Study of multihomogeneous polynomial systems via resultant matrices

Angelos Mantzaflaris[†]

Dept of Informatics & Telecoms,
University of Athens, Greece,
`amantzaf@di.uoa.gr`

Abstract. Resultants provide conditions for the solvability of polynomial equations and allow reducing polynomial system solving to linear algebra computations. Sparse resultants depend on the *Newton polytopes* of the input equations. This polytope is the convex hull of the exponent vectors corresponding to the nonzero monomials of the equations (viewed as lattice points in the Cartesian space of dimension equal to the number of variables). In this work we consider the case of *multihomogeneous* systems, previously studied in the case where all equations share the same Newton polytope. We generalize certain constructions to *mixed* systems, whose Newton polytopes are scaled copies of one polytope, thus taking a step towards systems with arbitrary supports. First, we specify matrices whose determinant equals the resultant and characterize the systems that admit such formulae. Bézout-type determinantal formulae do not exist, but we describe all possible Sylvester-type and hybrid formulae. We establish tight bounds for the corresponding degree vectors, as well as precise domains where these concentrate; the latter are new even for the unmixed case. Second, we make use of multiplication tables and strong duality theory to specify resultant matrices explicitly, in the general case. Our public-domain MAPLE implementation includes efficient storage of complexes in memory, and construction of resultant matrices.

1 Introduction

Resultants provide efficient ways for studying and solving polynomial systems by means of their matrices. This paper considers the sparse (or toric) resultant, which exploits *a priori* knowledge on the structure of the equations. We concentrate on systems where the variables can be partitioned into groups so that every polynomial is homogeneous in each group, i.e. *mixed multihomogeneous*, or mixed multigraded, systems; their study is a first step away from the theory of homogeneous and unmixed multihomogeneous systems, towards fully exploiting arbitrary sparse structure.

Resultant matrices are matrices whose nullspace encodes the solutions of a system of polynomial equations. They reduce the problem of finding common zeros to linear algebra computations that are well studied and fast implementations are available for their computation. The entries of these matrices are (functions on the) coefficients of the polynomials. If these coefficients are indeterminates,

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[†] Supervisor: Prof. Ioannis Z. Emiris, Dept of Informatics & Telecoms, University of Athens, Greece, `emiris@di.uoa.gr`

then the determinant of the resultant matrix is an irreducible polynomial in these indeterminates which vanishes iff the system has a common root, thus providing a condition of solvability of the system.

The construction of a resultant (matrix) starts from a system of $n+1$ equations in n variables. This is typically called an over-constrained affine system, in the sense that there are more constraints (equations) than variables. Such a system is expected to have no common solutions, hence the resultant matrix is expected to be of full rank, except from the case when there is a common zero, hence equivalently the rank of the matrix will drop.

In the univariate case, a resultant matrix of two homogeneous polynomials f_0, f_1 in one variable and of arbitrary degrees, i.e. polynomials of the projective space $\mathbb{P}^1$, is given by the classic Sylvester matrix. This matrix is said to be optimal for the system, in the sense that the dimension of it's nullspace is always equal to the number of solutions. This is a desired property that is usually not satisfied in multivariate settings.

For instance, a well known generalization of this matrix to multivariate systems is Macaulay's matrix for systems in $\mathbb{P}^n$, i.e. homogeneous polynomials in $n+1$ varibles. Here every equation f_i is assumed to have all the possible terms of total degree $\leq d_i$, $i = 0, 1, \ldots, n$. The disadvantage of this matrix is exactly that it is not optimal for the system. Nevertheless, optimal matrices are known to exist only for certain classes of polynomial systems, including some which we consider in this paper.

Sparse resultant matrices are of different types. On the one end of the spectrum are the *pure Sylvester-type* matrices, filled in by polynomial coefficients; such are Sylvester's and Macaulay's matrices. On the other end are the *pure Bézout-type* matrices, filled in by coefficients of the *Bezoutian* polynomial. Hybrid matrices contain blocks of both pure types.

There is a strong relation between resultants and abstract constructions of algebraic geometry. These constructions can as well be treated computationally and lead to matrices that express the resultant. More specifically, we provide a computational treatment of Weyman complexes (defined formally later on), which yield the resultant as the determinant of a complex. For the class of systems we consider, these complexes are parametrized by a *degree vector* $\boldsymbol{m}$; when the complex has two terms, its determinant is that of a matrix expressing the map between these terms, and equals the resultant. In this case, there is a *determinantal* formula, and the corresponding vector $\boldsymbol{m}$ is *determinantal*. The resultant matrix is then said to be *exact*, or *optimal*, in the sense that there is no extraneous factor in the determinant polynomial, thus the nullspace of the matrix corresponds describes the solutions of the system and only those. As is typical in all such approaches, including this paper, the polynomial coefficients are assumed to be sufficiently generic for the resultant.

We study multivariate over-constrained systems that have a certain structure, which generalizes the homogeneous structure as follows: We consider equations where the variables $\{x_1, x_2, \ldots, x_n\}$ are partitioned into r groups of $l_1, l_2, \ldots, l_r$ variables, with $n = l_1 + \cdots + l_r$. Also, the degree of every equation in the variable groups is a multiple of a base degree $\boldsymbol{d} = (d_0, \ldots, d_r) \in \mathbb{N}^r$, i.e. $\deg f_i = s_i \boldsymbol{d}$. We call this a scaled multihomogeneous system; it's structure is fully defined by $\boldsymbol{l}, \boldsymbol{d}$ and $\boldsymbol{s} = (s_0, \ldots, s_n) \in \mathbb{N}^{n+1}$.

Example 1. The data $\boldsymbol{l} = (1,1)$, $\boldsymbol{d} = (1,1)$, $\boldsymbol{s} = (1,1,2)$ define a system of two bilinear and one biquadratic equation.

$$f_0 = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2$$
$$f_1 = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1 x_2$$
$$f_2 = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_1 x_2 + c_4 x_1{}^2 + c_5 x_1{}^2 x_2 + c_6 x_2{}^2 + c_7 x_1 x_2{}^2 + c_8 x_1{}^2 x_2{}^2$$

Here the variable groups are $\{x_1\}$, $\{x_2\}$, and are equations are in de-homogenized (affine) form. If for every group we add a homogenizing variable, the $k-$th group has $l_k + 1$ homogeneous variables: $\{x_1, y_1\}$, $\{x_2, y_2\}$ e.g. $F_0 = a_0 y_1 y_2 + a_1 x_1 y_2 + a_2 y_1 x_2 + a_3 x_1 x_2$ is the corresponding homogenized equation for f_0.

We shall illustrate the use of our MAPLE implementation, discussed in Sect. 5, on this example. First, we check that this data is determinantal, using Thm. 2:
```
> has_deter( l, d, s);
```
$$true$$
Below we apply a search for all possible determinantal vectors, by examining all vectors in the boxes of Cor. 1. The condition used here is that the dimension of K_2 and K_{-1} is zero, which is both necessary and sufficient.
```
> allDetVecs( l, d, s) ;
```
$$[[2,0,4],[0,2,4],[3,0,6],[2,1,6],[2,-1,6],[1,2,6],[1,1,6],$$
$$[1,0,6],[0,3,6],[0,1,6],[-1,2,6],[3,1,8],[1,3,8],[1,-1,8],$$
$$[-1,1,8],[3,-1,10],[-1,3,10]]$$

The vectors are listed with matrix dimension, as the third coordinate. The search returned 17 vectors; the fact that the number of vectors is odd reveals that there exists a self-dual vector. The critical degree is $\boldsymbol{\rho} = (2,2)$, thus $\boldsymbol{m} = (1,1)$ yields the self-dual formula. Since the remaining 16 vectors come in dual pairs, we only mention one formula for each pair; finally, the first three formulae listed have a symmetric counterpart, due to the symmetries present to our data, so it suffices to list 6 distinct formulae.

Using Thm. 2 we can compute directly determinantal boxes:
```
> detboxes( l, d, s) ;
```
$$[[-1,1],[1,3]],[[1,3],[-1,1]]$$

Note that the determinantal vectors are exactly the vectors in these boxes. These intersect at $\boldsymbol{m} = (1,1)$ which yields the self-dual formula. In this example minimum dimension formulae correspond to the centers of the intervals, at $\boldsymbol{m} = (2,0)$ and $\boldsymbol{m} = (0,2)$ as noted in Conj. 1.

A pure Sylvester matrix comes from the vector
```
> m:= vector([d[1]*convert(op(s),'+')-1, -1]);
```
$$\boldsymbol{m} = (3,-1)$$
We compute the complex:
```
> K:= makeComplex(l,d,s,m):
> printBlocks(K); printCohs(K);
```
$$K_{1,2} \to K_{0,1}$$
$$H^1(1,-3) \oplus H^1(0,-4)^2 \to H^1(2,-2)^2 \oplus H^1(1,-3)$$

The dual vector $(-1,3)$ yields the same matrix transposed. The block type of the matrix is deduced by the first command, whereas `printCohs` returns the full description of the complex. The dimension is given by the multihomogeneous Bézout bound, cf. Lem. 1, which is equal to:
```
> mbezout( l, d, s) ;
```

It corresponds to a "twisted" Sylvester matrix:

```
> makematrix(l,d,s,m);
```

$$\begin{bmatrix}
-b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 & 0 \\
-b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 \\
0 & -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 \\
-c_4 & -c_5 & -c_8 & 0 & 0 & 0 & a_1 & 0 & a_3 & 0 \\
-c_1 & -c_3 & -c_7 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 \\
-c_0 & -c_2 & -c_6 & 0 & 0 & 0 & 0 & a_0 & 0 & a_2 \\
0 & 0 & 0 & -c_4 & -c_5 & -c_8 & b_1 & 0 & b_3 & 0 \\
0 & 0 & 0 & -c_1 & -c_3 & -c_7 & b_0 & b_1 & b_2 & b_3 \\
0 & 0 & 0 & -c_0 & -c_2 & -c_6 & 0 & b_0 & 0 & b_2
\end{bmatrix}$$

The rest of the matrices are presented in block format; the same notation is used for both the map and its matrix. Note that the dimension of these maps depend on $\boldsymbol{m}$, which we omit to write. Also, $B(x_k)$ stands for the partial Bézoutian wrt variables $\boldsymbol{x}_k$.

For $\boldsymbol{m} = (3,1)$ we get $K_{1,1} \oplus K_{1,2} \to K_{0,0}$, or $H^0(2,0)^2 \oplus H^1(0,-2)^2 \to H^0(3,1)$:

$$\begin{bmatrix}
M(f_0) \\
M(f_1) \\
B(x_2)
\end{bmatrix}$$

For $\boldsymbol{m} = (3,0)$, $K_{1,2} \to K_{0,0} \oplus K_{0,1}$ $H^1(1,-2) \oplus H^1(0,-3)^2 \to H^0(3,0) \oplus H^1(1,-2)^2$:

$$\left[\begin{array}{c|c}
\begin{matrix} 0 \\ M(f_0) \\ -M(f_1) \end{matrix} & B(x_2)
\end{array} \right]$$

For $\boldsymbol{m} = (2,1)$, we compute $K_{1,1} \oplus K_{1,3} \to K_{0,0}$, or $H^1(1,0)^2 \oplus H^2(-2,-3) \to H^0(2,1)$:

$$\left[\begin{array}{c|c}
\begin{matrix} M(f_1) \\ M(f_2) \end{matrix} & \Delta_{(0,1),(2,1)}
\end{array} \right]$$

If $\boldsymbol{m} = (1,1)$, $K_{1,1} \oplus K_{1,3} \to K_{0,0} \oplus K_{0,2}$, yielding $H^0(0,0)^2 \oplus H^2(-3,-3) \to H^0(1,1) \oplus H^2(-2,-2)^2$:

$$\begin{bmatrix}
f_0 & 0 \\
f_1 & \\
\Delta_{(1,1),(1,1)} & M(f_0) \quad -M(f_1)
\end{bmatrix}$$

We write here f_i instead of $M(f_i)$, since this matrix is just the 1×4 vector of coefficients of f_i.

For $\boldsymbol{m} = (2,0)$, we get $K_{1,2} \oplus K_{1,3} \to K_{0,0} \oplus K_{0,1}$, or $H^1(0,-2) \oplus H^2(-2,-4) \to H^0(2,0) \oplus H^1(0,-2)$:

$$\begin{bmatrix} B(x_2) & 0 \\ \Delta_{(2,0),(0,2)} & B(x_1) \end{bmatrix}$$

This is the minimum dimension determinantal complex, yielding a 4×4 matrix.

Multihomogeneous systems are encountered in several areas e.g. [3], [8]. Few foundational works exist, such as [13], where bigraded systems are analyzed, or [11], where straight-line programs are applied. Our work continues that of [7,14,16], where the unmixed case has been treated, and generalizes their results. As noticed above, we focus on systems whose Newton polytopes are scaled copies of one polytope, thus taking a step towards systems with arbitrary supports. This is the first work that treats *mixed* multihomogeneous equations, and provides explicit resultant matrices.

The main contributions of this paper are as follows: Firstly, we establish the analog of the bounds given in [7, Sect. 3]; in so doing, we simplify their proof in the unmixed case. We characterize the scaled systems that admit a determinantal formula, either pure or hybrid. If pure determinantal formulae exist, we explicitly provide the $\boldsymbol{m}$-vectors that correspond to them. In the search for determinantal formulae we discover box domains that consist of determinantal vectors thus improving the blind search for these vectors adopted in [7]. We conjecture that a formula of minimum dimension can be recovered from the centers of such boxes, analogous to the homogeneous case. We make the differentials in the Weyman complex explicit and provide details of the computation. Note that the actual construction of the matrix, given the terms of the complex, is nontrivial. Finally, we deliver a complete, publicly available MAPLE package for the computation of multihomogeneous resultant matrices. Based on the software of [7], it has been enhanced with new functions, including some which had not been implemented even for the unmixed case, such as the construction of resultant matrices and the efficient storage of complexes.

The rest of the paper is organized as follows. We start with sparse resultants and Weyman complexes in Sect. 2 below. Sect. 3 presents bounds on the coordinates of all determinantal vectors and classifies the systems that admit hybrid and pure determinantal formulae; explicit vectors are provided for pure formulae and minimum dimension choices are conjectured. In Sect. 4 we construct the actual matrices; we present Sylvester- and Bézout-type constructions that also lead to hybrid matrices. We conclude with a short presentation of our MAPLE implementation.

2 Resultants via complexes

We define the resultant, and connect it to complexes by homological constructions. Take the product $X := \mathbb{P}^{l_1} \times \cdots \times \mathbb{P}^{l_r}$ of projective spaces over an algebraically closed field $\mathbb{F}$ of characteristic zero, for $r \in \mathbb{N}$. Its dimension equals the number of affine variables $n = \sum_{k=1}^{r} l_k$. We consider polynomials over X of scaled degree: their multidegree is a multiple of a base degree $\boldsymbol{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r$, say $\deg f_i = s_i \boldsymbol{d}$. We assume $s_0 \leq \cdots \leq s_n$ and $\gcd(s_0, \ldots, s_n) = 1$, so that the data $\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s} = (s_0, \ldots, s_n) \in \mathbb{N}^{n+1}$ fully characterize the system. We denote by $S(\boldsymbol{d})$ the vector space of multihomogeneous forms of

degree $\boldsymbol{d}$ defined over X. These are homogeneous of degree d_k in the variables $\boldsymbol{x}_k$ for $k = 1, \ldots, r$. A system of type $(\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s})$ belongs to $V = S(s_0 \boldsymbol{d}) \oplus \cdots \oplus S(s_n \boldsymbol{d})$.

Definition 1. *Consider a generic scaled multihomogeneous system $\boldsymbol{f} = (f_0, \ldots, f_n)$ defined by the cardinalities $\boldsymbol{l} \in \mathbb{N}^r$, base degree $\boldsymbol{d} \in \mathbb{N}^r$ and $\boldsymbol{s} \in \mathbb{N}^{n+1}$. The multihomogeneous resultant $\mathcal{R}(f_0, \ldots, f_n) = \mathcal{R}_{\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s}}(f_0, \ldots, f_n)$ is the unique up to sign, irreducible polynomial of $\mathbb{Z}[V]$, which vanishes iff there exists a common root of $f_0, \ldots, f_n$ in X.*

This polynomial exists for any data l, d, s, since it is an instance of the sparse resultant. It is itself multihomogeneous in the coefficients of each f_i, with degree given by the multihomogeneous Bézout bound:

Lemma 1. *The resultant polynomial is homogeneous in the coefficients of each f_i, $i = 0, \ldots, n$, with degree*

$$\deg_{f_i} \mathcal{R} = \binom{n}{l_1, \ldots, l_r} \frac{d_1^{l_1} \cdots d_r^{l_r} s_0 \cdots s_n}{s_i}.$$

This yields the total degree of the resultant, that is, $\sum_{i=0}^{n} \deg_{f_i} \mathcal{R}$.

The rest of the section gives details on the underlying theory. The vanishing of the multihomogeneous resultant can be expressed as the failure of a complex of sheaves to be exact. This allows to construct a class of complexes of finite-dimensional vector spaces whose determinant is the resultant polynomial. This definition of the resultant was introduced by Cayley [10, App. A], [15].

For $\boldsymbol{u} \in \mathbb{Z}^r$, $H^q(X, \mathcal{O}_X(\boldsymbol{u}))$ denotes the q-th cohomology of X with coefficients in the sheaf $\mathcal{O}(\boldsymbol{u})$. Throughout this paper we write for simplicity $H^q(\boldsymbol{u})$, even though we also keep the reference to the space whenever it is different than X, for example $H^0(\mathbb{P}^{l_k}, u_k)$. To a polynomial system $\boldsymbol{f} = (f_0, \ldots, f_n)$ over V, we associate a finite complex of sheaves $K_\bullet$ on X :

$$0 \to K_{n+1} \to \cdots \xrightarrow{\delta_2} K_1 \xrightarrow{\delta_1} K_0 \xrightarrow{\delta_0} \cdots \to K_{-n} \to 0 \tag{1}$$

This complex (whose terms are defined in Def. 2 below) is known to be exact iff $f_0, \ldots, f_n$ share no zeros in X; it is hence generically exact. When passing from the complex of sheaves to a complex of vector spaces there exists a degree of freedom, expressed by a vector $\boldsymbol{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$. For every given $\boldsymbol{f}$ we specialize the differentials $\delta_i : K_i \to K_{i-1}$, $i = 1 - n, \ldots, n + 1$ by evaluating at $\boldsymbol{f}$ to get a complex of finite-dimensional vector spaces. The main property is that the complex is exact iff $\mathcal{R}(f_0, \ldots, f_n) \neq 0$ [15, Prop. 1.2].

The main construction that we study is this complex, which we define in our setting. It extends the unmixed case, where for given p the direct sum collapses to $\binom{n+1}{p}$ copies of a single cohomology group.

Definition 2. *For $\boldsymbol{m} \in \mathbb{Z}^r$, $\nu = -n, \ldots, n+1$ and $p = 0, \ldots, n+1$ set*

$$K_{\nu, p} = \bigoplus_{0 \leq i_1 < \cdots < i_p \leq n} H^{p-\nu}\left(\boldsymbol{m} - \sum_{\theta=1}^{p} s_{i_\theta} \boldsymbol{d}\right)$$

where the direct sum is over all possible indices $i_1 < \cdots < i_p$. The Weyman complex $K_\bullet = K_\bullet(\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s}, \boldsymbol{m})$ is generically exact and has terms $K_\nu = \bigoplus_{p=0}^{n+1} K_{\nu, p}$.

This generalizes the classic *Cayley-Koszul* complex. The determinant of the complex can be expressed as a quotient of products of minors from the δ_i. It is invariant under different choices of $\boldsymbol{m} \in \mathbb{Z}^r$ and equals the multihomogeneous resultant $\mathcal{R}(f_0, \ldots, f_n)$.

2.1   Combinatorics of $\mathbf{K}_\bullet$

We present a combinatorial description of the terms in our complex, applicable to the unmixed case as well. For details on the co-homological tools that we use, see [10].

By the Künneth formula, we have the decomposition

$$H^q\left(\boldsymbol{\alpha}\right) = \bigoplus_{\substack{j_1+\cdots+j_r=q}}^{j_k\in\{0,l_k\}} \bigotimes_{k=1}^r H^{j_k}\left(\mathbb{P}^{l_k}, \alpha_k\right), \tag{2}$$

where $q = p - \nu$ and the direct sum runs over all integer sums $j_1 + \cdots + j_r = q, j_k \in \{0, l_k\}$. In particular, $H^0(\mathbb{P}^{l_k}, \alpha_k)$ is isomorphic to $S(\alpha_k)$, the graded piece of S in degree α_k, or if you prefer the space of all homogeneous polynomials in $l_k + 1$ variables with total degree α_k, where $\boldsymbol{\alpha} = \boldsymbol{m} - z\boldsymbol{d} \in \mathbb{Z}^r$ for $z \in \mathbb{Z}$.

By Serre duality, for any $\boldsymbol{\alpha} \in \mathbb{Z}^r$, we know that

$$H^q(\boldsymbol{\alpha}) \simeq H^{n-q}(-\boldsymbol{l} - \mathbf{1} - \boldsymbol{\alpha})^*, \tag{3}$$

where * denotes dual, and $\mathbf{1} \in \mathbb{N}^r$ a vector full of ones. Furthermore, we identify $H^{l_k}(\mathbb{P}^{l_k}, \alpha_k)$ as the dual space $S(-\alpha_k - l_k - 1)^*$. This is the space of linear functions $\Lambda : S(\alpha_k) \to \mathbb{F}$. Sometimes we use the negative symmetric powers to interpret dual spaces, see also [16, p.576]. This notion of duality is naturally extended to the direct sum of cohomologies: the dual of a direct sum is the direct sum of the duals of the summands. The next proposition (Bott's formula) implies that this dual space is nontrivial iff $-\alpha_k - l_k - 1 \geq 0$.

Proposition 1. [2] *For any $\alpha \in \mathbb{Z}^r$ and $k \in \{1, \ldots, r\}$,*
(a) $H^j(\mathbb{P}^{l_k}, \alpha_k) = 0, \ \forall j \neq 0, l_k$,
(b) $H^{l_k}(\mathbb{P}^{l_k}, \alpha_k) \neq 0 \Leftrightarrow \alpha_k < -l_k$, $\dim H^{l_k}(\mathbb{P}^{l_k}, \alpha_k) = \binom{-\alpha_k-1}{l_k}$.
(c) $H^0(\mathbb{P}^{l_k}, \alpha_k) \neq 0 \Leftrightarrow \alpha_k \geq 0$, $\dim H^0(\mathbb{P}^{l_k}, \alpha_k) = \binom{\alpha_k+l_k}{l_k}$.

Definition 3. *Given $\boldsymbol{l}, \boldsymbol{d} \in \mathbb{N}^r$ and $\boldsymbol{s} \in \mathbb{N}^{n+1}$, define the* critical degree *vector $\rho \in \mathbb{N}^r$ by $\rho_k := d_k \sum_{\theta=0}^n s_\theta - l_k - 1$, for all $k = 1, \ldots r$.*

The Künneth formula (2) states that $H^q(\boldsymbol{\alpha})$ is a sum of products. We can give a better description:

Lemma 2. *If $H^q(\boldsymbol{\alpha})$ is nonzero, then it is equal to a product $H^{j_1}(\mathbb{P}^{l_k}, \alpha_k) \otimes \cdots \otimes H^{j_r}(\mathbb{P}^{l_k}, \alpha_k)$ for some integers $j_1, \ldots, j_r$ with $j_k \in \{0, l_k\}$, $\sum_{k=1}^r j_k = q$.*

Throughout this paper we denote $[u, v] := \{u, u+1, \ldots, v\}$. By Prop. 1, the set of integers z such that both $H^0(\mathbb{P}^{l_k}, m_k - zd_k)$ and $H^{l_k}(\mathbb{P}^{l_k}, m_k - zd_k)$ vanish is:

$$P_k := \left(\frac{m_k}{d_k}, \frac{m_k + l_k}{d_k} \right] \cap \mathbb{Z}.$$

All formulae (including determinantal ones) come in dual pairs, thus generalizing [7, Prop.4.4].

Lemma 3. *Assume $m, m' \in \mathbb{Z}^r$ satisfy $m + m' = \rho$, where ρ is the critical degree vector. Then, $K_\nu(m)$ is dual to $K_{1-\nu}(m')$ for all $\nu \in \mathbb{Z}$. In particular, m is determinantal iff m' is determinantal, yielding matrices of the same size, namely $\dim(K_0(m)) = \dim(K_1(m'))$.*

3 Determinantal formulae

Determinantal formulae occur only if there is exactly one nonzero differential, so the complex consists of two consecutive nonzero terms. The determinant of the complex is the determinant of this differential. We now specify this differential; for the unmixed case cf. [16, Lem.3.3].

Lemma 4. *If $m \in \mathbb{Z}^r$ is determinantal then the nonzero part of the complex is $\delta_1 : K_1 \to K_0$.*

3.1 Bounds for determinantal vectors

We generalize the bounds in [7, Sect.3] to the mixed case, for the coordinates of all determinantal $m-$vectors. We follow a simpler and more direct approach based on a global view of determinantal complexes.

Lemma 5. *If a vector $m \in \mathbb{Z}^r$ is determinantal then the corresponding $\bigcup_1^r P_k$ is contained in $[0, \sum_0^n s_i]$.*

We derive the folloing

Theorem 1. *For determinantal $m \in \mathbb{Z}^r$, for all k we have $\max\{-d_k, -l_k\} \leq$*

$$m_k \leq d_k \sum_0^n s_i - 1 + \min\{d_k - l_k, 0\}.$$

Our implementation in Sect. 5 conducts a search in the box defined by the above bounds. For each m in the box, the dimension of K_2 and K_{-1} is calculated; if both are zero the vector is determinantal. Finding these dimensions is time consuming; the following lemma provides a cheap necessary condition to check before calculating them.

Lemma 6. *If $m \in \mathbb{Z}^r$ is determinantal then there exist indices $k, k' \in [1, r]$ such that $m_k < d_k(s_{n-1} + s_n)$ and $m_{k'} \geq d_{k'} \sum_0^{n-2} s_i - l_{k'}$.*

3.2 Characterization and explicit vectors

A formula is determinantal iff it holds $K_2 = K_{-1} = 0$. In this section we provide necessary and sufficient conditions for the data l, d, s to admit a determinantal formula; we call this data *determinantal*. Also, we derive multidimensional integer intervals (boxes) that yield determinantal formulae and conjecture that minimum dimension formulae appear near the center of these intervals.

Let $\pi[k] := \sum_{\pi(i) \leq \pi(k)} l_i$. If $\pi = \mathrm{Id}$ this is $\mathrm{Id}[k] = l_1 + \cdots + l_k$. We now characterize determinantal data:

Theorem 2. *The data l, d, s admit a determinantal formula iff there exists $\pi :$ $[1, r] \to [1, r]$ s.t.*

$$d_k \sum_{n-\pi[k]+2}^{n} s_i - l_k < d_k \sum_{0}^{\pi[k-1]+1} s_i, \ \forall k.$$

Corollary 1. *For any permutation $\pi : [1, r] \to [1, r]$, the vectors $m \in \mathbb{Z}^r$ contained in the box*

$$d_k \sum_{n-\pi[k]+2}^{n} s_i - l_k \leq m_k \leq d_k \sum_{0}^{\pi[k-1]+1} s_i - 1$$

for $k = 1, \ldots, r$ are determinantal.

If $r = 1$ then a minimum dimension formula lies in the center of an interval [5]. We conjecture that a similar explicit choice also exists for $r > 1$. Experimental results indicate that minimum dimension formulae tend to appear near the *center* of the nonempty boxes:

Conjecture 1. If the data l, d, s is determinantal then determinantal formulae of minimum dimension lie *close* to the center of the nonempty boxes of Cor. 1.

3.3 Pure formulae

A determinantal formula is pure if it is of the form $K_{1,a} \to K_{0,b}$ for $a, b \in [0, n+1]$ with $a > b$. These formulae are either Sylvester- or Bézout-type, named after the matrices for the resultant of two univariate polynomials.

In the unmixed case both kinds of pure formulae exist exactly when for all $k \in [1, r]$ it holds that $\min\{l_k, d_k\} = 1$ [14,7]. The following theorem extends this characterization to the scaled case, by showing that only pure Sylvester formulae are possible and the only data that admit such formulae are univariate and bivariate-bihomogeneous systems.

Theorem 3. *If $s \neq 1$ a pure Sylvester formula exists iff $r \leq 2$ and $l = (1)$ or $l = (1, 1)$. If $l_1 = n = 1$ the degree vectors are given by*

$$m = d_1 \sum_{0}^{1} s_i - 1 \ \ and \ \ m' = -1$$

whereas if $l = (1, 1)$ the vectors are given by

$$\boldsymbol{m} = \left(-1, d_2 \sum_{0}^{2} s_i - 1\right) \ \ and \ \ \boldsymbol{m'} = \left(d_1 \sum_{0}^{2} s_i - 1, -1\right).$$

Pure Bézout determinantal formulae cannot *exist.*

If $s = 1$ pure determinantal formulae are possible for arbitrary n, r and a pure formula exists iff for all k, $l_k = 1$ or $d_k = 1$ [7, Thm. 4.5]; if a pure Sylvester formula exists for $a, b = a - 1$ then another exists for $a = 1, b = 0$ [7, p. 15]. Observe in the proof above that this is not the case if $s \neq 1, n = 2$, thus the construction of the corresponding matrices for $a \neq 1$ now becomes important and highly nontrivial, in contrast to [7].

4 Explicit matrix construction

In this section we provide algorithms for the construction of the resultant matrix expressed as the matrix of the differential δ_1 in the natural monomial basis and we clarify all the different morphisms that may be encountered.

Before we continue, let us justify the necessity of our matrices, using the data $\boldsymbol{l} = \boldsymbol{d} = (1,1)$ and $\boldsymbol{s} = (1,1,2)$, that is, a system of two bilinear and one biquadratic equation. It turns out that a (hybrid) resultant matrix of minimum dimension is of size 4×4. The standard Bézout-Dixon construction has size 6×6 but its determinant is identically zero, hence it does not express the resultant of the system.

The matrices constructed are unique up to row and column operations, reflecting the fact that monomial bases may be considered with a variety of different orderings. The cases of pure Sylvester or pure Bézout matrix can be seen as a special case of the (generally hybrid, consisting of several blocks) matrix we construct in this section.

In order to construct a resultant matrix we must find the matrix of the linear map $\delta_1 : K_1 \to K_0$ in some basis, typically the natural monomial basis, provided that $K_{-1} = 0$. In this case we have a generically surjective map with a maximal minor divisible by the sparse resultant. If additionally $K_2 = 0$ then $\dim K_1 = \dim K_0$ and the determinant of the square matrix is equal to the resultant, i.e. the formula is determinantal. We consider restrictions $\delta_{a,b} : K_{1,a} \to K_{0,b}$ for any direct summand $K_{1,a}$, $K_{0,b}$ of K_1, K_0 respectively. Every such restriction yields a block of the final matrix of size defined by the corresponding dimensions. Throughout this section the symbols a and b will refer to these indices.

4.1 Sylvester blocks

The Sylvester-type formulae we consider generalize the classical univariate Sylvester matrix and the multigraded Sylvester matrices of [14] by introducing multiplication matrices with block structure. Even though these Koszul morphisms are known to correspond to some Sylvester blocks since [16] (see Prop. 2 below), the exact interpretation of the morphisms into matrix formulae had not been made explicit until now.

By [16, Prop. 2.5, Prop. 2.6] we have the following

Proposition 2. [16] *If $a - 1 < b$ then $\delta_{a,b} = 0$. Moreover, if $a - 1 = b$ then $\delta_{a,b}$ is a Sylvester map.*

If $a = 1$ and $b = 0$ then every coordinate of $\boldsymbol{m}$ is non-negative and there are only zero cohomologies involved in $K_{1,1} = \bigoplus_i H^0(\boldsymbol{m} - s_i \boldsymbol{d})$ and $K_{0,0} = H^0(\boldsymbol{m})$. This map is a well known Sylvester map expressing the multiplication $(g_0, \ldots, g_n) \longmapsto \sum_{i=0}^{n} g_i f_i$. The entries of the matrix are indexed by the exponents of the basis monomials of $\bigoplus_i S(\boldsymbol{m} - s_i \boldsymbol{d})$ and $S(\boldsymbol{m})$ as well as the chosen polynomial f_i. Also, by Serre duality a block $K_{1,n+1} \to K_{0,n}$ corresponds to the dual of $K_{1,1} \to K_{0,0}$, i.e. to the degree vector $\boldsymbol{\rho} - \boldsymbol{m}$, and yields the same matrix transposed.

The following theorem constructs corresponding Sylvester-type matrix in the general case.

Theorem 4. *The entry of the transposed matrix of $\delta_{a,b} : K_{1,a} \to K_{0,a-1}$ in row $(I, \boldsymbol{\alpha})$ and column $(J, \boldsymbol{\beta})$ is*

$$\begin{cases} 0, & \text{if } J \not\subset I, \\ (-1)^{k+1}\mathrm{coef}\,(f_{i_k}, \boldsymbol{x^u}), & \text{if } I \setminus J = \{i_k\}, \end{cases}$$

where $I = \{i_1 < i_2 < \cdots < i_a\}$ and $J = \{j_1 < j_2 < \cdots < j_{a-1}\}$, $I, J \subseteq \{0, \ldots, n\}$. Moreover, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$ run through the exponents of monomial bases of $H^{a-1}(\boldsymbol{m} - \boldsymbol{d}\sum_{\theta=1}^{a} s_{i_\theta})$, $H^{a-1}(\boldsymbol{m} - \boldsymbol{d}\sum_{\theta=1}^{a-1} s_{j_\theta})$, and $\boldsymbol{u} \in \mathbb{N}^n$, with $u_t = |\beta_t - \alpha_t|$.

4.2 Bézout blocks

A Bézout-type block comes from a map of the form $\delta_{a,b} : K_{1,a} \to K_{0,b}$ with $a - 1 > b$. In the case $a = n + 1$, $b = 0$ this is a map corresponding to the Bezoutian of the system, whereas in other cases some Bézout-like matrices occur, from square subsystems obtained by hiding certain variables.

Consider the Bézoutian, or Morley form(cf. [12]), of $f_0, \ldots, f_n$. This is a polynomial of multidegree (ρ, ρ) in $\mathbb{F}[\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}]$ and can be decomposed as

$$\Delta := \sum_{u_1=0}^{\rho_1} \cdots \sum_{u_r=0}^{\rho_r} \Delta_u(\bar{x}) \cdot \bar{y}^u$$

where $\Delta_u(\bar{\boldsymbol{x}}) \in S$ has $\deg \Delta_u(\bar{\boldsymbol{x}}) = \boldsymbol{\rho} - \boldsymbol{u}$. Here $\bar{\boldsymbol{x}} = (\bar{\boldsymbol{x}}_1, \ldots, \bar{\boldsymbol{x}}_r)$ is the set of homogeneous variable groups and $\bar{\boldsymbol{y}} = (\bar{\boldsymbol{y}}_1, \ldots, \bar{\boldsymbol{y}}_r)$ a set of new variables with the same cardinalities.

The Bezoutian gives a linear map

$$\bigwedge{}^{n+1} V \to \bigoplus_{m_k \leq \rho_k} S(\boldsymbol{\rho} - \boldsymbol{m}) \otimes S(\boldsymbol{m}).$$

where the space on the left is the $(n + 1)$-th exterior algebra of $V = S(s_0\boldsymbol{d}) \oplus \cdots \oplus S(s_n\boldsymbol{d})$ and the direct sum runs over all vectors $\boldsymbol{m} \in \mathbb{Z}^r$ with $m_k \leq \rho_k$ for all $k \in [1, r]$.

In particular, the graded piece of Δ in degree $(\boldsymbol{\rho} - \boldsymbol{m}, \boldsymbol{m})$ in $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is

$$\Delta_{\boldsymbol{\rho}-\boldsymbol{m},\boldsymbol{m}} := \sum_{u_k=m_k} \Delta_u(\bar{\boldsymbol{x}}) \cdot \bar{\boldsymbol{y}}^{\boldsymbol{u}}$$

for all monomials $\bar{\boldsymbol{y}}^{\boldsymbol{u}}$ of degree $\boldsymbol{m}$ and coefficients in $\mathbb{F}[\bar{x}]$ of degree $\boldsymbol{\rho} - \boldsymbol{m}$. It yields a map $S(\boldsymbol{\rho} - \boldsymbol{m})^* \longrightarrow S(\boldsymbol{m})$ known as the Bézoutian in degree $\boldsymbol{m}$ of $f_0, \ldots, f_n$. The differential of $K_{1,n+1} \to K_{0,0}$ can be chosen to be exactly this map, since evidently $K_{0,0} = H^0(\boldsymbol{m}) \simeq S(\boldsymbol{m})$ and

$$K_{1,n+1} = H^n\left(\boldsymbol{m} - \sum_0^n s_i\boldsymbol{d}\right) \simeq S\left(-\boldsymbol{m} + \sum_0^n s_i\boldsymbol{d} + l + \boldsymbol{1}\right)^*$$

according to Serre duality recalled in Sect. 2.1, thus we get $K_{1,n+1} = S(\boldsymbol{\rho} - \boldsymbol{m})^*$.

The polynomial Δ defined above has $n + r$ homogeneous variables, hence it is not clear how it can be computed by matrix constructions. We show one construction of some part $\Delta_{\boldsymbol{\rho-m,m}}$ using an affine Bézoutian.

Denote $\boldsymbol{x}_k = (x_{k1}, \ldots, x_{k,l_k})$ the (dehomogenized) k-th variable group, and $\boldsymbol{y}_k = (y_{k1}, \ldots, y_{k,l_k})$. As a result the totality of variables is $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r)$ and $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_r)$.

We set $\boldsymbol{w}_t$, $t = 1, \ldots n - 1$ the conjunction of the first t variables of $\boldsymbol{y}$ and the last $n - t$ variables of $\boldsymbol{x}$.

If $a = n + 1, b = 0$ the affine Bézoutian construction follows from the expansion of

$$\begin{vmatrix} f_0(\boldsymbol{x}) & f_0(\boldsymbol{w}_1) & \cdots & f_0(\boldsymbol{w}_{n-1}) & f_0(\boldsymbol{y}) \\ \vdots & \vdots & & \vdots & \vdots \\ f_n(\boldsymbol{x}) & f_n(\boldsymbol{w}_1) & \cdots & f_n(\boldsymbol{w}_{n-1}) & f_n(\boldsymbol{y}) \end{vmatrix} \Bigg/ \prod_{k=1}^{r} \prod_{j=1}^{l_k} (x_{kj} - y_{kj})$$

as a polynomial in $\mathbb{F}[\boldsymbol{y}]$ with coefficients in $\mathbb{F}[\boldsymbol{x}]$. Hence the entry indexed $\boldsymbol{\alpha}, \boldsymbol{\beta}$ of the Bézoutian in some degree can be computed as the coefficient of $\boldsymbol{x}^{\alpha} \boldsymbol{y}^{\beta}$ of this polynomial.

5 Implementation

We have implemented the search for formulae and construction of the corresponding resultant matrices in Maple. Our code is based on that of [7, Sect. 8] and extends it to the scaled case. We also introduce new features, including construction of the matrices of Sect. 4; hence we deliver a full package for multihomogeneous resultants, available at `www-sop.inria.fr/galaad/amantzaf/soft.html`.

Our implementation has three main parts; given data $(\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s})$ it discovers all possible determinantal formula; this part had been implemented for the unmixed case in [7]. Moreover, for a specific $\boldsymbol{m}-$vector the corresponding resultant complex is computed and saved in memory in an efficient representation. As a final step the results of Sect. 4 are being used to output the resultant matrix coming from this complex. The main routines of our software are illustrated in Table 1.

routine	function
`Makesystem`	output polynomials of type $(\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s})$
`mBezout`	compute the m-Bézout bound
`allDetVecs`	enumerate all determinantal $\boldsymbol{m}-$vectors
`detboxes`	output the vector boxes of Cor. 1
`MakeComplex`	Compute the complex of an $m-$vector
`printBlocks`	print complex as $\oplus_a K_{1,a} \to \oplus_b K_{0,b}$
`printCohs`	print complex as $\oplus H^q(u) \to \oplus H^q(v)$
`multmap`	construct matrix $M(f_i) : S(u) \to S(v)$
`Sylvmat`	construct Sylv. matrix $K_{1,p} \to K_{0,p-1}$
`Bezoutmat`	construct Bézout matrix $K_{1,a} \to K_{0,b}$
`makeMatrix`	construct matrix $K_1 \to K_0$

Table 1. The main routines of our software.

The computation of all the $\boldsymbol{m}-$vectors can be done by searching the box defined in Thm. 1 and using the filter in Lem. 6. For every candidate, we check whether the terms K_2 and K_{-1} vanish to decide if it is determinantal.

For a vector $\boldsymbol{m}$, the resultant complex can be computed in an efficient data structure that captures its combinatorial information and allows us to compute the corresponding matrix. More specifically, a nonzero cohomology summand $K_{\nu,p}$ is represented as a list of pairs (c_q, e_p) where $c_q = \{k_1, \ldots, k_t\} \subseteq [1, r]$ such that $q = \sum_{i=1}^t l_{k_i} = p - \nu$ and $e_p \subseteq [0, n]$ with $\#e_p = p$ denotes a collection of polynomials (or a basis element in the exterior algebra). Furthermore, a term K_ν is a list of $K_{\nu,p}$'s and a complex a list of terms K_ν.

The construction takes place block by block. We iterate over all morphisms $\delta_{a,b}$ and after identifying each of them the corresponding routine constructs a Sylvester or Bézout block. Note that these morphisms are not contained in the representation of the complex, since they can be retrieved from the terms $K_{1,a}$ and $K_{0,b}$.

Example 2. **(Cont'd)** Let us present an illustration of how the matrices we constructed provide resultants that were previously not computable, even up to extraneous factors. Recall the instance $\boldsymbol{l} = \boldsymbol{d} = (1, 1)$ and $\boldsymbol{s} = (1, 1, 2)$ of Sect. 1. This data defines a system of two bilinear and one biquadratic equation. The resulting matrices for this particular example have already been presented in Sect. 1. We used `multires`[1] to try to compute the Macaulay matrix of this system, which is of size 28×28.

```
> read mhomo-scaled.mpl:
> l:=vector([1,1]): d:=l: s:= vector([1,1,2]):
> f:= Makesystem(l,d,s):
> read multires.mpl:
> M:=mresultant( f, [x_1, x_2] ): det(M);
```
$$0$$

Thus the determinant vanishes identically. We try again to compute the resultant using the standard Bézout-Dixon construction

```
> B:= mbezout( f, [x_1, x_2] ): det(B);
```
$$0$$

The size of this matrix is 6×6 but again its determinant is identically zero, due to the sparsity present to the supports, hence it neither expresses the multihomogeneous resultant, nor provides any information on the roots of the system. Instead, our software constructed optimal, generically non-singular matrices of dimensions ranging from 4×4 to 10×10.

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[1] Developed by team GALAAD, INRIA, http://www-sop.inria.fr/galaad/software/multires

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