Population Effects in Multiclass Processor Sharing Queues

Abdelghani Ben Tahar[1] Alain Jean-Marie[2]

[1]University Hassan I, Settat, Marocco
previously at the University of Rouen
Research funded by a CNRS Post-doctoral grant

[2]INRIA
LIRMM CNRS/Univ. Montpellier 2

Valuetools 2009, 20 October 2009

Outline

Progress

The Model

Customers arrive to a single-server, infinite buffer, Processor Sharing station.

They belong to *classes* which govern their service requirement and routing behavior.

When they complete some service, they may re-enter the queue, possibly with a different class, according to specified probabilities (Jackson-like routing).

The Questions

Questions we are interested in are related with *fluid limits*:
When either:

- the number of customers initially present
- or time

goes to infinity, *normalized* quantities

- of customers
- of work in progress

evolve according to deterministic differential equations.

As an application, we are interested in *fairness* issues between classes.

Related Literature

The Processor Sharing queue has a long history: see *e.g.* Yashkov & Yashkova (2007). However, multiclass results are rare.
The Discriminatory Processor Sharing has been actively studied lately, due to its applications in networking: see *e.g.* Avratchenkov *et al.*, Altman *et al.* (2005).

Related Literature

Fluid limits of the Processor Sharing queue have been studied before. Partial bibliography:
In single-class

- Robert and Jean-Marie (1994)
- Gromoll, Puha and Williams (2002), Puha and Williams (2004), Puha, Stolyar and Williams (2006)
- ...

With variants

- customers are impatient: Gromoll *et al.* (2006)
- server accepts a limited number of customers: Zhang *et al.* (2008)
- ...

Our analysis is a generalization of Puha, Stolyar and Williams (2006).

Purpose of this talk

The talk is aimed at:

- provide an overview of fluid results for the multiclass PS queue, including DPS
- provide some illustrations

Progress

Model Parameters

The principal parameters of the model are:

- the vector of *external* arrival rates α
- the routing matrix P
- the service time distribution in class k: σ_k with
  - measure ν_k
  - average β_k
  - Laplace transform $\widehat{B}_k(\cdot)$

Additional notation in the text

- the "vector of ones", e
- $Q = (I - P')^{-1}$
- vectors and diagonal matrices of per-class quantities

State representation

The state of the queue at time t is represented by:

- $A_k(t)$, amount of (fluid) arrivals of class k up to t
- $D_k(t)$, amount of (fluid) departures of class k up to t
- $\mu_k(t)$, distribution of residual workload in class k

In particular:

$Z_k = \langle 1, \mu_k \rangle$ is the quantity of customers of class k in queue

$\langle 1_{[x,\infty[}, \mu_k \rangle$ is the quantity of customers in queue with remaining service $\geq x$

$\langle \mathrm{id}, \mu_k \rangle$ is the total workload of customers of class k

Fluid equations

The input/output traffics satisfy:

$$A(t) \;=\; \alpha t + P' D(t)\ .$$

For every $k \in \mathcal{K}$, $x \in \mathbb{R}_+$, $t \geq 0$,

$$\langle 1, \mu_k(t) \rangle \;=\; \langle 1, \mu_k(0) \rangle + A_k(t) - D_k(t)$$

$$\begin{aligned} \langle 1_{[x,\infty)}, \mu_k(t) \rangle \;&=\; \langle 1_{[x,\infty)}(. - S(0,t)), \mu_k(0) \rangle \\ &+ \int_0^t \langle 1_{[x,\infty[}(. - (S(s,t))), \nu_k \rangle \mathrm{d}A_k(s)\ , \end{aligned}$$

where the cumulative service amount is:

$$S(s,t) \;=\; \int_s^t \frac{1}{\langle 1, e.\mu(u) \rangle}\, \mathrm{d}u\ .$$

The queueing model

Consider a sequence of (discrete stochastic) Processor sharing queues indexed by r. They are described by:

- i.i.d. inter-arrival sequences for each class $\{u_k^r(n); n = 1, 2, \ldots\}$; the arrival rate is α_k^r
- i.i.d. routing sequences $\varphi_{k\ell}^r$ with probabilities $p_{k\ell}^r$
- i.i.d. service time sequences,
- initial workload measures ν_k^{0r}.

Let $A_k^r(t)$ and $D_k^r(t)$ be the arrival and departure count processes, and $\mu_k^r(t)$ the residual workload measures.
Define

$$\bar{A}^r(t) = \frac{A^r(rt)}{r}, \quad \bar{D}^r(t) = \frac{D^r(rt)}{r}, \quad \bar{\mu}^r(t) = \frac{\mu^r(rt)}{r}.$$

Convergence result

Theorem

Assume that:

- *arrival rates* α_k^r *and routing frequencies* $\varphi_{k\ell}^r$ *converge*
- *initial measures converge in the appropriate sense.*

Then the normalized quantities $(\bar{A}^r, \bar{D}^r, \bar{\mu}^r)$ *converges in distribution to the solutions of the fluid model with initial measure* $\bar{\mu}_k(0)$ *when* $r \to \infty$*.*

Complete proof in Ben Tahar and Jean-Marie (2009).

Existence of solutions, general result

Theorem

There exists a unique solution to the system of fluid equations. It is given by

$$\begin{aligned}
A(t) &= \lambda t + QP'(Z(0) - Z(t)) \\
D(t) &= \lambda t + Q(Z(0) - Z(t)) \\
Z(t) &= \widetilde{Z}(T^{-1}(t)) \\
\widetilde{Z}(s) &= Q^{-1}(U * C)(s)Z(0) + Q^{-1}(U * (I - B) * (TI))(s)\lambda \\
T(s) &= (H * U_e)(s)
\end{aligned}$$

for some functions $C(\cdot)$, $H(\cdot)$, $U_e(\cdot)$, $U(\cdot)$ directly constructed from the data.

Time Range of the Solution

According to the results of Gromoll, Puha, Williams:

Lemma

Let $(A(t), D(t), \mu(t))$ *be a solution such that* $\mu(0) = \xi \neq 0$.

- *Let* $t^* = \inf\{t : e.\mu(t) = 0\}$. *Then,*

$$\begin{cases} t^* = +\infty & \text{if } \rho \geq 1 \\ t^* = \dfrac{e(\beta^0 + \beta Q P')Z(0)}{1-\rho} & \text{if } \rho < 1 \,. \end{cases}$$

- *The function* $S : [0, t^*) \to [0, \infty)$ *is well defined and strictly increasing. So is* $T \equiv S^{-1} : [0, \infty) \to [0, t^*)$.

Existence of solutions, supercritical case

Lemma

Assume that the queue is supercritical ($\rho > 1$). Then there exists a unique positive real number θ_0 solution to the equation:

$$\theta_0 \;=\; e\,(I - \widehat{B}(\theta_0))(I - P'\widehat{B}(\theta_0))^{-1}\,\alpha\;.$$

The Laplace transform in the RHS is that of the service of a "typical customer".

Define the vector $m = (m_1, \ldots, m_K)'$ as:

$$m \;=\; (I - \widehat{B}(\theta_0))(I - P'\widehat{B}(\theta_0))^{-1}\,\alpha\;.$$

This θ_0 is the global growth rate of the population. The vector m is its repartition among classes.

Existence of solutions, supercritical case (ctd.)

Define $p_k : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, for each $k \in \mathcal{K}$ by

$$p_k(x) \; = \; \frac{m_k}{1 - \widehat{B}_k(\theta_0)} \int_x^\infty \theta_0 e^{-\theta_0(y-x)} \mathrm{d}B_k(y) \; ,$$

and let $s_k \in \mathcal{M}_+$ be the measure:

$$s_k(x) \; = \; p_k(x) \, \mathrm{d}x \; .$$

Theorem

Assume that the system is supercritical, and let θ_0 be as above. Then the triple

$$(A, D, \mu)(t) \; = \; t \times \left((I - \widehat{B}(\theta_0))^{-1} m, (I - \widehat{B}(\theta_0))^{-1} \widehat{B}(\theta_0) m, s \right)$$

is the unique fluid solution of the model starting from the origin, that is, with $\mu(0) \equiv 0$. As a consequence, $Z(t) = mt$.

Asymptotic behavior, supercritical case

Assume that the data is supercritical ($\rho > 1$).

Theorem

Given a supercritical data (α, P, ν) and $\xi \in \mathcal{M}^{c,K}$, there holds:

$$\frac{\mu_k(t)}{t}(.) \implies s_k(.) \; .$$

As a consequence,

$$\lim_{t\to\infty} \frac{A(t)}{t} = \lambda - QP'm \qquad \lim_{t\to\infty} \frac{D(t)}{t} = \lambda - Qm \; .$$

These properties follow from the result of Athreya and Rama Murthy (1976) on systems of renewal equations.

Progress

Extension to the DPS

Under the Discriminatory Processor Sharing discipline with weights $(g_k)_k$, the service delivered to some customer of class k grows therefore as

$$\frac{g_k}{\sum_k g_k Z_k(t)} \;=\; \frac{g_k}{g.Z(t)} \;.$$

Then the cumulative of service per customer of class k can be expressed as:

$$S_k(s,t) = \int_s^t \frac{g_k}{\langle 1, g.\mu(u)\rangle} \mathrm{d}u \;.$$

The dynamics of the measure μ_k becomes:

$$\begin{aligned}\langle 1_{[x,\infty)}, \mu_k(t)\rangle \;&=\; \langle 1_{[x,\infty)}(. - S_k(t)), \mu_k(0)\rangle \\ &\quad + \int_0^t \langle 1_{[x,\infty)}(. - (S_k(s,t)), \nu_k\rangle \mathrm{d}A_k(s) \;.\end{aligned}$$

Extension to the DPS (ctd.)

Let G be the diagonal matrix obtained from g.

Theorem

The DPS Fluid solution can be constructed from an equivalent (egalitarian) PS Fluid solution with the following data (α^g, P^g, ν^g)*:*

$$\begin{cases} \alpha^g & = G\alpha \\ P^g & = GPG^{-1} \\ \nu_k^g(\cdot) & = \nu_k(g_k \times \cdot) \,. \end{cases}$$

Observe that GPG^{-1} is not necessarily stochastic, but has spectral radius less than 1.

In examples, everything works fine with this matrix!

Progress

Trajectories, stable case

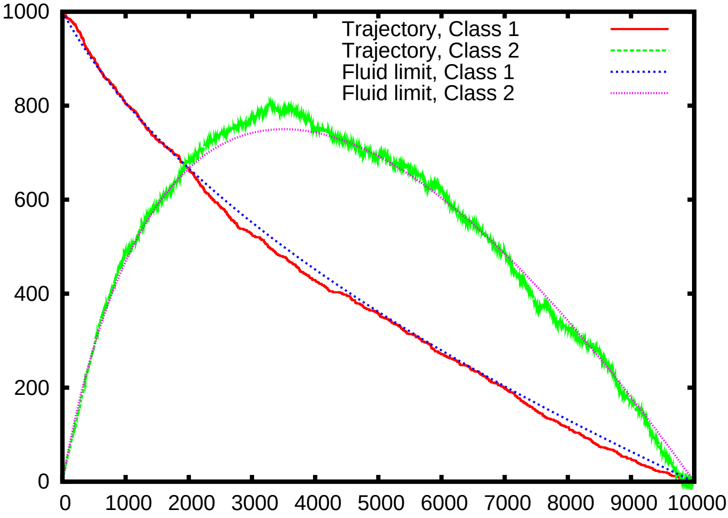
Two-class experiment:

- Class 1 customers have 0 arrival rate, and route to Class 2
  Services Expo mean 4.0
- Class 2 customers have 1/2 arrival rate, route to the outside
  Services Expo mean 1.0
- Load is 1/2, initial workload normalized to 1

The solution is given by:

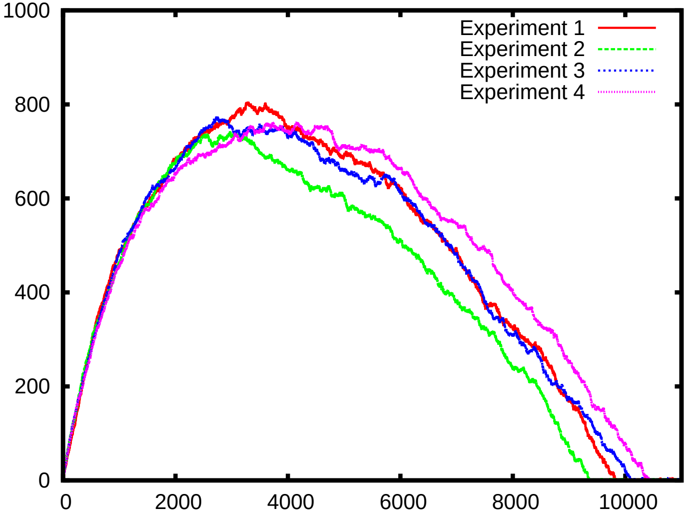
$$
\begin{aligned}
T(t) &= 10 - 16e^{-t/4} + 6e^{-t/2} \\
S(s) &= -4\log\left(\frac{4}{3} - \frac{\sqrt{4+6s}}{6}\right) \\
Z_1(s) &= \frac{4}{3} - \frac{\sqrt{4+6s}}{6} \\
Z_2(s) &= 3\ Z_1(s)\ (1 - Z_1(s))
\end{aligned}
$$

Trajectories, stable case (ctd.)



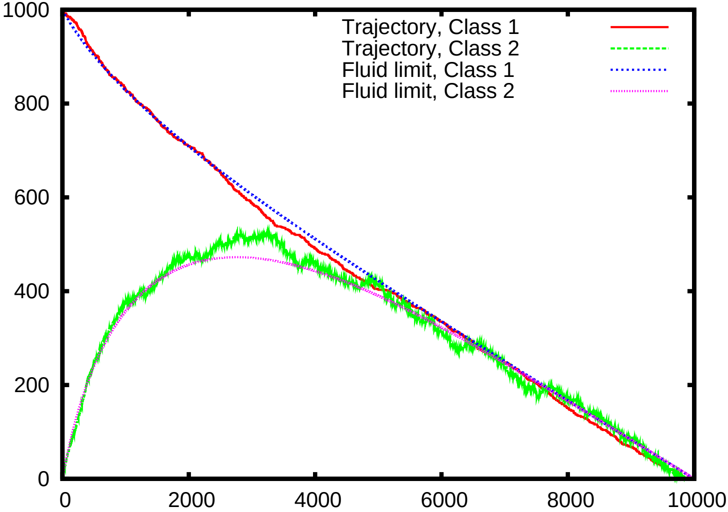
Trajectories, stable case (ctd.)

Not always so lucky...



Trajectories with DPS, stable case

Same situation with $G = 2$:



Closeness of the approximation

How to quantify the closeness of the approximation?

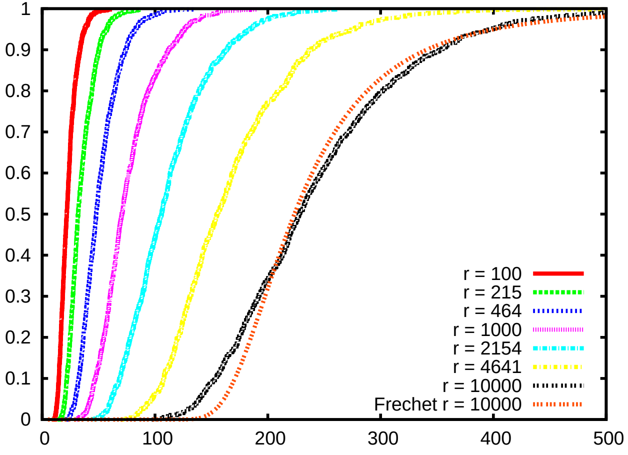
$\Longrightarrow$ maximum deviation between the real (random) trajectory $Z(t)$ and the fluid trajectory at the same scale.

Define the random variable:

$$M_k^{r,T} = \max_{0 \leq t \leq T} |rZ_k(t/r) - Z_k^r(t)| = r \max_{0 \leq u \leq T/r} |Z_k(u) - \bar{Z}_k^r(u)| .$$

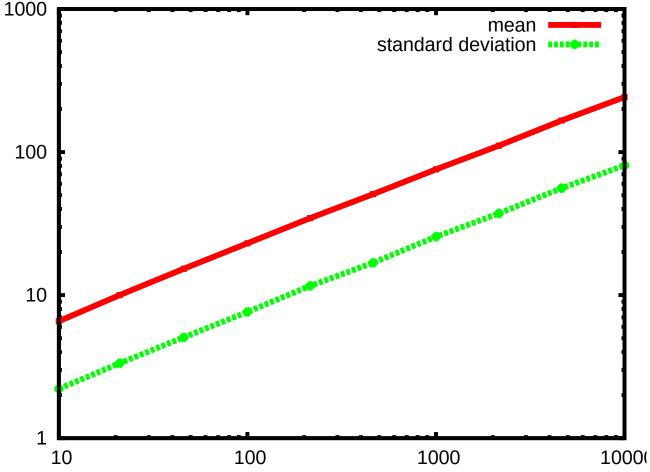
Closeness of the approximation

Empirical distribution of the metric $M_1^{r,T}$ for increasing r in the previous example, class 2.



Closeness of the approximation (ctd.)

Mean and std. dev. of the metric $M_1^{r,T}$ for increasing r: compatible with a $\sqrt{r}$ scaling.



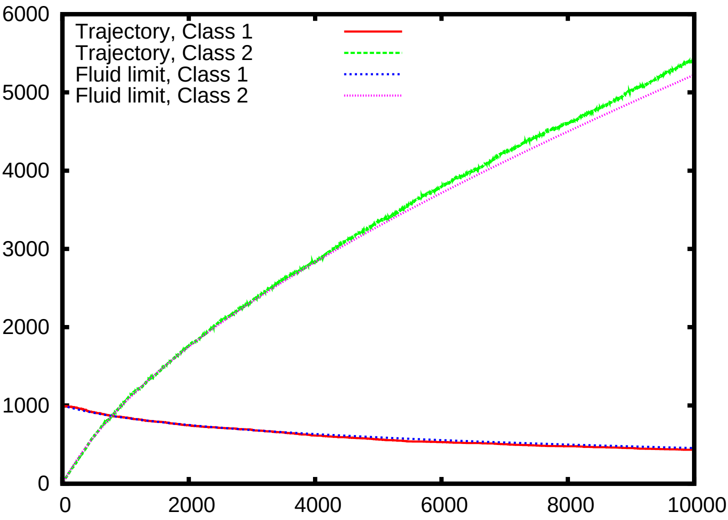
Trajectories, unstable case

Two-class experiment:

- Class 1 customers have 0 arrival rate, and route to Class 2
  Services Expo mean 4.0
- Class 2 customers have 5/4 arrival rate, route to the outside
  Services Expo mean 1.0
- Load is $5/4 > 1$, initial workload normalized to 1

$$
\begin{aligned}
Z_1(s) &= \frac{5}{4} + \frac{s}{16} - \frac{\sqrt{16 + 40s + s^2}}{16} \\
Z_2(s) &= 3\left(\frac{1}{Z_1(s)} - Z_1(s)\right)
\end{aligned}
$$

Trajectories, unstable case (ctd.)



Proportions of populations

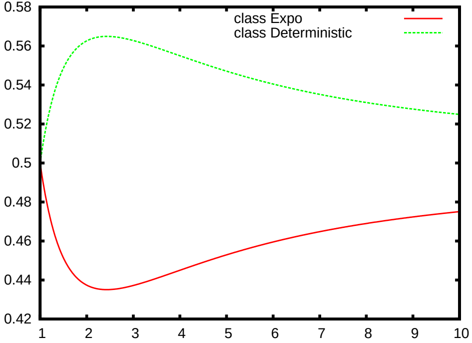
Principle of the experiment:

- In a “fair” unstable queue, proportions of customers in queue should be proportional to the arrival rate.
  This happens for FIFO.
- In a PS queue, there is a bias:
  - influence of the mean service, even within the same family of one-parameter distributions
  - influence of the distribution, within distributions with identical means.

Illustration: two classes of customers with identical arrival rates.

Proportions of populations

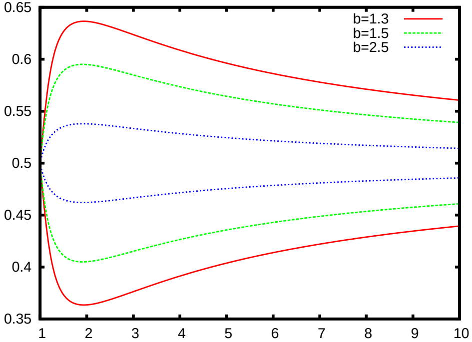
Example 2: exponential and deterministic distributions with same mean.



Proportion of customers of class 1 & 2 in queue, as a function of ρ.

Proportions of populations

Example 3: exponential and Pareto distributions with same mean.



Proportion of customers of class 1 (Exponential distribution, upper curves) & 2 (Pareto distribution, lower curves), as a function of ρ. Different Pareto shape parameters.

Questions?

Bibliography

References and surveys on the PS and DPS queues, and for the proofs

S. F. Yashkov and A. S. Yashkova, "Processor sharing: A survey of the mathematical theory", Automation and Remote Control, 68(9), pp. 1662–1731, 2007.

K. Avrachenkov, U. Ayesta, P. Brown, and R. Nuñez Queija, "Discriminatory processor sharing revisited", *Proc. INFOCOM'2005*, volume 2, pp. 784–795, 2005.

Altman, E., Avrachenkov, K. and Ayesta, U., "A survey on discriminatory processor sharing", *Queueing Syst. Theory Appl.* 53, 1-2, pp. 53-63, Jun. 2006.

A. Ben Tahar and A. Jean-Marie, "The fluid limit of the multiclass processor sharing queue", INRIA Research Report RR6867, April 2009. http://hal.inria.fr/inria-00368246_v2/.

Bibliography (ctd.)

Fluid limits and overloaded Processor Sharing

- Puha, A.L., Stolyar, A.L. and Williams R.J., "The Fluid Limit of an Overloaded Processor Sharing Queue", *Math. Ops. Res.*, Vol. 31, No. 2, pp. 316–350, 2006.
- Puha, A. L. and Williams, R. J., "Invariant states and rates of convergence for the fluid limit of a heavily loaded processor sharing queue", *Ann. Appl. Probab.*, Vol. 14, pp. 517–554, 2004.
- Gromoll, H. C., Puha, A. L. and Williams, R. J., "The fluid limit of a heavily loaded processor sharing queue", *Ann. Appl. Probab.*, 12, 797–859, 2002.
- Jean-Marie, A. and Robert, Ph., "On the transient behavior of the processor-sharing queue". *Queueing Systems, Theory and Applications*, 17:129-136, 1994.

Bibliography (ctd.)

Fluid limits, recent developments

H.C. Gromoll, Ph. Robert, B. Zwart, and R. Bakker. "The impact of reneging in processor sharing queues". *Sigmetrics Performance Evaluation Review*, 34(1):87-96, 2006.

J. Zhang, J.G. Dai and B. Zwart. "Limited Processor Sharing Queues". Georgia Tech, 2007.

Renewal theory

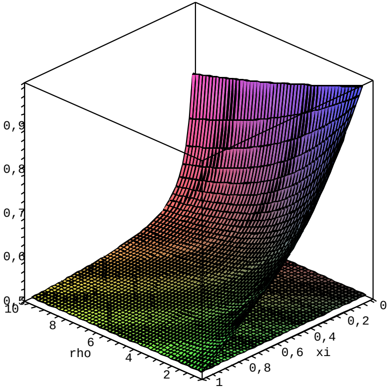
W. Feller, *An Introduction to Probability and Applications*, vol. 2.

K.B. Athreya and K. Rama Murthy, "Feller's Renewal Theorem for Systems of Renewal Equations", IISC, 1976.

Proportions of populations

Abdelghani Ben Tahar, Alain Jean-Marie

Example 1: exponential distributions with different means



Proportion of customers of class 1 in queue, as a function of ρ and $\xi = \mu_2/\mu_1$.