Librairy overview - a first guided tour

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Inria informatics mathematics — ihp Institut Henri Poincaré — MICROSOFT RESEARCH INRIA JOINT CENTRE

Outline

http://coqfinitgroup.gforge.inria.fr/ssreflect-1.3/

1. Walk around in natural numbers area
2. About finite objects

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   - ssrnat
   - div
   - prime
   - binomial
2. About finite objects

ssrnat: Type and Arithmetic Operators

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- "+" (`addn`), "-" (`subn`), "*" (`muln`), "^" (`expn`)
- Convertible to `plus`, `minus`, and `mult`.
- Locked to prevent simplification.
  Tactic: "`unlock addn.`"

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  - addnI: $p + m = p + n \Rightarrow m = n$ injectivity
    addIn: $m + p = n + p \Rightarrow m = n$

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- Boolean functions: `leq: nat -> nat -> bool`.
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- Reflection with standard Coq:
  - `leP: le` m $n \Leftrightarrow$ `leq` m n `= true`
  - `ltP: lt` m $n \Leftrightarrow$ `leq` m`.+1` n `= true`

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  - leq_trans: $m \leq p \Rightarrow p \leq n \Rightarrow m \leq n$ transitivity
  - ltn_trans: $m < p \Rightarrow p < n \Rightarrow m < n$
  - leq_ltn_trans: $m \leq p \Rightarrow p < n \Rightarrow m < n$

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But actually rewriting rules!

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Divisibility for natural numbers

- Operators:
  - "%/" (divn) quotient
  - "%%" (modn) remainder
  - "%|" (dvdn) divisor predicate

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  - "%/" (divn) quotient
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- Some properties:
  - divn_eq: $m = (m \mathbin{\%/} d) * d + (m \mathbin{\%\%} d)$
  - dvdn_eq: $(d \mathbin{\%|} m) = ((m \mathbin{\%/} d) * d == m)$

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More definitions

- Definitions using divisibility:
  - gcdn — A function computing the gcd of 2 numbers
  - coprime — `Definition coprime m n := gcdn m n == 1.`

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- The chinese remainder theorem

```
Lemma chinese: forall x y,
  (x == y %[mod m1 * m2]) =
  (x == y %[mod m1]) && (x == y %[mod m2]).
```

prime

- prime p — p is a prime.
- primes m — the sorted list of prime divisors of $m > 1$, else [::].
- prime_decomp m — the list of prime factors of $m > 1$, sorted by primes.
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```
Lemma phi_coprime : forall m n,
coprime m n -> phi (m * n)= phi m * phi n.
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  Lemma bin_factd :
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  ```
- ```
  Lemma prime_dvd_bin : forall k p,
    prime p -> 0 < k < p -> p %| 'C(p, k).
  ```

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   - seq
   - fintype
   - tuple
   - finfun
   - finset

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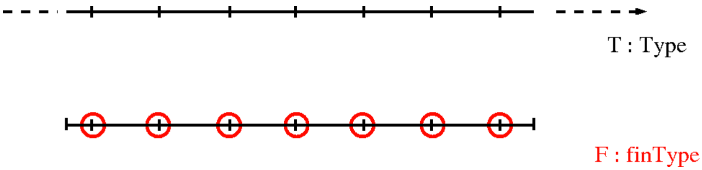
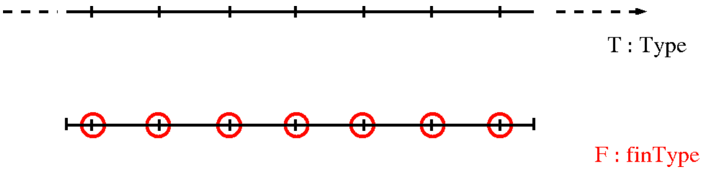
Always read the file header!

fintype



Types with finitely many elements,
supplying a duplicate-free sequence of all the elements.

fintype



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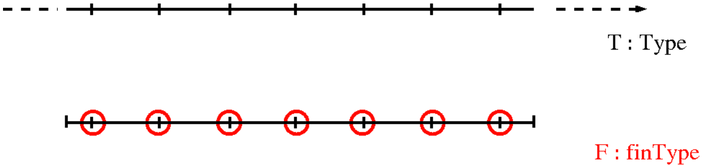

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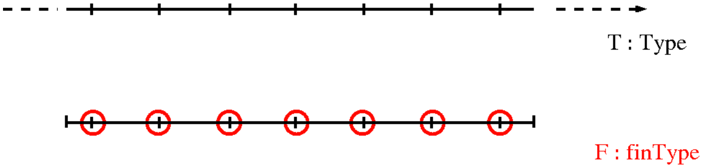

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- boolean version of quantifiers: forallb and existb
  with their reflection lemma forallP and existP

Ordinals

Fintype of natural numbers: "'I_n" is $\{k \mid k < n\}$.

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- `unsplit : 'I_m + 'I_n -> 'I_(m + n)`

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Lists with a fixed (known) length

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A central element for the definition of finite functions!

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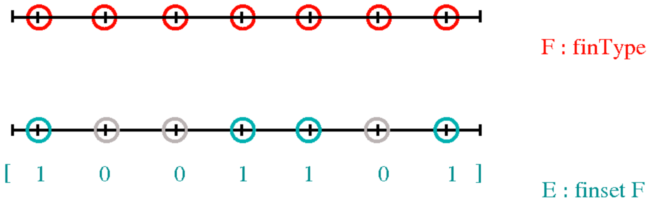
- `Lemma ffunE : forall f : aT -> rT, finfun f =1 f.`

finset

- Sets over a finite Type
- The finsets are finite functions with boolean values.



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- and a lot of lemmas
  (!! *naming conventions at the end of the file header* !!)