Examen Partiel 23/3/2011

Exercice 1

1) $A_i = \{$obtenir un "4" au tirage $i\}$

A_i et A_j $i \neq j$ sont indépendants

$P(\{\text{trois "4"}\}) = P(A_1 \cap A_2 \cap A_3) =$

$= P(A_1)\,P(A_2)\,P(A_3) = \boxed{\dfrac{1}{6^3}}$

2) $P(\{\text{au moins un "4"}\}) = 1 - P(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3})$

$= 1 - P(\overline{A_1})\,P(\overline{A_2})\,P(\overline{A_3}) = \boxed{\dfrac{91}{216}}$

3) $P(\{\text{deux "4"}\}) = P\big((A_1 \cap A_2 \cap \overline{A_3}) \cup$

$\cup\ (A_1 \cap \overline{A_2} \cap A_3) \cup (\overline{A_1} \cap A_2 \cap A_3)\big) =$

union de trois événements disjoints et équiprobables

$= 3\,P(A_1 \cap A_2 \cap \overline{A_3}) = 3 \cdot \dfrac{1}{6^2}\,\dfrac{5}{6} = \dfrac{15}{6^3} =$

$= \boxed{\dfrac{5}{72}}$

Exercice 2

R = { avoir une boule rouge }
B = { " " " blanche }
N = { " " " noire }.

1) tirage d'une boule

$$P(R|\bar{N}) = P(R|R\cup B) = \frac{P(R\cap(R\cup B))}{P(R\cup B)}$$

$$= \frac{P(R)}{P(R\cup B)} = \frac{P(R)}{P(R)+P(B)} = \frac{\frac{1}{5}}{\frac{1}{5}+\frac{1}{2}} = \boxed{\frac{2}{7}}$$

quand on tire une boule R, B et N sont disjoints.

2) N_i = { le i-ème tiré est noire }
R_i = { le i-ème tiré est rouge }

$$P(R|N_1) = P(R_1 \cup R_2 | N_1) =$$

disjoints

$$= P(R_1|N_1) + P(R_2|N_1) =$$

$$= 0 + \frac{2}{9} = \boxed{\frac{2}{9}}$$

deux rouges parmi 9 boules

3) $P(R \mid N) = \dfrac{P(R \cap N)}{P(N)} =$

$$= \frac{P\left((R_1 \cap N_2) \cup (R_2 \cap N_1)\right)}{1 - P(\bar{N})} =$$

$$= \frac{P(R_1 \cap N_2) + P(R_2 \cap N_1)}{1 - P(\bar{N}_1 \cap \bar{N}_2)} =$$

$$= \frac{2\,P(R_1 \cap N_2)}{1 - P(\bar{N}_1 \cap \bar{N}_2)} = \frac{2\,P(N_2 \mid R_1)\,P(R_1)}{1 - P(\bar{N}_2 \mid \bar{N}_1)\,P(\bar{N}_1)} =$$

$$= \frac{2 \cdot \frac{3}{9} \cdot \frac{2}{10}}{1 - \frac{6}{9} \cdot \frac{7}{10}} = \frac{12}{48} = \boxed{\frac{1}{4}}$$

Exercice 3

$M = \{\text{patient malade}\}$

$T_{1P} = \{\text{premier test donne indication positive}\}$

$P(M) = 0{,}5\%$

$P(T_{1P} \mid M) = 99\%$

$P(T_{1P} \mid \overline{M}) = 1\%$

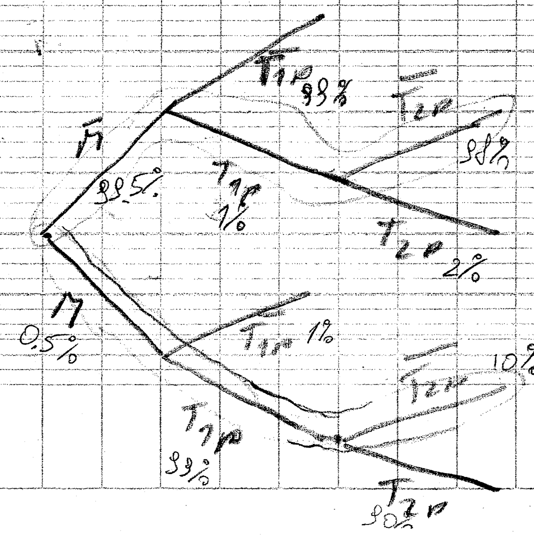
$P(M \mid T_{1P}) = ?$

$$P(M \mid T_{1P}) = \frac{P(M \cap T_{1P})}{P(T_{1P})} =$$

$$= \frac{P(T_{1P} \mid M)\, P(M)}{P(T_{1P} \mid M)\, P(M) + P(T_{1P} \mid \overline{M})\, P(\overline{M})} =$$

$$= \frac{0.99 \cdot 0{,}005}{0.99 \cdot 0.005 + 0{,}01 \cdot 0{,}995} \simeq 0{,}33$$

$T_{2P} = \{\text{deuxième test donne une indication positive}\}$



On est intéressé par les deux chemins en figure

$$P(T_{1P} \cap \overline{T_{2P}}) = P(M)\,P(T_{1P}|M)\,P(\overline{T_{2P}}|T_{1P}, M)$$
$$+ P(\overline{M})\,P(T_{1P}|\overline{M})\,P(\overline{T_{2P}}|T_{1P}, \overline{M})$$

$$= P(M)\,P(T_{1P}|M)\,P(\overline{T_{2P}}|M) +$$
$$+ P(\overline{M})\,P(T_{1P}|\overline{M})\,P(\overline{T_{2P}}|\overline{M}) =$$

$$= 0{,}005 \cdot 0{,}99 \cdot 0{,}1 + 0{,}995 \cdot 0{,}01 \cdot 0{,}98 =$$

$$\simeq \boxed{0{,}01024}$$

Exercice 4

Calculons les lois de probabilité marginales.

$$P_Y(y) = \begin{cases} \frac{1}{4} & y = 2 \\ \frac{3}{4} & y = 1 \end{cases}$$

$$P_X(x) = \begin{cases} \frac{2}{3} & x = 7 \\ \frac{1}{6} & x = 8 \\ \frac{1}{6} & x = 9 \end{cases}$$

On a $P_{XY}(x,y) = P_X(x)\,P_Y(y)$

donc X et Y sont indépendantes

$$E[Z] = E[X+Y] = E[X] + E[Y]$$

$$E[X] = \frac{2}{3} \cdot 7 + \frac{1}{6} \cdot 8 + \frac{1}{6} \cdot 9 = \frac{45}{6} = \frac{15}{2}$$

$$E[Y] = \frac{1}{4} \cdot 2 + \frac{3}{4} \cdot 1 = \frac{5}{4}$$

$$E[Z] = \frac{15}{2} + \frac{5}{4} = \frac{35}{4}$$

$$Var(Z) = Var(X) + Var(Y)$$

parce que X et Y sont indépendants.

$$Var(X) = \frac{2}{3}(7-7.5)^2 + \frac{1}{6}(8-7.5)^2 + \frac{1}{6}(9-7.5)^2 =$$

$$= \frac{1}{6} + \frac{1}{24} + \frac{9}{24} = \frac{14}{24} = \frac{7}{12}$$

$$Var(Y) = \frac{1}{4}\ \frac{3}{4} = \frac{3}{16} \qquad (Y \sim 1 + Bern(\tfrac{1}{4}))$$

$$Var(Z) = \frac{7}{12} + \frac{3}{16} = \frac{37}{48}$$

Exercice 5

X = # assistants à l'université

$X \sim Bin(10, \frac{1}{3})$

$$P(X > 7) = p_X(8) + p_X(9) + p_X(10) =$$

$$= \binom{10}{8}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^2 + 10\ \frac{1}{3^9}\ \frac{2}{3} + \frac{1}{3^{10}} =$$

$$= \frac{45 \cdot 4 + 10 \cdot 2 + 1}{3^{10}} = \frac{201}{3^{10}} = 0,342$$

Esercizio 6

$p_{NK}(n,k) = p_{N|K}(n|k)\, p_K(k)$

N \ K	1	2	3	4	$p_N(n)$
1	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{25}{48}$
2	0	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{13}{48}$
3	0	0	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{7}{48}$
4	0	0	0	$\frac{1}{16}$	$\frac{1}{16}$

1) $p_N(n) = \sum_k p_{NK}(n,k) =$ (see column $p_N(n)$)

2) $$p_{K|N}(k|n) = \frac{p_{KN}(k,n)}{p_N(n)}$$

$$p_{K|N}(k|n=2) = \frac{p_{KN}(k,2)}{p_N(2)} =$$

$$= \begin{cases} 0 & k=1 \\ \frac{6}{13} & k=2 \\ \frac{4}{13} & k=3 \\ \frac{3}{13} & k=4 \end{cases}$$

$$E[K|N=2] = \sum_k k\, p_{K|N}(k|N=2) =$$

$$= 1\cdot 0 + 2\cdot\frac{6}{13} + 3\cdot\frac{4}{13} + 4\cdot\frac{3}{13} = \frac{36}{13}$$

$$E[K^2 \mid N=2] = 1^2 \cdot 0 + 4 \frac{6}{13} + 9 \frac{4}{13} + 16 \cdot \frac{3}{13} =$$

$$= \frac{24 + 36 + 48}{13} = \frac{108}{13}$$

$$Var(K \mid N=2) = E[K^2 \mid N=2] - \left(E[K \mid N=2]\right)^2 =$$

$$= \frac{108}{13} - \left(\frac{36}{13}\right)^2 = \frac{108}{13^2}$$

3) $A = \{N=2 \cup N=3\}$

$$p_{K|A}(k) = \frac{P(\{K=k\} \cap A)}{P(A)} =$$

$$= \frac{P(\{K=k\} \cap \{N=2\}) + P(\{K=k\} \cap \{N=3\})}{P(\{N=2\}) + P(\{N=3\})} =$$

$$= \frac{p_{KN}(k,2) + p_{KN}(k,3)}{p_N(2) + p_N(3)} =$$

$$= \begin{cases} 0 & k=1 \\ \frac{3}{10} & k=2 \\ \frac{2}{5} & k=3 \\ \frac{3}{10} & k=4 \end{cases}$$

$$E[K \mid A] = \sum_k k\, p_{K|A}(k) = \frac{3}{10} \cdot 2 + \frac{2}{5} \cdot 3 + \frac{3}{10} \cdot 4 =$$

$$= \frac{6 + 12 + 12}{10} = 3$$

4) D = dépense $\quad p_i$ = prix du $i^{ème}$ livre acheté

$E[p_i] = 30$ euros

$$D = \sum_{i=1}^{N} p_i$$

$$E[D] = E_N[E_{D|N}[D]] =$$

$$= E[N]\,E[p_i]$$

$$E[N] = E_K[E_{N|K}[N]] =$$

$$= \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot\frac{3}{2} + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot\frac{5}{2} =$$

$$= \frac{7}{4}$$

$$\boxed{E[D] = \frac{7}{4}\cdot 30 = \frac{105}{2} = 52{,}5\ €}$$