Solution of the "Explosive Percolation" Quest

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"Explosive percolation" transition is actually continuous,
Phys. Rev. Lett. **105**, 255701 (2010); arXiv:1009.2534

D. Achlioptas, R. M. D'Souza, and J. Spencer, Science **323**, 1453 (2009).

simulation: 512,000 nodes

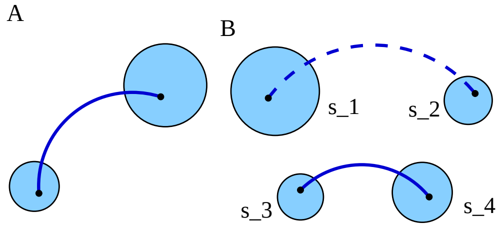
ordinary percolation:

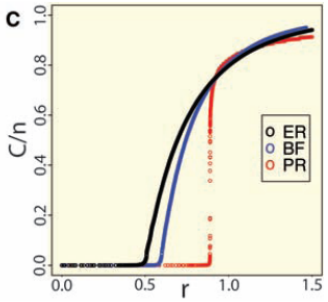
"explosive percolation":

$min(s_1 s_2, s_3 s_4)$

ordinary: —

explosive: —



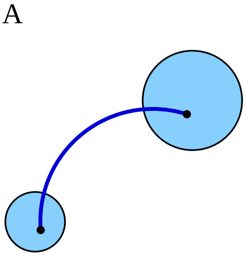


???

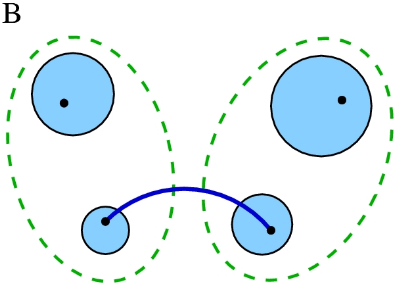
How can it be?:
discontinuity coexisting with critical power-law distributions and scaling above and below this transition - ???

Representative model of “explosive percolation”

ordinary percolation:

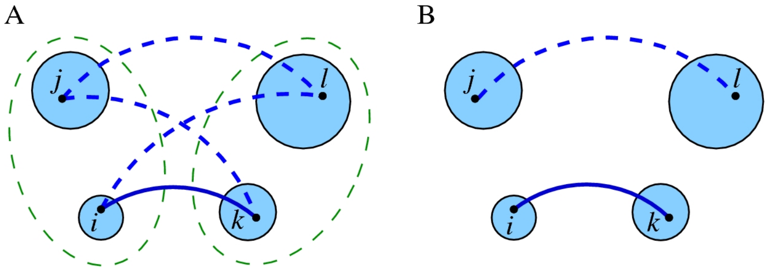


“explosive percolation” $(m = 2)$:



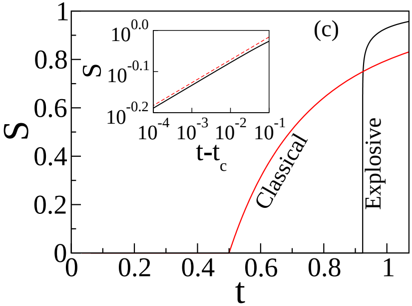
(arXiv:1009.2534)

Relation between models



(arXiv:1009.2534)

Simulations 2×10^9 nodes:



$$S \propto \delta^{\beta}$$

$\beta \approx 0.0476 \approx 1/18$, $\delta = t - t_c$

(arXiv:1009.2534)

Estimate

Suppose $\beta = 1/18$ and $N = 10^{18}$.
The smallest time interval is $1/N$.
Then a single step from the percolation threshold gives

$$S \sim (10^{-18})^{1/18} \sim 0.1$$

So simulations are virtually useless.
We must study the infinite system.

Distributions

$n(s)$ and $P(s)$ are for clusters

$$P(s) = sn(s)/\langle s \rangle, \qquad \sum_s P(s) = 1 - S$$

$Q(s)$ is for merging clusters, $\quad \sum_s Q(s) = 1 - S^2$

$$Q_{\text{cum}}(s) + S^2 = [P_{\text{cum}}(s) + S]^2$$

$$Q(s) = [P_{\text{cum}}(s) + P_{\text{cum}}(s+1) + 2S]P(s)$$
$$= [2 - 2P(1) - 2P(2) - \ldots - 2P(s{-}1) - P(s)]P(s)$$

(arXiv:1009.2534)

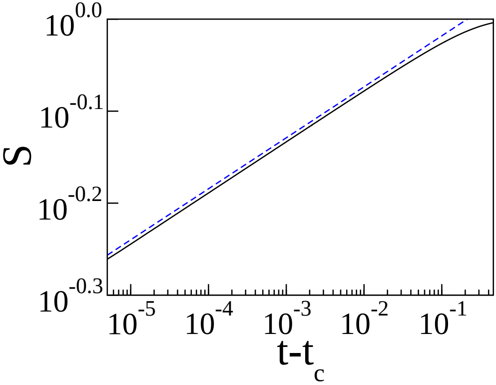
Equations

$$\frac{\partial P(s,t)}{\partial t} = s \sum_{u+v=s} Q(u,t)Q(v,t) - 2sQ(s,t)$$

exactly describe the evolution of the distributions in the full range of t for the infinite system.

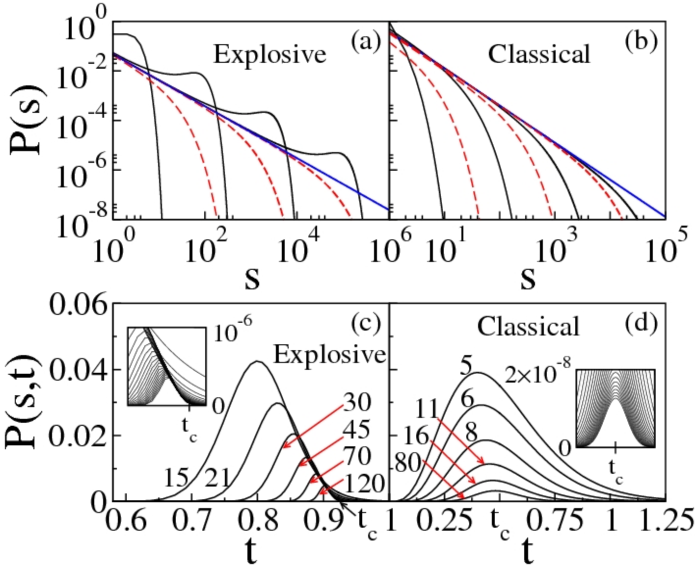
We solved numerically 10^6 equations, which gives precise description of the distributions for $s \leq 10^6$.

$s \leq 10^6$



Fitting by the law $S_0 + C\delta^\beta$ gives $S_0 < 0.005$.

$s \leq 10^6$



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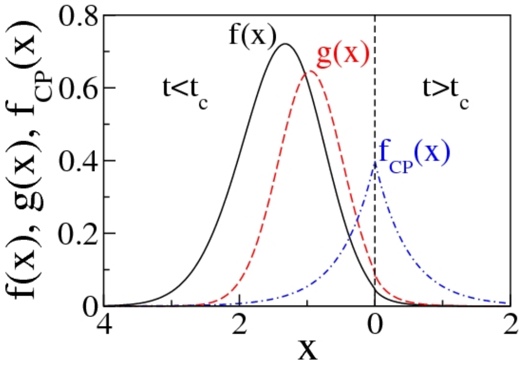
Scaling functions

$$P(s,t) = s^{1-\tau} f(s\delta^{1/\sigma})$$
$$Q(s,t) = s^{3-2\tau} g(s\delta^{1/\sigma})$$



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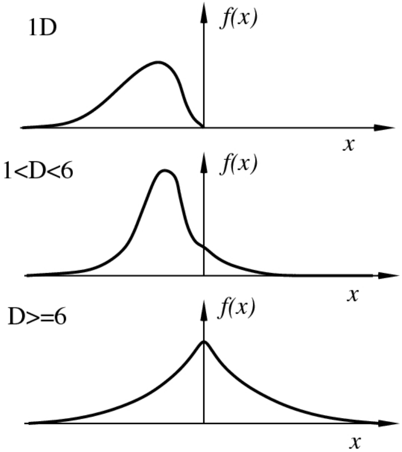
Ordinary percolation: scaling functions

$$P(s) = s^{1-\tau} f(s\delta^{1/\sigma})$$



(Nakanishi and Stanley, 1980)

Table: Percolation thresholds, critical exponents, and fractal and upper critical dimensions ($m = 2$)

	t_c	β	τ	σ	γ_P	γ_Q	d_f	d_u
Ordinary	1/2	1	5/2	1/2	1	—	4	6
Explosive	0.923207508(2)	0.0555(1)	2.04762(2)	0.857(3)	1.111(1)	1.0556(5)	2.333(1)	2.445(1)

Our more recent theory gave:

$m = 2 : \quad t_c = 0.92320750930(2), \quad \tau = 2.04763044(2)$

$$\langle s \rangle_P = \sum_s sP(s) \sim \delta^{-\gamma_P}, \qquad \langle s \rangle_Q = \sum_s sQ(s) \sim \delta^{-\gamma_Q}$$

$$\chi \sim \delta^{-\gamma}, \qquad \gamma = 1$$

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$$\tau = 1 + \beta/(1+3\beta), \qquad \sigma = 1/(1+3\beta),$$

$$\gamma_P = 1 + 2\beta, \qquad \gamma_Q = 1 + \beta, \qquad \gamma = 1.$$

if $d \geq d_u$, then always set $d \to d_u$, $\nu = 1/2$, $\eta = 0$, so

$$d_f = 2(1+3\beta), \qquad d_u = 2(1+4\beta)$$

$$t_c(\infty) - t_c(N) \sim N^{-2/d_u}$$

If the distributions are power-law at the critical point, then $S \sim \delta^{\beta}$.
Indeed, above t_c, we have

$$Q(s) \cong 2SP(s)$$

at large s.
So the equation for the asymptotics contains only $P(s)$ and $S(t)$. This equation is very similar to that for ordinary percolation.
Using $P(s, t_c) = f(0)s^{1-\tau}$ as an initial condition, we solve this equation and find critical exponents and scaling functions above t_c.

Relation between τ and t_c

$$P(s=1,t_c)\sum_s s^{1-\tau} \approx 1$$

$$P(s=1,t) = \frac{2}{1+e^{4t}}$$

$$\frac{2}{1+e^{4t_c}}\zeta(\tau-1) \approx 1$$

Substituting t_c gives $\tau - 2 \approx 0.05$.

(arXiv:1009.2534)

With increasing m, t_c approaches 1 and β rapidly decreases with m, but the transition remains continuous. E.g.,

$$m = 13: \quad \tau = 2.000000008489(2)$$

$$m = 17: \quad \tau = 2.00000000002776(2)$$

$$\beta \cong \tau - 2 \cong 1.05 e^{-1.43m}$$

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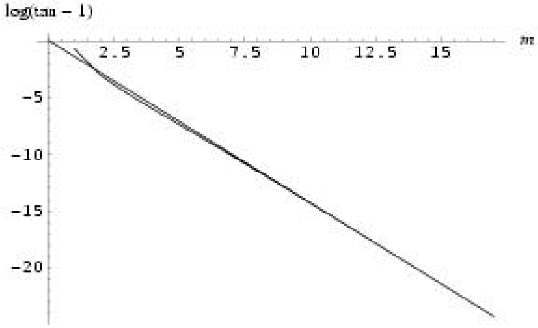
$\log(\tau - 2)$ vs. m



Question:

With so small β, how can I find experimentally that a phase transition is continuous?

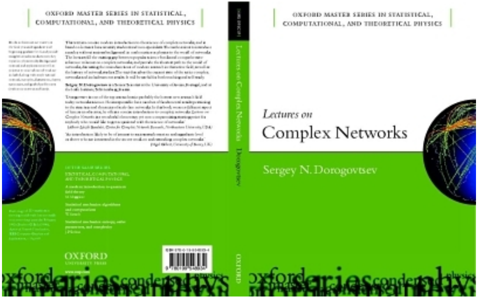
Answer:

Check if the susceptibility and the moments of cluster size distributions diverge.

Conclusion

There is no explosion in “explosive percolation”.

Lectures on
Complex Networks



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