} \left\{ \frac{\boldsymbol{\mu}_\gamma}{\left\| \boldsymbol{\mu}_\gamma \right\|_{\mathscr{V}}}, \gamma \text{ Lipschitz 1-rectifiable simple curve} \right\}.$$

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	0D	1D	2D
Geometry	Spikes	Curves	Sets
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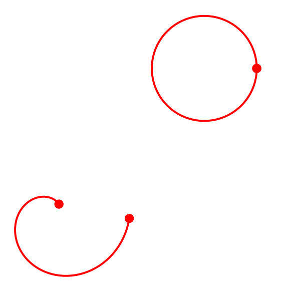
$\hookrightarrow$ perfect with our latter results!

We present the *Charge Sliding Frank-Wolfe* algorithm.

Synthetic problem



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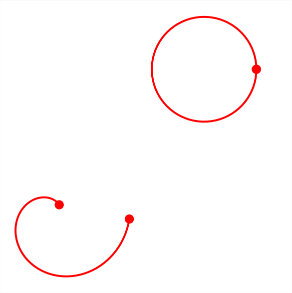
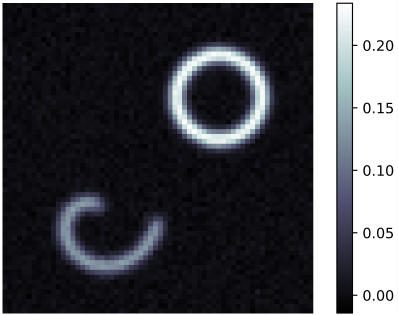

Figure 2: The source and its noisy acquired image I

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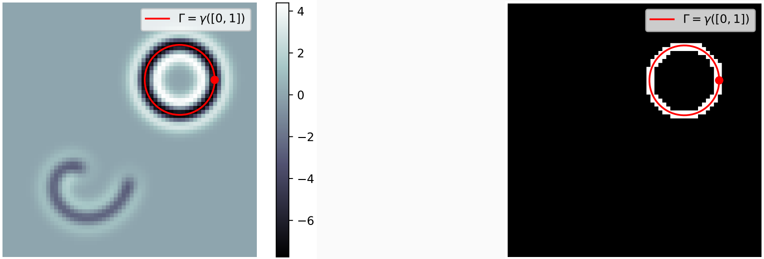

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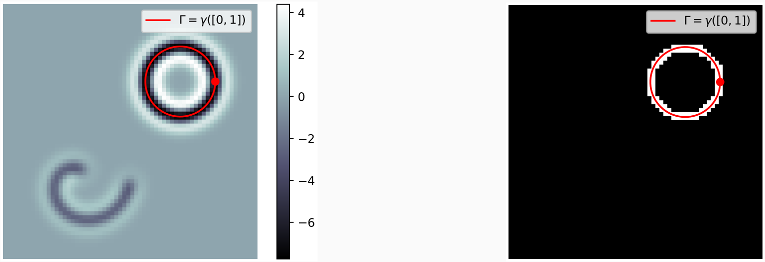

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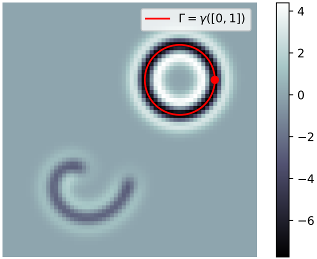

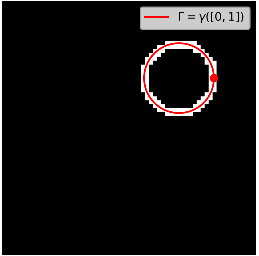

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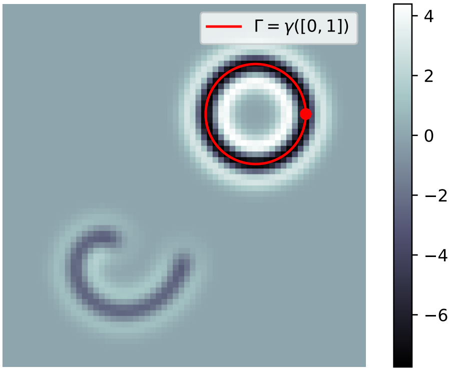

Figure 4: First step of first iteration: certificate and support of new curve estimated

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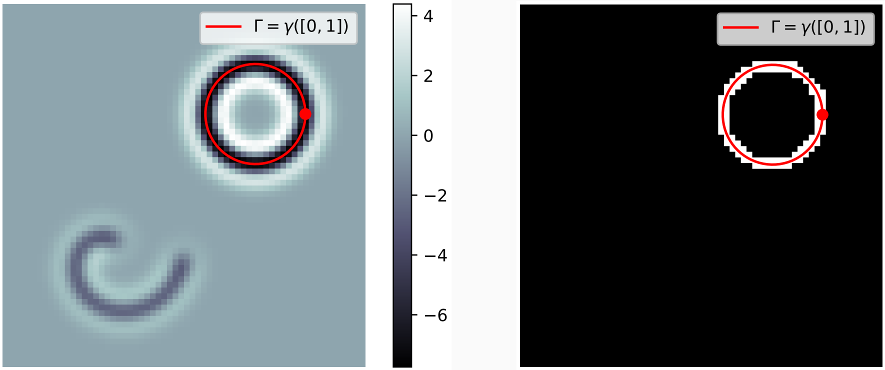

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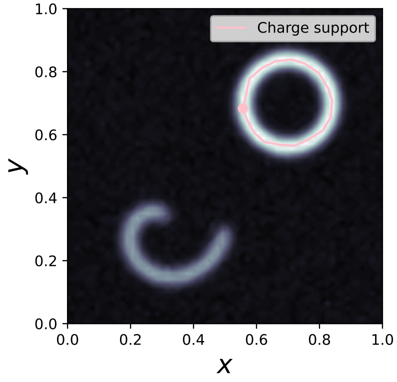

Amplitude optimisation

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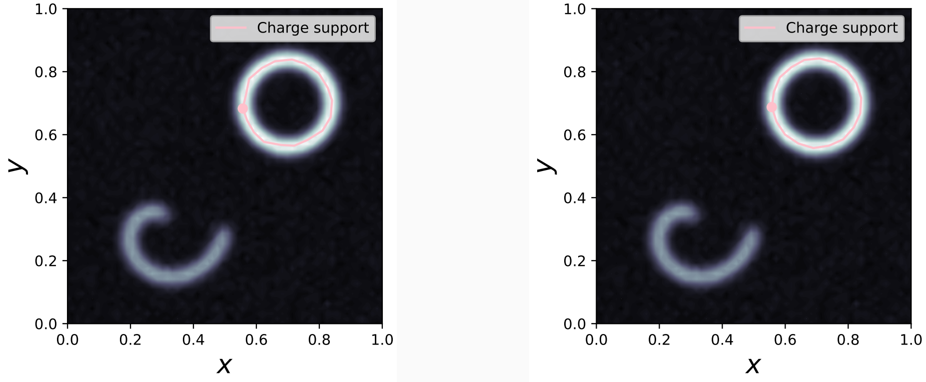

Amplitude optimisation

Both amplitude and position optimisation

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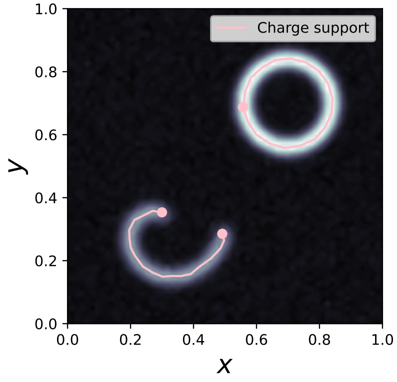

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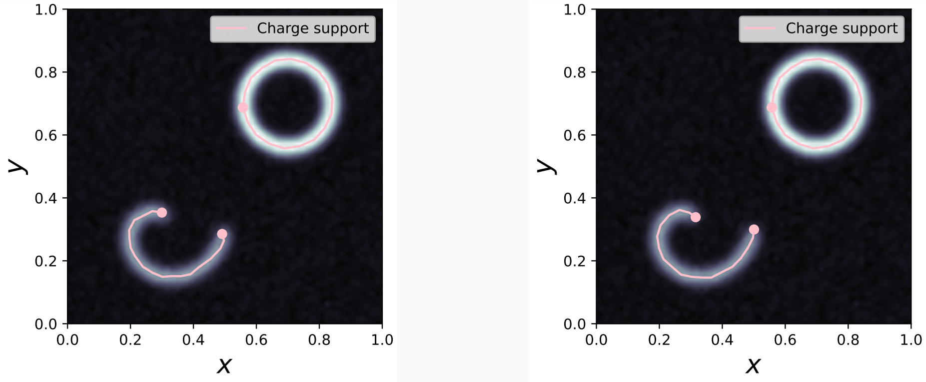

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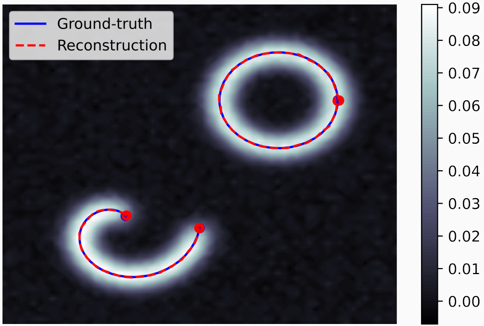

Reconstruction [Laville et al., 2023a].

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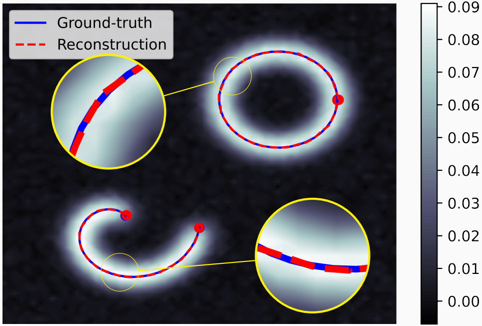

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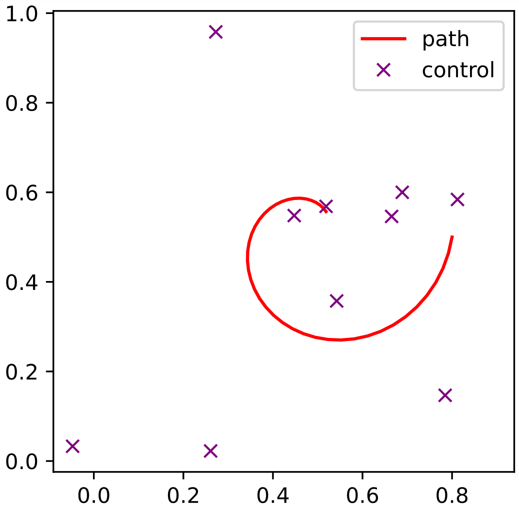

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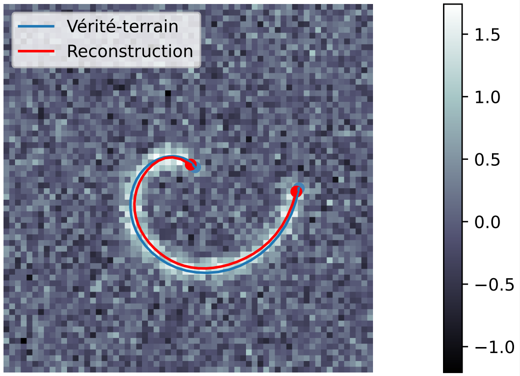
- Pro: always smooth curves. Cons: prone to shortening.



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Still, there is room for improvements:

- define a *scalar* operator, further enabling curve reconstruction in fluctuation microscopy;
- improve the support estimation step;
- tackle the curve crossing issue.

Dynamic curve untangling: lift and Riemannian regularisation

Bastien Laville

Théo Bertrand

João M. Machado

Laure B.-Féraud

Gilles Aubert

Related paper

Dynamic off-the-grid curves untangling by the Reeds-Shepp metric: theory, algorithm and a biomedical application. **Preprint, to appear 2024.**

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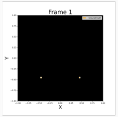

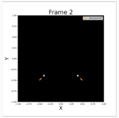

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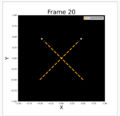

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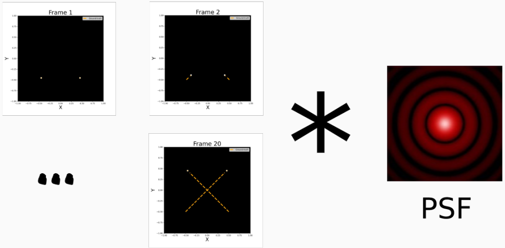

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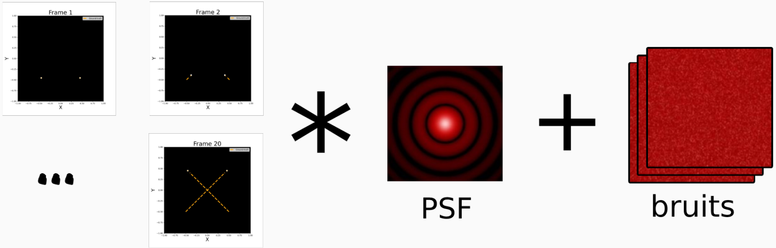

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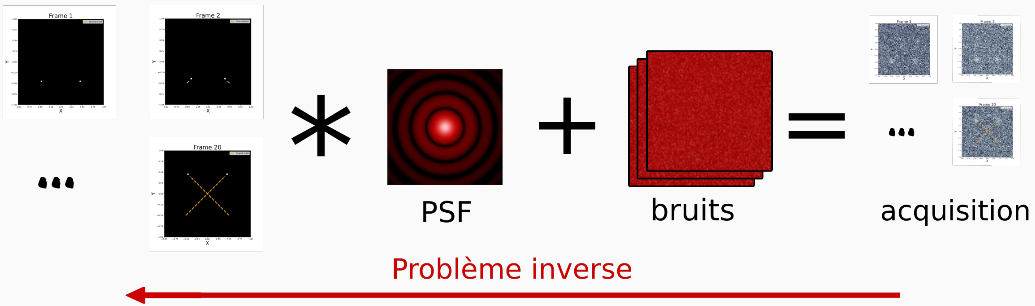

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- Let $X \stackrel{\text{def.}}{=} \mathcal{X} \times [0, T]$. How to recover the source measure $\rho \in \mathcal{M}(X)$?
- $\Gamma = \{\gamma = (h, \xi), \quad h \in C([0,1], \mathbb{R}), \quad \xi : [0,1] \to \mathcal{X}, \quad \xi_{|h \neq 0}$ continuous$\}$;
- with e_t the measurable map of evaluation at time t, $e_t(\gamma) = \gamma(t)$ i.e. $e_{t\sharp}\sigma \in \mathcal{M}(\Omega)$,

$$\underset{\sigma \in \mathcal{M}(\Gamma)}{\operatorname{argmin}} \sum_{i=1}^{T} \|y_{t_i} - \Phi e_{t_i}\sharp\sigma\|_{\mathcal{H}} + \alpha \int_0^T \underbrace{w(\gamma)}_{=\int_0^1 \|\dot{\gamma}(t)\|_g \, \mathrm{d}t} \mathrm{d}\sigma(\gamma).$$

Crossing curves may be **not** optimal in the sense we cannot infere them from the certificate.

A lift to the 'roto-translational space'

How to untangle crossing curves?

- Consider $\mathbb{S}_1 = [0, 2\pi)$ and the lifted space $\mathbb{R}^2 \times \mathbb{S}_1$ [Chambolle and Pock, 2019];
- we can separate objects with the same position but *different* local orientation.

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The separation prior is enforced by the relaxed Reeds-Shepp metric [Reeds and Shepp, 1990, Duits et al., 2018]. Let $(x, \theta) \in \mathbb{M}_2$ while $(\dot{x}, \dot{\theta}) \in T(\mathbb{M}_2)$ lies in the tangent bundle:

$$\begin{aligned} \|\dot{\gamma}(t)\|_g^2 &= \|(x, \theta)\|_g^2 \\ &= |\dot{x} \cdot e_\theta|^2 + \frac{1}{\varepsilon^2}|\dot{x} \wedge e_\theta|^2 + \xi^2|\dot{\theta}|^2. \end{aligned}$$

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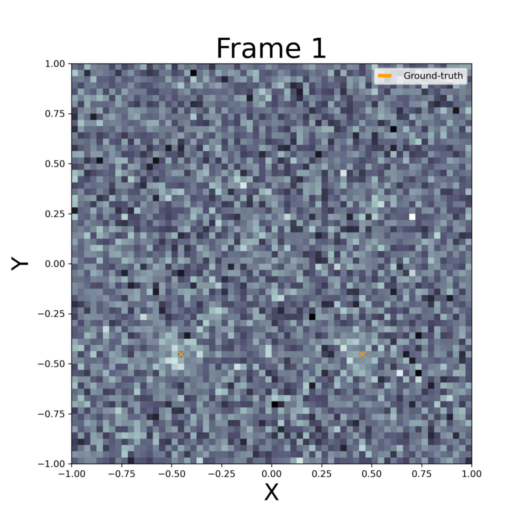
- $0 < \varepsilon < 1$ enforces the planarity of the curve,
- $\xi > 0$ penalises the local curvature.

A first example



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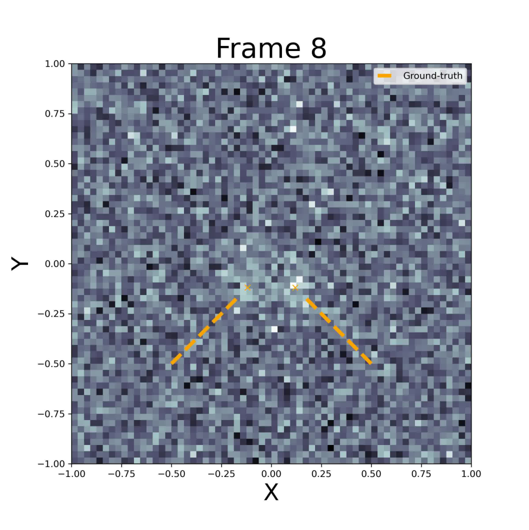

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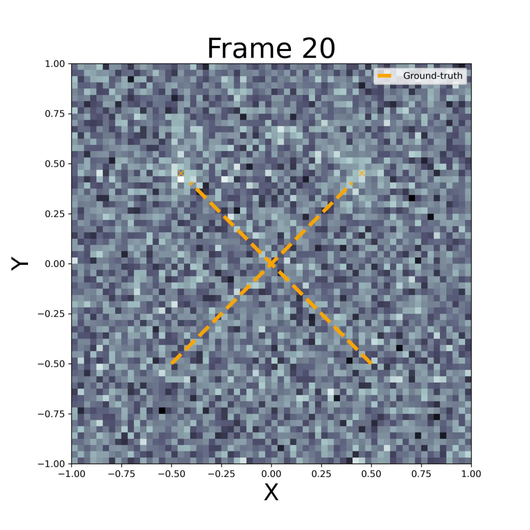

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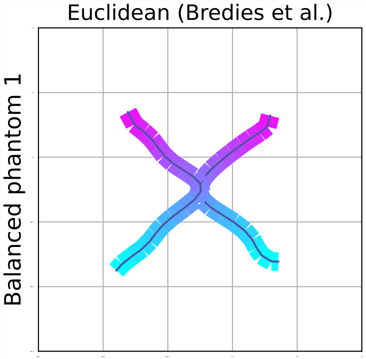

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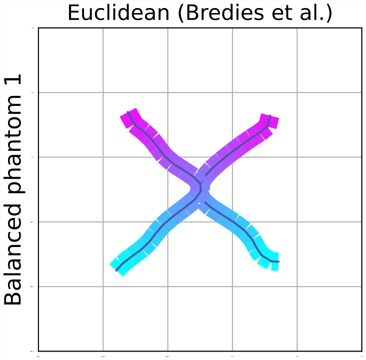

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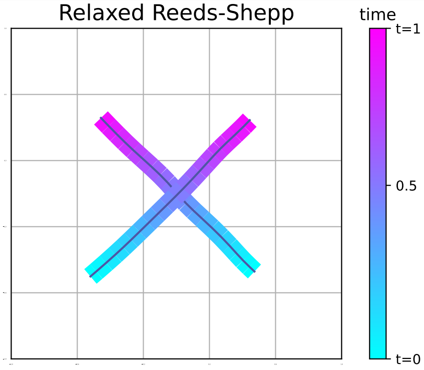
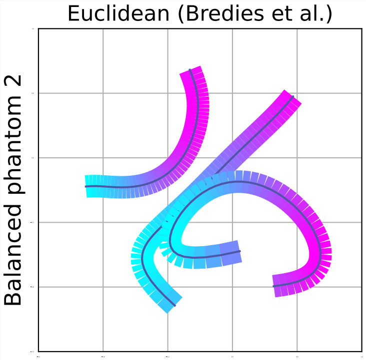


RS with $\beta = 10^{-3}$, $\varepsilon = 0.05$ and $\xi = 1$.

Other phantoms



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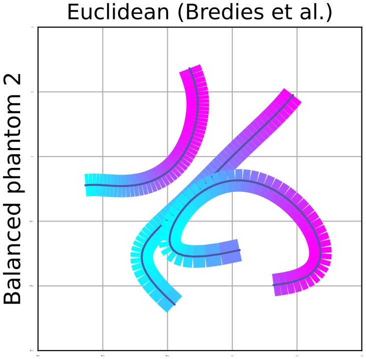

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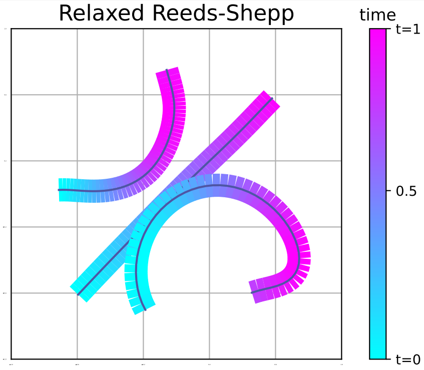

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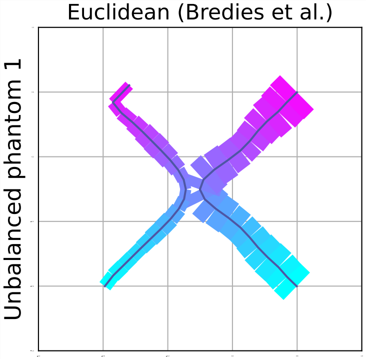

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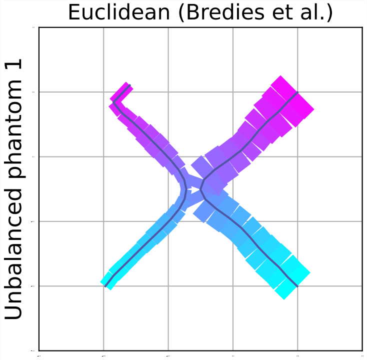

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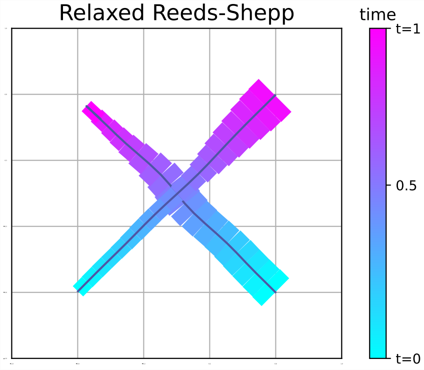

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Recap

- a roto-translational lift $\mathbb{R}^2 \times \mathbb{S}^1$ and a new metric regularisation with Reeds-Shepp metric;

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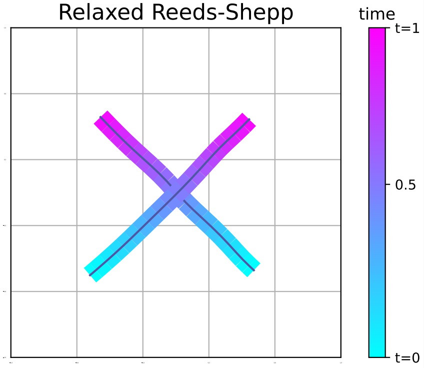
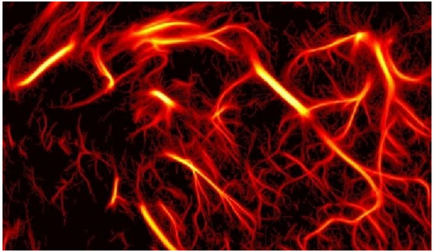


Recap

- a roto-translational lift $\mathbb{R}^2 \times \mathbb{S}^1$ and a new metric regularisation with Reeds-Shepp metric;
- Convincing first results;
- yet a work in progress: a Γ-convergence result for the more ubiquitous discretisations, test on real biological data (ULM), more lined up Riemannian optimisation, *etc.*



Ultrasound Localisation Microscopy (ULM)

Conclusion

Take home messages

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- off-the-grid methods yields compelling results (yet scarcely used by applicative researchers);
- we propose a way to bridge the gap in off-the-grid static curve reconstruction;
- we studied a new way to untangle trajectories, dynamic curve reconstruction;
- we believe there are connections between them two, improvements in the off-the-grid community may benefit both fields.

Measure v. current?

- a curve can be encoded in a *measure* $\boldsymbol{\mu}_\gamma$:

$$\langle \boldsymbol{\mu}_\gamma, \boldsymbol{f} \rangle = \int_0^1 \boldsymbol{f}(\gamma(t)) \cdot \dot{\gamma}(t) \, \mathrm{d}t.$$

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- but it can be encoded in a *current*, in the sense of *Georges de Rham*:

$$\begin{aligned} \langle T_\gamma, f \mathrm{d}\omega \rangle &= \int_0^1 f(\gamma(t)) \, \mathrm{d}\pi(\gamma(t)). \\ &= \int_0^1 f(\gamma(t)) \|\gamma(t)\|_g \mathrm{d}t. \end{aligned}$$

if $\mathrm{d}\pi$ is a 0-form, for a metric g.

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See our work and papers on

https://www-sop.inria.fr/members/Bastien.Laville/

Proof recipe I

First inclusion:

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) \supset \mathfrak{G}$$

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Let γ a simple Lipschitz curve and μ_γ the measure supported on this curve. By contradiction, let $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{B}^1_{\mathscr{V}}$ and for $\lambda \in (0,1)$:

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moreover, each *R* has *maximal* length implying $\operatorname{spt} \boldsymbol{R} = \operatorname{spt} \mu_\gamma$.

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Hence, each $\gamma_{\boldsymbol{R}}$ **is a reparametrisation of** γ **yielding** $\boldsymbol{R} = \frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathscr{V}}}$**, eventually:**

$$\boldsymbol{u}_i = \int_{\mathfrak{G}} \boldsymbol{R} \, \mathrm{d}\rho_i = \int_{\mathfrak{G}} \frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathscr{V}}} \, \mathrm{d}\rho_i = \frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathscr{V}}} \underbrace{\rho_i(\mathfrak{G})}_{=1} = \frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathscr{V}}}.$$

Contradiction, then μ_γ **is an extreme point.**

□

Proof recipe III

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$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) \subset \mathfrak{G}$$

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either ρ **is supported on a singleton of** $\mathfrak{G}$**, then there exists** μ_γ **s.t.** $\boldsymbol{T} = \dfrac{\boldsymbol{\mu}_{\boldsymbol{\gamma}}}{\|\boldsymbol{\mu}_{\boldsymbol{\gamma}}\|_{\mathscr{V}}}$

or there exists a Borel set $A \subset \mathfrak{G}$ **with arbitrary** $0 < \rho(A) < 1$ **and:**

$$\rho = |\rho|(A)\left(\frac{1}{|\rho|(A)}\rho \llcorner A\right) + |\rho|(A^c)\left(\frac{1}{|\rho|(A^c)}\rho \llcorner A^c\right).$$

Proof recipe IV

Then,

$$\boldsymbol{T} = |\rho|(A)\underbrace{\left[\int_{\mathfrak{G}} \frac{1}{|\rho|(A)} \boldsymbol{R} \,\mathrm{d}(\rho \llcorner A)(\boldsymbol{R})\right]}_{\overset{\text{def.}}{=} \boldsymbol{u_1}} + |\rho|(A^c)\underbrace{\left[\int_{\mathfrak{G}} \frac{1}{|\rho|(A^c)} \boldsymbol{R} \,\mathrm{d}(\rho \llcorner A^c)(\boldsymbol{R})\right]}_{\overset{\text{def.}}{=} \boldsymbol{u_2}}$$

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A **is chosen (up to a neighbourhood) as a convex set, hence** $\boldsymbol{u_1} = \int_A \boldsymbol{R}\,\mathrm{d}\rho(\boldsymbol{R})$ **belongs to** A**, while conversely** $\boldsymbol{u_2} \in A^c$**, thus** $\boldsymbol{u_1} \neq \boldsymbol{u_2}$**.**

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A is chosen (up to a neighbourhood) as a convex set, hence $\boldsymbol{u_1} = \int_A \boldsymbol{R}\,\mathrm{d}\rho(\boldsymbol{R})$ belongs to A, while conversely $\boldsymbol{u_2} \in A^c$, thus $\boldsymbol{u_1} \neq \boldsymbol{u_2}$. Eventually, thanks to Smirnov's decomposition:

$$\begin{aligned}\|\boldsymbol{u_1}\|_{\mathscr{V}} &\leq \int_{\mathfrak{G}} \frac{1}{|\rho|(A)}\underbrace{\|\boldsymbol{R}\|_{\mathscr{V}}}_{=1}\,\mathrm{d}(\rho \llcorner A)(\boldsymbol{R}) \\ &\leq \frac{|\rho|(A)}{|\rho|(A)} = 1.\end{aligned}$$

Proof recipe V

Then $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{B}^1_{\mathscr{V}}$ while $\boldsymbol{u_1} \neq \boldsymbol{u_2}$, thus reaching a non-trivial convex combination:

$$\boldsymbol{T} = \lambda \boldsymbol{u_1} + (1 - \lambda)\boldsymbol{u_2},$$

Proof recipe V

Then $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{B}^1_{\mathscr{Y}}$ while $\boldsymbol{u_1} \neq \boldsymbol{u_2}$, thus reaching a non-trivial convex combination:

$$\boldsymbol{T} = \lambda \boldsymbol{u_1} + (1 - \lambda) \boldsymbol{u_2},$$

thereby reaching a contradiction, and therefore concluding the proof.

$\square$